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Generalization of some Inequalities for Matrix Exponentials to Lie Groups

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Abstract. Kostant's pre-order on noncompact connected semisimple Lie groups is a generalization of log-majorization for matrices. We generalize some inequalities for matrix exponentials to Lie groups in terms of Kostant's pre-order.
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1. Introduction

Let G be a noncompact connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Kostant introduced a pre-order $\prec$ on G and obtained the following result [8, Theorem 6.3]:

$$e^{X+Y} \prec e^X e^Y \tag{1.1}$$

for all $X, Y \in \mathfrak{p}$. The relation (1.1) is a generalization of the famous Golden-Thompson trace inequality

$$\operatorname{tr} e^{A+B} \leqslant \operatorname{tr} e^A e^B, \tag{1.2}$$

where A and B are $n \times n$ Hermitian matrices. The inequality (1.2) was independently discovered by Golden [4], Symanzik [12], and Thompson [15] in the same year of 1965, all motivated by statistical mechanics. For historical aspects, one may see a recent paper by Forrester-Thompson [3].

Motivated by Kostant's pioneering paper [8], Tam [14] derived the following relations for all $X, Y \in \mathfrak{p}$

$$(e^{X/2}e^Y e^{X/2})^r \prec e^{rX/2}e^{rY}e^{rX/2}, \qquad \forall\, r \geqslant 1, \tag{1.3}$$

$$e^{rX/2}e^{rY}e^{rX/2} \prec (e^{X/2}e^Y e^{X/2})^r, \qquad \forall\, 0 \leqslant r \leqslant 1. \tag{1.4}$$

The relations (1.3) and (1.4) generalized the Araki-Lieb-Thirring inequalities [1]

$$\operatorname{tr}(A^{1/2}BA^{1/2})^{rq} \leqslant \operatorname{tr}(A^{r/2}B^r A^{r/2})^q, \qquad \forall\, q \geqslant 0, \forall\, r \geqslant 1, \tag{1.5}$$

$$\operatorname{tr}(A^{1/2}BA^{1/2})^{rq} \geqslant \operatorname{tr}(A^{r/2}B^r A^{r/2})^q, \qquad \forall\, q \geqslant 0, \forall\, 0 \leqslant r \leqslant 1, \tag{1.6}$$

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where A and B are $n \times n$ positive definite matrices.

There are other matrix inequalities that have been generalized to Lie groups in terms of Kostant's pre-order (see, for example, [13, 14, 9, 11]). The goal of this paper is to make more contributions in this direction. We will state some matrix inequalities in Section 2 and generalize them to Lie groups in Section 3.

2. Matrix Inequalities

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be in $\mathbb{R}^n$. Let $x^{\downarrow} = (x_{[1]}, x_{[2]}, \ldots, x_{[n]})$ denote a rearrangement of the components of x such that $x_{[1]} \geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}$. We say that x is *majorized* by y, denoted by $x \prec y$, if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1 \quad \text{and} \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

Among many equivalent conditions for majorization, the following one is suitable for generalization to Lie groups [6]:

$$x \prec y \quad \Leftrightarrow \quad \operatorname{conv} S_n \cdot x \subset \operatorname{conv} S_n \cdot y,$$

where $\operatorname{conv} S_n \cdot x$ denotes the convex hull of the orbit of x under the action of the symmetric group S_n. When x and y are nonnegative, we say that x is *log-majorized* by y, denoted by $x \prec_{\log} y$ if

$$\prod_{i=1}^{k} x_{[i]} \leqslant \prod_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1 \quad \text{and} \quad \prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}.$$

In other words, when x and y are positive, $x \prec_{\log} y$ if and only if $\log x \prec \log y$.

Let $\mathcal{M}_n$ be the linear algebra of all $n \times n$ complex matrices, let $\mathcal{H}_n$ be the real subspace of all Hermitian matrices, let $\mathcal{P}_n$ be the set of all positive definite matrices in $\mathcal{M}_n$, and let $\mathcal{N}_n$ be the set of all normal matrices in $\mathcal{M}_n$.

For any $A \in \mathcal{M}_n$ and $m \in \mathbb{N}$, let

$$\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$$

denote the vector of eigenvalues of A whose absolute values are in decreasing order, let

$$s(A) = (s_1(A), \ldots, s_n(A))$$

denote the vector of singular values of A in decreasing order, let

$$[s(A)]^m = ([s_1(A)]^m, \ldots, [s_n(A)]^m),$$

and let $|A| = (A^*A)^{1/2}$ so that $\lambda(|A|) = s(A)$.

Motivated by the Golden-Thompson inequality (1.2) and problems in linear-quadratic optimal feedback control, Bernstein [2] proved the following inequality for all $A \in \mathcal{M}_n$

$$\operatorname{tr} e^{A^*} e^{A} \leqslant \operatorname{tr} e^{A^*+A}, \tag{2.1}$$

where A^* denotes the Hermitian adjoint of A. The Bernstein inequality (2.1) was generalized in [10, Theorem 3.5] as for all $A \in \mathcal{M}_n$

$$\lambda(e^{A^*}e^{A}) \prec_{\log} \lambda(e^{A^*+A}). \tag{2.2}$$

As generalizations of the Golden-Thompson inequality (1.2) and the Araki-Lieb-Thirring inequalities (1.5) and (1.6) to normal matrices, Liu [10, Theorems 3.7 and 3.9] obtained the following log-majorization relations for all $A, B \in \mathcal{N}_n$:

$$\lambda(|e^{A+B}|) \prec_{\log} \lambda(|e^{A}| \cdot |e^{B}|), \tag{2.3}$$

and

$$\lambda((|e^{A/2}| \cdot |e^{B}| \cdot |e^{A/2}|)^r) \prec_{\log} \lambda(|e^{rA/2}| \cdot |e^{rB}| \cdot |e^{rA/2}|), \qquad \forall\, r \geqslant 1, \tag{2.4}$$

$$\lambda(|e^{rA/2}| \cdot |e^{rB}| \cdot |e^{rA/2}|) \prec_{\log} \lambda((|e^{A/2}| \cdot |e^{B}| \cdot |e^{A/2}|)^r), \qquad \forall\, 0 \leqslant r \leqslant 1. \tag{2.5}$$

Finally, the following relations are true for all $A, B \in \mathcal{N}_n$ and $m \in \mathbb{N}$ [10, Theorem 3.3]:

$$s((AB)^m) \prec_{\log} [s(AB)]^m \prec_{\log} s(A^m B^m),$$

which is equivalent to

$$\lambda(|(AB)^m|) \prec_{\log} \lambda(|AB|^m) \prec_{\log} \lambda(|A^m B^m|). \tag{2.6}$$

The point is that (2.1)–(2.6) can all be generalized to Lie groups.

3. Generalization to Lie Groups

We first recall some algebraic structures of semisimple Lie groups [5, 7]. Let G be a noncompact connected semisimple Lie group with Lie algebra $\mathfrak{g}$, let $\Theta\colon G \to G$ be a Cartan involution G, and let K be the fixed point set of Θ, which is an analytic subgroup of G. Let $\theta = d\Theta$ be the differential map of Θ. Then $\theta : \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition, where $\mathfrak{k}$ is the eigenspace of θ corresponding to the eigenvalue 1 (and also the Lie algebra of K) and $\mathfrak{p}$ is the eigenspace of θ corresponding to the eigenvalue -1 (and also an $\operatorname{Ad} K$-invariant subspace of $\mathfrak{g}$ complementary to $\mathfrak{k}$). The Killing form B on $\mathfrak{g}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, and the bilinear form B_θ defined by

$$B_\theta(X, Y) = -B(X, \theta Y), \quad X, Y \in \mathfrak{g}$$

is an inner product on $\mathfrak{g}$.

For each $X \in \mathfrak{g}$, let $e^X = \exp X$ be the exponential of X. Let $P = \{e^X : X \in \mathfrak{p}\}$. The map $K \times \mathfrak{p} \to G$, defined by $(k, X) \mapsto ke^X$, is a diffeomorphism [5, VI.Theorem 1.1]. So each $g \in G$ can be uniquely written as

$$g = kp = k(g)p(g) \tag{3.1}$$

with $k = k(g) \in K$ and $p = p(g) \in P$. The decomposition $G = KP$ is a Cartan decomposition of G [5, p.253] [7, p.362].

Let $* : G \to G$ be the diffeomorphism defined by $*(g) = g^* = \Theta(g^{-1})$. Obviously, $(fg)^* = g^* f^*$ for all $f, g \in G$. The differential of $*$, also denoted by $*$, is just $-\theta$. Similar to the group case, we denote $*(X) = X^*$ for all $X \in \mathfrak{g}$. Thus $\mathfrak{p}$ is the eigenspace of $* : \mathfrak{g} \to \mathfrak{g}$ associated with the eigenvalue 1, and hence $X^* + X \in \mathfrak{p}$ for all $X \in \mathfrak{g}$.

Because K is the fix point set of Θ and $\exp_{\mathfrak{g}} : \mathfrak{p} \to P$ is bijective, we see that $k^* = k^{-1}$ for all $k \in K$ and $p^* = p$ for all $p \in P$. By the Cartan decomposition (3.1), we have for all $g \in G$

$$p(g) = (g^* g)^{1/2}. \tag{3.2}$$

An element $g \in G$ (resp., $X \in \mathfrak{g}$) is said to be *normal* if $g^* g = g g^*$ (resp., $[X^*, X] = 0$). It follows that if $X \in \mathfrak{g}$ is normal, then e^X is normal in G.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let A be the analytic subgroup generated by $\mathfrak{a}$. It is true that $\mathfrak{p} = \mathrm{Ad}\,(K)\mathfrak{a}$ [7, p.378]. The Weyl group W of $(\mathfrak{g}, \mathfrak{a})$ acts simply transitively on the Weyl chambers of $\mathfrak{a}$ (and also of A through the exponential map $\exp : \mathfrak{a} \to A$).

An element $X \in \mathfrak{g}$ is called real semisimple (resp., nilpotent) if $\mathrm{ad}\, X$ is diagonalizable over $\mathbb{R}$ (resp., nilpotent). An element $g \in G$ is called *hyperbolic* (resp., *unipotent*) if $g = \exp X$ for some real semisimple (resp., nilpotent) $X \in \mathfrak{g}$; in either case X is unique and we write $X = \log g$. An element $g \in G$ is called *elliptic* if $\mathrm{Ad}\, g$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1. According to [8, Proposition 2.1], each $g \in G$ can be uniquely written as

$$g = ehu, \tag{3.3}$$

where e is elliptic, h is hyperbolic, u is unipotent, and the three elements e, h and u commute. The decomposition (3.3) is called the *complete multiplicative Jordan decomposition*, abbreviated as CMJD.

For any real semisimple $X \in \mathfrak{g}$, let $W(X)$ denote the set of elements in $\mathfrak{a}$ that are conjugate to X, i.e.,

$$W(X) = \mathrm{Ad}\, G(X) \cap \mathfrak{a}.$$

It is known that $W(X)$ is a single W-orbit in $\mathfrak{a}$ [8, Proposition 2.4]. Let $\mathrm{conv}\, W(X)$ be the convex hull in $\mathfrak{a}$ generated by $W(X)$. For each $g \in G$, define

$$A(g) = \exp \mathrm{conv}\, W(\log h(g)),$$

where $h(g)$ is the hyperbolic component of g in its CMJD.

Kostant's pre-order $\prec$ on G is defined by setting $f \prec g$ if [8, p.426]

$$A(f) \subset A(g).$$

This pre-order induces a partial order on the conjugacy classes of G. It does not depend on the choice of $\mathfrak{a}$ due to the following result.

Lemma 3.1. *(Kostant [8, Theorem 3.1]) Let $f, g \in G$. Then $f \prec g$ if and only if*

$$\rho(\pi(f)) \leqslant \rho(\pi(g))$$

for every irreducible finite dimensional representation π of G, where $\rho(\pi(g))$ denotes the spectral radius of the operator $\pi(g)$.

Example 3.2. See [14, Proposition 2.2] for the example $G = \mathrm{SL}_n(\mathbb{C})$ in which $\prec$ becomes log majorization. For $A, B \in \mathrm{SL}_n(\mathbb{C})$,

$$A \prec B \text{ if and only if } |\lambda(A)| \prec_{\log} |\lambda(B)|,$$

where $|\lambda(A)| = (|\lambda_1(A)|, \ldots, |\lambda_n(A)|)$ with the components in decreasing order. When $A, B \in P$ (i.e., A and B are positive definite), we have $\lambda(A) = s(A) > 0$ and $\lambda(B) = s(B) > 0$, so $A \prec B$ if and only if $\lambda(A) \prec_{\log} \lambda(B)$.

We first generalize the inequality (2.1) and (2.2) to Lie group G.

Theorem 3.3. *Let $X \in \mathfrak{g}$ be arbitrary. Then*

$$e^{X^*}e^X \prec e^{X^*+X}. \tag{3.4}$$

Moreover, if χ is the associated character of any irreducible representation of G, then

$$\chi(e^{X^*}e^X) \leqslant \chi(e^{X^*+X}). \tag{3.5}$$

Proof. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ be the complxcification of $\mathfrak{g}$. Since $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition,

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{u} + i\mathfrak{u} = (\mathfrak{k} + i\mathfrak{p}) + (\mathfrak{p} + i\mathfrak{k})$$

is a Cartan decomposition of $\mathfrak{g}_{\mathbb{C}}$ (viewed as a real semisimple Lie algebra), where $u = \mathfrak{k} + i\mathfrak{p}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$. Let $\pi : G \to \mathrm{Aut}\, V$ be any irreducible representation π of G. Then its differential $d\pi : \mathfrak{g} \to \mathrm{End}\, V$ is a representation of $\mathfrak{g}$. Because $\mathfrak{u}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$, there exists a unique (up to scalar) inner product $\langle \cdot, \cdot \rangle$ on V such that $d\pi(X)$ is skew-Hermitian for all $X \in \mathfrak{u}$ and $d\pi(Y)$ is Hermitian for all $Y \in i\mathfrak{u}$ [8, p.435]. We assume henceforth that V is given this inner product. Hence $\pi(k)$ is unitary for all $k \in K$ and $\pi(p)$ is positive definite for all $p \in P$ by naturality of the exponential map.

Note that for all $g \in G$,

$$\pi(g^*) = (\pi(g))^*,$$

where $(\pi(g))^*$ denotes the adjoint operator of $\pi(g)$. This is because if $g = kp$ with $k \in K$ and $p \in P$, then

$$\pi(g^*) = \pi(pk^{-1}) = \pi(p)(\pi(k))^{-1} = (\pi(p))^*(\pi(k))^* = (\pi(k)\pi(p))^* = (\pi(g))^*.$$

Consequently, $d\pi(X^*) = (d\pi(X))^*$ for all $X \in \mathfrak{g}$.

Note also that both sides of (3.4) are in P: $e^{X^*}e^X \in P$ because $e^{X^*} = (e^X)^*$, which follows from naturality of the exponential map, and $e^{X^*+X} \in P$ because $X^* + X \in \mathfrak{p}$.

Therefore, to prove (3.4), it suffices to show by Lemma 3.1 that

$$\rho(\pi(e^{X^*}e^X)) = \lambda_1(\pi(e^{X^*}e^X)) \leqslant \lambda_1(\pi(e^{X^*+X})) = \rho(\pi(e^{X^*+X})).$$

In fact,

$$\begin{aligned}\lambda_1(\pi(e^{X^*}e^X)) &= \lambda_1(\pi(e^{X^*})\pi(e^X))\\ &= \lambda_1(e^{(d\pi(X))^*}e^{d\pi(X)})\\ &\leqslant \lambda_1(e^{(d\pi(X))^*+d\pi(X)}) \qquad \text{(by (2.2))}\\ &= \lambda_1(e^{d\pi(X^*+X)})\\ &= \lambda_1(\pi(e^{X^*+X})).\end{aligned}$$

Thus (3.4) is valid. Then (3.5) follows from (3.4) by [8, Theorem 6.1]. ∎

Recall (3.2) that $p(g) = (g^*g)^{1/2}$ for each $g \in G$. So $p^2(g) = [p(g)]^2 = g^*g$. The next result generalizes (1.1) and (2.3).

Theorem 3.4. *If $X, Y \in \mathfrak{g}$ are normal, then*

$$p^2(e^{X+Y}) \prec p^2(e^X)p^2(e^Y). \tag{3.6}$$

In particular, if $X, Y \in \mathfrak{p}$, then (3.6) reduces to (1.1).

Proof. It is true that for all $g \in G$ and $m \in \mathbb{N}$ [14, Theorem 3.1]

$$(g^m)^*g^m = (g^*)^m g^m \prec (g^*g)^m. \tag{3.7}$$

Applying (3.7) on $g = e^{(X+Y)/m}$, we get

$$e^{(X+Y)^*}e^{X+Y} \prec \left(e^{(X+Y)^*/m}e^{(X+Y)/m}\right)^m, \qquad \forall m \in \mathbb{N}. \tag{3.8}$$

Then the Lie product formula and (3.8) and (1.1) yield

$$e^{(X+Y)^*}e^{X+Y} \prec e^{(X+Y)^*+(X+Y)} = e^{(X^*+X)+(Y^*+Y)} \prec e^{X^*+X}e^{Y^*+Y}.$$

Now because X and Y are normal,

$$p^2(e^{X+Y}) = e^{(X+Y)^*}e^{X+Y} \prec e^{X^*+X}e^{Y^*+Y} = e^{X^*}e^Xe^{Y^*}e^Y = p^2(e^X)p^2(e^Y).$$

If $X, Y \in \mathfrak{p}$, then $X + Y \in \mathfrak{p}$ and $p^2(e^X) = e^{2X}$, hence (3.6) reduces to (1.1). ∎

The next result generalizes (2.4) and (2.5).

Theorem 3.5. *If $X, Y \in \mathfrak{g}$ are normal, then*

$$\left(p(e^{X/2})p(e^Y)p(e^{X/2})\right)^r \prec p(e^{rX/2})p(e^{rY})p(e^{rX/2}), \qquad \forall r \geqslant 1, \tag{3.9}$$

$$p(e^{rX/2})p(e^{rY})p(e^{rX/2}) \prec \left(p(e^{X/2})p(e^Y)p(e^{X/2})\right)^r, \qquad \forall 0 \leqslant r \leqslant 1. \tag{3.10}$$

In particular, if $X, Y \in \mathfrak{p}$, then (3.9) and (3.10) reduce to (2.4) and (2.5), respectively.

Proof. Because X and Y are normal, $p(e^Y) = \left((e^Y)^*e^Y\right)^{1/2} = e^{(Y^*+Y)/2} \in P$ and $p(e^{X/2}) = e^{(X^*+X)/4} \in P$. Thus (3.9) and (3.10) follow from (2.4) and (2.5), respectively. If $X, Y \in \mathfrak{p}$, (3.9) and (3.10) are exactly (2.4) and (2.5). ∎

The next result generalizes (2.6).

Theorem 3.6. *If $f, g \in G$ are normal and $m \in \mathbb{N}$, then*

$$p((fg)^m) \prec p^m(fg) \prec p(f^m g^m). \tag{3.11}$$

Proof. Note that (3.11) is equivalent to

$$((fg)^m)^*(fg)^m \prec ((fg)^*(fg))^m \prec (f^m g^m)^*(f^m g^m). \tag{3.12}$$

This first relation in (3.12) follows by applying (3.7) on fg. To prove the second relation in (3.12), we apply the technique as in the proof of Theorem 3.3.

Let $\pi : G \to \operatorname{Aut} V$ be any irreducible representation π of G. We assume that V is an inner product such that $\pi(k)$ is unitary for all $k \in K$, $\pi(p)$ is positive definite for all $p \in P$, and $\pi(n)$ is normal for all normal $n \in G$. Since $f, g \in G$ are normal by assumption, $\pi(f)$ and $\pi(g)$ are normal matrices. Thus we have

$$\begin{aligned}
\rho(\pi(((fg)^*(fg))^m)) &= \lambda_1(\pi(((fg)^*(fg))^m)) \\
&= \lambda_1(((\pi(f)\pi(g))^*(\pi(f)\pi(g)))^m) \\
&\leqslant \lambda_1(((\pi(f))^m(\pi(g))^m)^*(\pi(f))^m(\pi(g))^m) \qquad \text{by (2.6)} \\
&= \lambda_1(\pi((f^m g^m)^*(f^m g^m))) \\
&= \rho(\pi((f^m g^m)^*(f^m g^m))).
\end{aligned}$$

Therefore, the second relation in (3.12) is valid by Lemma 3.1. ∎

Remark 3.7. The Cartan decomposition (3.1) is a generalization of the polar decomposition for matrices to Lie groups. There is another Lie group decomposition, also called Cartan decomposition, as a generalization of the singular value decomposition for matrices as follows. Let $\mathfrak{a}_+$ be a fixed closed Weyl chamber in $\mathfrak{a}$. Set $A_+ = \exp \mathfrak{a}_+$. Then every element in P is K-conjugate to a unique element in A_+ and each $g \in G$ can be written as

$$g = u a_+(g) v, \tag{3.13}$$

where $u, v \in K$ and $a_+(g) \in A_+$ is uniquely determined by g. The decomposition $G = KA_+K$ is also called Cartan decomposition [5, p.402].

There are majorization relations in terms of Kostant's pre-order expressed in the form of $a_+(g)$ [13]. We remark that Theorems 3.4, 3.5, and 3.6 can also be formulated in the form of $a_+(g)$, instead of $p(g)$. See [11] for some other relations about $p(g)$, which can be formulated in the form of $a_+(g)$ as well.

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