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Structure of Stratified Groups I. Product Decompositions

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Abstract. We examine how stratified Lie algebras decompose as direct sums of stratified Lie ideals and connected simply connected stratified Lie groups decompose as direct products. We study the corresponding groups of strata-preserving automorphisms.
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1. Introduction

The classical theory of Lie algebras considers decompositions of semisimple Lie algebras as sums of simple ideals, but typical textbooks say little about the structure of nilpotent Lie algebras and the corresponding Lie groups, at least in part because this appears to be considerably more complicated. Tanaka [13, 14, 15] and other Japanese geometers (see, e.g., [16]) studied the prolongations of stratified (they used the expression "graded fundamental") Lie algebras as part of a study of contact manifolds and their generalisations. Stratified Lie algebras and the corresponding groups appear in the study of subelliptic differential operators, starting with work such as that of Folland [3], Folland and Stein [4], and Rothschild and Stein [12]. They also arise as models for sub-Riemannian geometry, as evidenced by the labours of Mitchell [9] and Pansu [11], and continuing with work such as [1, 7, 10]; here the algebra is equipped with a norm or inner product, and the expression "Carnot group" is usual. Harmonic analysts have studied many aspects of analysis on these groups, see, for instance, [5, 2, 8].

Analysis and geometry on stratified groups has become an area of considerable activity, but most of the work in this area focusses on the applications of stratified groups and algebras rather than the algebraic structures. It seems timely to begin to rectify this situation, and in this short note we consider direct sum decompositions of stratified Lie algebras into ideals and the corresponding direct product decomposition of the associated groups. We also describe the au-

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tomorphism groups of direct sums of stratified Lie algebra and direct products of stratified Lie groups.

In this paper, we first treat the Lie algebras in some detail. Then derive the analogous results for the Lie groups, which follow immediately, and so we just state them. All Lie algebras and groups are assumed to be finite-dimensional throughout.

2. Stratified Lie algebras

Recall that a Lie algebra $\mathfrak{g}$ is said to be stratified of step ℓ if

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-\ell},$$

where $[\mathfrak{g}_{-j}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-j-1}$ when $1 \leq j \leq \ell$, while $\mathfrak{g}_{-\ell} \neq \{0\}$ and $\mathfrak{g}_{-\ell-1} = \{0\}$; this implies that $\mathfrak{g}$ is nilpotent. We assume that $\dim(\mathfrak{g})$ is at least 3 to avoid degenerate cases. Note that the choice of $\mathfrak{g}_{-1}$ determines the stratification, but the Lie algebra $\mathfrak{g}$ by itself need not do so.

We write π_j for the canonical projection of $\mathfrak{g}$ onto $\mathfrak{g}_{-j}$, $\mathfrak{Z}(\mathfrak{g})$ for the centre of $\mathfrak{g}$, and $\mathrm{Aut}(\mathfrak{g})$ for the group of automorphisms of $\mathfrak{g}$. In particular, for each $s \in \mathbb{R}^+$, the dilation $\delta_s \in \mathrm{Aut}(\mathfrak{g})$ is defined to be $\sum_{j=1}^{\ell} s^j \pi_j$. For a linear map of $\mathfrak{g}$, preserving all the subspaces $\mathfrak{g}_{-j}$ of the stratification is equivalent to commuting with dilations and to having a block-diagonal matrix representation. We call such maps "strata-preserving". We write $\mathrm{Aut}^\delta(\mathfrak{g})$ for the subset of $\mathrm{Aut}(\mathfrak{g})$ of strata-preserving automorphisms; these are determined by their action on $\mathfrak{g}_{-1}$.

Suppose that $\mathfrak{g}_{-1}$ admits a vector space direct sum decomposition $\mathfrak{g}_{-1} = \bigoplus_{k=1}^{K} \mathfrak{v}^k$ such that $[\mathfrak{v}^k, \mathfrak{v}^{k'}] = \{0\}$ if $k \neq k'$. Let $\mathfrak{g}^k$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{v}^k$ for all $k \in \{1, \ldots, K\}$; then repeated use of the Jacobi identity shows that $[\mathfrak{g}^k, \mathfrak{g}^{k'}] = \{0\}$ if $k \neq k'$, so the $\mathfrak{g}^k$ are pairwise commuting ideals in $\mathfrak{g}$, and

$$\mathfrak{g} = \bigoplus_{k=1}^{K} \mathfrak{g}^k;$$

we say that $\mathfrak{g}$ admits a direct sum decomposition into ideals. Of course, if $\mathfrak{g}$ admits such a direct sum decomposition into ideals, then they must commute pairwise, for if $X \in \mathfrak{g}^k$ and $X' \in \mathfrak{g}^{k'}$, then $[X, X'] \in \mathfrak{g}^k \cap \mathfrak{g}^{k'} = \{0\}$.

A stratified Lie algebra $\mathfrak{g}$ is said to be *totally nonabelian* if $\mathfrak{g}_{-1} \cap \mathfrak{Z}(\mathfrak{g}) = \{0\}$. A stratified Lie algebra that is neither abelian nor totally nonabelian is said to be *partly abelian.*

Theorem 2.1. *Suppose that $\mathfrak{g}$ is a partly abelian stratified Lie algebra and that $\mathfrak{v} = \mathfrak{g}_{-1} \cap \mathfrak{Z}(\mathfrak{g})$. Let $\mathfrak{w}$ be* any *complementary subspace of $\mathfrak{v}$ in $\mathfrak{g}_{-1}$, and let $\mathfrak{g}^{\mathfrak{w}}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{w}$. Then $\mathfrak{g}^{\mathfrak{w}}$ is totally nonabelian, and $\mathfrak{g}$ is a direct sum of ideals $\mathfrak{v} \oplus \mathfrak{g}^{\mathfrak{w}}$. The algebras $\mathfrak{g}^{\mathfrak{w}}$ and $\mathfrak{g}^{\mathfrak{w}'}$ corresponding to two different choices of complementary subspace $\mathfrak{w}$ and $\mathfrak{w}'$ are isomorphic, and their strata $\mathfrak{g}^{\mathfrak{w}}_{-j}$ and $\mathfrak{g}^{\mathfrak{w}'}_{-j}$ coincide when $j \geq 2$.*

Proof. The proof is evident. ∎

Observe that if $\mathfrak{g}_{-1}$ is equipped with an inner product, then there is a canonical choice of complementary subspace $\mathfrak{w}$.

Corollary 2.2. *Suppose that $\mathfrak{g}$ is a partly abelian stratified Lie algebra and that $\mathfrak{v} = \mathfrak{g}_{-1} \cap \mathfrak{z}(\mathfrak{g})$. Fix an arbitrary complementary subspace $\mathfrak{w}$ of $\mathfrak{v}$ in $\mathfrak{g}_{-1}$, and let $\mathfrak{g}^{\mathfrak{w}}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{w}$. The action of the automorphism group of $\mathfrak{g}$ is determined by its action on $\mathfrak{g}_{-1}$, and $\mathrm{Aut}^{\delta}(\mathfrak{g})$ may be identified with the group of all matrices of the form*

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A is an arbitrary element of $\mathrm{GL}(\mathfrak{v})$ and B is an arbitrary element of $\mathrm{Hom}(\mathfrak{w}, \mathfrak{v})$, while there is an automorphism of $\mathfrak{g}^{\mathfrak{w}}$ which may be identified with the matrix $D \in \mathrm{GL}(\mathfrak{w})$.

This result was certainly known before (see, e.g., [6]). This reduces the study of decompositions of stratified Lie algebras to consideration of the totally nonabelian case.

Theorem 2.3. *Suppose that $\mathfrak{g}$ is a totally nonabelian stratified Lie algebra. Then there exists a direct sum decomposition of $\mathfrak{g}$ into ideals:*

$$\mathfrak{g} = \bigoplus_{k=1}^{K} \mathfrak{g}^k,$$

which is finest in the sense that, given any direct sum decomposition into ideals $\mathfrak{g} = \bigoplus_{l=1}^{L} \tilde{\mathfrak{g}}^l$, there is a partition of the set $\{1, 2, \dots, K\}$ into disjoint subsets $I_1, I_2, \dots, I_L$ such that

$$\tilde{\mathfrak{g}}^l = \bigoplus_{k \in I_l} \mathfrak{g}^k \qquad \forall l \in \{1, \dots, L\}.$$

Proof. Take direct sum decompositions into ideals $\bigoplus_{m=1}^{M} \mathfrak{g}^m$ and $\bigoplus_{n=1}^{N} \tilde{\mathfrak{g}}^n$ of $\mathfrak{g}$, and let P^m and $\tilde{P}^n$ be the associated projections of $\mathfrak{g}$ onto $\mathfrak{g}^m$ and $\tilde{\mathfrak{g}}^n$; these projections are endomorphisms since the complements of $\mathfrak{g}^m$ and $\tilde{\mathfrak{g}}^n$ are ideals.

We claim that $P^m \tilde{P}^n = \tilde{P}^n P^m$. On the one hand, if $X, Y \in \mathfrak{g}$, then

$$P^m[X, Y] = [P^m X, P^m Y] = [P^m X, Y] = [X, P^m Y],$$

because P^m is an endomorphism and $\bigoplus_{m=1}^{M} \mathfrak{g}^m$ is a direct sum decomposition of $\mathfrak{g}$ into pairwise commuting ideals. The same holds when P^m is replaced by $\tilde{P}^n$. Hence, if $X, Y \in \mathfrak{g}$, then

$$\begin{aligned} P^m \tilde{P}^n[X, Y] - \tilde{P}^n P^m[X, Y] &= P^m[X, \tilde{P}^n Y] - \tilde{P}^n[P^m X, Y] \\ &= [P^m X, \tilde{P}^n Y] - [P^m X, \tilde{P}^n Y] \\ &= 0, \end{aligned}$$

so $P^m\tilde{P}^n - \tilde{P}^n P^m$ annihilates $\bigoplus_{j=2}^{\ell} \mathfrak{g}_{-j}$.

On the other hand, we see similarly that $[(P^m\tilde{P}^n - \tilde{P}^n P^m)X, Y] = 0$ for all $X, Y \in \mathfrak{g}_{-1}$. Since $\mathfrak{g}$ is totally nonabelian, $(P^m\tilde{P}^n - \tilde{P}^n P^m)X = 0$ for all $X \in \mathfrak{g}_{-1}$, and hence $P^m\tilde{P}^n = \tilde{P}^n P^m$, as claimed.

It follows immediately that the $P^m\tilde{P}^n$ are projections, and

$$\mathfrak{g} = \bigoplus_{m=1}^{M} \bigoplus_{n=1}^{N} P^m\tilde{P}^n \mathfrak{g} = \bigoplus_{m=1}^{M} \bigoplus_{n=1}^{N} \mathfrak{g}^m \cap \tilde{\mathfrak{g}}^n.$$

As $\mathfrak{g}$ is finite dimensional, the finest decomposition into ideals is the "intersection of all possible decompositions into ideals", of which there are finitely many. ∎

Given the decomposition into ideals $\mathfrak{g} = \bigoplus_{k=1}^{K} \mathfrak{g}^k$, we denote by $\mathrm{ProdAut}(\mathfrak{g})$ the subgroup of $\mathrm{Aut}^\delta(\mathfrak{g})$ defined as the direct product $\mathrm{Aut}^\delta(\mathfrak{g}^1) \times \cdots \times \mathrm{Aut}^\delta(\mathfrak{g}^K)$.

When $k, k' \in \{1, \dots, K\}$, we write $k \sim k'$ if and only if there is a strata-preserving isomorphism from $\mathfrak{g}^k$ to $\mathfrak{g}^{k'}$; then $\sim$ is an equivalence relation. For each equivalence class κ, choose a stratified Lie algebra $\mathfrak{g}^\kappa$ isomorphic to every $\mathfrak{g}^k$ when k is in the equivalence class, and for each $k \in \kappa$, choose a strata-preserving isomorphism I^k from $\mathfrak{g}^\kappa$ to $\mathfrak{g}^k$, whose inverse we write as I^{-k}.

When σ lies in $S_K^\sim$, the group of permutations of $\{1, \dots, K\}$ that preserve the equivalence classes of $\sim$, define $I^\sigma \in \mathrm{Aut}^\delta(\mathfrak{g})$ by first setting

$$I^\sigma(X) = I^{\sigma(k)} I^{-k}(X)$$

for all $X \in \mathfrak{g}^k$ and all $k \in \{1, \dots, K\}$, and then extending this definition to $\mathfrak{g}$ by linearity. It is easy to check that the map $\sigma \mapsto I^\sigma$ embeds $S_K^\sim$ in $\mathrm{Aut}^\delta(\mathfrak{g})$. We denote the image by $\mathrm{Perm}(\mathfrak{g})$.

Suppose that $\sigma \in S_K^\sim$ and that $(S^1, \dots, S^K) \in \mathrm{ProdAut}(\mathfrak{g})$. If $X \in \mathfrak{g}^k$, then

$$(S^1, \dots, S^K) \circ I^\sigma(X) = I^{\sigma(k)} \circ I^{-k} \circ I^k \circ I^{-\sigma(k)} \circ S^{\sigma(k)} \circ I^{\sigma(k)} \circ I^{-k}(X).$$

Define $(T^1, \dots, T^K) \in \mathrm{ProdAut}(\mathfrak{g})$ by

$$T^k = I^k \circ I^{-\sigma(k)} \circ S^{\sigma(k)} \circ I^{\sigma(k)} \circ I^{-k};$$

by linearity,

$$(S^1, \dots, S^K) \circ I^\sigma = I^\sigma \circ (T^1, \dots, T^K).$$

It follows that the set of all products $I^\sigma \circ (S^1, \dots, S^K)$, where $I^\sigma \in \mathrm{Perm}(\mathfrak{g})$ and $(S^1, \dots, S^K) \in \mathrm{ProdAut}(\mathfrak{g})$, is closed under multiplication and inversion, and so is a group, which is the semidirect product $\mathrm{Perm}(\mathfrak{g}) \ltimes \mathrm{ProdAut}(\mathfrak{g})$. It is not hard to see that this group does not depend on the choices of $\mathfrak{g}^\kappa$ or the isomorphisms I^k.

Corollary 2.4. *Let $\bigoplus_{k=1}^{K} \mathfrak{g}^k$ be the finest direct sum decomposition of $\mathfrak{g}$ into ideals, as in Theorem 2.3. Then $\mathrm{Aut}^\delta(\mathfrak{g})$ may be identified with*

$$\mathrm{Perm}(\mathfrak{g}) \ltimes \mathrm{ProdAut}(\mathfrak{g}).$$

Proof. Any automorphism T of $\mathfrak{g}$ must send the decomposition $\bigoplus_{k=1}^{K} \mathfrak{g}^k$ to another equivalent decomposition of $\mathfrak{g}$. Hence there exists $\sigma \in S_K^{\sim}$ such that $T\mathfrak{g}^k = \mathfrak{g}^{\sigma(k)}$. By composing T with the inverse of I^σ, we obtain an automorphism of $\mathfrak{g}$ that preserves each of the $\mathfrak{g}^k$, and hence lies in $\mathrm{ProdAut}(\mathfrak{g})$. ■

Example 2.5. Let $\mathfrak{h}_1 = \mathrm{span}\{X, Y, Z\}$ be the 3-dimensional Heisenberg algebra, with $[X, Y] = Z$ being the only non-zero bracket. Define $\mathfrak{g} = \mathbb{R}^2 \oplus \mathfrak{h}_1$ and let $\{e_1, e_2\}$ be a basis of $\mathbb{R}^2$. Then $\mathfrak{g}$ is partly abelian, because $\mathfrak{v} = \mathfrak{g}_{-1} \cap \mathfrak{z}(\mathfrak{g}) = \mathrm{span}\{e_1, e_2\}$ for every choice of $\mathfrak{g}_{-1}$. If $\mathfrak{g}_{-1} = \mathrm{span}\{e_1, e_2, X, Y\}$, then the infinite choices of complementary subspaces of $\mathfrak{v}$ in $\mathfrak{g}_{-1}$ have the form $\mathfrak{w} = \mathrm{span}\{X + \sum_{i=1}^2 a_i e_i, Y + \sum_{j=1}^2 b_j e_j\}$, for some $a_i, b_j \in \mathbb{R}$. The Lie algebras generated by every such $\mathfrak{w}$ are totally nonabelian and mutually isomorphic. A straightforward computation shows that $\mathrm{Aut}^\delta(\mathfrak{g})$ may be identified with the group of all matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A and D are arbitrary matrices in $\mathrm{GL}(2, \mathbb{R})$, and B is an arbitrary matrix in $\mathfrak{gl}(2, \mathbb{R})$.

Example 2.6. Define $\mathfrak{g} = \mathfrak{h}_1^1 \oplus \cdots \oplus \mathfrak{h}_1^K$ to be the direct sum of K copies of the Heisenberg algebra $\mathfrak{h}_1$. This decomposition into ideals is the finest in the sense of Theorem 2.3, and $\mathfrak{g}$ is totally nonabelian. Since the components $\mathfrak{h}_1^j$, for $j = 1, \ldots, K$, are mutually isomorphic, there is only one equivalence class. Then $\mathrm{Aut}^\delta(\mathfrak{g}) = \mathrm{Perm}(\mathfrak{g}) \ltimes \mathrm{ProdAut}(\mathfrak{g})$, where $\mathrm{Perm}(\mathfrak{g})$ is in bijection with the full group of permutations of $\{1, \ldots, K\}$.

Example 2.7. Let $\mathfrak{h}_n$ denote the $2n+1$-dimensional Heisenberg algebra. Define

$$\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_l,$$

where for every $j = 1, \ldots, l$ we denote by $\mathfrak{a}_j$ the direct sum of K_j copies of $\mathfrak{h}_{n_j}$, and we choose $n_i \neq n_j$ if $i \neq j$. Then $\mathfrak{g}$ is totally nonabelian, it has l equivalence classes, and $\mathrm{Aut}^\delta(\mathfrak{g}) = \mathrm{Perm}(\mathfrak{g}) \ltimes \mathrm{ProdAut}(\mathfrak{g})$. In particular, $\mathrm{Perm}(\mathfrak{g}) = \mathrm{Perm}(\mathfrak{a}_1) \times \cdots \times \mathrm{Perm}(\mathfrak{a}_l)$, where $\mathrm{Perm}(\mathfrak{a}_i)$ is in bijection with the group of permutations of $\{1, \ldots, K_i\}$, for every $i = 1, \ldots, l$.

3. Stratified Lie groups

Let G be a stratified Lie group of step ℓ. This means that G is connected and simply connected, and its Lie algebra $\mathfrak{g}$ is stratified with ℓ layers. The identity of G is written e, and we view the Lie algebra $\mathfrak{g}$ as the tangent space at the identity.

Since G is nilpotent, connected and simply connected, the exponential map $\exp$ is a bijection from $\mathfrak{g}$ to G, with inverse $\log$. We also write δ_s for the automorphism of G given by $\exp \circ \delta_s \circ \log$. The differential $T \mapsto (T_*)_e$ is a one-to-one correspondence between automorphisms of G and of $\mathfrak{g}$, and $T = \exp \circ T_{*e} \circ \log$.

We denote by $\mathrm{Aut}(G)$ the group of automorphisms of G, and by $\mathrm{Aut}^{\delta}(G)$ the subgroup of automorphisms that commute with dilations.

A stratified connected simply connected Lie group G is called totally nonabelian or partly abelian or a direct product if its Lie algebra is totally nonabelian or partly abelian or a direct sum. The exponential mapping exp is a bijection of the Lie algebra $\mathfrak{g}$ of G onto G, with inverse log, and consequently all the results above have immediate analogues for stratified Lie groups. We omit the proofs of the following results, which follow immediately.

Theorem 3.1. *Suppose that G is a partly abelian stratified Lie group with Lie algebra $\mathfrak{g}$ and that $\mathfrak{v} = \mathfrak{g}_{-1} \cap \mathfrak{Z}(\mathfrak{g})$. Let $\mathfrak{w}$ be* any *complementary subspace of $\mathfrak{v}$ in $\mathfrak{g}_{-1}$, let $\mathfrak{g}^{\mathfrak{w}}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{w}$, and let $\tilde{G}$ be $\exp(\mathfrak{g}^{\mathfrak{w}})$. Then $G^{\mathfrak{w}}$ is totally nonabelian, and G is a direct product $V \times G^{\mathfrak{w}}$.*

Corollary 3.2. *Suppose that G is a partly abelian stratified Lie group with Lie algebra $\mathfrak{g}$ and that $\mathfrak{v} = \mathfrak{g}_{-1} \cap \mathfrak{Z}(\mathfrak{g})$. Let $\mathfrak{w}$ be* any *complementary subspace of $\mathfrak{v}$ in $\mathfrak{g}_{-1}$, let $\mathfrak{g}^{\mathfrak{w}}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{w}$, and let $\tilde{G}$ be $\exp(\mathfrak{g}^{\mathfrak{w}})$. Then $\mathrm{Aut}^{\delta}(G)$ may be identified with the group of all matrices of the form*

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A is an arbitrary element of $\mathrm{GL}(\mathfrak{v})$ and B is an arbitrary element of $\mathrm{Hom}(\mathfrak{w}, \mathfrak{v})$, while there is an automorphism of $G^{\mathfrak{w}}$ which may be identified with the matrix $D \in \mathrm{GL}(\mathfrak{w})$.

Theorem 3.3. *Suppose that G is a totally nonabelian stratified Lie group. Then there exists a direct product decomposition of G into normal subgroups:*

$$G = G^1 \times \cdots \times G^K,$$

which is the finest in the sense that, given any direct product decomposition into normal subgroups $G = \tilde{G}^1 \times \cdots \times \tilde{G}^L$, there is a partition of the set $\{1, \ldots, K\}$ into disjoint subsets $I_1, I_2, \ldots, I_L$ such that $\tilde{G}^l = \prod_{k \in I_l} G^k$ for all $l \in \{1, \ldots, L\}$.

We denote by $\mathrm{Perm}(G)$ and $\mathrm{ProdAut}(G)$ the spaces of those automorphisms of G whose differentials are in $\mathrm{Perm}(\mathfrak{g})$ and $\mathrm{ProdAut}(\mathfrak{g})$.

Corollary 3.4. *Let $G^1 \times \cdots \times G^K$ be the finest direct product decomposition of G into normal subgroups, as in Theorem 3.3. Then $\mathrm{Aut}^{\delta}(G)$ may be identified with*

$$\mathrm{Perm}(G) \ltimes \mathrm{ProdAut}(G).$$

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