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Diameters of the Commuting Graphs of Simple Lie Algebras

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Abstract. Let L be a Lie algebra with center $Z(L)$. The commuting graph $\Gamma(L)$ of L is a graph with vertex set $L \setminus Z(L)$, two distinct vertices x and y are adjacent if and only if x and y commute, i.e., $[x, y] = 0$. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic zero. In this paper, we study the diameter of $\Gamma(\mathfrak{g})$.
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1. Introduction

To date much research has concerned the isomorphisms between commuting graphs (see, e.g., [1, 2, 15]) and the determination of the diameters of commuting graphs over groups, semigroups, rings or associative algebras (see, e.g., [3-8 10-11, 14, 16-19]). Here we particularly mention the commuting graphs of rings or associative algebras. Let R be a non-commutative ring or an associative algebra and $Z(R)$ be its center. The commuting graph of R was defined in [4] to be the graph $\Gamma(R)$ whose vertex set is $R \setminus Z(R)$ and two distinct vertices x, y are joint by an edge whenever $xy = yx$, or equivalently $[x, y] = xy - yx = 0$. It was proved in [5] that if $n \geq 3$ and F is an algebraically closed field, then the diameter of $\Gamma(M_n(F))$, the commuting graph of the matrix algebra $M_n(F)$ of all $n \times n$ matrices over F, is always four and if F is not algebraically closed, then either the commuting graph is disconnected or the diameter is between four and six, and it was conjectured that the diameter of $\Gamma(M_n(F))$ is at most five. When $n = 2$, Remark 8 in [6] showed that the commuting graph of $M_n(F)$ is always disconnected. Although the commuting graphs over groups, semigroups, rings or associative algebras have attracted extensive attention, no paper dealt with the commuting graphs of Lie algebras until the present paper (as far as we know). In this paper we study the commuting graph of simple Lie algebras.

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Let L be a Lie algebra over a field F with bracket product $[*,*]$. The center of L is

$$Z(L) = \{x \in L : [x,y] = 0,\ \forall\, y \in L\}.$$

The maximal solvable subalgebras of L are called Borel subalgebras of L. If $[x,y] = 0$ we say that x commutes with y in L. The commuting graph of L, written as $\Gamma(L)$, is defined to be the graph with vertex set $L \setminus Z(L)$ and two distinct vertices x, y are joint by an edge whenever x, y commute, i.e., $[x,y] = 0$. Let R be an associative algebra over a field F. If we define $[x,y] = xy - yx$ for $x, y \in R$, then R becomes a Lie algebra, which we denote by $L(R)$. In this case, the commuting graph $\Gamma(R)$ of R is the same as the commuting graph $\Gamma(L(R))$ over the Lie algebra $L(R)$. In view of this point, the notion of commuting graphs of Lie algebras can be viewed as natural imitation of the notion of commuting graphs of associative algebras.

In a graph Γ, a path P is a sequence of distinct vertices $v_1 \sim v_2 \sim \cdots \sim v_{k+1}$ in which every two consecutive vertices are adjacent. The number k is called the length of P. A graph Γ is called connected if there exists a path between every two distinct vertices. For two vertices u and v in a graph Γ, the distance between u and v, denoted by $d(u,v)$, is the length of a shortest path between u and v. The diameter of a connected graph Γ is defined as

$$\mathrm{diam}(\Gamma) = \{\sup d(u,v) : u \neq v \in V(\Gamma)\}.$$

In this paper, we study the diameters of the commuting graphs of finite-dimensional simple Lie algebras $\mathfrak{g}$ of rank l over an algebraically closed field of characteristic zero.

Remark 1.1. If $l = 1$, then $\mathfrak{g}$ is the Lie algebra consisting of all 2×2 matrices of trace zero. In this case $\Gamma(\mathfrak{g})$ is not connected since all diagonal matrices in $\mathfrak{g}$ forms a connected component.

In the remainder we will always suppose that $l \geq 2$. The following remark can be easily derived from a result of [5] which says that if $n \geq 3$ and F is an algebraically closed field, then the diameter of $\Gamma(M_n(F))$, the commuting graph of the matrix algebra $M_n(F)$ of all $n \times n$ matrices over F, is always 4.

Remark 1.2. The diameter of $\Gamma(\mathfrak{g})$ is 4 if $\mathfrak{g}$ has type A_l with $l \geq 2$.

Proof. If $\mathfrak{g}$ has type A_l then $\mathfrak{g}$ is the Lie algebra consisting of all trace zero square matrices over F of order $l+1$. Let σ be the mapping from $M_{l+1}(F)$ to $\mathfrak{g}$ which sends any $X \in M_{l+1}(F)$ to $X - \frac{\mathrm{tr}(X)}{l+1} I$, where $\mathrm{tr}(X)$ denotes the trace of X and I denotes the identity matrix of order $l+1$. Clearly σ is an epimorphism from $M_{l+1}(F)$ to $\mathfrak{g}$, with kernel all scalar matrices in $M_{l+1}(F)$ (exactly is the the center of $M_{l+1}(F)$). Thus we have $\frac{M_{l+1}(F)}{Z(M_{l+1}(F))} \cong \mathfrak{g}$. It is easy to see that the diameter of $\frac{M_{l+1}(F)}{Z(M_{l+1}(F))}$ equals the diameter of $M_{l+1}(F)$, because a sequence $\overline{X_1} \sim \overline{X_2} \sim \cdots \sim \overline{X_m}$ is a longest path in $\Gamma(\frac{M_{l+1}(F)}{Z(M_{l+1}(F))})$ if and only if $X_1 \sim X_2 \sim \cdots \sim X_m$ is a longest path in $\Gamma(M_{l+1}(F))$, where

$\overline{X_i} = X_i + Z(M_{l+1}(F))$. Consequently, the diameter of $\Gamma(\mathfrak{g})$ is 4 since $\Gamma(M_{l+1}(F))$ has diameter 4. ■

Our main results are the following:

Theorem 1.3. *(i) The diameter of* $\Gamma(\mathfrak{g})$ *is either* 3 *or* 4 *if* $l > 2$ *and the type of* $\mathfrak{g}$ *is not* F_4.
(ii) If $\mathfrak{g}$ *is of type* F_4, *then the diameter of* $\Gamma(\mathfrak{g})$ *is between* 3 *and* 5.
(iii) If $\mathfrak{g}$ *is of type* C_2 *or* G_2, *then the diameter of* $\Gamma(\mathfrak{g})$ *is between* 3 *and* 6.

Theorem 1.4. *The diameter of* $\Gamma(\mathfrak{g})$ *is exactly* 4 *if* $\mathfrak{g}$ *has type* B_l *or* C_l *with* $l > 2$.

In Section 2, we introduce the notations concerning simple Lie algebras; In Section 3 and Section 4, we prove Theorem 1.3 and Theorem 1.4, respectively.

2. Notation concerning simple Lie algebras

Our notation concerning simple Lie algebras are mainly as in [9, 12]. Let F be an algebraically closed field of characteristic zero, $\mathfrak{g}$ an arbitrary finite-dimensional simple Lie algebra over F of rank l, $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$, $\Phi \subseteq \mathfrak{h}^*$ the corresponding root system of $\mathfrak{g}$, Δ a fixed base of Φ, Φ^+ (resp.,Φ^-) the set of positive (resp., negative) roots relative to Δ. The roots in Δ are called *simple* (see Page 58 of [12]).

A root β can be written as $\beta = \sum_{\pi\in\Delta} k_\pi \pi$ with $k_\pi \in \mathbb{Z}$, the integer $\sum_{\pi\in\Delta} k_\pi$ is called the *height* of β and is denoted by $\mathrm{ht}(\beta)$. Φ has a unique maximal root (relative to the height of roots), which we denote by θ_0. For the explicit expression of θ_0 as a linear combination of the simple roots in Δ, one can see Page 66, Table 2 in [12] for details. If two root lengths occur in Φ, then θ_0 is a long root ([12], 10.4, Lemma D) and Φ also has a unique maximal short root, which we denote by θ_s (see [12], Page 66, Table 2). For $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ be the root space of $\mathfrak{g}$ relative to α, and set $\mathfrak{n} = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$, $\mathfrak{n}_- = \sum_{\alpha\in\Phi^-} \mathfrak{g}_\alpha$. Let $\mathfrak{b}_0 = \mathfrak{h} \oplus \mathfrak{n}$, which is a standard Borel subalgebra of $\mathfrak{g}$. We denote by $\mathrm{Ker}\ \alpha$, for $\alpha \in \Phi$, the kernel of α in $\mathfrak{h}$. For each $\alpha \in \Phi^+$, let e_α be a non-zero element of $\mathfrak{g}_\alpha$, then there is a unique element $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $e_\alpha, e_{-\alpha}, h_\alpha = [e_\alpha, e_{-\alpha}]$ span a three-dimensional simple subalgebra of $\mathfrak{g}$ isomorphic to $sl(2, F)$ via

$$e_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The set $\{h_\pi, e_\beta, e_{-\beta} : \pi \in \Delta, \beta \in \Phi^+\}$ forms a basis of $\mathfrak{g}$. If $\alpha, \beta, \alpha+\beta \in \Phi$, then $[e_\alpha, e_\beta]$ is a nonzero scalar multiple of $e_{\alpha+\beta}$ since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. We define $N_{\alpha,\beta}$ by $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$. All $N_{\alpha,\beta}$ are called the *structure constants* of $\mathfrak{g}$. We can choose a basis $\{h_\pi, e_\beta, e_{-\beta} : \pi \in \Delta, \beta \in \Phi^+\}$ of $\mathfrak{g}$ such that all structure constants of $\mathfrak{g}$ are integers. A basis of $\mathfrak{g}$ chosen in such way is called a *Chevalley basis* of $\mathfrak{g}$. Throughout, the set

$$\{h_\pi, e_\beta, e_{-\beta} : \pi \in \Delta, \beta \in \Phi^+\}$$

will always denote a Chevalley basis of $\mathfrak{g}$. A symmetric bilinear form $(*,*)$ is defined on the $l-$dimensional real vector space spanned by Φ, which is dual to the Killing form on $\mathfrak{h}$. For $\alpha,\beta\in\Phi$, let $\langle\beta,\alpha\rangle = 2(\beta,\alpha)/(\alpha,\alpha)$. If $\alpha\neq\pm\beta$, let p,q be the greatest non-negative integers for which $\beta-p\alpha, \beta+q\alpha\in\Phi$, then

$$\langle\beta,\alpha\rangle = p-q; \quad \text{and if } \alpha+\beta\in\Phi, \text{ then } N_{\alpha,\beta}=\pm(p+1). \tag{1}$$

The *inner derivation* ad x of $\mathfrak{g}$ induced by $x\in\mathfrak{g}$ is defined by $y\mapsto[x,y]$ for any $y\in\mathfrak{g}$. For an ad-nilpotent element x in $\mathfrak{g}$, the map exp(ad x) is an automorphism of $\mathfrak{g}$. The group generated by all such automorphisms is called the inner automorphism group of $\mathfrak{g}$, which we denote by $\mathrm{Int}(\mathfrak{g})$, and each element in it is called an *inner automorphism* of $\mathfrak{g}$. In particular, for any $\alpha\in\Phi$ and any $t\in F$, te_α is ad-nilpotent in $\mathfrak{g}$, so the map exp(ad te_α), denoted by $\sigma_\alpha(t)$, belongs to $\mathrm{Int}(\mathfrak{g})$. We denote by G the subgroup of $\mathrm{Int}(\mathfrak{g})$ generated by the elements $\sigma_\alpha(t)$ for all $\alpha\in\Phi$, $t\in F$. It is well known that G coincides with $\mathrm{Int}(\mathfrak{g})$ (see [14], page 288). For $\alpha\in\Phi$, let X_α be the subgroup of G consisting of the elements $\sigma_\alpha(t)$ for all $t\in F$, which we call the *root subgroup* of G relative to α. Let $\langle X_\alpha, X_{-\alpha}\rangle$ be the subgroup of G generated by X_α and $X_{-\alpha}$. For $\alpha\in\Phi^+$, there exists a homomorphism ϕ_α from the special linear group $SL(2,F)$ onto $\langle X_\alpha, X_{-\alpha}\rangle$, sending $\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}$ to $\sigma_\alpha(a)$ and $\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}$ to $\sigma_{-\alpha}(b)$. Set

$$\chi_\alpha(c)=\phi_\alpha\begin{pmatrix}c & 0\\ 0 & c^{-1}\end{pmatrix} \text{ with } c\in F^*, \quad \omega_\alpha=\phi_\alpha\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Then

$$\omega_\alpha=\sigma_\alpha(1)\sigma_{-\alpha}(-1)\sigma_\alpha(1).$$

We define H to be the subgroup of G generated by the elements $\chi_\alpha(c)$ for all $\alpha\in\Phi$, $c\in F^*$, N the subgroup of G generated by H together with the elements ω_α for all $\alpha\in\Phi$. We denote by $\mathscr{W}$ the Weyl group of $\mathfrak{g}$ generated by the reflections w_α, $\alpha\in\Phi$. There exists a homomorphism from N onto $\mathscr{W}$ with kernel H (see [9], Theorem 7.2.2). Thus N/H is isomorphic to $\mathscr{W}$.

The following two well known results about Lie algebras will be applied in our proof.

Lemma 2.1. ([12], Page 84, Theorem 16.4) *The Borel subalgebras of an arbitrary Lie algebra L are conjugate under $\mathcal{E}(L)$, a subgroup of* $\mathrm{Int}(L)$.

Lemma 2.2. ([12], Page 53, Lemma C) *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over an algebraically closed field F of characteristic zero, Φ be its root system. Then at most two root lengths occur in Φ, and all roots of a given length are conjugate under $\mathscr{W}$.*

3. Bounds for diam($\Gamma(\mathfrak{g})$)

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over an algebraically closed field F of characteristic zero, Φ be its root system. The Dynkin diagrams of Φ are drawn in Figure 1 (see Page 58 of [12]).

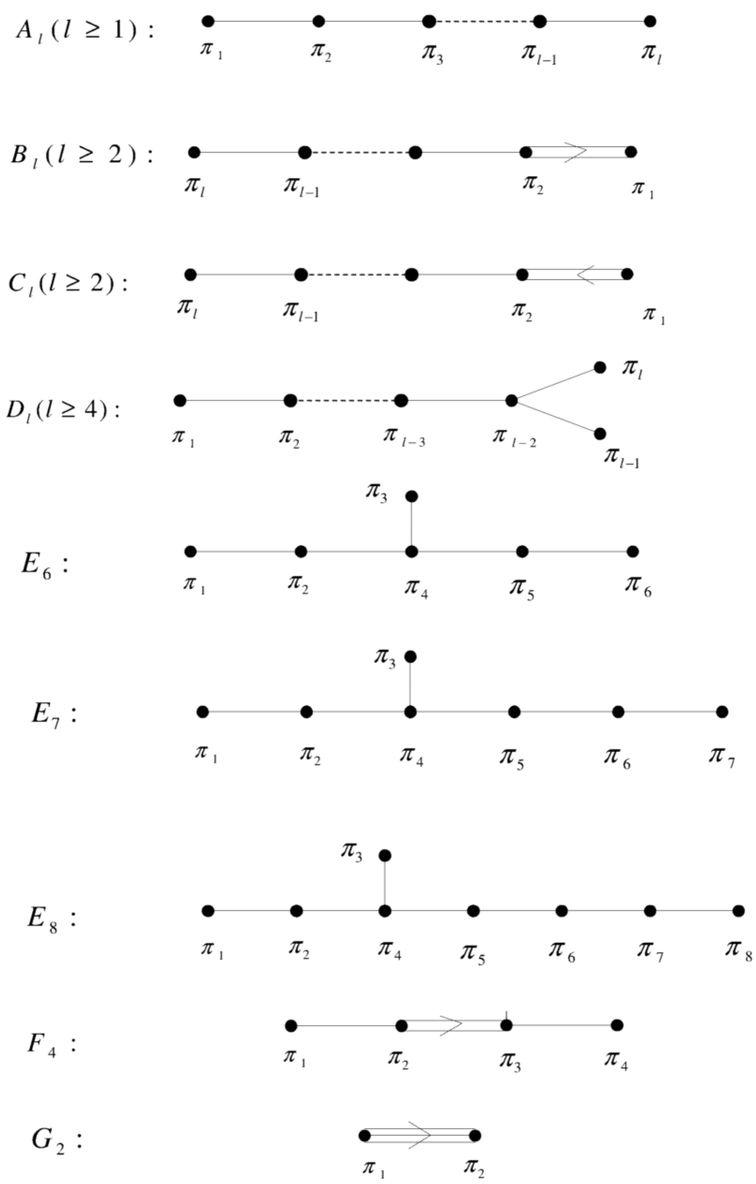

Figure 1: Dynkin diagrams of Φ

Recall that θ_s denotes the unique maximal short root in Φ with respect to the height of roots if two root lengths occur in Φ. According to Page 66, Table 2 of [12], we know that $\theta_s = \sum_{i=1}^{l} \pi_i$ if $\mathfrak{g}$ is of type B_l; $\theta_s = \pi_1 + 2\sum_{i=2}^{l-1} \pi_i + \pi_l$ if $\mathfrak{g}$ is of type C_l and $\theta_s = \pi_1 + 2\pi_2 + 3\pi_3 + 2\pi_4$ if $\mathfrak{g}$ is of type F_4. Set

$$\mathcal{C}(\theta_s) = \{\alpha \in \Phi : \alpha + \theta_s \notin \Phi \cup \{0\}\}.$$

Lemma 3.1. *If two root lengths occur in Φ and $l > 2$, then $|\mathcal{C}(\theta_s)| \geq |\Phi^+|$.*

Proof. Since $|\mathcal{C}(\theta_s)| = |\Phi^+ \cap \mathcal{C}(\theta_s)| + |\Phi^- \cap \mathcal{C}(\theta_s)|$ and $|\Phi^+| = |\Phi^+ \cap \mathcal{C}(\theta_s)| + |\Phi^+ \setminus \mathcal{C}(\theta_s)|$, it suffices to prove that $|\Phi^- \cap \mathcal{C}(\theta_s)| \geq |\Phi^+ \setminus \mathcal{C}(\theta_s)|$.

Case 1. $\mathfrak{g}$ is of type B_l.

In this case, $\theta_s = \sum_{i=1}^{l} \pi_i$ and $\Phi^+ \setminus \mathcal{C}(\theta_s) = \{\sum_{i=1}^{k} \pi_i : 1 \leq k \leq l-1\}$ has $l-1$ roots. Note that for distinct β, γ in $\Phi^+ \setminus \mathcal{C}(\theta_s)$, $-(\beta+\gamma)$ is a negative root satisfying $-(\beta+\gamma) \in \mathcal{C}(\theta_s)$. So $\Phi^- \cap \mathcal{C}(\theta_s)$ has at least $\frac{(l-1)(l-2)}{2}$ roots. If $l \geq 4$, then $\frac{(l-1)(l-2)}{2} \geq l-1$, and we are done. If $l = 3$, then $\Phi^- \cap \mathcal{C}(\theta_s) = \{-(\pi_2 + 2\pi_1), -\pi_2\}$, the assertion also holds.

Case 2. $\mathfrak{g}$ is of type C_l.

In this case, $\theta_s = \pi_1 + 2\sum_{i=2}^{l-1} \pi_i + \pi_l$ and $\Phi^+ \setminus \mathcal{C}(\theta_s) = \{\pi_l\}$ has only one root. Since $-\pi_1 \in \Phi^- \cap \mathcal{C}(\theta_s)$, the assertion holds true.

Case 3. $\mathfrak{g}$ is of type F_4.

In this case, $\theta_s = \pi_1 + 2\pi_2 + 3\pi_3 + 2\pi_4$ and $\Phi^+ \setminus \mathcal{C}(\theta_s) = \{\pi_3, \pi_2 + \pi_3, \pi_1 + \pi_2 + \pi_3\}$ has three roots. Since $\Phi^- \cap \mathcal{C}(\theta_s)$ contains the set $\{-(\beta+\gamma) : \beta \neq \gamma \in \Phi^+ \setminus \mathcal{C}(\theta_s)\}$, consisting of three roots, the assertion holds true. ∎

Lemma 2.1 does not hold for the case that $l = 2$. For example, if $\mathfrak{g}$ is of type C_2, then $\theta_s = \pi_2$, $\mathcal{C}(\theta_s) = \{\pi_2, \pi_1 + 2\pi_2, -\pi_1\}$, which has three roots, less than the number of roots in Φ^+. If $\mathfrak{g}$ is of type G_2, then $\theta_s = \pi_1 + 2\pi_2$, $\mathcal{C}(\theta_s) = \{\pi_1, \pi_1 + 2\pi_2, \pi_1 + 3\pi_2, 2\pi_1 + 3\pi_2, -\pi_1\}$, which has five roots, less than the number of roots in Φ^+.

For $x \in \mathfrak{g}$, we denote by x' the set of elements in $\mathfrak{g}$ that commutes x, i.e.,

$$x' = \{y \in \mathfrak{g} : [x, y] = 0\}.$$

Lemma 3.2. *(i)* Dim $(e_{\theta_0})' \geq |\Phi^+| + l - 1$.
(ii) If two root lengths occur in Φ *and* $l > 2$, *then* Dim $(e_{\theta_s})' \geq |\Phi^+| + l - 1$.

Proof. (i) follows immediately from the observation that Ker $\theta_0 + \mathfrak{n} \subseteq (e_{\theta_0})'$.

Let $x = h + \sum_{\alpha \in \Phi} b_\alpha e_\alpha \in \mathfrak{g}$ with $h \in \mathfrak{h}$. It is easy to see that $x \in (e_{\theta_s})'$ if and only if $b_\alpha = 0$ for each $\alpha \in \Phi \setminus \mathcal{C}(\theta_s)$ and $h \in$ Ker θ_s, from which it follows that $(e_{\theta_s})' =$ Ker $\theta_s + \sum_{\alpha \in \mathcal{C}(\theta_s)} \mathfrak{g}_\alpha$. By Lemma 3.1, we have Dim $(e_{\theta_s})' = l - 1 + |\mathcal{C}(\theta_s)| \geq |\Phi^+| + l - 1$. ∎

Let $\Delta = \{\pi_i : 1 \leq i \leq l\}$ be the base of Φ corresponding to Figure 1. Choose d_l from $\mathfrak{h}$ such that $\pi_i(d_l) = 0$ if $i \neq l$ and $\pi_l(d_l) = 1$. Let Φ_1 be the subset of Φ consisting of the roots which are the combination of the simple roots in $\Delta \setminus \{\pi_l\}$. Thus, Φ_1 forms a new root system with rank $l-1$. The type of Φ_1 are listed in Figure 2.

Now we consider the dimension of $(d_l)'$.

Lemma 3.3. *(i)* Dim $(d_l)' = l + |\Phi_1|$.
(ii) If $l > 2$, *then* Dim $(d_l)' > \frac{l}{2} + |\Phi^+|$ *except for the case when* $\mathfrak{g}$ *has type* F_4.

Proof. Let $x = h + \sum_{\beta \in \Phi} b_\beta e_\beta$ be an element of $\mathfrak{g}$ with $h \in \mathfrak{h}$ and $b_\beta \in F$. Clearly, $[x, d_l] = 0$ if and only if $b_\beta = 0$ for $\beta \notin \Phi_1$. Hence, $(d_l)' = \mathfrak{h} + \sum_{\alpha \in \Phi_1} \mathfrak{g}_\alpha$, from which it follows that Dim $(d_l)' = l + |\Phi_1|$. By the tables listed in Figure 2, we find that $|\Phi_1| > |\Phi^+| - \frac{l}{2}$ except for the case when Φ is of type F_4. Hence, if Φ is not of type F_4, then Dim $(d_l)' > \frac{l}{2} + |\Phi^+|$. ∎

Type	A_l	B_lC_l	D_l	E_6	E_7	E_8	F_4	G_2
Number of roots	$l(l+1)$	$2l^2$	$2(l^2-l)$	72	126	240	48	12

Φ	A_l	B_l	C_l	D_l	E_6	E_7	E_8	F_4	G_2
Φ_1	A_{l-1}	B_{l-1}	C_{l-1}	A_{l-1}	D_5	E_6	E_7	B_3	A_1

Figure 2: Number of roots and types of Φ_1

Note that when $l = 2$ or Φ has type F_4, Lemma 3.3 fails to hold. For example, if Φ has type F_4 then Φ_1 has type B_3, in this case Dim $(d_l)' = 4+18 < 24 = |\Phi^+|$.

Define G^+ to be the subgroup of Int$(\mathfrak{g})$ generated by all $\sigma_\alpha(t)$ for $\alpha \in \Phi^+$, $t \in F$.

For a given $x \in \mathfrak{b}_0$, we can write x as $x = h + n$, where $h \in \mathfrak{h}$ and $n \in \mathfrak{n}$. Further, we write n as $n = \sum_{\alpha\in\Phi^+} b_\alpha e_\alpha$. Define

$$\Omega_x = \{\alpha \in \Phi^+ \mid b_\alpha \cdot \alpha(h) \neq 0\};$$

$$\Psi_x = \{\alpha \in \Phi^+ \mid b_\alpha \cdot \alpha(h) = 0\}.$$

Then $\Omega_x \cup \Psi_x = \Phi^+$ and $\Omega_x \cap \Psi_x = \emptyset$. If $\Omega_x \neq \emptyset$, i.e., $[h, n] \neq 0$, we call the minimal height of the roots in Ω_x the *Jordan degree* of x, and we denote it by D_x. Otherwise, if $\Omega_x = \emptyset$, i.e., $[h, n] = 0$, we set the *Jordan degree* of x to be $\mathrm{ht}(\theta_0) + 1$.

Lemma 3.4. *(i) For an element x in $\mathfrak{b}_0$, there exists some $\sigma \in G^+$, such that the Jordan degree of $\sigma(x)$ is $\mathrm{ht}(\theta_0)+1$, i.e., $\Omega_{\sigma(x)} = \emptyset$.*
(ii) For any element x in $\mathfrak{g}$, there exists $\sigma \in$ Int$(\mathfrak{g})$ such that $\sigma(x) = h + n$, where $h \in \mathfrak{h}$, $n \in \mathfrak{n}$ satisfy $[h, n] = 0$.

Proof. Let $G^+(x)$ denote the set of $\sigma(x)$ for all $\sigma \in G^+$. We now use decreasing induction on the Jordan degree of the elements in $G^+(x)$ to prove the lemma. If D_x takes the maximal value $\mathrm{ht}(\theta_0)+1$, then we choose σ to be the identity element in G^+, and $\sigma(x)$ is as desired. Suppose that $k := D_x$, is strictly smaller than $\mathrm{ht}(\theta_0)+1$, and assume that $x = h+n$ with $h \in \mathfrak{h}$, $n \in \mathfrak{n}$. Since $D_x = k$, we can express n in the form

$$n = \sum_{\beta\in\Psi_x} b_\beta e_\beta + \sum_{\mathrm{ht}(\alpha)=k,\alpha\in\Omega_x} b_\alpha e_\alpha + \sum_{\mathrm{ht}(\alpha)\geq k+1,\alpha\in\Omega_x} b_\alpha e_\alpha.$$

Choose $\sigma_1 = \prod_{\mathrm{ht}(\alpha)=k,\alpha\in\Omega_x} \sigma_\alpha(b_\alpha \cdot \alpha(h)^{-1}) \in G^+$, where the product is taken according to any fixed order. Considering the action of σ_1 on x, we have

$$\sigma_1(x) = \sigma_1(h) + \sigma_1(\sum_{\beta\in\Psi_x} b_\beta e_\beta) + \sigma_1(\sum_{\mathrm{ht}(\alpha)=k,\alpha\in\Omega_x} b_\alpha e_\alpha) + \sigma_1(\sum_{\mathrm{ht}(\alpha)\geq k+1,\alpha\in\Omega_x} b_\alpha e_\alpha). \quad (2)$$

For a positive integer i, we denote $\sum_{\mathrm{ht}(\alpha)\geq i}\mathfrak{g}_\alpha$ by $\mathfrak{n}_i^+$. It is easy to see that

$$\sigma_1(h) \equiv h - \sum_{\mathrm{ht}(\alpha)=k,\alpha\in\Omega_x} b_\alpha e_\alpha \pmod{\mathfrak{n}_{k+1}^+}, \tag{3}$$

$$\sigma_1(\sum_{\beta\in\Psi_x} b_\beta e_\beta) \equiv \sum_{\beta\in\Psi_x} b_\beta e_\beta \pmod{\mathfrak{n}_{k+1}^+}. \tag{4}$$

$$\sigma_1(\sum_{\mathrm{ht}(\alpha)=k,\alpha\in\Omega_x} b_\alpha e_\alpha) \equiv \sum_{\mathrm{ht}(\alpha)=k,\alpha\in\Omega_x} b_\alpha e_\alpha \pmod{\mathfrak{n}_{k+1}^+}, \tag{5}$$

$$\sigma_1(\sum_{\mathrm{ht}(\alpha)\geq k+1,\alpha\in\Omega_x} b_\alpha e_\alpha) \in \mathfrak{n}_{k+1}^+. \tag{6}$$

Substituting (3)-(6) into (2) we have that

$$\sigma_1(x) \equiv h + \sum_{\beta\in\Psi_x} b_\beta e_\beta \pmod{\mathfrak{n}_{k+1}^+}. \tag{7}$$

From (7) we find that $D_{\sigma_1(x)} > D_x$. Then we conclude, by induction, that there exists $\sigma_2 \in G^+$ such that the Jordan degree of $\sigma_2\sigma_1(x)$ is $\mathrm{ht}(\theta_0)+1$. In other words, $\Omega_{\sigma_2\sigma_1(x)} = \emptyset$. Taking σ to be $\sigma_2\sigma_1$ we complete the proof of (i).

If $x \in \mathfrak{g}$, then x is contained in a Borel subalgebra of $\mathfrak{g}$. By Lemma 2.1 there exists $\sigma_0 \in \mathrm{Int}(\mathfrak{g})$ such that $\sigma_0(x) \in \mathfrak{b}$. By (i) of this lemma we can find $\sigma_1 \in G^+ \leq \mathrm{Int}(\mathfrak{g})$ such that $\Omega_{\sigma_1\sigma_0(x)} = \emptyset$. In other words, if we assume that $\sigma_1\sigma_0(x) = h+n$ with $h \in \mathfrak{h}$, $n \in \mathfrak{n}$, then $[h,n]=0$. ■

Proof of Theorem 1.3

Let $\{d_\pi | \pi \in \Delta\}$ be the dual basis of Δ, i.e., $d_\pi \in h$ such that $\pi(d_\pi) = 1$ and $\pi'(d_\pi) = 0$ if $\pi', \pi \in \Delta$ are distinct. Choose a nonzero element h from $\mathrm{Ker}(\theta_0)$. Then there exists $\pi \in \Delta$ such that $\pi(h) \neq 0$ (if $\pi(h) = 0$ for all $\pi \in \Delta$, then $h = 0$). Since

$$[\sum_{\pi\in\Delta} d_\pi, e_{\theta_0}] = \mathrm{ht}(\theta_0)e_{\theta_0} \neq 0,$$

$$[\sum_{\pi\in\Delta} d_\pi, \sum_{\pi\in\Delta} e_\pi] = \sum_{\pi\in\Delta} e_\pi \neq 0,$$

$$[h, \sum_{\pi\in\Delta} e_\pi] = \sum_{\pi\in\Delta} \pi(h)e_\pi \neq 0,$$

the length of the path $\sum_{\pi\in\Delta} d_\pi \sim h \sim e_{\theta_0} \sim \sum_{\pi\in\Delta} e_\pi$ is 3. So the distance between $\sum_{\pi\in\Delta} d_\pi$ and $\sum_{\pi\in\Delta} e_\pi$ is at most 3. If $x = h + \sum_{\alpha\in\Phi} c_\alpha e_\alpha \in \mathfrak{g}$ with $h \in \mathfrak{h}$ satisfies $d \sim x \sim x_0$, where $d = \sum_{\pi\in\Delta} d_\pi$ and $x_0 = \sum_{\pi\in\Delta} e_\pi$, we need to prove $x = 0$. Indeed, from $[d,x] = 0$ it follows that $\sum_{\alpha\in\Phi} \alpha(d)c_\alpha e_\alpha = 0$ and thus $\sum_{\alpha\in\Phi} c_\alpha e_\alpha = 0$ (note that $\alpha(d) = \mathrm{ht}(\alpha) \neq 0$ for any $\alpha \in \Phi$), $x = h \in \mathfrak{h}$. Further, it follows from $[x, x_0] = 0$ that $\sum_{\pi\in\Delta} \pi(h)e_\pi = 0$ and $\pi(h) = 0$ for any $\pi \in \Delta$. Hence $x = h = 0$. So there is no vertex in $\Gamma(\mathfrak{g})$ which is adjacent to both d and x_0, and thus the distance between $\sum_{\pi\in\Delta} d_\pi$ and $\sum_{\pi\in\Delta} e_\pi$ is not 2. Thus the distance between $\sum_{\pi\in\Delta} d_\pi$ and $\sum_{\pi\in\Delta} e_\pi$ is 3, and the diameter of $\mathfrak{g}$ is at least 3.

Let $x, y \in \mathfrak{g}$ be distinct vertices in $\Gamma(\mathfrak{g})$. By Lemma 3.4, there exists $\sigma \in \mathrm{Int}(\mathfrak{g})$ such that $\sigma(x) = h + n$, where $h \in \mathfrak{h}$ commutes $n \in \mathfrak{n}$.

(i) Suppose $\mathfrak{g}$ is not of type F_4 and $l > 2$. In this case, it suffices to find a path of length at most 4 between x and y. Firstly, we claim that there is $u_0 \in \mathfrak{g}$ such that $[x, u_0] = 0$ and Dim $(u_0)' > \frac{l}{2} + |\Phi^+|$. To complete the proof of the claim, two different cases are taken into consideration.

Case 1 There is $\alpha \in \Phi^+$ such that $\alpha(h) = 0$. Suppose that $\gamma \in \Phi^+$ is a maximal root (with a maximal height) such that $\gamma(h) = 0$. It is not difficult to prove that $[\sigma(x), e_\gamma] = 0$. Indeed, if $n = 0$, the assertion is obvious; otherwise, assume that $n = \sum_{\alpha \in \Psi_{\sigma(x)}} b_\alpha e_\alpha$, where $b_\alpha \neq 0$ for at least one $\alpha \in \Psi_{\sigma(x)}$ (here, $\Psi_{\sigma(x)}$ is defined as in Lemma 3.4). By the choice of γ, we have $\gamma + \alpha \notin \Phi^+$ for any $\alpha \in \Psi_{\sigma(x)}$ with $b_\alpha \neq 0$ (otherwise, $\gamma + \alpha$ is a root which has larger height than that of γ and $(\gamma + \alpha)(h) = 0$). Hence $[n, e_\gamma] = 0$, from which it follows that $[\sigma(x), e_\gamma] = 0$. By Lemma 2.2, there is certain $w \in \mathscr{W}$ such that $w(\gamma) = \theta$, where $\theta = \theta_0$ if γ is long, or $\theta = \theta_s$ if γ is short. Let $\omega \in N$ be preimage of w under the epimorphism from N to $\mathscr{W}$, then $\omega(e_\gamma) = e_\theta$. Let $u_0 = \sigma^{-1}\omega^{-1}(e_\theta)$. Then x commutes u_0. It follows from $(u_0)' = \sigma^{-1}\omega^{-1}(e'_\theta)$ that Dim $(u_0)'$=Dim $(e_\theta)'$. Further, we have Dim $(u_0)' \geq l - 1 + |\Phi^+| > \frac{l}{2} + |\Phi^+|$ (using Lemma 3.2), which proves the claim.

Case 2 $\alpha(h) \neq 0$ for all $\alpha \in \Phi^+$. In this case, $n = 0$ and $\sigma(x) = h$. Clearly, $[\sigma(x), d_l] = 0$. Choose u_0 to be $\sigma^{-1}(d_l)$, then $[x, u_0] = 0$ and Dim $(u_0)' > \frac{l}{2} + |\Phi^+|$ (using Lemma 3.3).

The claim is proved.

Similarly, for the vertex y, there is $v_0 \in \mathfrak{g}$ such that $[y, v_0] = 0$ and Dim $(v_0)' > \frac{l}{2} + |\Phi^+|$. Since

$$\mathrm{Dim}\ (u_0)' + \mathrm{Dim}\ (v_0)' > l + |\Phi| = \mathrm{Dim}\ \mathfrak{g},$$

we have $(u_0)' \cap (v_0)' \neq \{0\}$. Let $0 \neq z \in (u_0)' \cap (v_0)'$. Then we obtain

$$[x, u_0] = [u_0, z] = [z, v_0] = [v_0, y] = 0,$$

from which one obtains a path $x \sim u_0 \sim z \sim v_0 \sim y$ of length at most 4 between x and y.

(ii) The type of $\mathfrak{g}$ is F_4. In this case, it suffices to find a path of length at most 5 between x and y. Firstly, we claim that there are $u, s \in \mathfrak{g}$ such that $[x, u] = [u, s] = 0$ and Dim $s' = 36$. If there is an $\alpha \in \Phi^+$ such that $\alpha(h) = 0$, let $\gamma \in \Phi^+$ be a maximal root such that $\gamma(h) = 0$, then $[\sigma(x), e_\gamma] = 0$, in this case, let $u = \sigma^{-1}(e_\gamma)$ and $s = \sigma^{-1}(e_{\theta_0})$, then we have $[x, u] = [u, s] = 0$. Since $\{\beta \in \Phi^- : \beta + \theta_0 \notin \Phi \cup \{0\}\}$ has 9 roots,

$$\mathrm{Dim}\ s' = \mathrm{Dim}\ (e_{\theta_0})' = |\Phi^+| + (l - 1) + 9 = 36.$$

If $\alpha(h) \neq 0$ for all $\alpha \in \Phi^+$, then $\sigma(x) = h$ and so $[\sigma(x), d_l] = 0$. We also have $[d_l, e_{\pi_1}] = 0$. Since π_1 is a long root, there exists an $\omega \in N$ such that $\omega(e_{\pi_1}) = e_{\theta_0}$ (by Lemma 2.2). Let $u = \sigma^{-1}(d_l)$, $s = \sigma^{-1}\omega^{-1}(e_{\theta_0})$, then $[x, u] = [u, s] = 0$ and Dim $s' = 36$. The claim is proved.

For the vertex y, there exists $\sigma_1 \in \mathrm{Int}(\mathfrak{g})$ such that $\sigma_1(y) = h_1 + n_1$, where $h_1 \in \mathfrak{h}$ commutes $n_1 \in \mathfrak{n}$. We claim that there is $v \in \mathfrak{g}$ such that $[v, y] = 0$ and Dim $v' \geq 22$. If there is certain $\alpha \in \Phi^+$ such that $\alpha(h_1) = 0$, suppose that $\gamma_1 \in \Phi^+$ is a maximal root such that $\gamma_1(h_1) = 0$. Then $[\sigma_1(y), e_{\gamma_1}] = 0$. Suppose $\omega_1 \in N$ such that $\omega_1(e_{\gamma_1}) = e_\theta$, where $\theta = \theta_0$ or $\theta = \theta_s$. Let $v = \sigma_1^{-1}\omega_1^{-1}(e_\theta)$. Then $[y, v] = 0$ and Dim $v' =$ Dim $(e_\theta)' \geq l - 1 + |\Phi^+| = 27 \geq 22$. If $\alpha(h_1) \neq 0$ for any $\alpha \in \Phi^+$, then $[\sigma_1(y), d_l] = 0$. In this case, let $v = \sigma_1^{-1}(d_l)$, then $[y, v] = 0$ and Dim $v' =$ Dim $(d_l)' = l + |\Phi_1| = 4 + 18 = 22$.

Since Dim s' + Dim $v' \geq 36 + 22 >$ Dim $\mathfrak{g}$, we have $s' \cap v' \neq \{0\}$. Let $0 \neq z \in s' \cap v'$. Then we obtain $[x, u] = [u, s] = [s, z] = [z, v] = [v, y] = 0$, which implies that there is a path $x \sim u \sim s \sim z \sim v \sim y$ of length at most 5 between x and y. The proof of (ii) is completed.

(iii) The type of $\mathfrak{g}$ is C_2 or G_2. It suffices to find a path of length at most 6 between x and y. If $n = 0$, set $u = \sigma^{-1}(\mathrm{Ker}(\theta_0))$ and $v = \sigma^{-1}(e_{\theta_0})$, then $x \sim u \sim v$ and Dim $v' =$ Dim $(e_{\theta_0})' = |\Phi^+| + 2$. If $n \neq 0$, then there is certain $\alpha \in \Phi^+$ such that $\alpha(h) = 0$, suppose that $\beta \in \Phi^+$ is a maximal root such that $\beta(h) = 0$, then $[\sigma(x), e_\beta] = 0$ and $[e_\beta, e_{\theta_0}] = 0$. In this case, let $u = \sigma^{-1}(e_\beta), v = \sigma^{-1}(e_{\theta_0})$, also we have $x \sim u \sim v$, where Dim $v' = |\Phi^+| + 2$. A similar discussion shows that there are $z, w \in \mathfrak{g}$ such that $w \sim z \sim y$ and Dim $w' = |\Phi^+| + 2$. Since Dim v' + Dim $w' = 2|\Phi^+| + 4 >$ Dim $\mathfrak{g}$, there exists a nonzero element, say $q \in \mathfrak{g}$, such that $q \in v' \cap w'$. Thus we have a path $x \sim u \sim v \sim q \sim w \sim z \sim y$ of length at most 6 between x, y, which completes the proof of (iii). □

4. Diameter of the commuting graph of simple Lie algebra of type A_l, B_l or C_l

In this section we give a unified method to determine the diameter of the commuting graph of simple Lie algebra of type A_l, B_l or C_l. Indeed, the diameter of $\Gamma(\mathfrak{g})$ has been determined in Remark 1.1 and 1.2 if $\mathfrak{g}$ has type A_l.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over an algebraically closed field F of characteristic zero, Φ be the root system of $\mathfrak{g}$ with Δ a base of Φ. For a nonzero integer $k \in [-\mathrm{ht}(\theta_0), \mathrm{ht}(\theta_0)]$, denote by Φ_k the set of roots of height k. Based on Φ_k, we define a graph Γ_k (possibly with loops) with Φ_k as its vertex set, and there is an edge between distinct $\alpha, \beta \in \Phi_k$ if and only if there exist $\delta_1, \delta_2 \in \Delta$ such that $\alpha + \delta_1 = \beta + \delta_2 \in \Phi$ and $(\alpha + \delta_1) - \pi \notin \Phi$ for any $\pi \in \Delta \setminus \{\delta_1, \delta_2\}$. Note that there is a loop at a vertex $\alpha \in \Phi_k$ if and only if there is certain $\pi \in \Delta$ such that $\alpha + \pi \in \Phi$ and $(\alpha + \pi) - \delta \notin \Phi$ for any $\delta \in \Delta$ different from π.

Now we provide a lemma showing what happens for the coefficients of an element $x \in \mathfrak{g}$ at those e_α with $\alpha \in \Phi_k$, if x commutes $x_0 = \sum_{\pi \in \Delta} e_\pi$ and Γ_k is connected.

Lemma 4.1. *Let $x = h + \sum_{\alpha \in \Phi} b_\alpha e_\alpha \in \mathfrak{g}$ with $h \in \mathfrak{h}$, $b_\alpha \in F$, and let $x_0 = \sum_{\pi \in \Delta} e_\pi$. If $[x, x_0] = 0$ and Γ_k is connected for some integer $k \in [-\mathrm{ht}(\theta_0), -2] \cup [1, \mathrm{ht}(\theta_0)]$, then either $b_\alpha = 0$ for all $\alpha \in \Phi_k$ or $b_\alpha \neq 0$ for all $\alpha \in \Phi_k$. Furthermore, if there is a loop at a vertex of Γ_k, then $b_\alpha = 0$ for all $\alpha \in \Phi_k$.*

Proof. If $b_\alpha = 0$ for all $\alpha \in \Phi_k$, then the result holds. Otherwise, suppose $b_\beta \neq 0$ for certain $\beta \in \Phi_k$, we need to prove that $b_\alpha \neq 0$ for any other $\alpha \in \Phi_k$. Since Γ_k is connected, there is a path, say $\beta \sim \alpha_1 \sim \alpha_2 \sim \cdots \sim \alpha_s \sim \alpha$, between α and β, where $\alpha_i \in \Phi_k$. Since β and α_1 are adjacent, we have $\beta+\delta_1 = \alpha_1+\delta_2 \in \Phi$ for certain $\delta_1, \delta_2 \in \Delta$ and $\gamma + \delta \neq \beta + \delta_1$ for any $\gamma \in \Phi_k \setminus \{\beta, \alpha_1\}$ and for any $\delta \in \Delta$. Express $[x, x_0]$ as the linear combinations of the Chevalley basis, it is clear that the coefficient at $e_{\beta+\delta_1}$ is $N_{\beta,\delta_1} b_\beta + N_{\alpha_1,\delta_2} b_{\alpha_1}$. From $[x, x_0] = 0$ it follows that

$$N_{\beta,\delta_1} b_\beta + N_{\alpha_1,\delta_2} b_{\alpha_1} = 0.$$

Since $b_\beta \neq 0$, we have $b_{\alpha_1} \neq 0$. Similarly, from $b_{\alpha_1} \neq 0$ we have $b_{\alpha_2} \neq 0$. After a finite number of steps, we have $b_\alpha \neq 0$.

If a vertex, say β, endows with a loop, then there is certain $\pi \in \Delta$ such that $\beta + \pi \in \Phi$ and $\alpha + \delta \neq \beta + \pi$ for any $\alpha \in \Phi_k$ different from β and for any $\delta \in \Delta$. Express $[x, x_0]$ as the linear combinations of the Chevalley basis, it is clear that the coefficient at $e_{\beta+\pi}$ is $N_{\beta,\pi} b_\beta$. From $[x, x_0] = 0$ it follows that $N_{\beta,\pi} b_\beta = 0$. Consequently, $b_\beta = 0$. Since Γ_k is connected, we have $b_\alpha = 0$ for all $\alpha \in \Phi_k$. ∎

We now study when Γ_k is connected and when Γ_k has vertices with loops.

Lemma 4.2. *If $\mathfrak{g}$ is of type A_l, B_l or C_l with $l \geq 2$, then Γ_k is connected for any $k \in [-\mathrm{ht}(\theta_0), -2] \cup [1, \mathrm{ht}(\theta_0)]$, and Γ_k has at least one vertex with a loop when $k \in [-\mathrm{ht}(\theta_0), -2]$.*

Proof. Case 1. $\mathfrak{g}$ is of type A_l $(l \geq 2)$.

In this case, $\mathrm{ht}(\theta_0) = l$. If $1 \leq k \leq l$, Γ_k is a path:

$$\sum_{i=1}^{k} \pi_i \sim \sum_{i=2}^{k+1} \pi_i \sim \cdots \sim \sum_{i=l-k}^{l-1} \pi_i \sim \sum_{i=l-k+1}^{l} \pi_i.$$

If $-l \leq -k \leq -2$, Γ_{-k} is a path:

$$(-\sum_{i=1}^{k} \pi_i) \sim (-\sum_{i=2}^{k+1} \pi_i) \sim \cdots \sim (-\sum_{i=l-k}^{l-1} \pi_i) \sim (-\sum_{i=l-k+1}^{l} \pi_i).$$

Case 2. $\mathfrak{g}$ is of type B_l $(l \geq 2)$.

We only check the result for the special case when $l = 5$ and omit the analogous (however much more involved) discussion for the general l. One will see that Γ_k are always pathes.

$$\Gamma_1 : \pi_5 \sim \pi_4 \sim \pi_3 \sim \pi_2 \sim \pi_1;$$

$$\Gamma_2 : \pi_5 + \pi_4 \sim \pi_4 + \pi_3 \sim \pi_3 + \pi_2 \sim \pi_2 + \pi_1;$$

$$\Gamma_3 : \pi_5 + \pi_4 + \pi_3 \sim \pi_4 + \pi_3 + \pi_2 \sim \pi_3 + \pi_2 + \pi_1 \sim \pi_2 + 2\pi_1;$$

$$\Gamma_4 : \pi_5 + \pi_4 + \pi_3 + \pi_2 \sim \pi_4 + \pi_3 + \pi_2 + \pi_1 \sim \pi_3 + \pi_2 + 2\pi_1;$$

$$\Gamma_5 : \pi_5 + \pi_4 + \pi_3 + \pi_2 + \pi_1 \sim \pi_4 + \pi_3 + \pi_2 + 2\pi_1 \sim \pi_3 + 2\pi_2 + 2\pi_1;$$

$$\Gamma_6 : \pi_5 + \pi_4 + \pi_3 + \pi_2 + 2\pi_1 \sim \pi_4 + \pi_3 + 2\pi_2 + 2\pi_1;$$
$$\Gamma_7 : \pi_5 + \pi_4 + \pi_3 + 2\pi_2 + 2\pi_1 \sim \pi_4 + 2\pi_3 + 2\pi_2 + 2\pi_1;$$
$$\Gamma_8 : \pi_5 + \pi_4 + 2\pi_3 + 2\pi_2 + 2\pi_1;$$
$$\Gamma_9 : \pi_5 + 2\pi_4 + 2\pi_3 + 2\pi_2 + 2\pi_1;$$

$$\Gamma_{-2} : -(\pi_5 + \pi_4) \sim -(\pi_4 + \pi_3) \sim -(\pi_3 + \pi_2) \sim -(\pi_2 + \pi_1);$$
$$\Gamma_{-3} : -(\pi_5 + \pi_4 + \pi_3) \sim -(\pi_4 + \pi_3 + \pi_2) \sim -(\pi_3 + \pi_2 + \pi_1) \sim -(\pi_2 + 2\pi_1);$$
$$\Gamma_{-4} : -(\pi_5 + \pi_4 + \pi_3 + \pi_2) \sim -(\pi_4 + \pi_3 + \pi_2 + \pi_1) \sim -(\pi_3 + \pi_2 + 2\pi_1);$$
$$\Gamma_{-5} : -(\pi_5 + \pi_4 + \pi_3 + \pi_2 + \pi_1) \sim -(\pi_4 + \pi_3 + \pi_2 + 2\pi_1) \sim -(\pi_3 + 2\pi_2 + 2\pi_1);$$
$$\Gamma_{-6} : -(\pi_5 + \pi_4 + \pi_3 + \pi_2 + 2\pi_1) \sim -(\pi_4 + \pi_3 + 2\pi_2 + 2\pi_1);$$
$$\Gamma_{-7} : -(\pi_5 + \pi_4 + \pi_3 + 2\pi_2 + 2\pi_1) \sim -(\pi_4 + 2\pi_3 + 2\pi_2 + 2\pi_1);$$
$$\Gamma_{-8} : -(\pi_5 + \pi_4 + 2\pi_3 + 2\pi_2 + 2\pi_1);$$
$$\Gamma_{-9} : -(\pi_5 + 2\pi_4 + 2\pi_3 + 2\pi_2 + 2\pi_1).$$

Case 3. $\mathfrak{g}$ is of type C_l $(l > 2)$.

We only check the result for the special case when $l = 4$ and omit the involved discussion for the general case. One will see that Γ_k are always pathes.

$$\Gamma_1 : \pi_4 \sim \pi_3 \sim \pi_2 \sim \pi_1;$$

$$\Gamma_2 : \pi_4 + \pi_3 \sim \pi_3 + \pi_2 \sim \pi_2 + \pi_1;$$
$$\Gamma_3 : \pi_4 + \pi_3 + \pi_2 \sim \pi_3 + \pi_2 + \pi_1 \sim 2\pi_2 + \pi_1;$$
$$\Gamma_4 : \pi_4 + \pi_3 + \pi_2 + \pi_1 \sim \pi_3 + 2\pi_2 + \pi_1;$$
$$\Gamma_5 : \pi_4 + \pi_3 + 2\pi_2 + \pi_1 \sim 2\pi_3 + 2\pi_2 + \pi_1;$$
$$\Gamma_6 : \pi_4 + 2\pi_3 + 2\pi_2 + \pi_1;$$
$$\Gamma_7 : 2\pi_4 + 2\pi_3 + 2\pi_2 + \pi_1.$$

$$\Gamma_{-2} : -(\pi_4 + \pi_3) \sim -(\pi_3 + \pi_2) \sim -(\pi_2 + \pi_1);$$
$$\Gamma_{-3} : -(\pi_4 + \pi_3 + \pi_2) \sim -(\pi_3 + \pi_2 + \pi_1) \sim -(2\pi_2 + \pi_1);$$
$$\Gamma_{-4} : -(\pi_4 + \pi_3 + \pi_2 + \pi_1) \sim -(\pi_3 + 2\pi_2 + \pi_1);$$
$$\Gamma_{-5} : -(\pi_4 + \pi_3 + 2\pi_2 + \pi_1) \sim -(2\pi_3 + 2\pi_2 + \pi_1);$$
$$\Gamma_6 : -(\pi_4 + 2\pi_3 + 2\pi_2 + \pi_1);$$
$$\Gamma_7 : -(2\pi_4 + 2\pi_3 + 2\pi_2 + \pi_1).$$

In all cases, if $k \in [-\mathrm{ht}(\theta_0), -2]$, then the vertex in Γ_k which contains $-\pi_l$ has a loop. ∎

Note that Lemma 4.2 fails to hold for the cases when $\mathfrak{g}$ is of type $D_l (l \geq 4)$, E_6, E_7, E_8, or F_4. When $\mathfrak{g}$ is of type D_l with $l \geq 4$, Γ_{l-1} has $\sum_{i=1}^{l-1} \pi_i$ as an isolated vertex, because if $(\sum_{i=1}^{l-1} \pi_i) + \pi \in \Phi$ for certain $\pi \in \Delta$ then π must be π_l, however

$$(\sum_{i=1}^{l-1} \pi_i) + \pi_l = (\pi_l + \sum_{i=1}^{l-2} \pi_i) + \pi_{l-1} = (\sum_{i=2}^{l} \pi_i) + \pi_1,$$

where $\sum_{i=1}^{l-1} \pi_i$, $\pi_l + \sum_{i=1}^{l-2} \pi_i$ and $\sum_{i=2}^{l} \pi_i$ are three roots in Φ_{l-1}, from which it follows that $\sum_{i=1}^{l-1} \pi_i$ is an isolated vertex of Γ_{l-1}. When $\mathfrak{g}$ is of type E_6, E_7, E_8 or F_4, then by a similar discussion one will see that Γ_4 has $\sum_{i=1}^{4} \pi_i$ as an isolated vertex.

Lemma 4.3. *Let $\mathfrak{g}$ be a simple Lie algebra of type A_l, B_l or C_l. For a positive integer $k \leq \mathrm{ht}(\theta_0)$, the elements in $\{h_\beta : \beta \in \Phi_k\}$ are linearly independent.*

Proof. For $\gamma \in \Phi$, let t_γ be the unique element in $\mathfrak{h}$ corresponding to γ such that $\alpha(t_\gamma) = (\alpha, \gamma)$ for $\alpha \in \Phi$ (where $(*;*)$ is the dual of the Killing form, see [12] page 37). Then $h_\gamma = \frac{2t_\gamma}{(\gamma,\gamma)}$ for $\gamma \in \Phi$. To complete the proof it suffices to prove that the elements in $\{t_\beta : \beta \in \Phi_k\}$ are linearly independent. To achieve this goal, we first need to prove that the elements in Φ_k are linearly independent. Suppose that $\sum_{\beta \in \Phi_k} a_\beta \beta = 0$. We need to prove $a_\beta = 0$ for all $\beta \in \Phi_k$. If $\mathfrak{g}$ is of type A_l, B_l or C_l then there is a unique root, say β_1, in Φ_k containing π_l (obtained by checking the root system Φ), then from $\sum_{\beta \in \Phi_k} a_\beta \beta(d_l) = a_{\beta_1} \beta_1(d_l) = 0$ and $\beta_1(d_l) \neq 0$ we have $a_{\beta_1} = 0$, where $\{d_1, d_2, \ldots, d_l\}$ denotes the dual basis of $\{\pi_1, \pi_2, \ldots, \pi_l\}$. Furthermore, since only one root, say β_2, in $\Phi_k \setminus \{\beta_1\}$ (if it is not empty) contains π_{l-1}, we have $a_{\beta_2} = 0$. Proceeding in this way we have $a_\beta = 0$ for all $\beta \in \Phi_k$. Label the roots in Φ_k as $\Phi_k = \{\beta_1, \beta_2, \ldots, \beta_m\}$, where $m = |\Phi_k|$. Since the elements in Φ_k are linearly independent, the $m \times m$ Gram matrix A (with respect to Φ_k) with (β_i, β_j) at the (i,j)-position is nonsingular. Now we return to prove that $\{t_\beta : \beta \in \Phi_k\}$ are linearly independent. Suppose that $\sum_{\beta \in \Phi_k} b_\beta t_\beta = 0$. Then,

$$\sum_{\beta \in \Phi_k} b_\beta \gamma(t_\beta) = 0 \quad \text{for any } \gamma \in \Phi_k.$$

Since $\gamma(t_\beta) = (\gamma, \beta)$, we have

$$\sum_{\beta \in \Phi_k} b_\beta (\gamma, \beta) = 0 \quad \text{for any } \gamma \in \Phi_k,$$

from which we have $b_\beta = 0$ for all $\beta \in \Phi_k$ (recalling that the Gram matrix with respect to Φ_k is nonsingular). ■

Lemma 4.4. *Let $\mathfrak{g}$ be a simple Lie algebra of type A_l, B_l or C_l with $l \geq 2$, and let $x_0 = \sum_{\pi \in \Delta} e_\pi$, $y_0 = \sum_{\pi \in \Delta} e_{-\pi}$. Then $d(x_0, y_0) = 4$.*

Proof. We first consider what form does a nonzero element $z \in x_0'$ have. Suppose that $0 \neq z = h_z + \sum_{\alpha\in\Phi} b_\alpha e_\alpha$ with $h_z \in \mathfrak{h}$ commutes x_0, and we denote $\sum_{\mathrm{ht}(\alpha)=k} b_\alpha e_\alpha$ by z_k for each $k \in [-\mathrm{ht}(\theta_0), -1] \cup [1, \mathrm{ht}(\theta_0)]$. Then

$$z = h_z + \sum_{k=-\mathrm{ht}(\theta_0)}^{-1} z_k + \sum_{k=1}^{\mathrm{ht}(\theta_0)} z_k.$$

From $[z, x_0] = 0$ it follows that

$$[h_z, x_0] = 0, \text{ and } [z_k, x_0] = 0 \text{ for any } k \in [-\mathrm{ht}(\theta_0), -1] \cup [1, \mathrm{ht}(\theta_0)],$$

since $[h_z, x_0] \in \mathfrak{g}_1, [z_k, x_0] \in \mathfrak{g}_{k+1}$ for $k \neq -1$ and $[z_{-1}, x_0] \in \mathfrak{h}$, where $\mathfrak{g}_k = \sum_{\alpha\in\Phi_k} F_q e_\alpha$ for $k \neq 0$. From $[h_z, x_0] = 0$ we have $\pi(h_z) = 0$ for any $\pi \in \Delta$, which further implies that $h_z = 0$. Applying Lemma 4.1 and Lemma 4.2, we have $z_k = 0$ for $k \in [-\mathrm{ht}(\theta_0), -2]$. From $[z_{-1}, x_0] = [\sum_{\pi\in\Delta} b_{-\pi} e_{-\pi}, \sum_{\pi\in\Delta} e_\pi] = -\sum_{\pi\in\Delta} b_{-\pi} h_\pi = 0$, we have $b_{-\pi} = 0$ for any $\pi \in \Delta$. Thus $z_{-1} = 0$. When $k \in [1, \mathrm{ht}(\theta_0)]$, by Lemma 4.1 and Lemma 4.2 we know either $b_\alpha = 0$ for all $\alpha \in \Phi_k$ or $b_\alpha \neq 0$ for all $\alpha \in \Phi_k$. Let p be the least positive integer such that $z_p \neq 0$ (among all z_k).

Analogously, we suppose that $u = h_u + \sum_{\alpha\in\Phi} c_\alpha e_\alpha$ is a nonzero element in $\mathfrak{g}$ commuting with y_0, and we denote $\sum_{\mathrm{ht}(\alpha)=k} c_\alpha e_\alpha$ by u_k for each $k \in [-\mathrm{ht}(\theta_0), -1] \cup [1, \mathrm{ht}(\theta_0)]$. Then, a similar argument as above we have that $h_u = u_k = 0$ for $k \in [1, \mathrm{ht}(\theta_0)]$, and when $-k \in [-\mathrm{ht}(\theta_0), -1]$ either $c_\alpha = 0$ for all $\alpha \in \Phi_{-k}$ or $c_\alpha \neq 0$ for all $\alpha \in \Phi_{-k}$. Let $-q$ be the least negative integer such that $u_{-q} \neq 0$ (among all u_{-k}).

Now we are in a position to show that $[z, u] \neq 0$.

We first consider the case when $q \geq p$. If $q = p$, then by Lemma 4.3

$$[z_p, u_{-p}] = [\sum_{\alpha\in\Phi_p} b_\alpha e_\alpha, \sum_{-\alpha\in\Phi_{-p}} c_{-\alpha} e_{-\alpha}] = \sum_{\alpha\in\Phi_p} b_\alpha c_{-\alpha} h_\alpha \neq 0,$$

and so $[z, u] \neq 0$ since $[z, u] \equiv [z_p, u_{-p}] \pmod{\mathfrak{n}}$. If $q > p$, since $[x_0, z] = 0$, we have

$$\begin{aligned}(\mathrm{ad}\ x_0)^{q-p}([z, u]) &= [z, (\mathrm{ad}\ x_0)^{q-p}(u)] = [\sum_{i=p}^{\mathrm{ht}(\theta_0)} z_i, \sum_{j=-q}^{-1} (\mathrm{ad}\ x_0)^{q-p}(u_j)] \\ &\equiv [z_p, (\mathrm{ad}\ x_0)^{q-p}(u_{-q})] \pmod{\mathfrak{n}}.\end{aligned}$$

We claim that $(\mathrm{ad}\ x_0)^{q-p}(u_{-q})$ is a nonzero element in $\mathfrak{n}_{-p} = \sum_{\alpha\in\Phi_{-p}} \mathfrak{g}_\alpha$. Indeed, since $u_{-q} \neq 0$ we have $(\mathrm{ad}\ x_0)(u_{-q}) = [x_0, u_{-q}]$ is a nonzero element in $\mathfrak{n}_{-(q-1)}$ (otherwise, by $[x_0, u_{-q}] = 0$ and using Lemma 4.1 and Lemma 4.2 we have $u_{-q} = 0$). Similarly, it follows from $(\mathrm{ad}\ x_0)(u_{-q}) \neq 0$, we have $(\mathrm{ad}\ x_0)^2(u_{-q}) = (\mathrm{ad}\ x_0)(\mathrm{ad}\ x_0)(u_{-q}) \neq 0$. After a finite number of steps, we arrive at $(\mathrm{ad}\ x_0)^{q-p}(u_{-q}) \neq 0$. Since $(\mathrm{ad}\ x_0)^{q-p}(u_{-q}) \in \mathfrak{n}_{-p}$, the claim is proved. Now suppose that $(\mathrm{ad}\ x_0)^{q-p}(u_{-q}) = \sum_{-\alpha\in\Phi_{-p}} a_{-\alpha} e_{-\alpha}$. By the claim, at least one

$a_{-\alpha} \neq 0$. Because $b_\alpha \neq 0$ for all $\alpha \in \Phi_p$ and there is at least one $-\alpha \in \Phi_{-p}$ such that $a_{-\alpha} \neq 0$, we have

$$[z_p, (\text{ad } x_0)^{q-p}(u_{-q})] = [\sum_{\alpha\in\Phi_p} b_\alpha e_\alpha, \sum_{-\alpha\in\Phi_{-p}} a_{-\alpha}e_{-\alpha}] = \sum_{\alpha\in\Phi_p} b_\alpha a_{-\alpha} h_\alpha \neq 0,$$

the last inequality follows from Lemma 4.3. Consequently, we have $[z, u] \neq 0$.

If $p > q$, by considering $(\text{ad } y_0)^{p-q}([z, u])$ we can also prove $[z, u] \neq 0$. The parallel discussion is omitted.

The fact (just proved) that $[z, u] \neq 0$ for any $0 \neq z \in x'_0$ and $0 \neq u \in y'_0$ implies that $d(x_0, y_0) \geq 4$. It is easy to see that $x_0 \sim e_{\theta_0} \sim h \sim e_{-\theta_0} \sim y_0$ is a path from x_0 to y_0, where $0 \neq h \in \text{Ker } \theta_0$. Hence, $d(x_0, y_0) = 4$. ■

Proof of Theorem 1.4

Let $\mathfrak{g}$ be a simple Lie algebra of type A_l, B_l or C_l with $l > 2$. By Theorem 1.3, the diameter of $\Gamma(\mathfrak{g})$ is at most 4. On the other hand, there exist two vertices x_0, y_0 in $\Gamma(\mathfrak{g})$ such that $d(x_0, y_0)$=4. Hence, the diameter of $\Gamma(\mathfrak{g})$ is exactly 4. ■

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