

On Functions of Negative Type on the Olshanski Spherical Pair $(SL(\infty), SU(\infty))$

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Abstract. In this paper, using a generalized Bochner type representation for Olshanski spherical pairs, we prove the boundedness of every $SU(\infty)$ -biinvariant continuous function of negative type on the infinite dimensional special linear group $SL(\infty)$.

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1. Introduction

The continuous functions of negative type have been introduced by Schoenberg (cf. [18]) in order to determine the invariant Hilbert metrics on a group. The importance of such functions lies in their applications in the theory of limit theorems for independent and identically distributed random variables (cf. [14]). In 1990, G. Olshanski elaborated a spherical analysis theory related to the inductive limits of increasing sequences of Gelfand pairs (cf. [13], [6], [8] and [16]). In this new framework, many results concerning the continuous functions of negative type had been obtained (cf. [4], [5] and [17]).

In this paper, we consider the spherical pair (G_∞, K_∞) , which is the inductive limit of the increasing sequence of Gelfand pairs (G_n, K_n) where $G_n = SL(n, \mathbb{C})$ and $K_n = SU(n)$ are respectively the special linear group and its unitary subgroup of order n . The infinite dimensional special linear group $G_\infty = SL(\infty)$ consists in unimodular infinite invertible complex matrices (g_{ij}) with a finite number of coefficients $g_{ij} \neq \delta_{ij}$ and $K_\infty = SU(\infty)$ is the group of unimodular infinite unitary matrices (u_{ij}) with complex coefficients such that $u_{ij} = \delta_{ij}$ for $i + j$ large enough. The group G_∞ is equipped with the inductive limit topology and the subgroup K_∞ is closed.

A function $\varphi : G_\infty \rightarrow \mathbb{C}$ is said to be of positive type if the kernel

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defined on $G_\infty \times G_\infty$ by $(g_1, g_2) \mapsto \varphi(g_2^{-1}g_1)$ is of positive type, i.e. for all $g_1, g_2, \dots, g_n \in G_\infty$ and all $c_1, c_2, \dots, c_n \in \mathbb{C}$,

$$\sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j \varphi(g_j^{-1}g_i) \geq 0.$$

Every function φ of positive type on G_∞ is Hermitian, i.e. for all $g \in G_\infty$, $\overline{\varphi(g)} = \varphi(g^{-1})$. In addition, the function φ is bounded : $|\varphi(g)| \leq \varphi(e)$, where e is the origin of G_∞ . Besides, a function φ defined on G_∞ is said to be K_∞ -biinvariant if it holds that $\varphi(k_1 g k_2) = \varphi(g)$, for all $k_1, k_2 \in K_\infty$ and all $g \in G_\infty$. A function ψ , defined on G_∞ , with complex values is said to be of *negative type* if $\psi(e) \geq 0$, $\psi(g^{-1}) = \overline{\psi(g)}$ and, for all $g_1, \dots, g_N \in G_\infty$ and all $c_1, \dots, c_N \in \mathbb{C}$ such that $\sum_{i=1}^N c_i = 0$,

$$\sum_{i,j=1}^N c_i \bar{c}_j \psi(g_j^{-1}g_i) \leq 0.$$

A spherical function for the Olshanski spherical pair (G_∞, K_∞) is a K_∞ -biinvariant continuous function φ on G_∞ which is normalized at the origin of the group: $\varphi(e) = 1$, and such that

$$\lim_{n \rightarrow \infty} \int_{K_n} \varphi(xky) \alpha_n(dk) = \varphi(x)\varphi(y),$$

where α_n denotes the normalized Haar measure on K_n (cf. [13] and [9]). In section 2, using the explicit expression of the spherical functions of positive type relatively to the spherical pair (G_∞, K_∞) given by N.I. Nessonov in [12], we prove that the spherical dual of the pair (G_∞, K_∞) can be identified with a set \mathbb{X} of parameters. This enables us to obtain a parametrized version of the generalized Bochner theorem ([15], Theorem 7), which represents the key theorem used, in the last section, to prove the boundedness of all continuous functions of negative type, relatively to the pair (G_∞, K_∞) . The method that we use follows an idea given by C. Berg, J.P. Christensen and P. Ressel in [2] and the work of M. Bouali in [5] and [4].

2. Functions of class \mathfrak{B} : definition and convergence

In his paper [12], N.I. Nessonov gives a complete classification of the representations of G_∞ containing the trivial representation of the unitary group K_∞ . This class of representations contains those of spherical representations. In fact, the trivial representation of K_∞ is contained in a unitary representation (π, \mathcal{H}) of G_∞ if and only if the subspace \mathcal{H}_{K_∞} consisting of K_∞ -invariant vectors in \mathcal{H} is not reduced to zero. On the other hand, every spherical representation is associated via the G.N.S. construction (Gelfand-Naimark-Segal) to a spherical function of positive type, i.e. an extreme element of the convex set of continuous functions of positive type on G_∞ which are biinvariant by the subgroup K_∞ and equal to 1 at the origin. Thus, this representation possesses a cyclic vector which is in addition K_∞ -invariant. It follows that the result of N.I. Nessonov, that one finds at

the end of his paper (cf. [12], Theorem 4.5), allows us to give an explicit expression of spherical functions of positive type relatively to the pair (G_∞, K_∞) .

Definition 2.1. A function of class \mathfrak{B} of parameter $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{R}^p$ with $p \in \mathbb{N}$ is defined on \mathbb{R} by :

$$\Pi(\alpha, \lambda) := \prod_{j=1}^p (\cosh(\lambda) - i\alpha_j \sinh(\lambda))^{-1}.$$

Remark 2.2. If $p = 0$, then $\alpha = \emptyset$ and so $\Pi(\emptyset; 0, \lambda) \equiv 1$.

A function $\Pi(\alpha, \lambda)$ of class \mathfrak{B} is of positive type on \mathbb{R} . In fact, for $\alpha \in \mathbb{R}$, the function $(\cosh(\lambda) - i\alpha \sinh(\lambda))^{-1}$ is the Fourier transform of the probability measure μ_α on \mathbb{R} given by

$$\mu_\alpha = g_\alpha(t) dt \quad g_\alpha(t) = \frac{e^{t \arctan \alpha}}{2\sqrt{1 + \alpha^2} \cosh(\pi t/2)}.$$

This can be proved by the residue theorem via a contour integral of the function

$$(\cosh(\pi z) - i\alpha \sinh(\pi z))^{-1} e^{-2\pi itz},$$

around the rectangle of vertices $\pm R$, $\pm R + i$ and letting $R \rightarrow +\infty$. Then, the function of positive type $\Pi(\alpha, \lambda)$ is the Fourier transform of the convolution measure $\mu_{\alpha_1} \star \dots \star \mu_{\alpha_p}$, where $(\alpha_1, \dots, \alpha_p) \in \mathbb{R}^p$.

We consider, on the set of functions of class \mathfrak{B} , the topology of uniform convergence on compact sets of \mathbb{R} . The resulting topological space is metrizable and complete. This topology can be expressed in terms of the set \mathbb{X} of parameters $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{R}^p$ with $p \in \mathbb{N}$. Let us introduce, for $p \in \mathbb{N}^*$, the sets

$$\mathbb{X}_p = \{\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{R}^p \mid \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_p\}.$$

Since, the class \mathfrak{B} should contain the trivial function 1, one has to define the space of one element $\mathbb{X}_0 = \{\emptyset\}$ with the discrete topology. In consequence, the space of parameters \mathbb{X} becomes the disjoint union of \mathbb{X}_p with $p \in \mathbb{N}$. For every $p \in \mathbb{N}$, the set \mathbb{X}_p is both closed and open in \mathbb{X} . Hence, it is locally compact with a countable basis.

The set of functions of class \mathfrak{B} is then parametrized by the set \mathbb{X} and every function $\Pi(\alpha, \lambda)$ have the following logarithmic derivative:

$$\frac{\Pi'(\alpha, \lambda)}{\Pi(\alpha, \lambda)} = i \sum_{m=0}^{\infty} \tilde{p}_m(\alpha) (i \tanh(\lambda))^m, \tag{1}$$

where p_m is the Newton power sum function : for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p) \in \mathbb{R}^p$ and $m \in \mathbb{N}$,

$$p_m(\alpha) = \sum_{k=1}^p \alpha_k^m.$$

$$\tilde{p}_m(\alpha) = p_{m+1}(\alpha) + p_{m-1}(\alpha) \quad (m \in \mathbb{N}), \quad (2)$$

with the convention that

$$p_{-1}(\alpha) = 0.$$

Proposition 2.3. *The mapping $\alpha \mapsto \Pi(\alpha, \cdot)$ is a bijection of $\mathbb{X}^* = \mathbb{X} \setminus \mathbb{X}_0$ to the set of functions of class \mathfrak{B} .*

Proof. For $\lambda \in \mathbb{R}$, $\alpha \in \mathbb{X}_p$ and $\beta \in \mathbb{X}_q$, let us assume that

$$\Pi(\alpha, \lambda) = \Pi(\beta, \lambda).$$

Since,

$$\lim_{\lambda \rightarrow +\infty} e^{p\lambda} \Pi(\alpha, \lambda) = 2^p \prod_{k=1}^p (1 - i\alpha_k)^{-1},$$

it follows that

$$p = q.$$

Using the power series expansion (1), we conclude that $\alpha = \beta$. ■

A function of class \mathfrak{B} of parameter α is the Fourier transform of a probability measure μ_α . Let $\mathfrak{M}_{\mathbb{X}}$ be the set of these measures : $\mathfrak{M}_{\mathbb{X}} = \{\mu_\alpha | \Pi(\alpha, \lambda) = \widehat{\mu_\alpha}(\lambda)\}$. We consider on $\mathfrak{M}_{\mathbb{X}}$ the weak topology of measures.

Proposition 2.4. *The topology of \mathbb{X} is equivalent to the topology of $\mathfrak{M}_{\mathbb{X}}$.*

Proof. (i) Assume that $\alpha^{(n)}$ converges to α in the topology of \mathbb{X} . Then $\alpha \in \mathbb{X}_p$, for some p and so, there exists n_0 such that $\alpha^{(n)} \in \mathbb{X}_p$, for $n \geq n_0$. It follows that $\Pi(\alpha^{(n)}, \lambda)$ converges uniformly on compact sets of \mathbb{R} to $\Pi(\alpha, \lambda)$. Finally, by applying the Lévy-Cramer theorem, one can prove that $\mu_{\alpha^{(n)}}$ converges weakly to μ_α .

(ii) Assume that $\mu_{\alpha^{(n)}}$ converges weakly to μ_α . This implies that $\Pi(\alpha^{(n)}, \lambda)$ converges uniformly on compact sets in \mathbb{R} to $\Pi(\alpha, \lambda)$. Let $\lambda_0 > 0$. Since, the function $\Pi(\alpha^{(n)}, \lambda)$ is continuous, non-zero on \mathbb{R} and satisfies $\Pi(\alpha^{(n)}, 0) = 1$, there exists $0 < C < 1$ and $n_1 \in \mathbb{N}$ such that, for every $n \geq n_1$,

$$\Re(\Pi(\alpha^{(n)}, \lambda_0)) \geq C,$$

and so,

$$|\Pi(\alpha^{(n)}, \lambda_0)| \geq C.$$

In consequence,

$$\prod_{j=1}^{p_n} (\cosh^2(\lambda_0) + \alpha_j^{(n)2} \sinh^2(\lambda_0))^{-1} \geq C^2.$$

Hence, we get

$$\sinh^2(\lambda_0) \sum_{j=1}^{p_n} \alpha_j^{(n)2} \leq \prod_{j=1}^{p_n} (1 + \alpha_j^{(n)2} \sinh^2(\lambda_0)) \leq \frac{1}{C^2},$$

or

$$\sum_{j=1}^{p_n} \alpha_j^{(n)2} \leq \frac{1}{C^2 \sinh^2(\lambda_0)}.$$

This implies that, for all $n \geq n_1$ and all $1 \leq j \leq p_n$,

$$|\alpha_j^{(n)}| \leq \frac{1}{C \sinh(\lambda_0)}. \tag{3}$$

Moreover,

$$\prod_{j=1}^{p_n} \cosh^2(\lambda_0) \leq \frac{1}{C^2}.$$

As a result, for all $n \geq n_1$,

$$p_n \leq -\frac{\log C}{\log(\cosh(\lambda_0))}. \tag{4}$$

Since, by (4), the sequence p_n is bounded, we can assume, upon passing to a subsequence, that $p_n = p$. Further, by (3), there exists a subsequence n_k such that $\alpha^{(n_k)}$ converges to some $\beta \in \mathbb{X}_p$. It follows from the first part of the proof that the corresponding sequence of measures $\mu_{\alpha^{(n_k)}}$ converges weakly to μ_β . Thus, we get $\mu_\alpha = \mu_\beta$ and so $\alpha = \beta$. Since any accumulation point of the sequence $\alpha^{(n)}$ in \mathbb{X}_p is necessary equal to α , it follows that the sequence $\alpha^{(n)}$ itself converges to α in \mathbb{X} . ■

Corollary 2.5. *The mapping $\alpha \mapsto \Pi(\alpha, \cdot)$ is a homeomorphism between \mathbb{X}^* and \mathfrak{B} .*

Proof. By Proposition 2.3, the function $\alpha \mapsto \Pi(\alpha, \cdot)$ is a bijection between \mathbb{X}^* and \mathfrak{B} . Besides, by Proposition 2.4, the topology of uniform convergence on compact sets of \mathbb{R} defined on \mathfrak{B} is equivalent to the topology defined on the set of parameters \mathbb{X}^* . In consequence, the application $\alpha \mapsto \Pi(\alpha, \cdot)$ defines a homeomorphism between \mathbb{X}^* and \mathfrak{B} . ■

3. Parametrized Bochner Theorem on (G_∞, K_∞)

Let \mathcal{P} be the convex set of K_∞ biinvariant continuous functions of positive type on G_∞ . The topology defined on the set of its extremal points $\text{ext}(\mathcal{P})$ can be expressed in terms of the set \mathbb{X} . This enables us to prove a parametrized version of the generalized Bochner theorem ([15], Theorem 7).

The group G_∞ can be seen as the group of unimodular infinite invertible matrices $g = [g_{ij}]_{i,j=1}^\infty$ with a finite number of coefficients $g_{ij} \neq \delta_{ij}$. Every matrix g in G_∞ can be written under the form $g = u \text{diag}(e^{\lambda_1}, e^{\lambda_2}, \dots) v$, with $u, v \in K_\infty$, $\lambda_1, \lambda_2, \dots \in \mathbb{R}$, $\sum_j \lambda_j = 0$ and $\lambda_j = 0$ for j large enough. We say that a K_∞ -biinvariant function φ on G_∞ which is normalized by the condition $\varphi(e) = 1$, is multiplicative if

$$\varphi(\text{diag}(a_1, a_2, \dots)) = \Phi(a_1)\Phi(a_2)\dots,$$

where Φ is a function defined on \mathbb{R} such that $\Phi(0) = 1$. In other words, for all $g \in G_\infty$,

$$\varphi(g) = \Phi(a_1)\Phi(a_2)\dots,$$

where $a_1, a_2, \dots \in \mathbb{R}$ are the elements of the diagonal matrix in the decomposition of g .

Theorem 3.1. ([8], Theorem 5.2, page 228) *Let φ be a K_∞ -biinvariant continuous function of positive type on G_∞ such that $\varphi(e) = 1$. Then, φ is spherical (or extremal) if and only if it is multiplicative.*

By the preceding theorem, the set $\text{ext}(\mathcal{P})$ consists of the multiplicative functions of \mathcal{P} . The following theorem gives these elements via the functions of class \mathfrak{B} .

Theorem 3.2. *The extremal points of the set \mathcal{P} are the functions φ_α defined for all $g = u \text{diag}(e^{\lambda_1}, e^{\lambda_2}, \dots) v \in G_\infty$, with $u, v \in K_\infty$, $\lambda_1, \lambda_2, \dots \in \mathbb{R}$, $\sum_j \lambda_j = 0$ and $\lambda_j = 0$, for j large enough, by*

$$\varphi_\alpha(g) = \varphi_\alpha(\text{diag}(e^{\lambda_1}, e^{\lambda_2}, \dots)) = \prod_{j=1}^{\infty} \Pi(\alpha, \lambda_j).$$

Proof. a) By the preceding theorem, a function in \mathcal{P} defined, for all $g \in G_\infty$, by $\varphi_\alpha(g) = \prod_j \Pi(\alpha, \lambda_j)$ is extremal in \mathcal{P} since it is multiplicative.

b) Let $\varphi \in \text{ext}(\mathcal{P})$. It is a spherical function. By the result of Nessonov (cf. [12], Theorem 4.5), the function φ is a product of functions of class \mathfrak{B} . ■

Remark 3.3. Remark that the trivial spherical function is isolated in $\text{ext}(\mathcal{P})$. This follows immediately from the fact that there is no sequence $\alpha^{(n)}$ for which $\Pi(\alpha^{(n)}, \cdot)$ converges to 1.

Proposition 3.4. *The correspondence $\mathbb{X} \leftrightarrow \text{ext}(\mathcal{P})$ is an isomorphism between two standard spaces.*

Proof. Since the set \mathbb{X} is locally compact, separable, metrizable and complete, it represents, in consequence, a standard space. In addition, the proof of the generalized Bochner theorem ([15], Theorem 7), shows that $\text{ext}(\mathcal{P})$ is a Borel subset of a standard space. Hence, it is standard by ([7], Appendix B, B 20). Furthermore, the correspondence $\text{ext}(\mathcal{P}) \rightarrow \mathbb{X}, \varphi_\alpha \mapsto \alpha$ is Borelian and one-to-one. In consequence, by ([7], Appendix B, B22), it is an isomorphism between two standard spaces. ■

Hence, we can get a parametrized version of the generalized Bochner theorem ([15], Theorem 7):

Theorem 3.5. *Let φ be a K_∞ -biinvariant continuous function of positive type on G_∞ . Then, there exists a unique positive and bounded measure μ defined on \mathbb{X} such that, for every $g \in G_\infty$,*

$$\varphi(g) = \int_{\mathbb{X}} \varphi_\alpha(g) \mu(d\alpha).$$

The following proposition follows from the uniqueness of the representing measure in the generalized Bochner Theorem.

Proposition 3.6. *Let μ_1 and μ_2 be two positive and bounded measures on \mathbb{X} satisfying*

$$\int_{\mathbb{X}} (1 - \varphi_\alpha(g)) \mu_1(d\alpha) = \int_{\mathbb{X}} (1 - \varphi_\alpha(g)) \mu_2(d\alpha). \tag{5}$$

Then,

$$\mu_1 = \mu_2 \quad \text{on} \quad \mathbb{X}^*.$$

Proof. Since the function φ_α is spherical, it satisfies, for all $g, h \in G_\infty$,

$$\lim_{n \rightarrow \infty} \int_{K_n} \varphi_\alpha(gkh) \beta_n(dk) = \varphi_\alpha(g) \varphi_\alpha(h), \tag{6}$$

where β_n is the normalized Haar measure of the group K_n . Since, the measures β_n and μ_1 are bounded, the function

$$(\alpha, k) \mapsto |1 - \varphi_\alpha(gkh)|$$

is integrable with respect to the product measure $\mu_1 \times \beta_n$. Hence, by the Fubini theorem, we get that

$$\int_{K_n} \int_{\mathbb{X}} (1 - \varphi_\alpha(gkh)) \mu_1(d\alpha) \beta_n(dk) = \int_{\mathbb{X}} \int_{K_n} (1 - \varphi_\alpha(gkh)) \beta_n(dk) \mu_1(d\alpha).$$

Using the fact that $|1 - \varphi_\alpha(gkh)| \leq 2$ and that the measure β_n is a probability measure, we conclude that the function

$$\alpha \mapsto \int_{K_n} (1 - \varphi_\alpha(gkh)) \beta_n(dk)$$

is dominated by 2. Since, the measure μ_1 is positive and bounded, the dominated convergence theorem and the equation (6), imply that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{X}} \int_{K_n} (1 - \varphi_\alpha(gkh)) \beta_n(dk) \mu_1(d\alpha) = \int_{\mathbb{X}} (1 - \varphi_\alpha(g)\varphi_\alpha(h)) \mu_1(d\alpha),$$

and so,

$$\lim_{n \rightarrow \infty} \int_{K_n} \int_{\mathbb{X}} (1 - \varphi_\alpha(gkh)) \mu_1(d\alpha) \beta_n(dk) = \int_{\mathbb{X}} (1 - \varphi_\alpha(g)\varphi_\alpha(h)) \mu_1(d\alpha).$$

In consequence, by equation (5), we get that

$$\int_{\mathbb{X}} (1 - \varphi_\alpha(g)\varphi_\alpha(h)) \mu_1(d\alpha) = \int_{\mathbb{X}} (1 - \varphi_\alpha(g)\varphi_\alpha(h)) \mu_2(d\alpha).$$

By substituting, in the previous equation, g by g^{-1} , and using the fact that φ_α is Hermitian, we get

$$\int_{\mathbb{X}} (1 - \varphi_\alpha(h)\Re\varphi_\alpha(g)) \mu_1(d\alpha) = \int_{\mathbb{X}} (1 - \varphi_\alpha(h)\Re\varphi_\alpha(g)) \mu_2(d\alpha).$$

By substituting g by h in equation (5), and considering the difference with the previous equation, we get, for all $g, h \in G_\infty$,

$$\int_{\mathbb{X}} \varphi_\alpha(h)(1 - \Re\varphi_\alpha(g)) \mu_1(d\alpha) = \int_{\mathbb{X}} \varphi_\alpha(h)(1 - \Re\varphi_\alpha(g)) \mu_2(d\alpha).$$

Now, let us consider the function $\tilde{\varphi}$ defined on G_∞ by

$$\tilde{\varphi}(h) = \int_{\mathbb{X}} \varphi_\alpha(h)(1 - \Re\varphi_\alpha(g)) \mu_1(d\alpha).$$

Since, for all $g \in G_\infty$, we have $1 - \Re\varphi_\alpha(g) \geq 0$, the function $\tilde{\varphi}$ is of positive type. It is also K_∞ -biinvariant and continuous on G_∞ . Moreover, the function $\tilde{\varphi}$ is bounded, since it is dominated by a μ_1 -integrable function on \mathbb{X} :

$$|\varphi_\alpha(h)(1 - \Re\varphi_\alpha(g))| \leq 1 - \Re\varphi_\alpha(g).$$

So, for all $g \in G_\infty$,

$$\int_{\mathbb{X}} \varphi_\alpha(h) \mu_{1,g}(d\alpha) = \int_{\mathbb{X}} \varphi_\alpha(h) \mu_{2,g}(d\alpha).$$

where,

$$\mu_{i,g}(d\alpha) = (1 - \Re\varphi_\alpha(g)) \mu_i(d\alpha) \quad \text{for } i = 1, 2.$$

By uniqueness of the representing measure in the generalized Bochner theorem (Theorem 3.5), we get that, for all $g \in G_\infty$,

$$\mu_{1,g} = \mu_{2,g} \quad \text{on } \mathbb{X}.$$

Since, we have $|\Pi(\alpha, \lambda)| < 1$, for $\alpha \in \mathbb{X}^*$ and $\lambda \neq 0$, we get, for $g_0 = \text{diag}(e^1, e^{-1}, 1, \dots)$, that $1 - \Re\varphi_\alpha(g_0) > 0$. Hence

$$\mu_1 = \mu_2 \quad \text{on } \mathbb{X}^*. \quad \blacksquare$$

4. Functions of negative type on (G_∞, K_∞)

In this section, using the generalized Bochner theorem (Theorem 3.5), we establish the boundedness of every continuous function of negative type on the pair (G_∞, K_∞) . The method that we follow is inspired from the work of M. Bouali in [5] and [4]. If φ is a function of positive type, then $\psi(g) = \varphi(e) - \varphi(g)$ is a bounded function of negative type. The functions of negative type and those of positive type are related by the following property :

Proposition 4.1. (Schoenberg ([18], page 527) and ([3], Theorem 7.8))
The function ψ is of negative type if and only if $\psi(e) \geq 0$, and, for all $t \geq 0$, $e^{-t\psi}$ is of positive type.

Theorem 4.2. *Every K_∞ -biinvariant continuous function ψ of negative type on G_∞ is bounded and can be uniquely written as $\psi(g) = \psi(e) + \varphi(e) - \varphi(g)$, where φ is a function of positive type relatively to the spherical pair (G_∞, K_∞) .*

Proof. Let ψ be a K_∞ -biinvariant continuous function of negative type on G_∞ . Since $\psi(g) - \psi(e)$ is also K_∞ -biinvariant, continuous and of negative type,

we can assume, without loss of generality, that $\psi(e) = 0$. For $t \geq 0$, the function $e^{-t\psi}$ is K_∞ -biinvariant, continuous and of positive type on G_∞ . Hence, by the generalized Bochner theorem (Theorem 3.5), there exists a probability measure μ_t on \mathbb{X} such that

$$e^{-t\psi(g)} = \int_{\mathbb{X}} \varphi_\alpha(g) \mu_t(d\alpha).$$

It follows that, for all $t > 0$,

$$e^{-t\Re\psi(g)} \cos(t\Im\psi(g)) = \int_{\mathbb{X}} \Re\varphi_\alpha(g) \mu_t(d\alpha), \tag{7}$$

and

$$e^{-t\Re\psi(g)} \sin(t\Im\psi(g)) = - \int_{\mathbb{X}} \Im\varphi_\alpha(g) \mu_t(d\alpha).$$

i) The equation (7) implies that

$$\frac{1 - e^{-t\Re\psi(g)} \cos(t\Im\psi(g))}{t} = \int_{\mathbb{X}} (1 - \Re\varphi_\alpha(g)) \frac{\mu_t}{t}(d\alpha).$$

In addition, we have

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1 - e^{-t\Re\psi(g)} \cos(t\Im\psi(g))}{t} &= \Re\psi(g), \\ \lim_{t \rightarrow +\infty} \frac{1 - e^{-t\Re\psi(g)} \cos(t\Im\psi(g))}{t} &= 0. \end{aligned}$$

For g fixed, the last expression is a continuous function in t on $]0, +\infty[$ which tends to 0 as t tends to ∞ . Thus, there exists a constant $C(g) \geq 0$ such that

$$0 \leq \frac{1 - e^{-t\Re\psi(g)} \cos(t\Im\psi(g))}{t} \leq C(g).$$

Therefore,

$$\int_{\mathbb{X}} (1 - \Re\varphi_\alpha(g)) \frac{\mu_t}{t}(d\alpha) \leq C(g).$$

In particular, for $g_0 = \text{diag}(e^1, e^{-1}, 1, \dots)$, we have

$$\int_{\mathbb{X}} (1 - \Re\varphi_\alpha(g_0)) \frac{\mu_t}{t}(d\alpha) \leq C(g_0) = M. \tag{8}$$

Let ν_t denote the positive and bounded measure defined on \mathbb{X} by

$$\nu_t = (1 - \Re\varphi_\alpha(g_0)) \frac{\mu_t}{t}.$$

Since the set $\{\nu_t \mid t > 0\}$ is uniformly bounded by the constant M , it is relatively compact for the weak topology $\sigma(\mathcal{M}(\mathbb{X}), \mathcal{C}_0(\mathbb{X}))$, where $\mathcal{M}(\mathbb{X})$ is the set of positive

and bounded measures on \mathbb{X} and $\mathcal{C}_0(\mathbb{X})$ is the set of continuous functions on \mathbb{X} , vanishing at ∞ . In consequence, there exists a sequence (t_j) in $]0, +\infty[$ converging to 0, such that the measures ν_{t_j} weakly converge to a positive and bounded measure ν , i.e. for all $f \in \mathcal{C}_0(\mathbb{X})$,

$$\lim_{j \rightarrow \infty} \int_{\mathbb{X}} f(\alpha) \nu_{t_j}(d\alpha) = \int_{\mathbb{X}} f(\alpha) \nu(d\alpha).$$

Since $\varphi_\alpha(g_0) = 1$ for $\alpha_0 = \{\emptyset\}$, the measure ν_t has no mass at the point α_0 and therefore $\nu(\{\emptyset\}) = 0$. On another hand, we have

$$\begin{aligned} \frac{1 - e^{-t_j \Re \psi(g)} \cos(t_j \Im \psi(g))}{t_j} &= \int_{\mathbb{X}^*} \left[\frac{1 - \Re \varphi_\alpha(g)}{1 - \Re \varphi_\alpha(g_0)} - 1 \right] \nu_{t_j}(d\alpha) \\ &+ \frac{1 - e^{-t_j \Re \psi(g_0)} \cos(t_j \Im \psi(g_0))}{t_j}. \end{aligned} \tag{9}$$

For $g \neq e$ fixed, let us consider the function f defined on \mathbb{X}^* by

$$f(\alpha) = \frac{1 - \Re \varphi_\alpha(g)}{1 - \Re \varphi_\alpha(g_0)} - 1.$$

The function f is well defined since $\Re \varphi_\alpha(g_0) \neq 1$, for all $\alpha \in \mathbb{X}^*$. It is also continuous, by Corollary 2.5 and Theorem 3.2. In addition, for every $g \in G_\infty$ such that $\lambda_j \neq 0$ for at least one j , we have

$$0 < |\varphi_\alpha(g)| \leq \prod_{j=1}^{\infty} \prod_{k=1}^p (\cosh^2(\lambda_j) + \alpha_k^2 \sinh^2(\lambda_j))^{-\frac{1}{2}}.$$

It follows that, for $\alpha \in \mathbb{X}_p$ and p_0 large enough independent of α ,

$$|\varphi_\alpha(g)| \leq \cosh^{-p}(\lambda_j) < \varepsilon.$$

Besides, for $1 \leq p \leq p_0$ and for α in \mathbb{X}_p such that $\|\alpha\|$ is large enough,

$$|\varphi_\alpha(g)| \leq \frac{1}{\|\alpha\| |\sinh(\lambda_j)|} < \varepsilon.$$

In consequence, $|\varphi_\alpha(g)|$ tends to 0 as α tends to ∞ in the locally compact space \mathbb{X} . Therefore, the function f belongs to $\mathcal{C}_0(\mathbb{X}^*)$. Now, as j tends to ∞ in the equation (9), one gets, for all $g \neq e$,

$$\Re \psi(g) = \int_{\mathbb{X}^*} f(\alpha) \nu(d\alpha) + \Re \psi(g_0) = \int_{\mathbb{X}^*} \left(\frac{1 - \Re \varphi_\alpha(g)}{1 - \Re \varphi_\alpha(g_0)} - 1 \right) \nu(d\alpha) + \Re \psi(g_0).$$

Since

$$\left| \frac{1 - \Re\varphi_\alpha(g)}{1 - \Re\varphi_\alpha(g_0)} - 1 \right| \leq \frac{2}{1 - \Re\varphi_\alpha(g_0)} + 1,$$

with $\Re\varphi_\alpha(g_0) \leq \cosh^{-2p}(1)$, the integrand in the last integral is dominated and we can apply the dominated convergence theorem to get, as g tends to e ,

$$\nu(\mathbb{X}^*) = \Re\psi(g_0).$$

So,

$$\Re\psi(g) = \int_{\mathbb{X}^*} (1 - \Re\varphi_\alpha(g))\mu(d\alpha),$$

where μ is the measure defined on \mathbb{X}^* by

$$\mu = \frac{1}{1 - \Re\varphi_\alpha(g_0)}\nu.$$

By Corollary 2.5, the function

$$\alpha \mapsto \frac{1}{1 - \Re\varphi_\alpha(g_0)}$$

is continuous on \mathbb{X}^* . Moreover, this function tends to 1 as α tends to ∞ and it has as limit $\cosh^{-2p}(1)$ at 0. It follows that it is bounded and so is the measure μ .

ii) Similarly, we prove that the function h defined on \mathbb{X}^* by

$$h(\alpha) = \frac{\Im\varphi_\alpha(g)}{1 - \Re\varphi_\alpha(g_0)},$$

belongs to $\mathcal{C}_0(\mathbb{X}^*)$. As j tends to ∞ in the relation

$$e^{-t_j \Re\psi(g)} \frac{\sin(t_j \Im\psi(g))}{t_j} = - \int_{\mathbb{X}^*} \frac{\Im\varphi_\alpha(g)}{1 - \Re\varphi_\alpha(g_0)} \nu_{t_j}(d\alpha),$$

we get

$$\Im\psi(g) = - \int_{\mathbb{X}^*} \frac{\Im\varphi_\alpha(g)}{1 - \Re\varphi_\alpha(g_0)} \nu(d\alpha) = - \int_{\mathbb{X}^*} \Im\varphi_\alpha(g) \mu(d\alpha).$$

Finally,

$$\psi(g) = \int_{\mathbb{X}^*} (1 - \varphi_\alpha(g))\mu(d\alpha) = \varphi(e) - \varphi(g),$$

where

$$\varphi(g) = \int_{\mathbb{X}^*} \varphi_\alpha(g) \mu(d\alpha).$$

Remark that, by Proposition 3.6, the measure μ is unique. ■

Remark 4.3. In [11], Kazhdan introduced the notion of the property (T) as follows: a locally compact topological group G has the property (T) if the trivial representation ι_G is isolated in the unitary dual \widehat{G} which is the set of all unitary irreducible representations of G equipped with the Fell topology. The notion of property (T) was extended to any Hausdorff topological group and was also related to other topological properties such as the property (FH). One can refer to [1] for extensive reading. Property (T) is related to the functions of negative type since it holds that if a Hausdorff topological group G has property (T), then every continuous function of negative type on G is bounded. A question immediately arises : does $SL(\infty)$ have property (T)?

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