Journal of Lie Theory
Volume **27** (2016) 193–215

On the Construction of Simply Connected Solvable Lie Groups

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Communicated by P. Olver

Abstract. Let $\omega_{\mathfrak{g}}$ be a Lie algebra valued differential 1-form on a manifold M satisfying the structure equations $d\omega_{\mathfrak{g}} + \frac{1}{2}\omega_{\mathfrak{g}} \wedge \omega_{\mathfrak{g}} = 0$, where $\mathfrak{g}$ is a solvable real Lie algebra. We show that the problem of finding a smooth map $\rho\colon M \to G$, where G is an n-dimensional solvable real Lie group with Lie algebra $\mathfrak{g}$ and left invariant Maurer-Cartan form τ, such that $\rho^*\tau = \omega_{\mathfrak{g}}$ can be solved by quadratures and the matrix exponential. In the process, we give a closed form formula for the vector fields in Lie's third theorem for solvable Lie algebras. A further application produces the multiplication map for a simply connected n-dimensional solvable Lie group using only the matrix exponential and n quadratures. Applications to finding first integrals for completely integrable Pfaffian systems with solvable symmetry algebras are also given.
Mathematics Subject Classification 2010: Primary 22E25; Secondary 58A15, 58J70, 34A26.
Key Words and Phrases: Solvable Lie algebras, solvable Lie groups, Lie's third theorem, first integrals.

1. Introduction

Let M be an m-dimensional manifold and let ω^i, $i = 1, \ldots, n$ be a collection of n differential 1-forms on M satisfying

$$d\omega^i + \frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k = 0, \tag{1}$$

where C^i_{jk} are the structure constants of an n-dimensional real Lie algebra $\mathfrak{g}$. Given $x \in M$ there exists an open set U about x and a smooth map $\rho : U \to G$, where G is a Lie group with a basis of left invariant forms τ^i satisfying (1) on G, such that

$$\omega^i = \rho^* \tau^i. \tag{2}$$

If M is simply connected, then U can be taken to be all of M. These facts and many others related to equations (1) and (2) are proven in Chapter 3 of [15].

ISSN 0949–5932 / \$2.50

Equation (1) occurs in a number of applications such as the integration of Pfaffian systems of finite type with symmetry (see [3], [5] and [4]), the reconstruction problem for quotients of Pfaffian systems (see [1] and [5]), and the method of moving frames [10]. Equation (1) also occurs in Lie's third theorem. One version of this theorem (see Section 4) states that, given a Lie algebra $\mathfrak{g}$ with structure constants C^i_{jk}, there exist n pointwise linearly independent 1-forms ω^i on $\mathbb{R}^n$ satisfying equation (1).

This article addresses how quadrature and matrix algebra can be used to find the map $\rho : M \to G$ in equation (2) when $\mathfrak{g}$ is a solvable Lie algebra. The solution to this problem leads to an algorithm that can be used to explicitly construct simply connected solvable Lie groups from their Lie algebra using matrix exponentiation and quadrature. By quadrature we mean the following (see [4]). Given a closed differential 1-form ω on a simply connected manifold M, then $\omega = df$ for a smooth function $f : M \to \mathbb{R}$, where f can be determined from ω by integration (quadrature) along a path from a fixed point (the Poincare Lemma for 1-forms).

Our starting point is Theorem 2.2 in Section 2 which shows that if a Lie algebra $\mathfrak{g}$ admits a codimension one ideal $\mathfrak{k} \subset \mathfrak{g}$, then the structure equations in (1) can be reduced to $\mathfrak{k} \times \mathbb{R}$ using one quadrature and the matrix exponential. If $\mathfrak{g}$ is a solvable Lie algebra, a simple induction argument can then be applied which reduces the structure equations in (1) to $d\tilde{\omega}^i = 0$ (Corollary 2.4).

In Section 3 we show how the results in Section 2 can be used to construct first integrals for a completely integrable Pfaffian system with a solvable symmetry algebra using quadrature and matrix exponentiation.

In Section 4 we reverse the argument from Corollary 2.4 in Section 2 to produce a proof of Lie's Third theorem for vector fields. This proof gives an *explicit closed form* for the vector fields (see Theorem 4.2 and Corollary 4.3). In comparison, the proof of the vector field version of Lie's third theorem given by Cartan, which is valid for any Lie algebra, uses ODE existence theorems (see [8]) and does not give closed form formulas.

In Section 5 our main goal is to show that the multiplication map $\mu : G \times G \to G$ for any simply connected solvable Lie group G can be constructed by n quadratures and the matrix exponential (Theorem 5.7). This is done in 3 steps.

The first step is given in Section 5.1, where we show that $\mathbb{R}^n$ with the solution to Lie's third theorem for vector fields from Section 4 can be given a global Lie group structure such that the 1-forms $\{\tau^i\}_{1\leq i\leq n}$ in Theorem 4.2 are a basis for the left invariant 1-forms. This provides an alternate proof to the group version of Lie's third theorem for solvable algebras along with an alternate proof that a simply connected solvable Lie group is topologically $\mathbb{R}^n$.

The second step is given in Section 5.2, where we return to the problem of finding a map $\rho : M \to G$ satisfying (2) when G is solvable and M is simply connected. Again, the function ρ can be found using only quadrature and the matrix exponential as shown in Theorem 5.4.

The last step is given in Section 5.3, where we solve the problem of finding the multiplication map μ for the simply connected solvable Lie group G, whose existence in guaranteed from step one. We construct (algebraically) a set of n

1-forms ω^i on $G \times G$ which satisfy equation (1) with the property that the map $\rho : G \times G \to G$ from Theorem 5.4 in Section 5.2 is the multiplication map (Theorem 5.7). This gives an algorithm that produces the global multiplication map $\mu : G \times G \to G$ for any n-dimensional simply connected solvable Lie group G using n quadratures and the product of matrix exponentials.

The calculations for this paper were performed using the Maple DifferentialGeometry package. Interested readers can access http://digitalcommons.usu.edu/dg/ to download the current release of the DifferentialGeometry package, the worksheets which implement Theorems 4.2, 5.4, 5.7, and the examples.

2. The Reduction Theorem

Let $\mathfrak{g}$ be an n-dimensional real Lie algebra and suppose that $\mathfrak{k} \subset \mathfrak{g}$ is a codimension one ideal. With this hypothesis on $\mathfrak{g}$, Theorem 2.2 below uses one quadrature and the matrix exponential to modify a set of differential 1-forms satisfying equation (1) so that the resulting forms satisfy equation (1) with the structure constants for the Lie algebra $\mathfrak{k} \times \mathbb{R}$.

Let $\mathfrak{k} \subset \mathfrak{g}$ denote a codimension one ideal in the n-dimensional Lie algebra $\mathfrak{g}$. A basis $\beta = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}, \boldsymbol{e}_n\}$ for $\mathfrak{g}$ such that $\hat{\beta} = \{\boldsymbol{e}_a\}_{1 \le a \le n-1}$ is a basis for $\mathfrak{k}$ is **adapted to the codimension one ideal** $\mathfrak{k} \subset \mathfrak{g}$. In this case the structure constants are

$$[\boldsymbol{e}_a, \boldsymbol{e}_b] = C^c_{ab}\boldsymbol{e}_c, \quad [\boldsymbol{e}_n, \boldsymbol{e}_a] = C^b_{na}\boldsymbol{e}_b \quad , 1 \le a, b, c \le n. \tag{3}$$

Note that C^a_{bc} in equation (3) are the structure constants of the $(n-1)$-dimensional Lie algebra $\mathfrak{k}$ in the basis $\hat{\beta}$.

This leads to a well-known lemma (see [4]).

Lemma 2.1. *Let $\mathfrak{k} \subset \mathfrak{g}$ be a codimension one ideal and let $\beta = \{\boldsymbol{e}_a, \boldsymbol{e}_n\}_{1 \le a \le n}$ be an adapted basis. Let $\tilde{\omega}^i$ be n differential forms on a manifold M satisfying equation* (1) *for the Lie algebra $\mathfrak{g}$, where the structure constants $\tilde{C}^i_{jk}$ are given in a basis $\tilde{\beta} = \{\mathbf{f}_i\}_{1 \le i \le n}$, and let $[P^i_j]$ be the change of basis matrix*

$$\mathbf{f}_j = P^i_j \mathbf{e}_i.$$

Then $\omega^i = P^i_j \tilde{\omega}^j$ satisfy the structure equations

$$\begin{aligned} d\omega^a + \frac{1}{2}C^a_{bc}\omega^b \wedge \omega^c + C^a_{nb}\omega^n \wedge \omega^b = 0 \quad \text{and} \\ d\omega^n = 0, \end{aligned} \tag{4}$$

where C^a_{bc} and C^n_{nb} are the structure constants of the Lie algebra $\mathfrak{g}$ in the basis β (equation (3)*).*

Let ω^i be n differential 1-forms on a manifold M satisfying the structure equation (1), where $\mathfrak{g}$ admits a codimension one ideal. Lemma 2.1 shows how to find a constant linear change in the forms ω^i so that the resulting forms satisfy equations (4). Equation (4) is then the starting point for the following reduction theorem which is fundamental in this article.

Theorem 2.2. *Suppose $\mathfrak{g}$ is an n-dimensional Lie algebra with codimension one ideal $\mathfrak{k} \subset \mathfrak{g}$, and let $\beta = \{\boldsymbol{e}_a, \boldsymbol{e}_n\}_{1\le a\le n-1}$ be a basis adapted to the codimension one ideal. Let $\boldsymbol{\omega} = [\omega^1, \ldots, \omega^n]^T$ be an n-vector of differential 1-forms on a simply connected manifold M satisfying (4), where C^i_{jk} are the structure constants of $\mathfrak{g}$ in the basis β.*

Let $f : M \to \mathbb{R}$ so that ω^n in (4) satisfies

$$\omega^n = df \tag{5}$$

and let $\hat{\boldsymbol{\omega}}$ be the n-vector of 1-forms on M defined by

$$\hat{\boldsymbol{\omega}} = \mathrm{e}^{[\mathrm{ad}(\mathbf{e}_n)]f} \cdot \boldsymbol{\omega}, \tag{6}$$

where $[\mathrm{ad}(\mathbf{e}_n)]$ is the n by n matrix representation of $\mathrm{ad}(\mathbf{e}_n)$ in the basis β. Then the structure equations for $\hat{\boldsymbol{\omega}}$ are

$$d\hat{\omega}^a + \frac{1}{2} C^a_{bc} \hat{\omega}^b \wedge \hat{\omega}^c = 0 \quad \text{and} \quad d\hat{\omega}^n = 0, \tag{7}$$

where C^a_{bc} are the structure constants of the Lie algebra $\mathfrak{k}$ in the basis $\hat{\beta} = \{\mathbf{e}_a\}_{1\le a\le n-1}$.

Proof. The matrix representation of the derivation $\mathrm{ad}(\mathbf{e}_n) : \mathfrak{g} \to \mathfrak{g}$ is computed in the adapted basis from equation (3) to be

$$\mathrm{ad}(\mathbf{e}_n)(\boldsymbol{e}_b) = [\boldsymbol{e}_n, \boldsymbol{e}_b] = C^a_{nb}\boldsymbol{e}_a, \quad \mathrm{ad}(\mathbf{e}_n)(\boldsymbol{e}_n) = [\boldsymbol{e}_n, \boldsymbol{e}_n] = 0. \tag{8}$$

Let $\widetilde{\mathrm{ad}}(\boldsymbol{e}_n) : \mathfrak{k} \to \mathfrak{k}$ be the restriction of $\mathrm{ad}(\mathbf{e}_n)$ to the invariant subspace $\mathfrak{k}$. The $n-1 \times n-1$ matrix representation $[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]$ of $\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)$ in the basis β is computed from (8) to be

$$[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]^a_b = C^a_{nb}. \tag{9}$$

The $n \times n$ matrix valued function $\mathrm{e}^{[\mathrm{ad}(\mathbf{e}_n)]f}$ in M in equation (6) can then be written as

$$\mathrm{e}^{[\mathrm{ad}(\mathbf{e}_n)]f} = \begin{bmatrix} \mathrm{e}^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]f} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}, \tag{10}$$

where $\mathbf{0}$ is the $n-1$ zero vector and $\mathbf{0}^T$ is its transpose. The forms $\hat{\boldsymbol{\omega}}$ in equation (6) can then be written using equation (10) (and matrix vector multiplication) as

$$[\hat{\omega}^a] = \mathrm{e}^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]f}[\omega^b], \quad \hat{\omega}^n = \omega^n. \tag{11}$$

The verification of equation (7) in Theorem 2.2 can now be checked directly by taking the exterior derivative of the forms $\hat{\omega}^a$ in equation (11). In order to do this, we first compute the exterior derivative of $\mathrm{e}^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]f}$, where $[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]$ is defined in equation (9), to be

$$d\left(\mathrm{e}^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]f} \right) = df\, \mathrm{e}^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]f} [\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]. \tag{12}$$

We now compute $d\hat{\omega}^a$ using equations (11), (12), (4), (9) and (5) to be

$$\begin{aligned}[][d\hat{\omega}^a] &= \mathrm{e}^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]f}[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)][-\omega^b \wedge df] + \mathrm{e}^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]f}[d\omega^b] \\ &= \mathrm{e}^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]f}[-C^c_{nb}\omega^b \wedge \omega^n] - \mathrm{e}^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]f}[\frac{1}{2}C^b_{cd}\omega^c \wedge \omega^d + C^b_{cn}\omega^c \wedge \omega^n] \\ &= \mathrm{e}^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]f}[-\frac{1}{2}C^b_{cd}\omega^c \wedge \omega^d] \\ &= [-\frac{1}{2}C^a_{cd}\hat{\omega}^c \wedge \hat{\omega}^d],\end{aligned} \tag{13}$$

where the last line follows because $\mathrm{e}^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]f}$ is the matrix representation of an automorphism of $\mathfrak{k}$ for each $x \in M$. ■

The computation leading to Theorem 2.2 uses one quadrature in equation (5) and the matrix exponential (6). We also note that the change of forms in equation (6) results in the elimination of the third term C^a_{nb} in the first structure equation in (4). Since the constants C^a_{bc} in equation (7) are the structure constants for $\mathfrak{k}$ in the basis $\{\boldsymbol{e}_a\}_{1\le a\le n-1}$, equation (6) can then be thought of as reducing the Lie algebra $\mathfrak{g}$ to $\mathfrak{k} \times \mathbb{R}$. (See also Remark 2.6 below.)

We now apply Theorem 2.2 to the case where $\mathfrak{g}$ admits a sequence of subalgebras $\mathfrak{k}_s \subset \mathfrak{g}$, $s = 0, \dots, r$ that satisfy

$$\mathfrak{k}_r \subset \mathfrak{k}_{r-1} \subset \dots \mathfrak{k}_1 \subset \mathfrak{k}_0 = \mathfrak{g}, \tag{14}$$

where each $\mathfrak{k}_s \subset \mathfrak{k}_{s-1}$, $1 \le s \le r$ is a codimension one ideal. We call the sequence in equation (14) **a sequence of relative ideals of codimension one**. Note that $\dim \mathfrak{k}_s = \dim \mathfrak{g} - s = n - s$. A basis $\beta = \{\boldsymbol{e}_i\}_{1\le i\le n}$ for $\mathfrak{g}$ is said to be **adapted to a sequence of relative ideals of codimension one** $\mathfrak{k}_s \subset \mathfrak{k}_{s-1}, s = 1, \dots, r$, if

$$\mathfrak{k}_s = \mathrm{span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-s}\}, \quad \text{for} \quad s = 0, \dots, r. \tag{15}$$

Given the basis $\beta = \{\boldsymbol{e}_i\}_{1\le i\le n}$ adapted to the sequence of relative ideals of codimension one $\mathfrak{k}_s \subset \mathfrak{k}_{s-1}, s = 1, \dots, n$, we let $\mathrm{ad}_s(\boldsymbol{e}_{n-s}) \in \mathrm{Der}(\mathfrak{k}_s)$ be the restriction of $\mathrm{ad}(\boldsymbol{e}_{n-s})$ to the invariant subspace $\mathfrak{k}_s$,

$$\mathrm{ad}_s(\boldsymbol{e}_{n-s}) = \mathrm{ad}(\boldsymbol{e}_{n-s})|_{\mathfrak{k}_s}, \quad 0 \le s \le r-1. \tag{16}$$

In particular, $\mathrm{ad}_0(\boldsymbol{e}_n) = \mathrm{ad}(\boldsymbol{e}_n)$.

Theorem 2.2 easily extends to the case of a sequence of relative ideals of codimension one giving the following corollary.

Corollary 2.3. *Suppose the n-dimensional Lie algebra $\mathfrak{g}$ in Theorem 2.2 admits a sequence of relative ideals of codimension one $\mathfrak{k}_s \subset \mathfrak{k}_{s-1}$, $s = 1, \dots, r$, and let $\beta = \{\boldsymbol{e}_i\}_{1\le i\le n}$ be a basis adapted to the sequence. Let ω^i be n differential 1-forms on a simply connected manifold M satisfying (1), where the structure constants are computed in the basis β. Then there exist r functions $f^t : M \to \mathbb{R}$, $t = n - r + 1, \dots, n$ and $n - r$ differential 1-forms $\{\check{\omega}^\alpha\}_{1\le\alpha\le n-r}$*

obtained inductively from ω^i *by* r *quadratures and the exponential of matrices such that*

$$\begin{bmatrix} \mathrm{e}^{[\mathrm{ad}_{r-1}(\boldsymbol{e}_{n-r+1})]f^{n-r+1}} & \mathbf{0}_{r-1} \\ \mathbf{0}_{r-1}^T & \mathrm{I}_{r-1} \end{bmatrix} \cdots \begin{bmatrix} \mathrm{e}^{[\mathrm{ad}_1(\boldsymbol{e}_{n-1})]f^{n-1}} & \mathbf{0}_1 \\ \mathbf{0}_1^T & 1 \end{bmatrix} \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_n)]f^n} \begin{bmatrix} \omega^1 \\ \vdots \\ \omega^n \end{bmatrix} = \begin{bmatrix} \check{\omega}^1 \\ \vdots \\ \check{\omega}^{n-r} \\ df^{n-r+1} \\ \vdots \\ df^n \end{bmatrix}, \tag{17}$$

where $[\mathrm{ad}_s(\boldsymbol{e}_{n-s})]$ *is the* $n-s$ *by* $n-s$ *matrix representation of* $\mathrm{ad}_s(\boldsymbol{e}_{n-s}) \in \mathrm{Der}(\mathfrak{k}_s)$ *defined in* (16) *in the basis* (15) *for* $\mathfrak{k}_s$, I_s *is the* s *by* s *identity matrix, and* $\mathbf{0}_s$ *is the* $n-s$ *by* s *zero matrix.*

Furthermore the $n-r$ *differential* 1*-forms* $\check{\omega}^\alpha$ *satisfy*

$$d\check{\omega}^\alpha + \frac{1}{2} C^\alpha_{\beta\gamma} \check{\omega}^\beta \wedge \check{\omega}^\gamma = 0, \tag{18}$$

where $C^\alpha_{\beta\gamma}$ *are the structure constants for the* $(n-r)$*-dimensional Lie algebra* $\mathfrak{k}_r$ *in the basis* $\check{\beta} = \{\boldsymbol{e}_\alpha\}_{1 \le \alpha \le n-r}$.

Proof. Using Theorem 2.2 we inductively compute the functions f^t, $n-r+1 \le t \le n$. First we let $f^n : M \to \mathbb{R}$ be the function obtained in Theorem 2.2 using $\mathfrak{k} = \mathfrak{k}_1$ so that $\omega^n = df^n$. The first term in equation (17) is

$$\mathrm{e}^{[\mathrm{ad}(\mathbf{e}_n)]f^n} \begin{bmatrix} \omega^1 \\ \vdots \\ \omega^n \end{bmatrix} = \begin{bmatrix} \hat{\omega}^1 \\ \vdots \\ \hat{\omega}^{n-1} \\ df^n \end{bmatrix}. \tag{19}$$

Because $\mathfrak{k}_2 \subset \mathfrak{k}_1$ is an ideal in $\mathfrak{k}_1$ of codimension one, and the basis $\{\boldsymbol{e}_i\}_{1 \le i \le n}$ is adapted to the sequence (14), we now have

$$d\hat{\omega}^{n-1} = 0.$$

Again letting $f^{n-1} : M \to \mathbb{R}$ so that $\hat{\omega}^{n-1} = df^{n-1}$, and applying Theorem 2.2 to the case of $\mathfrak{k}_2 \subset \mathfrak{k}_1$, we get

$$\mathrm{e}^{[\mathrm{ad}_1(\boldsymbol{e}_{n-1})]f^{n-1}} \begin{bmatrix} \hat{\omega}^1 \\ \vdots \\ \hat{\omega}^{n-1} \end{bmatrix} = \begin{bmatrix} \tilde{\omega}^1 \\ \vdots \\ \tilde{\omega}^{n-2} \\ df^{n-1} \end{bmatrix}. \tag{20}$$

The $n-2$ differential forms $\tilde{\omega}^u$ then satisfy

$$d\tilde{\omega}^u + \frac{1}{2} C^u_{vw} \tilde{\omega}^v \wedge \tilde{\omega}^w = 0,$$

where C^u_{vw} are the structure constants for $\mathfrak{k}_2$ in the basis $\tilde{\beta} = \{\boldsymbol{e}_u\}_{1 \le u \le n-2}$. The use of equations (20) and (19) produces the rightmost two matrices on the left-hand side of equation (17).

Continuing by induction, this produces the $n-r$ functions $f^t : M \to \mathbb{R}$ and the matrices in equation (17) using $n-r$ quadratures and matrix exponentiation. Furthermore, the forms $\breve{\boldsymbol{\omega}}$ in equation (17) satisfy the structure equation (18). ■

If $\mathfrak{g}$ is an n-dimensional solvable Lie algebra, it is well known that $\mathfrak{g}$ admits a sequence of relative ideals of codimension one, $\mathfrak{k}_s \subset \mathfrak{k}_{s-1}$, $s = 1, \ldots, n$, (Corollary 3.7.5 in [17]). For example, if $\beta = \{\boldsymbol{e}_i\}_{1\le i\le n}$ is a basis adapted to the derived series of $\mathfrak{g}$, then $\mathfrak{k}_s = \text{span}\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-s}\}, s = 0, \ldots, n$ is a sequence of relative ideals of codimension 1 and β is a basis adapted to the sequence $\mathfrak{k}_s$. Note that $\dim \mathfrak{k}_n = 0$ and that the subalgebras $\mathfrak{k}_s \subset \mathfrak{g}$ need not be ideals when $s > 1$.

Corollary 2.4. *Let $\mathfrak{g}$ be an n-dimensional solvable Lie algebra, and let $\beta = \{\boldsymbol{e}_i\}_{1\le i\le n}$ be a basis adapted to a sequence of relative ideals of codimension one $\mathfrak{k}_s \subset \mathfrak{k}_{s-1}$, $s = 1, \ldots, n$. Let ω^i be n differential 1-forms on a simply connected manifold M satisfying equation* (1)*, where the structure constants are computed in the basis β.*

There exist n functions $f^i : M \to \mathbb{R}$, $i = 1, \ldots, n$, obtained inductively from the forms ω^i using n quadratures and the exponential of matrices so that

$$\begin{bmatrix} \mathrm{e}^{[\mathrm{ad}_{n-2}(\boldsymbol{e}_2)]f^2} & \mathbf{0}_{n-2} \\ \mathbf{0}_{n-2}^T & \mathrm{I}_{n-2} \end{bmatrix} \cdots \begin{bmatrix} \mathrm{e}^{[\mathrm{ad}_1(\boldsymbol{e}_{n-1})]f^{n-1}} & \mathbf{0}_1 \\ \mathbf{0}_1^T & 1 \end{bmatrix} \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_n)]f^n} \begin{bmatrix} \omega^1 \\ \vdots \\ \omega^n \end{bmatrix} = \begin{bmatrix} df^1 \\ \vdots \\ df^n \end{bmatrix}, \quad (21)$$

where for $s = 0, \ldots, n-2$, $[\mathrm{ad}_s(\boldsymbol{e}_{n-s})]$ is the $n-s$ by $n-s$ matrix representation of $\mathrm{ad}_s(\boldsymbol{e}_{n-s}) \in \mathrm{Der}(\mathfrak{k}_s)$ defined in (16) *in the basis* (15) *for $\mathfrak{k}_s$, I_s is the s by s identity matrix, and $\mathbf{0}_s$ is the $n-s$ by s zero matrix.*

Remark 2.5. The function $f : M \to \mathbb{R}$ in Theorem 2.2 is uniquely determined if the value of f is prescribed at a point. This is also true for the functions in Corollaries 2.3 and 2.4.

Remark 2.6. Theorem 2.2 can also be described as follows. Given differential 1-forms $\{\omega^i\}_{1\le i\le n}$ on a manifold M satisfying (1) and a basis $\beta = \{\boldsymbol{e}_i\}_{1\le i\le n}$ for the Lie algebra $\mathfrak{g}$, define the $\mathfrak{g}$ valued differential 1-form $\boldsymbol{\omega}_\mathfrak{g}$ as

$$\boldsymbol{\omega}_\mathfrak{g} = \omega^i \otimes \boldsymbol{e}_i.$$

Equation (1) is then written

$$d\boldsymbol{\omega}_\mathfrak{g} + \frac{1}{2}\boldsymbol{\omega}_\mathfrak{g} \wedge \boldsymbol{\omega}_\mathfrak{g} = 0, \quad (22)$$

where the wedge product $\boldsymbol{\omega}_\mathfrak{g} \wedge \boldsymbol{\omega}_\mathfrak{g}$ is defined as

$$\boldsymbol{\omega}_\mathfrak{g} \wedge \boldsymbol{\omega}_\mathfrak{g}(X, Y) = [\boldsymbol{\omega}_\mathfrak{g}(X), \boldsymbol{\omega}_\mathfrak{g}(Y)]_\mathfrak{g} \qquad \text{for all } X, Y \in TM. \quad (23)$$

Suppose now that $\mathfrak{k} \subset \mathfrak{g}$ is a codimension one ideal and that $\boldsymbol{\omega}_\mathfrak{g}$ is a Lie algebra valued form on a simply connected manifold M satisfying the structure equations $d\boldsymbol{\omega}_\mathfrak{g} + \frac{1}{2}\boldsymbol{\omega}_\mathfrak{g} \wedge \boldsymbol{\omega}_\mathfrak{g} = 0$. Let $A : M \to \mathrm{Aut}(\mathfrak{g})$ be $A(x) = \exp\left(\mathrm{ad}(\mathbf{e}_n) f(x)\right)$, where $f : M \to \mathbb{R}$ is defined in (5). Theorem 2.2 shows that $\hat{\boldsymbol{\omega}}_{\mathfrak{k}\times\mathbb{R}} = A \circ \omega$ takes values in the Lie algebra $\mathfrak{k} \times \mathbb{R}$ and satisfies

$$d\,\hat{\boldsymbol{\omega}}_{\mathfrak{k}\times\mathbb{R}} + \frac{1}{2}\hat{\boldsymbol{\omega}}_{\mathfrak{k}\times\mathbb{R}} \wedge \hat{\boldsymbol{\omega}}_{\mathfrak{k}\times\mathbb{R}} = 0.$$

3. First integrals for completely integrable Pfaffian systems with a solvable symmetry algebra

Let $I \subset T^*M$ be a rank n Pfaffian system. A function $f : M \to \mathbb{R}$ such that $df(x) \in I_x$ for all $x \in M$ is called a **first integral** of I (see [4] and [3]). If $f^i : M \to \mathbb{R}$ are n first integrals satisfying $I = \operatorname{span}\{df^i\}$, then the functions f^i are a complete set of first integrals.

Suppose $I = \operatorname{span}\{\omega^i\}_{1\le i\le n}$, where ω^i are differential 1-forms which satisfy the structure equations (1). If $\mathfrak{g}$ satisfies the conditions in Theorem 2.2 or its corollaries, we show how first integrals of I can be computed using only quadrature and matrix exponentiation. In particular Theorem 2.2 states that if $\mathfrak{k} \subset \mathfrak{g}$ is a codimension ideal, then one quadrature produces the first integral f in equation (5). Corollary 2.4 implies the following corollary.

Corollary 3.1. *Let $I \subset T^*M$ be a completely integrable rank n Pfaffian system on a simply connected manifold M. Suppose $I = \operatorname{span}\{\omega^i\}_{1\le i\le n}$, where the differential 1-forms ω^i satisfy*

$$d\omega^i + \frac{1}{2}C^i_{jk}\omega^j \wedge \omega^k = 0$$

and C^i_{jk} are the structure constants of a solvable Lie algebra $\mathfrak{g}$. The n functions $f^i : M \to \mathbb{R}$ constructed sequentially by quadratures and matrix exponentials through formula (21) are a complete set of first integrals for I.

We now show how Corollary 3.1 is used in practice (see also [4]). Let $\Gamma = \operatorname{span}\{Z_i\}_{1\le i\le n}$ be an n-dimensional Lie algebra of vector fields on M which are infinitesimimal symmetries of a completely integrable rank n Pfaffian system $I = \operatorname{span}\{\theta^j\}_{1\le j\le n}$. Assume $\boldsymbol{\Gamma}_p = \operatorname{span}\{Z_i(p)\}$ is an n-dimensional subspace of T_pM for each $p \in M$ which satisfies the transversality condition

$$\boldsymbol{\Gamma}_p \cap \operatorname{ann}(I_p) = 0, \qquad \text{for all } p \in M. \tag{24}$$

The transversality condition (24) implies that the $n \times n$ matrix $(P^i_j) = (\theta^i(Z_j))$ is invertible, and a simple computation (see [4] or [5]) shows that the differential forms

$$\omega^i = (P^{-1})^i_j\theta^j, \qquad \text{with} \quad P^i_j = \theta^i(Z_j), \tag{25}$$

satisfy $I = \operatorname{span}\{\omega^i\}_{1\le i\le n}$ and

$$d\omega^i + \frac{1}{2}C^i_{jk}\omega^j \wedge \omega^k = 0,$$

where $[X_j, X_k] = C^i_{jk}X_i$. If the Lie algebra Γ is solvable, then Corollary 2.4 may be used to produce n first integrals for I using quadratures and matrix exponentiation. This is similar to Theorem 3 on page 39 in [4], but with the important distinction that Theorem 3 of [4] requires solving functional equations while Corollary 2.4 does not.

Example 3.2. Consider the third order ODE from pg. 152 in [12] given by

$$u_{xxx} = \frac{3u_{xx}^2}{u_x} + \frac{u_{xx}^3}{u_x^5}. \tag{26}$$

On the manifold $M = \{(x, u, u_x, u_{xx}) \in \mathbb{R}^4 \mid u_x \neq 0\ \}$, equation (26) defines the Pfaffian system $I = \mathrm{span}\{\theta^1, \theta^2, \theta^3\}$, where

$$\theta^1 = du - u_x dx, \quad \theta^2 = du_x - u_{xx} dx, \quad \theta^3 = du_{xxx} - \left(3u_{xx}^2 u_x^{-1} + u_{xx}^3 u_x^{-5}\right) dx. \tag{27}$$

The Pfaffian system (27) admits the infinitesimal symmetries

$$Z_1 = \partial_x,\ Z_2 = \partial_u,\ Z_3 = u\partial_x - u_x^2 \partial_{u_x} - 3u_x u_{xx} \partial_{u_{xx}}\ , \tag{28}$$

and the only non-vanishing bracket is

$$[Z_2, Z_3] = Z_1. \tag{29}$$

Let $\{\ \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3\ \}$ be a basis for the 3-dimensional Lie algebra isomorphic to the Lie algebra of vector fields $\{Z_1, Z_2, Z_3\}$ so that $[\boldsymbol{e}_2, \boldsymbol{e}_3] = \boldsymbol{e}_1$. We have the sequence of relative ideals of codimension one $\mathrm{span}\{\boldsymbol{e}_1\} \subset \mathrm{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2\} \subset \mathfrak{g}$.

On the set $M^0 = (x, u, u_x \neq 0, u_{xx} \neq 0)$ the forms $\{\omega^i\}_{1 \leq i \leq 3}$ are computed from equation (25) using equations (27) and (28) to be

$$\begin{aligned} \omega^1 &= dx + \frac{1}{(u_x u_{xx})^2}(3u_x^6 + 3u_{xx} u u_x^4 + u_{xx}^2 u) du_x - \frac{u_x^3}{u_{xx}^3}(u_x^2 + u_{xx} u) du_{xx}, \\ \omega^2 &= du + \frac{3u_x^5}{u_{xx}^2} du_x - \frac{u_x^6}{u_{xx}^3} du_{xx}, \quad \omega^3 = -\frac{(3u_x^4 + u_{xx})}{u_{xx} u_x^2} du_x + \frac{u_x^3}{u_{xx}^2} du_{xx}\ . \end{aligned} \tag{30}$$

Now $d\omega^3 = 0$ and we may write $\omega^3 = df^3$ where

$$f^3 = \frac{1}{u_x} - \frac{u_x^3}{u_{xx}}. \tag{31}$$

Then by exponentiating the matrix

$$[\mathrm{ad}(\boldsymbol{e}_3) f^3] = \begin{bmatrix} 0 & -f^3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{32}$$

we can implement the first step in the algorithm of Corollary 2.4 as

$$\begin{bmatrix} 1 & -f^3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix} = \begin{bmatrix} \hat{\omega}^1 \\ \hat{\omega}^2 \\ df^3 \end{bmatrix}, \tag{33}$$

where

$$\begin{bmatrix} \hat{\omega}^1 \\ \hat{\omega}^2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{u_x^3}{u_{xx}} - \frac{1}{u_x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} dx + \frac{u_x^4 - u_{xx}}{u_x u_{xx}} du + \frac{3u_x^{10} + 3uu_{xx}^2 u_x^4 + uu_{xx}^3}{u_{xx}^3 u_x^2} du_x - \frac{u_x^3(u_x^6 + uu_{xx}^2)}{u_{xx}^4} du_{xx} \\ du + \frac{3u_x^5}{u_{xx}^2} du_x - \frac{u_x^6}{u_{xx}^3} du_{xx} \end{bmatrix}.$$

Now

$$d\hat{\omega}^1 = 0, \quad d\hat{\omega}^2 = 0.$$

The corresponding first integrals $f^1, f^2 : M^0 \to \mathbb{R}$ satisfying the equations $\hat{\omega}^1 = df^1, \hat{\omega}^2 = df^2$ are given by

$$f^1 = x + \frac{u_x^{10} + 3uu_{xx}^2 u_x^4 - 3uu_{xx}^3}{3u_x u_{xx}^3}, \quad f^2 = u + \frac{u_x^6}{2u_{xx}^2}. \tag{34}$$

4. Lie's Third Theorem: The Vector Field Case

The following theorem is the vector field version of Lie's third theorem.

Theorem 4.1. *(Lie's third theorem for vector fields) Let $\mathfrak{g}$ be an n-dimensional real Lie algebra. There exist n pointwise linearly independent vector fields on $\mathbb{R}^n$ whose span form a Lie algebra of vector fields that is isomorphic to $\mathfrak{g}$.*

Starting with a solvable Lie algebra $\mathfrak{g}$, we give an algebraic proof of this version of Lie's third theorem by constructing the vector fields using only the matrix exponential. The proof follows almost directly from Corollary 2.4. First we give the dual version.

Theorem 4.2. *Let $\mathfrak{g}$ be an n-dimensional solvable Lie algebra and let $\beta = \{\boldsymbol{e}_i\}_{1\le i\le n}$ be a basis for $\mathfrak{g}$ adapted to a sequence of relative ideals of codimension one $\mathfrak{k}_s \subset \mathfrak{k}_{s-1}$, $s = 1, \ldots, n$ in $\mathfrak{g}$. Let $\mathrm{ad}_s(\boldsymbol{e}_{n-s}) \in \mathrm{Der}(\mathfrak{k}_s), s = 0, \ldots, n-2$ be the derivation of $\mathfrak{k}_s$ defined in (16) by*

$$\mathrm{ad}_s(\boldsymbol{e}_{n-s}) = \mathrm{ad}(\boldsymbol{e}_{n-s})|_{\mathfrak{k}_s}, \quad s = 0, \ldots, n-2. \tag{35}$$

Let τ^i be the n pointwise linear independent 1-forms on $\mathbb{R}^n$ defined by

$$\begin{bmatrix} \tau^1 \\ \vdots \\ \tau^n \end{bmatrix} = \mathrm{e}^{-[\mathrm{ad}(\boldsymbol{e}_n)]x^n} \begin{bmatrix} \mathrm{e}^{-[\mathrm{ad}_1(\boldsymbol{e}_{n-1})]x^{n-1}} & \mathbf{0}_1 \\ \mathbf{0}_1^T & 1 \end{bmatrix} \cdots \begin{bmatrix} \mathrm{e}^{-[\mathrm{ad}_{n-2}(\boldsymbol{e}_2)]x^2} & \mathbf{0}_{n-2} \\ \mathbf{0}_{n-2}^T & \mathrm{I}_{n-2} \end{bmatrix} \begin{bmatrix} dx^1 \\ \vdots \\ dx^n \end{bmatrix}, \tag{36}$$

where $[\mathrm{ad}_s(\boldsymbol{e}_{n-s})]$ is the $n-s$ by $n-s$ matrix representation of $\mathrm{ad}_s(\boldsymbol{e}_{n-s})$ in the basis (15) for $\mathfrak{k}_s$, $s = 0, \ldots, n-2$, I_s is the s by s identity matrix, and $\mathbf{0}_s$ is the $n-s$ by s zero matrix. Then τ^i satisfy

$$d\tau^i + \frac{1}{2} C^i_{jk} \tau^j \wedge \tau^k = 0, \tag{37}$$

where C^i_{jk} are the structure constants for $\mathfrak{g}$ in the basis β.

Before presenting the proof of Theorem 4.2, let $X_i, 1 \le i \le n$ be the dual frame to τ^i in equation (36). This produces the n linearly independent vector fields on $\mathbb{R}^n$, given by

$$\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_n)]^T x^n} \begin{bmatrix} \mathrm{e}^{[\mathrm{ad}_1(\boldsymbol{e}_{n-1})]^T x^{n-1}} & \mathbf{0}_1 \\ \mathbf{0}_1^T & 1 \end{bmatrix} \cdots \begin{bmatrix} \mathrm{e}^{[\mathrm{ad}_{n-2}(\boldsymbol{e}_2)]^T x^2} & \mathbf{0}_{n-2} \\ \mathbf{0}_{n-2}^T & \mathrm{I}_{n-2} \end{bmatrix} \begin{bmatrix} \partial_{x^1} \\ \vdots \\ \partial_{x^n} \end{bmatrix}, \quad (38)$$

where T is the transpose. We then have the following corollary (Lie's third theorem for vector fields).

Corollary 4.3. *The n vector fields X_i on $\mathbb{R}^n$ defined in* (38) *are pointwise linearly independent and satisfy*

$$[X_i, X_j] = C^k_{ij} X_k, \quad (39)$$

where C^i_{jk} are the structure constants for $\mathfrak{g}$ in the basis β of Theorem 4.2. Therefore, the vector fields in (38) *produce a solution to Lie's third Theorem (for vector fields) using only the matrix exponential. The linear map $\phi : \mathfrak{g} \to \mathrm{span}\{X_i\}$ induced by*

$$\phi(\mathbf{e}_i) = X_i \quad (40)$$

is a Lie algebra isomorphism.

Proof. (Theorem 4.2) The idea of the proof of Theorem 4.2 is to solve equation (21) of Corollary 2.4 for ω^i. This gives equation (36) if we substitute $f^i = x^i$ and $\omega^i = \tau^i$.

The proof goes by induction. Consider the differential forms on $\mathbb{R}^n$ given by

$$\begin{bmatrix} \hat{\tau}^1 \\ \vdots \\ \hat{\tau}^n \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-[\mathrm{ad}_1(\boldsymbol{e}_{n-1})]x^{n-1}} & \mathbf{0}_1 \\ \mathbf{0}_1^T & 1 \end{bmatrix} \cdots \begin{bmatrix} \mathrm{e}^{-[\mathrm{ad}_{n-2}(\boldsymbol{e}_2)]x^2} & \mathbf{0}_{n-2} \\ \mathbf{0}_{n-2}^T & \mathrm{I}_{n-2} \end{bmatrix} \begin{bmatrix} dx^1 \\ \vdots \\ dx^n \end{bmatrix}. \quad (41)$$

By the induction hypothesis, the forms $\{\hat{\tau}^a\}_{1 \le a \le n-1}$ from equation (41) satisfy,

$$d\hat{\tau}^a + \frac{1}{2} C^a_{bc} \hat{\tau}^b \wedge \hat{\tau}^c = 0,$$

where C^a_{bc} are the structure constants of the subalgebra $\mathfrak{k}_1 \subset \mathfrak{g}$ in the basis $\hat{\beta} = \{\boldsymbol{e}_a\}_{1 \le a \le n-1}$.

The fact that the differential forms in equation (36) satisfy equation (37) follows simply by reversing the computation in the proof of Theorem 2.2 with $\boldsymbol{\tau} = \exp(-[\mathrm{ad}(\boldsymbol{e}_n)]x^n)\hat{\boldsymbol{\tau}}$ (in particular using equations (10) through (13)). This shows that the forms τ^i in (36) satisfy equation (37). ■

Example 4.4. Consider the 5-dimensional solvable Lie algebra $A_{5,ab}$ from [14], with multiplication table

$$\begin{aligned} &[\boldsymbol{e}_1, \boldsymbol{e}_4] = b\,\boldsymbol{e}_1,\ [\boldsymbol{e}_1, \boldsymbol{e}_5] = a\,\boldsymbol{e}_1,\ [\boldsymbol{e}_2, \boldsymbol{e}_4] = \boldsymbol{e}_2,\ [\boldsymbol{e}_2, \boldsymbol{e}_5] = -\boldsymbol{e}_3, \\ &[\boldsymbol{e}_3, \boldsymbol{e}_4] = \boldsymbol{e}_3, \quad [\boldsymbol{e}_3, \boldsymbol{e}_5] = \boldsymbol{e}_2, \end{aligned} \quad (42)$$

where the two real parameters a and b satisfy $a^2+b^2\neq 0$. The basis $\{\boldsymbol{e}_i\}_{1\leq i\leq 5}$ is adapted to the sequence of relative ideals of codimension one,

$$\mathfrak{k}_s = \mathrm{span}\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_{5-s}\} \text{ for } s=0,\dots,5.$$

We find the non-zero $\mathrm{ad}_s(\boldsymbol{e}_{n-s})$ matrices defined in equation (35), using equations (42), to be

$$[\mathrm{ad}(\boldsymbol{e}_5)] = \begin{bmatrix} -a & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad [\mathrm{ad}_1(\boldsymbol{e}_4)] = \begin{bmatrix} -b & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{43}$$

The coframe $\boldsymbol{\tau}$ on $\mathbb{R}^5$ is computed from (36) using equation (43) and that $\mathrm{ad}_3(\boldsymbol{e}_2)=0,\ \mathrm{ad}_2(\boldsymbol{e}_3)=0$, to be

$$\begin{bmatrix} \tau^1 \\ \tau^2 \\ \tau^3 \\ \tau^4 \\ \tau^5 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{ax^5} & 0 & 0 & 0 & 0 \\ 0 & \cos x^5 & \sin x^5 & 0 & 0 \\ 0 & -\sin x^5 & \cos x^5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{bx^4} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{x^4} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{x^4} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \\ dx^5 \end{bmatrix}$$
$$= \begin{bmatrix} \mathrm{e}^{ax^5+bx^4b}dx^1 \\ \mathrm{e}^{x^4}\cos x^5\, dx^2 + \mathrm{e}^{x^4}\sin x^5\, dx^3 \\ -\mathrm{e}^{x^4}\sin x^5\, dx^2 + \mathrm{e}^{x^4}\cos x^5\, dx^3 \\ dx^4 \\ dx^5 \end{bmatrix}. \tag{44}$$

The structure equations are then confirmed to be

$$d\tau^1 = -b\,\tau^1\wedge\tau^4 - a\,\tau^1\wedge\tau^5,\ d\tau^2 = -\tau^2\wedge\tau^4 - \tau^3\wedge\tau^5,\ d\tau^3 = \tau^2\wedge\tau^5 - \tau^3\wedge\tau^4,$$
$$d\tau^4 = 0,\ d\tau^5 = 0.$$

5. Lie's Third Theorem and the Construction of Simply Connected Solvable Lie Groups

The following theorem is the group version of Lie's third theorem.

Theorem 5.1. (Lie's third theorem) *Let $\mathfrak{g}$ be an n-dimensional real Lie algebra. There exists an n-dimensional Lie group G whose Lie algebra of left invariant vector fields is isomorphic to $\mathfrak{g}$.*

One of our goals in this section is to give a simple proof of Lie's third theorem for solvable Lie algebras which will enable us to explicitly construct the group multiplication for a simply connected solvable Lie group using n quadratures and matrix exponentiation (Theorem 5.7). As an intermediate step in the construction of the multiplication map for these groups, we show how to construct the map $\rho: M\to G$ in equation (2) for solvable Lie algebras using quadrature and matrix exponentiation.

5.1. Lie's third theorem for solvable groups. We begin by showing that given the vector fields $\{X_i\}_{1\leq i\leq n}$ on $\mathbb{R}^n$ in equation (38), we can give a global Lie group structure on $\mathbb{R}^n$ such that $\{X_i\}_{1\leq i\leq n}$ are a basis for the left invariant vector fields. This proves Lie's third theorem (Theorem 5.1) for solvable Lie groups while also showing that a simply connected solvable Lie group is (topologically) $\mathbb{R}^n$. The key aspect of the proof, which is used in Sections 5.2 and 5.3, is that we have a closed form formula for the left invariant vector fields. See [16] for a recent discussion of Lie's third theorem.

Theorem 5.2. *Let $\mathfrak{g}$ be an n-dimensional solvable Lie algebra and let $\beta = \{\mathbf{e}_i\}_{1\leq i\leq n}$ be a basis for $\mathfrak{g}$ adapted to a sequence of relative ideals of codimension one, $\mathfrak{k}_s \subset \mathfrak{k}_{s-1}$, $s = 1, \ldots, n$. Let $\{X_i\}_{1\leq i\leq n}$ be the vector fields on $\mathbb{R}^n$ given in equation* (38). *Given a point $x_0 \in \mathbb{R}^n$, there exists a smooth multiplication map $\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ so that $\mathbb{R}^n$ is a solvable Lie group with identity x_0 and with Lie algebra of left invariant vector fields given by $\mathfrak{g} = \mathrm{span}\{X_i\}_{1\leq i\leq n}$.*

The proof of Theorem 5.2 follows directly from Theorem 8.7 in [15] (see also [16]) and the next lemma.

Lemma 5.3. *Every vector field in the span of the vector fields $\{X_i\}_{1\leq i\leq n}$ on $\mathbb{R}^n$ in equation* (38) *is complete.*

Proof. We prove this by induction. Assume that the span of $n-1$ vector fields $\{\tilde{X}_r\}_{1\leq r\leq n-1}$ on $\mathbb{R}^{n-1}$ with coordinates $x^1, \ldots, x^{n-1}$ given by

$$\begin{bmatrix} \tilde{X}_1 \\ \vdots \\ \tilde{X}_{n-1} \end{bmatrix} = \begin{bmatrix} e^{[\mathrm{ad}_1(\boldsymbol{e}_{n-1})]x^{n-1}} & \mathbf{0}_1 \\ \mathbf{0}_1^T & 1 \end{bmatrix} \cdots \begin{bmatrix} e^{[\mathrm{ad}_{n-2}(\boldsymbol{e}_2)]^T x^2} & \mathbf{0}_{n-2} \\ \mathbf{0}_{n-2}^T & \mathrm{I}_{n-2} \end{bmatrix} \begin{bmatrix} \partial_{x^1} \\ \vdots \\ \partial_{x^{n-1}} \end{bmatrix} \tag{45}$$

consists of complete vector fields. The vector fields $\{\tilde{X}_r\}_{1\leq r\leq n-1}$ are globally defined on $\mathbb{R}^{n-1}$ and satisfy equation (39), with the structure constants for the codimension one sub-algebra $\mathfrak{k}_1 \subset \mathfrak{g}$ defined in Theorem 4.2. By [15] (or [16]), given $\tilde{x}_0 \in \mathbb{R}^{n-1}$ there exists a Lie group structure on $\mathbb{R}^{n-1}$ with $\tilde{x}_0$ as the identity such that $\{\tilde{X}_r\}_{1\leq r\leq n-1}$ are a basis for the left invariant vector fields on $\mathbb{R}^{n-1}$. Denote this Lie group by H and let $\mathfrak{h} = \mathrm{span}\{\tilde{X}_r\}_{1\leq r\leq n-1}$ be the corresponding Lie algebra.

Now the n vector fields X_i in (38) can be written in terms of the vector fields in (45) (considered as vector fields on $\mathbb{R}^n$) as

$$\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = e^{[\mathrm{ad}(\boldsymbol{e}_n)]^T x^n} \begin{bmatrix} \tilde{X}_1 \\ \vdots \\ \tilde{X}_{n-1} \\ \partial_{x^n} \end{bmatrix}. \tag{46}$$

Let $\tilde{\xi}_r^s$, $s, r = 1, \ldots, n-1$ be the coefficients of the vector fields $\tilde{X}_r$ and let η_i^j, $i, j = 1, \ldots, n$ be the coefficients of the vector fields X_i. From equations (46) and (10) we have

$$\eta^s_r = \left[e^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]x^n} \right]^u_r \tilde{\xi}^s_u, \quad \eta^s_n = 0, \quad \eta^n_r = 0, \quad \eta^n_n = 1. \tag{47}$$

The flow equations for the vector field on $\mathbb{R}^n$ given by $c^i X_i$, $c^i \in \mathbb{R}$, are determined from (47) to be

$$\begin{aligned} \frac{dx^s}{dt} &= c^r \left[e^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)]x^n} \right]^u_r \tilde{\xi}^s_u, \\ \frac{dx^n}{dt} &= c^n. \end{aligned} \tag{48}$$

Integrating the last equation in (48) gives $x^n = c^n t + c^0$, where $c^0, c^n \in \mathbb{R}$. Therefore the first equation in (48) can be written as

$$\frac{dx^s}{dt} = A^u(t)\tilde{\xi}^s_u, \quad s = 1, \ldots, n-1, \tag{49}$$

where

$$A^u(t) = \left[e^{[\widetilde{\mathrm{ad}}(\boldsymbol{e}_n)](c^n t + c^0)} \right]^u_r c^r.$$

Equation (49) is the equation of Lie-type for the function $\sigma : \mathbb{R} \to \mathfrak{h}$, given by

$$\sigma(t) = A^u(t)\tilde{X}_u.$$

Equations of Lie type admit global solutions in t ([13] page 37), and hence $c^r X_r$ is complete. ∎

In Section 5.3 we construct the multiplication map $\mu : G \times G \to G$, for the group G whose existence is guaranteed by Theorem 5.2, using matrix exponentiation and quadrature. The multiplication function μ will be obtained as a consequence of solving the problem, as stated in the introduction, of finding $\rho : M \to G$ in equation (2) for solvable Lie groups.

5.2. Constructing the map $\rho : M \to G$ for solvable G. Let $\{\omega^i\}_{1 \le i \le n}$ be n differential forms on a simply connected manifold M satisfying equations (1) (written as $d\boldsymbol{\omega}_{\mathfrak{g}} + \frac{1}{2}\boldsymbol{\omega}_{\mathfrak{g}} \wedge \boldsymbol{\omega}_{\mathfrak{g}} = 0$ in Remark 2.6). As remarked in the introduction, it is well known that there exists a function $\rho : M \to G$ such that $\omega^i = \rho^* \tau^i$, where τ^i are the left invariant Maurer Cartan forms on a Lie group G with Lie algebra $\mathfrak{g}$ (see [15]). Furthermore, ρ is unique up to left translation by an element of G. We now show how Corollary 2.4, combined with Theorem 5.2, determines ρ when $\mathfrak{g}$ is solvable.

Theorem 5.4. *Let $\mathfrak{g}$ be an n-dimensional solvable real Lie algebra and let $\beta = \{\mathbf{e}_i\}_{1 \le i \le n}$ be a basis for $\mathfrak{g}$ adapted to a sequence of relative ideals of codimension one $\mathfrak{k}_s \subset \mathfrak{g}$, $s = 0, \ldots, n$. Let $(x^1, \ldots, x^n)$ be coordinates on the corresponding simply connected solvable Lie group G, with basis of left invariant forms $\{\tau^i\}_{1 \le i \le n}$ given in* (36).

Let $\{\omega^i\}_{1 \le i \le n}$ be n differential forms on a simply connected manifold M satisfying the structure equations (1)*, where C^i_{jk} are computed in the basis β for*

$\mathfrak{g}$. *Also let* $f^i : M \to \mathbb{R}$ *be the functions from Corollary 2.4. Then the function* $\rho : M \to G$ *given by*

$$\rho(p) = (x^1 = f^1(p), \dots, x^n = f^n(p)) \quad \text{for all } p \in M \tag{50}$$

satisfies

$$\rho^* \tau^i = \omega^i. \tag{51}$$

Proof. Let $f^i : M \to \mathbb{R}$ be the functions in Corollary 2.4 so that equation (21) holds. Solving for ω^i in equation (21) gives

$$\begin{bmatrix} \omega^1 \\ \vdots \\ \omega^n \end{bmatrix} = \mathrm{e}^{-[\mathrm{ad}(\boldsymbol{e}_n)]f^n} \begin{bmatrix} \mathrm{e}^{-[\mathrm{ad}_1(\boldsymbol{e}_{n-1})]f^{n-1}} & \mathbf{0}_1 \\ \mathbf{0}_1^T & 1 \end{bmatrix} \cdots \begin{bmatrix} \mathrm{e}^{-[\mathrm{ad}_{n-2}(\boldsymbol{e}_2)]f^2} & \mathbf{0}_{n-2} \\ \mathbf{0}_{n-2}^T & \mathrm{I}_{n-2} \end{bmatrix} \begin{bmatrix} df^1 \\ \vdots \\ df^n \end{bmatrix}. \tag{52}$$

The map $\rho : M \to G$ in equation (50) defined in coordinates by $x^i = f^i(p)$ for all $p \in M$, satisfies equation (51). This follows at once by substituting $x^i = f^i$ in the expression for τ^i in equation (36), which produces the expression for ω^i in equation (52). ■

Remark 5.5. Theorem 5.4 shows that determining the map $\rho : M \to G$ in equation (2) for a solvable Lie group G can be found by quadrature and the matrix exponential and is, in fact, found trivially from the n functions f^i in Corollary 2.4.

Example 5.6. In Example 3.2 we have the three 1-forms $\omega^1, \omega^2, \omega^3$ on M^0 given in equation (30), which satisfy

$$d\omega^1 = -\omega^2 \wedge \omega^3, \quad d\omega^2 = 0, \quad d\omega^3 = 0.$$

The 3-dimensional simply connected Lie group G with Lie algebra $[\boldsymbol{e}_2, \boldsymbol{e}_3] = \boldsymbol{e}_1$, constructed as in Theorem 5.2 with coordinates (x^1, x^2, x^3), has as a basis of left invariant forms (see (36))

$$\tau^1 = dx^1 + x^3 dx^2, \quad \tau^2 = dx^2, \quad \tau^3 = dx^3. \tag{53}$$

These 1-forms satisfy $d\tau^1 = -\tau^2 \wedge \tau^3$.

The functions f^1, f^2, and f^3 are given in equations (31) and (34) in Example 3.2. They give the map $\rho : M^0 \to G$ in equation (50) in Theorem 5.4 by

$$\begin{aligned} x^1 &= f^1 = x + \frac{u_x^{10} + 3uu_{xx}^2 u_x^4 - 3uu_{xx}^3}{3u_x u_{xx}^3}, \\ x^2 &= f^2 = u + \frac{u_x^6}{2u_{xx}^2}, \qquad x^3 = f^3 = \frac{1}{u_x} - \frac{u_x^3}{u_{xx}}. \end{aligned} \tag{54}$$

It is easy to check $\rho^* \tau^i = \omega^i$, $i = 1, 2, 3$, with ω^i in (30) and with τ^i in equation (53).

We now demonstrate the equivariant nature of ρ. First let G be the simply connected 3 dimensional Lie group where τ^i in (53) are a basis of *right* invariant forms. Let X_1, X_2, X_3 be the right invariant vector fields on G dual to τ^i G. Then $\rho_*(Z_i) = X_i$ and $\rho_* : \Gamma \to \mathfrak{g}$ is an isomorphism, where $\mathfrak{g}$ is the Lie algebra of right invariant vector fields and $\Gamma = \operatorname{span}\{Z_i\}$ are the infinitesimal generators of the action of G on M. By an application of Theorem 13.1 in [2], the function ρ is G-equivariant. Note that the infinitesimal generators of the action of G on itself on the left are the right invariant vector fields.

To explicitly show the equivariance of ρ, we make the change of coordinates $(a = x^1 + x^2x^3, b = x^2, c = x^3)$ on G so that the map $\rho : M^0 \to G$ from (54) is given in these new coordinates as

$$a = x + \frac{u_x^5}{2u_{xx}^2} - \frac{u_x^9}{6u_{xx}^3}, \quad b = u + \frac{u_x^6}{2u_{xx}^2}, \quad c = \frac{1}{u_x} - \frac{u_x^3}{u_{xx}}. \tag{55}$$

With the multiplication map on G given by $(a, b, c) \cdot (a'b'c') = (a + a' + cb', b + b', c + c')$, the (local) action of G on M^0 is

$$\mu(a, b, c; x, u, u_x, u_{xx}) = \left(x + a + cu, \; u + b, \; \frac{u_x}{cu_x + 1}, \frac{u_{xx}}{(cu_x + 1)^3} \right). \tag{56}$$

From equations (55) and (56), the equivariance condition $\rho(\mu(a, b, c; x, u, u_x, u_{xx})) = (a, b, c) \cdot \rho(x, u, u_x, u_{xx})$ is easily checked.

Finally, we point out that ρ is the moving frame (see [7]) for the action of G on M^0 given in (56) that is determined from the cross-section $K = \rho^{-1}(0, 0, 0)$ to the orbits of G. Since K corresponds to the level set of a complete set of first integrals, the cross-section K is a solution to the ODE in (26). The solution is determined from $\rho^{-1}(0, 0, 0)$, using equation (55), and is (the prolongation of) $u = -\frac{1}{2}(3x)^{\frac{2}{3}}$.

5.3. Constructing the multiplication function for simply connected solvable G. The results from Section 2, along with Theorem 5.4 in Section 5.2, will now be used to construct the multiplication map μ for a simply connected solvable Lie group G from its Lie algebra using only n quadratures and matrix exponentials. This is given by the following theorem.

Theorem 5.7. *Let $\mathfrak{g}$ be a real n-dimensional solvable Lie algebra and let $\beta = \{\mathbf{e}_i\}_{1 \le i \le n}$ be a basis for $\mathfrak{g}$ adapted to a sequence of relative ideals of codimension one $\mathfrak{k}_s \subset \mathfrak{k}_{s-1}$, $s = 1, \ldots, n$. Let $\boldsymbol{\tau} = [\tau^1, \ldots, \tau^n]^T$ be the n-vector of left invariant 1-forms defined in equation (36) on the corresponding simply connected Lie group G, with identity chosen to be $\mathbf{0} \in \mathbb{R}^n$ (Theorem 5.2).*

On $G \times G$ with coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$, define the n-vector of differential 1-forms $\boldsymbol{\omega} = [\omega^1, \ldots, \omega^n]^T$ as

$$\boldsymbol{\omega} = \mathrm{e}^{-[\mathrm{ad}(\mathbf{e_n})]y^n} \cdots \mathrm{e}^{-[\mathrm{ad}(\mathbf{e_1})]y^1} \pi_1^* \boldsymbol{\tau} + \pi_2^* \boldsymbol{\tau}, \tag{57}$$

where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ are the two projection maps on $G \times G$.
[i] *The forms ω^i satisfy*

$$d\omega^i + \frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k, \tag{58}$$

where C^i_{jk} *are the structure constants of* $\mathfrak{g}$ *in the basis* β*.*
[ii] *Let* $\mu : G \times G \to G$ *with* $\mu(\mathbf{0},\mathbf{0}) = \mathbf{0}$ *be the map constructed in Theorem 5.4, so that* $\omega^i = \mu^*\tau^i$*. Then* $\mu : G \times G \to G$ *is the multiplication map for the Lie group* G*, with basis of left invariant forms* $\{\tau^i\}_{1\le i\le n}$ *and identity* $\mathbf{0} \in \mathbb{R}^n$*.*

Part **[ii]** in Theorem 5.7 states that if (z^i) denote coordinates on the target G (with τ^i in equation (36) given in the z^i coordinates) and $f^i(x,y)$ are the functions determined by applying Corollary 2.4 to $\boldsymbol{\omega}$ in equation (57), then the multiplication map is given explicitly by $z^i = f^i(x,y)$. This is demonstrated in Example 5.13 below. The choice of identity element $\mathbf{0}$ in Theorem 5.7 was made for simplicity.

In order to prove Theorem 5.7 we first recall how to construct the multiplication map $\mu : G \times G \to G$ using integral manifolds. The proof of the following theorem is given in Section 7 of [9].

Theorem 5.8. *Let* G *be a simply connected Lie group and let* $\{\tau^i\}_{1\le i\le n}$ *be a basis of left invariant* 1*-forms on* G*. On* $G\times G$ *define the rank* n *Pfaffian system* $I = \operatorname{span}\{\theta^1,\dots,\theta^n\}$*, where*

$$\theta^i = \pi_2^*\tau^i - \pi_1^*\tau^i. \tag{59}$$

The Pfaffian system I *is completely integrable. If* $(x_0,y_0) \in G\times G$ *and* $\mathcal{L}_{(x_0,y_0)}$ *is the maximal integral manifold through* (x_0,y_0)*, then*

$$\mathcal{L}_{(x_0,y_0)} = \{\ (x, L_{y_0x_0^{-1}}x), \quad x \in G\ \}, \tag{60}$$

where $L_{y_0x_0^{-1}}$ *is left multiplication by* $\mu(y_0,x_0^{-1})$*.*

We now determine a set of generators for $I = \operatorname{span}\{\theta^1,\dots,\theta^n\}$, with G given in Theorem 5.7 and θ^i given in equation (59), that will allow us to determine the maximal integral manifolds of I by quadratures and matrix exponentiation. This will determine the multiplication map using equation (60) (see Lemma 5.10). The details are contained in the next two lemmas.

Lemma 5.9. *Let* $\mathfrak{g}$ *be an* n*-dimensional solvable Lie algebra and let* $\beta = \{\mathbf{e}_i\}_{1\le i\le n}$ *be a basis for* $\mathfrak{g}$ *adapted to a sequence of relative ideals of codimension one* $\mathfrak{k}_s \subset \mathfrak{k}_{s-1}$*,* $s = 1,\dots,n$*. Let* $\boldsymbol{\tau} = [\tau^1,\dots,\tau^n]^T$ *be the vector of left invariant* 1*-forms on* G *defined in equation* (36)*. Let* $\tilde\theta^i$ *be the* n *1-forms on* $G\times G$ *given by*

$$\tilde{\boldsymbol{\theta}} = \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_1)]x^1}\cdots \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_n)]x^n}\left(\pi_2^*\boldsymbol{\tau} - \pi_1^*\boldsymbol{\tau}\right). \tag{61}$$

These form a basis for the Pfaffian system I *in Theorem 5.8,* $I = \operatorname{span}\{\tilde\theta^1,\dots,\tilde\theta^n\}$*, where the structure equations for* $\tilde\theta^i$ *are*

$$d\tilde\theta^i + \frac{1}{2}C^i_{jk}\tilde\theta^j\wedge\tilde\theta^k = 0. \tag{62}$$

The generators $\tilde{\boldsymbol{\theta}}$ for I in (61) and Theorem 5.4 can now be used to construct the multiplication map.

Lemma 5.10. *Suppose the hypothesis of Lemma 5.9 on* $\mathfrak{g}$, β *and* $\boldsymbol{\tau}$ *are satisfied and let* $\tilde{\theta}^i$ *be the* 1*-forms on* $G \times G$ *from equation* (61) *which satisfy* (62). *Let* $\rho : G \times G \to G$ *be the unique map constructed using Theorem 5.4 so that* $\rho(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ *and*

$$\rho^* \tau^i = \tilde{\theta}^i.$$

Then

$$\rho(x, y) = \mu(y, x^{-1}), \tag{63}$$

where $\mu : G \times G \to G$ *is the multiplication map with* $\mathbf{0} \in G$ *as the identity.*

We will now prove Lemmas 5.9 and 5.10.

Proof. (Lemma 5.9) On $G \times G$, consider the vector of $2n$ differential forms given by

$$\boldsymbol{\Omega} = [\theta^1, \dots, \theta^n, \pi_1^* \tau^1, \dots \pi_1^* \tau^n]^T = \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\tau}_1 \end{bmatrix}, \tag{64}$$

where (from equation (59)) $\theta^i = \pi_2^* \tau^i - \pi_1^* \tau^i$ and $\boldsymbol{\tau}_1 = \pi_1^* \boldsymbol{\tau}$. The structure equations for $\boldsymbol{\Omega}$ follow from equation (37), and are

$$\begin{aligned} d\theta^i + \frac{1}{2} C^i_{jk} \theta^j \wedge \theta^k + C^i_{jk} \theta^j \wedge \tau_1^k &= 0, \\ d\tau_1^i + \frac{1}{2} C^i_{jk} \tau_1^j \wedge \tau_1^k &= 0. \end{aligned} \tag{65}$$

The $2n$ forms in (64) are a basis of left invariant forms on $G \times G$ which are dual to the basis for $\mathfrak{g} \times \mathfrak{g}$ of left invariant vector fields given by

$$\gamma = \{ \ Y_1, \dots, Y_n, X_1 + Y_1, X_2 + Y_2, \dots, X_n + Y_n \ \}, \tag{66}$$

where the set of vector fields $\{X_1, \dots, X_n, Y_1 \dots, Y_n\}$ is the dual frame to the coframe $\{\pi_1^* \tau^1, \dots, \pi_1^* \tau^n, \pi_2^* \tau^1, \dots, \pi_2^* \tau^n\}$ on $G \times G$.

Define subalgebras $\mathfrak{l}_s \subset \mathfrak{g}$, $s = 0, \dots, 2n$ by

$$\mathfrak{l}_a = \text{span}\{ \ Y_1, \dots, Y_n, X_1 + Y_1, X_2 + Y_2, \dots, X_{n-a} + Y_{n-a} \ \}, \quad 0 \le a \le n-1 \tag{67}$$

and

$$\mathfrak{l}_{n+a} = \text{span}\{ \ Y_1, \ Y_2, \dots, \ Y_{n-a} \ \}, \quad 0 \le a \le n. \tag{68}$$

It is then simple to show that $\mathfrak{l}_s \subset \mathfrak{l}_{s-1}$, $s = 1, \dots, 2n$ forms a sequence of relative ideals of codimension one for the solvable Lie algebra $\mathfrak{g} \times \mathfrak{g}$. Furthermore, by definition, the basis (66) is adapted to the sequence of relative ideals of codimension one $\mathfrak{l}_s \subset \mathfrak{l}_{s-1}, s = 1 \dots 2n$.

We will now apply Corollary 2.3 using the first n terms in the sequence, $\mathfrak{l}_a, a = 0, \dots, n$. This will result in a change in the coframe $[\boldsymbol{\theta}, \boldsymbol{\tau}_1]^T$ on $G \times G$, where the terms $C^i_{jk} \theta^j \wedge \tau_1^k$ on the first line of equation (65) don't appear (see the paragraph following the proof of Theorem 2.2).

In order to apply Corollary 2.3 we need to compute

$$\text{ad}_a(X_{n-a} + Y_{n-a}) : \mathfrak{l}_a \to \mathfrak{l}_a, \quad 0 \le a \le n$$

in terms of the basis (66). This is easily done here:

$$[\mathrm{ad}_a(X_{n-a}+Y_{n-a})] = \begin{bmatrix} [\mathrm{ad}(\boldsymbol{e}_{n-a})] & 0 \\ 0 & [\mathrm{ad}_a(\boldsymbol{e}_{n-a})] \end{bmatrix},$$

where $[\mathrm{ad}_a(\boldsymbol{e}_{n-a})]$ is the $n-a$ by $n-a$ matrix representation of the derivation defined in equation (16) in the basis $\beta = \{\boldsymbol{e}_i\}_{1\leq i\leq n}$ for $\mathfrak{g}$. We have also used the isomorphism (40).

We determine the rightmost matrix in equation (17) by using $\Omega^{2n} = dx^n$ from equation (64) and letting $f^n = x^n$. In this case, the first reduction (which is equivalent to one application of Theorem 2.2) is given by

$$\hat{\boldsymbol{\Omega}} = \begin{bmatrix} \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_n)]x^n} & \mathbf{0}_n \\ \mathbf{0}_n^T & \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_n)]x^n} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\tau}_1 \end{bmatrix}. \tag{69}$$

Now, using equation (36) and the fact that $\mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_n)]x^n}\mathrm{e}^{-[\mathrm{ad}(\boldsymbol{e}_n)]x^n} = \mathrm{I}_n$, equation (69) simplifies to

$$\hat{\boldsymbol{\Omega}} = \begin{bmatrix} \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_n)]x^n}\boldsymbol{\theta} \\ A_{n-1}(x^{n-1})\dots A_2(x^2)\mathbf{dx}. \end{bmatrix}, \tag{70}$$

where

$$A_{n-a}(x^{n-a}) = \begin{bmatrix} \mathrm{e}^{-[\mathrm{ad}_a(\boldsymbol{e}_{n-a})]x^{n-a}} & \mathbf{0}_a \\ \mathbf{0}_a^T & \mathrm{I}_a \end{bmatrix}, \quad a = 0,\dots,n-2 \tag{71}$$

are the matrices in equation (36) and (70).

Continuing by induction, we have that $f^i = x^i$ and that after n steps Corollary 2.3 produces

$$\begin{bmatrix} \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_1)]x^1} & \mathbf{0} \\ \mathbf{0}^T & \mathrm{I}_n \end{bmatrix} \begin{bmatrix} \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_2)]x^2} & \mathbf{0} \\ \mathbf{0}^T & A_2(-x^2) \end{bmatrix} \cdots \begin{bmatrix} \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_n)]x^n} & \mathbf{0} \\ \mathbf{0}^T & A_n(-x^n) \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\tau} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{\theta}} \\ \mathbf{dx} \end{bmatrix},$$

where again $A_s(x^s)$ are given in (71) and $\tilde{\boldsymbol{\theta}}$ are the sought-after forms in equation (61). By construction, the structure equations (62) hold. This proves the lemma.

■

We now turn to the proof of Lemma 5.10.

Proof. (Lemma 5.10) Theorem 5.4 applies to $\tilde{\boldsymbol{\theta}}$ in equation (61), and so let $\rho : G\times G \to G$ be the unique function satisfying

$$\rho^*\tau^i = \tilde{\theta}^i \tag{72}$$

and $\rho(\mathbf{0},\mathbf{0}) = \mathbf{0}$, where $\mathbf{0}$ is the identity in G. Let $(x_0, y_0) \in G\times G$, $z_0 = \rho(x_0, y_0)$ and let

$$\mathcal{N} = \{(x,y) \in G\times G \mid \rho(x,y) = z_0\} = \rho^{-1}(z_0). \tag{73}$$

The functions $\rho^1,\dots,\rho^n$ are n functionally independent first integrals of I, so the embedded manifold $\mathcal{N}$ is an integral manifold of I of dimension n which contains (x_0, y_0).

Let $\mathcal{L}$ be the maximal integral manifold of I through (x_0, y_0) (Theorem 5.8). Theorem 5.8 shows that the projection map $\pi_1 : G\times G$ restricted to $\mathcal{L}$

is a diffeomorphism, while the connectivity of $\mathcal{L}$ implies $\mathcal{L} \subset \mathcal{N}$. Therefore, $\pi_1 : \mathcal{N} \to G$ is a covering map and hence a diffeomorphism. Consequently $\mathcal{N} = \mathcal{L}$, while a similar argument also shows that $\pi_2 : \mathcal{N} \to G$ is a diffeomorphism.

Let $\iota : G \to G \times G$ be the function $\iota(a) = (\mathbf{0}, a)$ (where $\mathbf{0}$ is the identity in G). Then $\rho \circ \iota$ satisfies $\rho \circ \iota(\mathbf{0}) = \mathbf{0}$ and, from equations (72) and (61), $(\rho \circ \iota)^* \tau^i = \tau^i$. Therefore, by the uniqueness of such functions, $\rho \circ \iota$ is the identity function. Now $\iota(y_0 x_0^{-1}) = (0, y_0 x_0^{-1}) \in \mathcal{L} = \mathcal{N} = \rho^{-1}(z_0)$. Therefore $z_0 = \rho \circ \iota(y_0 x_0^{-1}) = y_0 x_0^{-1}$, which holds for any (x_0, y_0) in $G \times G$. ■

Corollary 5.11. *Let $\mathfrak{g}$ be a solvable Lie algebra with basis $\beta = \{\boldsymbol{e}_i\}_{1 \le i \le n}$ adapted to a sequence of relative ideals of codimension one $\mathfrak{k}_s \subset \mathfrak{k}_{s-1}$, $s = 1, \ldots, n$. Let G be the corresponding Lie group in Theorem 5.7 with basis of left-invariant forms $\{\tau^i\}_{1 \le i \le n}$ given in equation* (36) *in the coordinates $(x^i)_{1 \le i \le n}$ for G. Then*

$$[\mathrm{Ad}(x)] = \mathrm{e}^{[\mathrm{ad}(\mathbf{e_1})]x^1} \cdots \mathrm{e}^{[\mathrm{ad}(\mathbf{e_n})]x^n}, \tag{74}$$

where $[\mathrm{Ad}(x)]$ is the matrix representation of $\mathrm{Ad}(x)$, $x \in G$ in the basis β.

Proof. Let $\rho : G \times G \to G$ be $\rho(x, y) = \mu(y, x^{-1})$. In components, using the basis β, we have

$$\rho^* \boldsymbol{\tau} = [Ad(x)] \pi_2^* \boldsymbol{\tau} - \pi_1^* \boldsymbol{\tau}_R, \tag{75}$$

where $\boldsymbol{\tau}_R$ are the right invariant 1-forms on G which are equal to the left invariant 1-forms $\boldsymbol{\tau}$ at the identity $\mathbf{0}$. From Lemma 5.10 we also have $\rho^* \boldsymbol{\tau} = \tilde{\boldsymbol{\theta}}$. Using the expression in equation (61) for $\tilde{\boldsymbol{\theta}}$ and equation (75) gives

$$\mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_1)]x^1} \cdots \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_n)]x^n} (\pi_2^* \boldsymbol{\tau} - \pi_1^* \boldsymbol{\tau}) = [Ad(x)] \pi_2^* \boldsymbol{\tau} - \pi_1^* \boldsymbol{\tau}_R,$$

leading to the expression for $[\mathrm{Ad}(x)]$ in equation (74). ■

We will now prove Theorem 5.7.

Proof. (Theorem 5.7) The proof relies on equation (74) in Corollary 5.11, written as

$$[\mathrm{Ad}(y^{-1})] = \mathrm{e}^{-[\mathrm{ad}(\mathbf{e_n})]y^n} \cdots \mathrm{e}^{-[\mathrm{ad}(\mathbf{e_1})]y^1}.$$

Therefore, equation (57) can be written

$$\boldsymbol{\omega} = [\mathrm{Ad}(y^{-1})] \pi_1^* \boldsymbol{\tau} + \pi_2^* \boldsymbol{\tau}. \tag{76}$$

Proposition 4.10 in [15] and equation (76) show that

$$\omega^i = \mu^* \tau^i, \tag{77}$$

where $\mu : G \times G \to G$ is the multiplication map. By taking the exterior derivative of equation (77), we find $\boldsymbol{\omega}$ in (57) satisfy the structure equations (58). This proves part [**i**].

To prove part [**ii**], let $\rho : G \times G \to G$ be constructed by Theorem 5.4, where f^i are chosen to satisfy $f^i(\mathbf{0}, \mathbf{0}) = 0$. Theorem 8.7 of [15] shows that the multiplication function, $\mu : G \times G \to G$, is the unique map satisfying $\mu^* \tau^i = \omega^i$ and $\mu(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. Therefore, the function ρ is the multiplication map μ. ■

Remark 5.12. If, instead of the forms $\boldsymbol{\omega}$ in equation (57) of Theorem 5.7, we take

$$\boldsymbol{\omega} = \pi_1^* \boldsymbol{\tau} + \mathrm{e}^{-[\mathrm{ad}(\mathbf{e_n})]x^n} \cdots \mathrm{e}^{-[\mathrm{ad}(\mathbf{e_1})]x^1} \pi_2^* \boldsymbol{\tau} \tag{78}$$

in the construction of $\rho : G \times G \to G$ with $\rho^* \tau^i = \omega^i$, then ρ is the multiplication map where the forms τ^i are *right* invariant.

Example 5.13. We continue with Example 4.4 and produce the multiplication map using Theorem 5.7 for the corresponding Lie groups where, according to Theorem 5.2, $\tau^i, i = 1, \dots, 5$ in equation (44) form a basis for the left invariant forms. First we need the forms ω^i, $i = 1, \dots, 5$ in equation (57) in Theorem 5.7. We compute the matrix $[\mathrm{Ad}(y^{-1})]$ in equation (57), using the structure constants in equation (42) (or see equation (76)), to be

$$[\mathrm{Ad}(y^{-1})] = \mathrm{e}^{-[\mathrm{ad}(\mathbf{e_5})]y^5} \cdots \mathrm{e}^{-[\mathrm{ad}(\mathbf{e_1})]y^1} =$$
$$\begin{bmatrix} \mathrm{e}^{a\,y^5 + b\,y^4} & 0 & 0 & -b\,y^1 \mathrm{e}^{a\,y^5+b\,y^4} & -a\,y^1 \mathrm{e}^{a\,y^5 a + b\,y^4} \\ 0 & \mathrm{e}^{y^4}\cos y^5 & \mathrm{e}^{y^4}\sin y^5 & -\mathrm{e}^{y^4}(y^2\cos y^5 + y^3 \sin y^5) & \mathrm{e}^{y^4}(y^2 \sin y^5 - y^3 \cos y^5) \\ 0 & -\mathrm{e}^{y^4}\sin y^5 & \mathrm{e}^{y^4}\cos y^5 & \mathrm{e}^{y^4}(y^2 \sin y^5 - y^3 \cos y^5) & \mathrm{e}^{y^4}(y^2\cos y^5 + y^3 \sin y^5) \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{79}$$

We multiply the vector of forms in equation (44) by the matrix in (79) to produce the forms $\boldsymbol{\omega}$ in equation (57),

$$\begin{aligned} \omega^1 &= \mathrm{e}^{a\,y^5 + b\,y^4}\left(dy^1 + \mathrm{e}^{a\,x^5+b\,x^4}dx^1 - a\,y^1 dx^5 - by^1 dx^4\right), \\ \omega^2 &= \mathrm{e}^{y^4}\Big(\mathrm{e}^{x^4}(\cos(x^5+y^5)dx^2 + \sin(x^5+y^5)dx^3) + \cos y^5 dy^2 + \sin y^5 dy^3 \\ &\qquad - (y^2\cos y^5 + y^3 \sin y^5)dx^4 + (y^2 \sin y^5 - y^3\cos y^5)dx^5\Big), \\ \omega^3 &= \mathrm{e}^{y^4}\Big(\mathrm{e}^{x^4}(\cos(x^5+y^5)dx^3 - \sin(x^5+y^5)dx^2) + \cos y^5 dy^3 - \sin y^5 dy^2 \\ &\qquad (y^2\sin y^5 - y^3 \cos y^5)dx^4 + (y^2 \cos y^5 + y^3\sin y^5)dx^5\Big), \\ \omega^4 &= dx^4 + dy^4, \\ \omega^5 &= dx^5 + dy^5\,. \end{aligned} \tag{80}$$

We now find, by quadratures, the functions in Corollary 2.4 which satisfy $f^i(\mathbf{0},\mathbf{0}) = 0$.

We find f^4, f^5 from $d\omega^4 = 0$ and $d\omega^5 = 0$ in the last two lines of equation (80) to be

$$f^5 = x^5 + y^5 \quad \text{and} \quad f^4 = x^4 + y^4. \tag{81}$$

By applying Theorem 2.2 twice to equation (80), we reduce $\boldsymbol{\omega}$ as follows

$$\hat{\boldsymbol{\omega}} = \begin{bmatrix} \mathrm{e}^{[\mathrm{ad}_1(\boldsymbol{e}_4)](x^4+y^4)} & \mathbf{0}_1 \\ \mathbf{0}_1^T & 1 \end{bmatrix} \mathrm{e}^{[\mathrm{ad}(\boldsymbol{e}_5)](x^5+y^5)} \boldsymbol{\omega}$$

where the matrices $[\mathrm{ad}_1(\boldsymbol{e}_4)]$ and $[\mathrm{ad}(\boldsymbol{e}_5)]$ are given in equation (43). This gives

$$
\begin{aligned}
\hat{\omega}^1 &= dx^1 + \mathrm{e}^{-a\,x^5 - b\,x^4}\left(dy^1 - b\,y^1 dx^4 - a\,y^1 dx^5\right),\\
\hat{\omega}^2 &= dx^2 + \mathrm{e}^{-x^4}\big(\cos x^5 dy^2 - \sin x^5 dy^3 - (y^2\cos x^5 - y^3 \sin x^5)dx^4\\
&\qquad\qquad - (y^3\cos x^5 + y^2\sin x^5)dx^5\big),\\
\hat{\omega}^3 &= dx^3 + \mathrm{e}^{-x^4}\big(\sin x^5 dy^2 + \cos x^5 dy^3 - (y^3\cos x^5 + y^2 \sin x^5)dx^4\\
&\qquad\qquad + (y^2\cos x^5 - y^3\sin x^5)dx^5\big),\\
\hat{\omega}^4 &= dx^4 + dy^4,\\
\hat{\omega}^5 &= dx^5 + dy^5.
\end{aligned}
\tag{82}
$$

Since $[\mathrm{ad}_3(\boldsymbol{e}_2)]$ and $[\mathrm{ad}_2(\boldsymbol{e}_3)]$ are zero matrices, there is no more reduction to be done in equation (21) so $d\hat{\omega}^3 = 0, d\hat{\omega}^2 = 0$ and $d\hat{\omega}^1 = 0$ in equation (82). This produces the final three functions ($\hat{\omega}^i = df^i, f^i(\mathbf{0},\mathbf{0}) = 0$) by quadratures,

$$
\begin{aligned}
f^3 &= x^3 + \mathrm{e}^{-x^4}(y^2\sin x^5 + y^3\cos x^5),\\
f^2 &= x^2 + \mathrm{e}^{-x^4}(y^2\cos x^5 - y^3\sin x^5),\\
f^1 &= x^1 + \mathrm{e}^{-a\,x^5 - b\,x^4} y^1.
\end{aligned}
\tag{83}
$$

In accordance with Theorem 5.7, the multiplication map $z^i = f^i(x, y)$ is given using equations (81) and (83) as

$$
\begin{aligned}
z^1 &= x^1 + \mathrm{e}^{-a\,x^5 - b\,x^4} y^1,\\
z^2 &= x^2 + \mathrm{e}^{-x^4}(y^2\cos x^5 - y^3\sin x^5),\\
z^3 &= x^3 + \mathrm{e}^{-x^4}(y^2\sin x^5 + y^3\cos x^5),\\
z^4 &= x^4 + y^4,\\
z^5 &= x^5 + y^5.
\end{aligned}
\tag{84}
$$

Equation (84) is the multiplication map $z = \mu(x, y)$ for the simply connected 5-dimensional Lie group G, with basis of left invariant forms in (80) and $\mathbf{0}$ as the identity.

6. Conclusion

It seems possible that an explicit formula for the multiplication map for a simply connected solvable Lie group could be given in terms of the structure constants of the Lie algebra by using the techniques developed in this paper, but the author has not been able to do this.

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Received December 16, 2015
and in final form June 17, 2016