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Schur-Weyl Duality for Special Orthogonal Groups

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Abstract. Classical Schur-Weyl duality is between the group algebras of the general linear group, $GL_m(\mathbb{C})$, and the symmetric group, S_r; both acting on the rth tensor power of the space $\mathbb{C}^m$. To get an analogue of this duality for orthogonal groups, Brauer described the so-called Brauer algebra which surjects onto the commutant of the group algebra of the orthogonal group. He also proved a Schur-Weyl duality for orthogonal groups over $\mathbb{C}$ which was later extended by Doty and Hu to all infinite fields of characteristic not two. In this paper, we prove the analogous duality for the special orthogonal groups over any infinite field of characteristic not two.
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1. Introduction

Let K denote an infinite field, let Γ be a group and let V be a finite-dimensional K-representation of Γ. Whenever $V^{\otimes r}$ is semisimple under the diagonal action of Γ, it is natural to ask if one can understand its decomposition into irreducible subrepresentations. For that, it has been useful to compute the commutant $\mathrm{End}_{K\Gamma}(V^{\otimes r})$ of $K\Gamma$, the group ring of Γ over the field K, in the ring of endomorphisms of $V^{\otimes r}$ and to see if its commutant is equal to the image of $K\Gamma$ in $\mathrm{End}(V^{\otimes r})$.

Consider the classical case where Γ is $GL(V)$, the group of linear automorphisms of a finite dimensional vector space V over K, acting naturally on V. The natural action of $GL(V)$ on V induces the corresponding diagonal action of $GL(V)$ on a tensor power $V^{\otimes r}$. Let S_r denote the symmetric group on r letters. Note that S_r also acts (on the left) on $V^{\otimes r}$ by permuting the factors of a simple r-tensor:

$$\sigma(u_1 \otimes u_2 \otimes \cdots \otimes u_r) = u_{\sigma^{-1}(1)} \otimes u_{\sigma^{-1}(2)} \otimes \cdots \otimes u_{\sigma^{-1}(r)}$$

where $\sigma \in S_r$ and $u_i \in V$. These actions give rise to natural maps

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$$\Psi : KGL(V) \to \mathrm{End}_K(V^{\otimes r}) \text{ and } \Phi : KS_r \to \mathrm{End}_K(V^{\otimes r}).$$

The action of S_r on $V^{\otimes r}$ clearly commutes with the action of $GL(V)$. Therefore, we have that $\Phi(KS_r) \subseteq \mathrm{End}_{KGL(V)}(V^{\otimes r})$ and $\Psi(KGL(V)) \subseteq \mathrm{End}_{KS_r}(V^{\otimes r})$. In fact, we have:

Theorem 1.1 (Schur-Weyl duality)**.** *Let K be an infinite field and let V be a finite dimensional vector space over K. If Φ and Ψ are maps as defined above then*

$$\Psi(KGL(V)) = \mathrm{End}_{KS_r}(V^{\otimes r}); \qquad \Phi(KS_r) = \mathrm{End}_{KGL(V)}(V^{\otimes r}).$$

Moreover, for $\dim V \geq r$ the map Φ is injective.

For $K = \mathbb{C}$, this result was proved in a classical paper of Schur ([13]). The proof of the result for an arbitrary infinite field seems to be well-known since 1980, and has appeared in print in the works of Green and De Concini-Procesi. Green proved that $\Psi(KGL(V))$ is the commutant of KS_r in $\mathrm{End}_K(V^{\otimes r})$ ([10, Theorem 2.6c]) whereas the other part about the map Φ was proved by De Concini and Procesi ([6, Theorem 4.1]). Most of the cases of double centralizer theorem were also proved in [5].

Doty ([7]) provided a new proof of Schur-Weyl duality over arbitrary infinite fields as well as a new proof for the injectivity result stated above when $\dim V$ is bigger than or equal to r. He proved surjectivity of the maps Φ and Ψ by proving that the dimensions of the algebras $\Psi(KGL(V))$, $\mathrm{End}_{KS_r}(V^{\otimes r})$, $\Phi(KS_r)$ and $\mathrm{End}_{KGL(V)}(V^{\otimes r})$ as vector spaces over K are independent of the infinite field K after which the result would follow by surjectivity over $K = \mathbb{C}$. This strategy was suggested by S. Koenig. We use a variant of this strategy in proving the results of this paper.

There are analogues of Schur-Weyl duality for other groups. The main point in these analogous results is to find the commutant of the group algebra acting on the representation space and then to prove a version of the double commutant theorem that the commutant of the commutant is the image of the group algebra.

Let us now assume that the vector space V is equipped with a quadratic form q, represented by $\langle 1, 1, \ldots, 1\rangle$. That is, we assume that there is a basis $\{v_1, v_2, \ldots, v_m\}$ of the space V consisting of orthonormal vectors with respect to the bilinear form corresponding to q. We then consider the groups $O(V, q)$ and $SO(V, q)$ and their natural actions on $V^{\otimes r}$. The commutants of $O(V, q)$ and $SO(V, q)$ in the ring of endomorphisms of $V^{\otimes r}$ were first studied by R. Brauer ([2, Section 5]). Brauer defined a certain algebra, by giving generators and relations for it, and defined an action of this algebra on $V^{\otimes r}$. He proved that this algebra, over $K = \mathbb{C}$, surjects onto the commutant of $O(V, q)$ in $\mathrm{End}_K(V^{\otimes r})$. We follow the standard way of denoting this algebra by $B_r(m)$, and calling it a Brauer algebra, where r is the tensor power and m is the dimension of the underlying space. In [8], Doty and Hu proved (analogue of) Schur-Weyl duality between the Brauer algebra $B_r(m)$ and the orthogonal group $O(V, q)$ over an arbitrary infinite field K of odd characteristic. This involves defining the analogues of the maps Φ and Ψ, and proving their surjectivity on certain commutants. We give more details on this in Section 2.

The commutant of the group $SO(V,q)$ in $\mathrm{End}_K(V^{\otimes r})$ differs from that of the group $O(V,q)$ only in the case when $m = 2n$ and $n \leq r$. The commutant in this case was first studied by Brauer in the aforementioned paper. However, he only gave a generating set for it and gave an action of the generators on $V^{\otimes r}$ but did not describe the multiplication algorithm in terms of the generators. We denote it by $D_r(2n)$ and call it an even Brauer algebra. In [12], Grood computed the multiplication of the generators given by Brauer so that the map defined by Brauer turns out to be an action of the algebra $D_r(2n)$ on $V^{\otimes r}$. She also studied Schur-Weyl duality between the special orthogonal group $SO(V,q)$ and the even Brauer algebra over $K = \mathbb{C}$. We give more details on this in Section 2, however, we remark here that $D_r(2n)$ is a non-associative algebra. Consider the map $\Psi : KSO(V,q) \to \mathrm{End}_K(V^{\otimes r})$ which is defined by the natural action of $SO(V,q)$ on $V^{\otimes r}$.

In this paper, we prove the analogue of Schur-Weyl duality for special orthogonal groups over an infinite field K of characteristic not two. Our main theorem is:

Theorem 1.2. *Let K be an infinite field of characteristic not two, let V be a vector space of dimension $2n$ over K together with the standard quadratic form $q = \langle 1, 1, \ldots, 1\rangle$. Let $n \leq r$ and Φ, Ψ be maps mentioned above. Then*

1. $\Psi(KSO(V,q)) = \mathrm{End}_{D_r(2n)}(V^{\otimes r})$.

2. $\Phi(D_r(2n)) = \mathrm{End}_{KSO(V,q)}(V^{\otimes r})$.

3. *If $n \leq r \leq 2n$ and if $V^{\otimes r}$ is a semisimple module for the group $SO(V,q)$, then the largest associative semisimple quotient of $D_r(2n)$ is isomorphic to* $\mathrm{End}_{KSO(V,q)}(V^{\otimes r})$.

We remark here that although Doty and others have studied the Brauer algebra over a general field K, the quadratic form they work with is not the same as our form q. So we need to define the Brauer algebra for q over a general K and also the map Φ. We do this in Section 2. This is done by observing that the multiplication rule in $B_r(m)$ and the definition of Φ given by Brauer make sense over $\mathbb{Z}$ from which we just base change to K. We do the same for the even Brauer algebra $D_r(2n)$ as well in the same section. We also prove that $\Phi(D_r(2n))$ is contained in $\mathrm{End}_{KSO(V,q)}(V^{\otimes r})$.

Section 3 is devoted to proving the main result. We first show that the dimensions of the four algebras involved are invariant under a base change. So it is enough to prove that Φ and Ψ are surjective over $\overline{K}$. Any two non-degenerate quadratic forms on $V \otimes \overline{K}$ are isomorphic and so the corresponding special orthogonal groups are conjugate in $GL(V \otimes \overline{K})$. Now we can use Schur-Weyl duality due to Doty and Hu for the orthogonal group over $\overline{K}$. We finally conclude the proof of the main result by comparing dimensions of the algebras involved. We give more details on this at the beginning of Section 3.

2. The Brauer algebra and the even Brauer algebra

Let us fix the notations. We denote by K an infinite field of characteristic not two and by V a vector space of dimension m over K. We work with the quadratic form $q = \langle 1, 1, \ldots, 1\rangle$ on V. This means that we can (and will) assume that V admits an orthonormal basis with respect to the bilinear form corresponding to q. We fix this basis, say $\{v_1, v_2, \ldots, v_m\}$.

For $K = \mathbb{C}$, Brauer defined an algebra $B_r(m)$ and a surjective map from $B_r(m)$ onto $\mathrm{End}_{\mathbb{C}O(V,q)}(V^{\otimes r})$, and Grood defined an algebra $D_r(m)$, when $m = 2n$ with $n \leq r$, which surjects onto $\mathrm{End}_{\mathbb{C}SO(V,q)}(V^{\otimes r})$. We show that these constructions carry over to integers and hence over any commutative ring, although we are going to need the constructions only over an infinite field K. Let us review these constructions first.

2.1. The Brauer algebra. Define linear transformations E_i on $V^{\otimes r}$, for $i = 1, \ldots, r-1$, by specifying

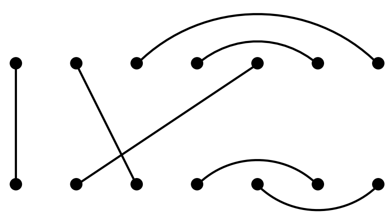
$$E_i(v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_r}) = \delta_{j_i, j_{i+1}} \sum_{l=1}^{m} v_{j_1} \otimes \cdots \otimes v_{j_{i-1}} \otimes (v_l \otimes v_l) \otimes v_{j_{i+2}} \otimes \cdots \otimes v_{j_r}.$$

Brauer proved:

Theorem 2.1 (Brauer [2]). *The algebra* $\mathrm{End}_{\mathbb{C}O(V,q)}(V^{\otimes r})$ *is the subalgebra of* $\mathrm{End}_{\mathbb{C}}(V^{\otimes r})$ *generated by* $\mathrm{End}_{\mathbb{C}GL(V)}(V^{\otimes r})$ *and the transformations* E_i *for* $i = 1, \ldots, r-1$.

Brauer then defined an algebra, described by a set of generators and relations, in terms of certain combinatorial objects known as r-diagrams.

An *r-diagram* is a graph with $2r$ vertices, written in two rows each containing r vertices, such that each vertex has degree one. The diagram given below is an example of a 7-diagram.

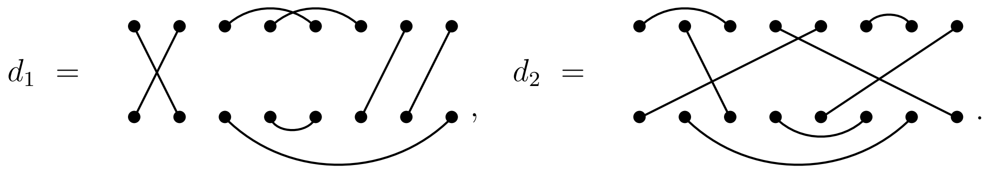


The Brauer algebra $B_r(m)$ is defined to be the $\mathbb{C}$-linear span of the set of r-diagrams with a product that we describe below. The dimension of this algebra can be easily seen to be

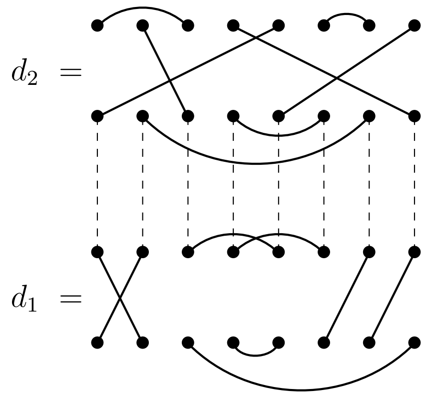
$$\dim(B_r(m)) = (2r-1)!! := (2r-1)(2r-3)\cdots 3 \cdot 1.$$

The product of two r-diagrams d_1 and d_2 is obtained as follows: we first place d_2 above d_1 and then connect each vertex in the bottom row of d_2 by a dotted line with the corresponding one from the top row of d_1. The product is defined to be the resulting r-diagram, whose edges are determined by the paths of this attachment, multiplied by the scalar m^c, where c is the number of cycles in the attachment.

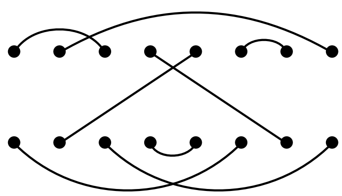
We illustrate this by means of an example. Assume that we are given following two r-diagrams (here $r = 8$)



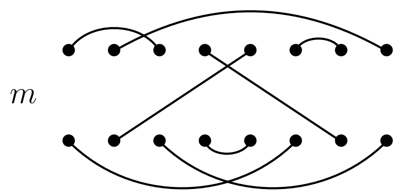
In the product $d_1 d_2$, the bars in the top row of d_2 and those in the bottom row of d_1 remain undisturbed. The conjoined graph looks as follows:



There are four rows in the conjoined graph. Observe that there is a path connecting the fourth vertex in the first row to the eighth vertex in second row, to the eighth vertex in the third row, to the seventh vertex in the fourth row. Thus, the resulting product will have an edge connecting the fourth vertex of the top row to the seventh vertex of the bottom row and similarly, an edge connecting fifth vertex of the top row to the second vertex of the bottom row. Traversing the paths in the attachment, we obtain an additional bar in the top row connecting second vertex to its eighth vertex. Similarly, an additional bar appears in the bottom row connecting first vertex to its sixth vertex. After completing all the paths, we get the following 8-diagram:

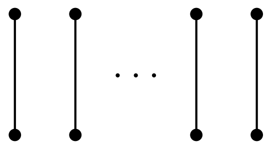


Observe that there is a cycle in the attachment joining fourth vertex in the second row to the sixth vertex of the second row, to the sixth vertex of the third row, to the fourth vertex of the third row, finally to the fourth vertex of the second row, completing the cycle. Thus, the final product is obtained by multiplying the above graph by the scalar m^c, where c is the number of cycles in the conjoined graph. Thus, the product $d_1 d_2$ is as follows:

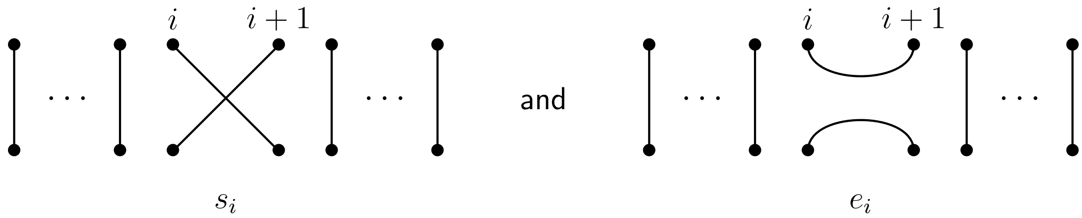


We remark here that the product defined by Brauer has the diagram d_1 above the diagram d_2 and so our product actually describes the opposite algebra to the one defined by Brauer. This enables us to get a left action of $B_r(m)$ on $V^{\otimes r}$ whereas in Brauer's notation we would have had to write $B_r(m)^{op}$ all the time.

The r-diagram shown below is the identity element of the algebra $B_r(m)$.



Brauer observed that $B_r(m)$ is generated by the r-diagrams $\{s_i, e_i : i = 1, 2, \ldots, r-1\}$, where s_i and e_i are the diagrams given below:



After defining this algebra, Brauer defined a map Φ on the generators $\{e_i, s_i\}$ of $B_r(m)$ that sends e_i to E_i and s_i to the transformation in $\mathrm{End}_{\mathbb{C}}(V^{\otimes r})$ corresponding to the transposition $(i, i+1)$ in S_r, i.e., the map which takes a simple tensor of the form $(v_{j_1} \otimes \cdots \otimes v_{j_r})$ to the simple tensor $(v_{j_1} \otimes \cdots \otimes v_{j_{i-1}} \otimes v_{j_{i+1}} \otimes v_{j_i} \otimes v_{j_{i+2}} \otimes \cdots \otimes v_{j_r})$. With our definition of the product in $B_r(m)$, this is an algebra homomorphism from $B_r(m)$ and thus we get a left action of $B_r(m)$ on $V^{\otimes r}$.

The natural action of $O(V,q)$ on $V^{\otimes r}$ induces a map $\Psi : O(V,q) \to \mathrm{End}_{\mathbb{C}}(V^{\otimes r})$ which is nothing but the restriction of the map Ψ defined on $GL(V)$.

Theorem 2.2 (Brauer [2], Brown [3, 4])**.** *Let V be an m-dimensional vector space over $\mathbb{C}$. We consider the quadratic form q on V whose bilinear form is given by the matrix $[\delta_{ij}]$ with respect to some basis. Let Φ and Ψ be the maps described above. The natural action of $O(V,q)$ on $V^{\otimes r}$ commutes with the left action of $B_r(m)$ under the map Φ. Further*

1. $\Psi(\mathbb{C}O(V,q)) = \mathrm{End}_{B_r(m)}(V^{\otimes r})$,
2. $\Phi(B_r(m)) = \mathrm{End}_{\mathbb{C}O(V,q)}(V^{\otimes r})$ *and,*
3. *if* $m \geq r$*, then* Φ *is injective and hence an isomorphism onto* $\mathrm{End}_{\mathbb{C}O(V,q)}(V^{\otimes r})$.

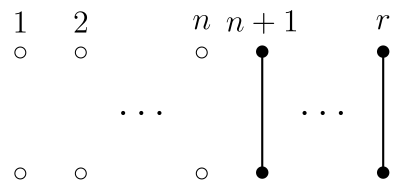
2.2. The even Brauer algebra. We work with K, V, q and the orthonormal basis $\{v_i\}$ of V as defined in the beginning of this section, and now we consider the natural action of the group $SO(V,q)$ on $V^{\otimes r}$. We have that $\mathrm{End}_{O(V,q)}(V^{\otimes r}) \subseteq \mathrm{End}_{SO(V,q)}(V^{\otimes r})$, and, since $SO(V,q)$ is an index two subgroup of $O(V,q)$ one would think that these two commutant algebras must be closely related. Indeed, we have:

Theorem 2.3 (Brauer [2]). *Let $K = \mathbb{C}$ and V, q be as above.*

1. *If m, the dimension of V, is odd then* $\mathrm{End}_{KO(V,q)}(V^{\otimes r}) = \mathrm{End}_{KSO(V,q)}(V^{\otimes r})$.

2. *If $m = 2n$, then* $\mathrm{End}_{KO(V,q)}(V^{\otimes r}) = \mathrm{End}_{KSO(V,q)}(V^{\otimes r})$ *if and only if $n > r$.*

One sees, by the work of De Concini-Procesi ([6, Theorem 5.6]), that the above result holds over a general infinite field K also. We, therefore, restrict our focus to the case when the dimension of V is even, say $m = 2n$, with $n \leq r$. Brauer in [2] described a possible candidate for an analogue of the algebra $B_r(2n)$ that would surject onto the commutant of $SO(V,q)$. He described a linear endomorphism T_0 of $V^{\otimes r}$ which would generate $\mathrm{End}_{\mathbb{C}SO(V,q)}(V^{\otimes r})$ together with $\mathrm{End}_{\mathbb{C}O(V,q)}(V^{\otimes r})$. He also suggested a diagram t_0 depicted below, which would correspond to this element T_0 and claimed that this diagram together with $B_r(2n)$ would generate an algebra which would be the analogue of the Brauer algebra for the group $SO(V,q)$. Brauer stated that the diagram t_0 corresponds to the invariant determinant of the group $SO(V,q)$ such that the determinant of the $2n$ unconnected vectors appears as a factor in the corresponding invariant [2, Page 871]. He also suggested how the other diagrams would look like, however, he did not describe the full multiplication algorithm in this new algebra.

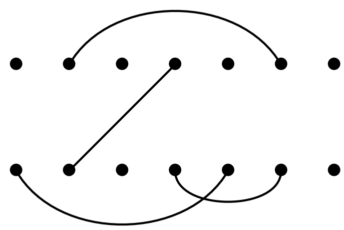


t_0

C. Grood studied this algebra in her thesis and defined a multiplication as well as a left action of this algebra on $V^{\otimes r}$. We need to introduce the terminology of an $r\backslash n$-diagram to describe this algebra.

Recall that we have $n \leq r$. An *$r\backslash n$-diagram* is a graph with $2r$ vertices, written in two rows each containing r vertices, such that exactly $2n$ vertices have degree 0 and the remaining $2r - 2n$ vertices have degree 1.

The diagram below is an example of a $7\backslash 3$-diagram.



Let $D_r(2n)$ be the vector space over $\mathbb{C}$ with a basis consisting of the set of r-diagrams and $r\backslash n$-diagrams. The dimension of $D_r(2n)$ can be easily seen to be ([12, Page 685])

$$\dim(D_r(2n)) = (2r-1)!! + \binom{2r}{2n}(2r-2n-1)!!$$

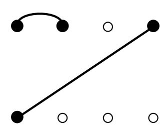
with the convention that $(-1)!! = 1$. To each basis diagram d of $D_r(2n)$ Brauer associated a transformation $\Phi(d)$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes r})$. We consider the basis $\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} | 1 \leq i_1, \ldots, i_r \leq 2n\}$ of $V^{\otimes r}$ where $\{v_i\}$ is the basis of V fixed at the beginning of the section. Then

$$\Phi(d)(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}) := \sum_{1 \leq j_1, \ldots, j_r \leq 2n} C_{I,J}\ v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_r},$$

where $C_{I,J}$ is a constant depending on $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_r)$; and it can be determined by the following algorithm:

Label the vertices in the top row of d by $i_1, \ldots, i_r$. Label the vertices in the bottom row of d by $j_1, \ldots, j_r$. If there is an edge connecting vertex l and vertex k, label it by $\delta_{l,k}$, the Kronecker delta. Let s be the number of unconnected vertices in the top row of d and let t be the number of unconnected vertices in the bottom row of d. Then the scalar $C_{I,J}$ is the product of all the edge labels and the term $\epsilon(i_{l_1}, \ldots, i_{l_s}, j_{k_1}, \ldots, j_{k_t})$ where $i_{l_1}, \ldots, i_{l_s}$ are the labels of the unconnected vertices in the top row read from left to right, $j_{k_1} \ldots j_{k_t}$ are the labels of the unconnected vertices in the bottom row read from left to right $(s + t = 2n)$ and $\epsilon(i_{l_1}, \ldots, i_{l_s}, j_{k_1}, \ldots, j_{k_t})$ is the sign of the set $\{i_{l_1}, \ldots, i_{l_s}, j_{k_1}, \ldots, j_{k_t}\}$ which is 0 if there are any repetitions in the set and is the sign of the corresponding permutation in S_{2n} otherwise.

Let us look at an example. Consider the diagram:

d =


We see that $d \in D_4(4)$. Then the image of the basis vector $v_1 \otimes v_1 \otimes v_2 \otimes v_3$, for example, is given by:

$$\Phi(d)(v_1 \otimes v_1 \otimes v_2 \otimes v_3) = \sum_{1 \leq j_1, j_2, j_3 \leq 4} \epsilon(2, j_1, j_2, j_3)\ v_3 \otimes v_{j_1} \otimes v_{j_2} \otimes v_{j_3}.$$

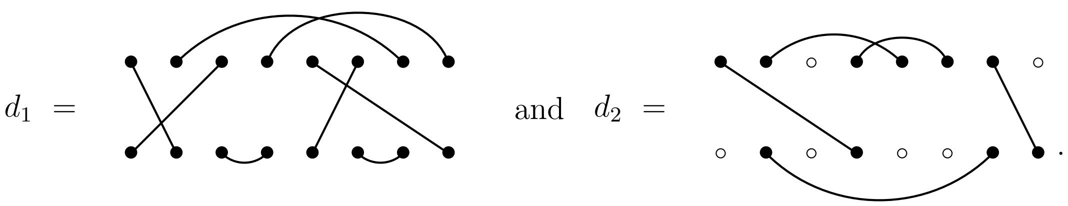
We now describe the multiplication algorithm on $D_r(2n)$ defined by Grood which would make Φ an algebra homomorphism from $D_r(2n)$ to $\mathrm{End}_{\mathbb{C}}(V^{\otimes r})$. Following Grood, we call this algebra *an even Brauer algebra*.

We treat $B_r(2n)$ as a subalgebra of $D_r(2n)$, therefore it is enough to describe the product of two basis diagrams of $D_r(2n)$ at least one of which is an $r\backslash n$ -diagram.

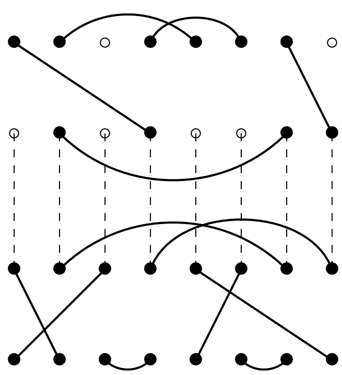
Let d_1 and d_2 be two basis diagrams of $D_r(2n)$. We place d_2 above d_1 to obtain the product $d_1 d_2$, as done in the case of Brauer algebra, and connect each vertex in the bottom row of d_2 with the corresponding one from the top row of d_1 by a dotted line. In the conjoined graph, a vertex has degree $\tilde{0}$ if it had degree 0 before the two diagrams were connected by dotted lines. We make two cases depending on whether both the diagrams are $r\backslash n$-diagrams or not.

Case 1. Here we assume that one of the two diagrams is an r-diagram and the other is an $r\backslash n$-diagram. Therefore all the degree $\tilde{0}$ vertices belong to only one diagram. A path in this graph is called a *quasi-cycle* if it begins and ends with distinct vertices of degree $\tilde{0}$. If a quasi-cycle exists, we define the product $d_1 d_2$ to be 0. Let us now assume that there is no quasi-cycle in the above graph, obtained from d_1 and d_2. Label the vertices of degree $\tilde{0}$ with the numbers $1, 2, \ldots, 2n$, taking first those in the top row and then those in the bottom row, from left to right. As in the $B_r(2n)$ product, let d_3 be the diagram determined by the paths in this attachment. If a path p ends in a vertex of degree $\tilde{0}$ which is labeled i, then the vertex in d_3 that corresponds to the initial vertex of p is defined to have degree 0 and is labeled i. Note that in the diagram d_3 there would be exactly $2n$ vertices of degree 0. Consider the permutation $\pi \in S_{2n}$ in two-row form, the second row of which is obtained by reading off the labels of degree 0 vertices of d_3 from left to right, top to bottom. Then, the product $d_1 d_2$ is defined to be $\operatorname{sgn}(\pi)(2n)^c d_3$ where c is the number of cycles in the conjoined graph.

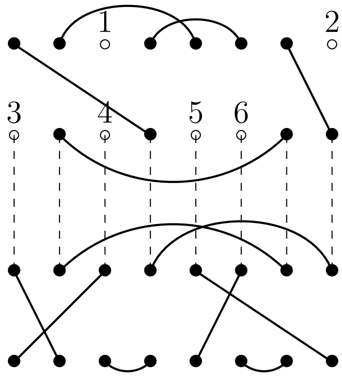
Let us look at an illustration with $r = 8$. Consider the following diagrams in $D_8(6)$:



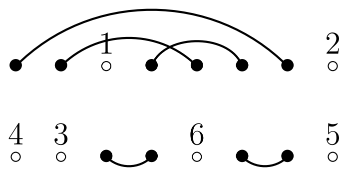
Let us concatenate the two diagrams by placing d_2 above d_1.



We observe that there is no quasi-cycle in the attachment. We now label the degree $\tilde{0}$ vertices in the increasing order from left to right, top to bottom:



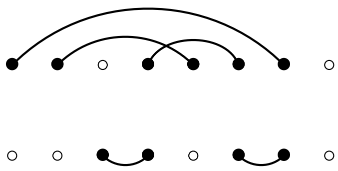
By traversing the paths in this attachment, we get the following diagram:

$d_3 =$ 

Note that there is a cycle in the attachment starting from the second vertex of the second row of d_2. Also, the permutation π is given by:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 3 & 6 & 5 \end{pmatrix}.$$

Thus, the scalar factor $\operatorname{sgn}\pi \cdot 6^c = 1 \cdot 6 = 6$ is multiplied to the diagram d_3 to obtain the product $d_1 d_2$:

6 

Case 2. Now we assume that both d_1 and d_2 are $r\backslash n$-diagrams. Here, we represent the vertices of degree 0 in d_2 with squares instead of circles. Since both the diagrams have degree $\tilde{0}$ vertices, a quasi-cycle connects either two degree $\tilde{0}$ vertices from the same diagram or two degree $\tilde{0}$ vertices from different diagrams. If a quasi-cycle connects two degree $\tilde{0}$ vertices from the same diagram, we define the product $d_1 d_2$ to be 0. So, let us now assume that there is no quasi-cycle in the conjoined graph that connects two degree zero vertices from the same diagram.

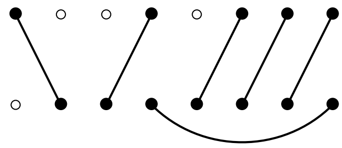
Using the conjoined graph, we let c be the number of its cycles and a be the number of its quasi-cycles that connect vertices of degree $\tilde{0}$ from different diagrams. Label the vertices of degree $\tilde{0}$ from d_1 with the numbers $1, 2, \ldots, 2n$ from left to right, top to bottom, as before. We will now label the vertices of degree $\tilde{0}$ in d_2 with the numbers $1, 2, \ldots, 2n$ in the following way. First, consider the vertices of degree $\tilde{0}$ that are contained in a quasi-cycle; label each one with the same number as its associated vertex in d_1. Now let A be the set which consists of the $2n - a$ numbers which have not yet been used to label vertices in d_2, and use them in increasing order to label these remaining vertices of degree $\tilde{0}$. Let τ be the permutation in S_{2n} written in two-row form, the second row of which is given by reading off the labels of the vertices of degree $\tilde{0}$ in d_2 from left to right, top to bottom.

As before, let d_3 be the diagram determined by the paths in the conjoined graph. If a path p ends in a square (or circular) vertex of degree $\tilde{0}$ which is labeled i, then the vertex in d_3 that corresponds to the initial vertex of p will be defined to be a square (respectively, circular) vertex of degree 0 and will be labeled i. For $\sigma \in S_A$, let $\sigma * d_3$ represent the r-diagram which is formed by drawing an edge between the round vertex of d_3 labeled i and the square vertex labeled $\sigma(i)$. The product $d_1 d_2$ is then the linear combination of r-diagrams given by

$$\operatorname{sgn}(\tau) a! (2n)^c \Big(\sum_{\sigma \in S_A} \operatorname{sgn}(\sigma) \sigma * d_3 \Big).$$

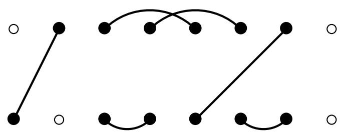
Let us consider the following two diagrams in $D_8(4)$:

$d_1 =$

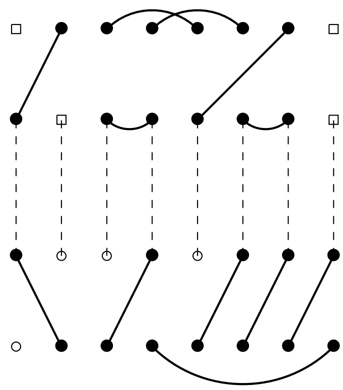


and

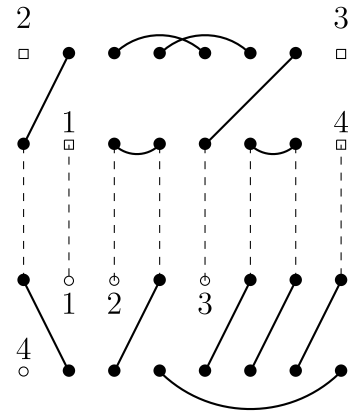
$d_2 =$



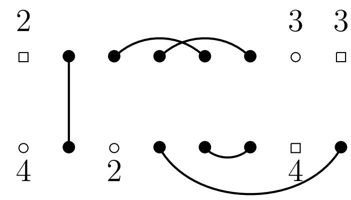
Let us obtain the conjoined graph:



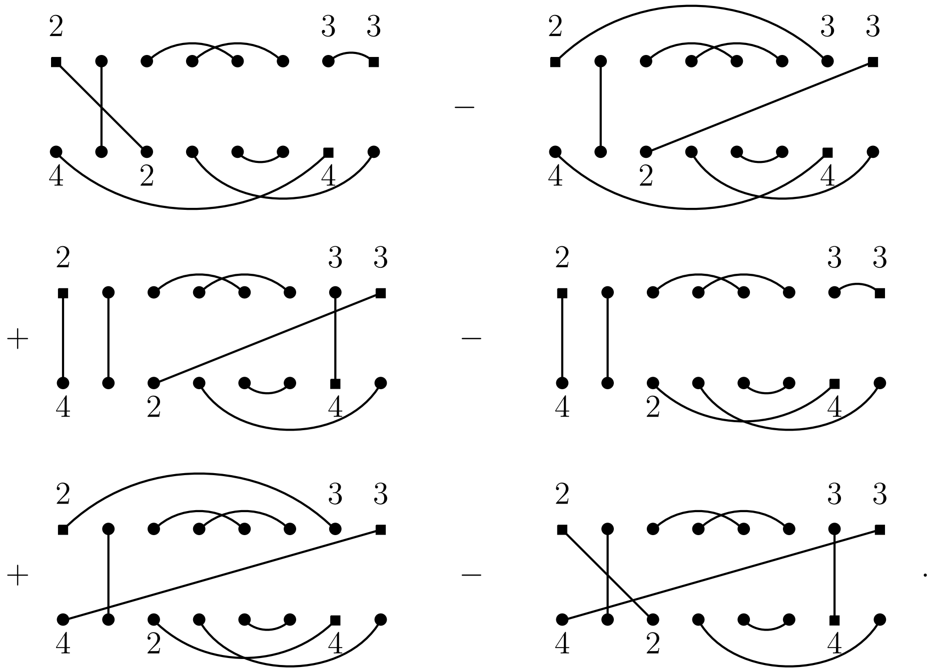
The attachment has no quasi-cycle connecting two degree $\tilde{0}$ vertices from the same diagram. Hence, the product $d_1 d_2$ is non-zero. However, there is one quasi-cycle connecting a degree $\tilde{0}$ vertex from d_2 to a degree $\tilde{0}$ vertex from d_1, hence $a = 1$. There is no cycle in the attachment, hence, $c = 0$. We now label the degree zero vertices and obtain the following graph:



Then $A = \{2, 3, 4\}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$. Traversing the paths in the attachment, we obtain the following graph:



The scalar factor $\mathrm{sgn}(\tau)a!(4)^c = 1 \cdot 1! \cdot 4^0 = 1$. Hence, the product $d_1 d_2$ is as follows:



The multiplicative identity of $D_r(2n)$ is the same as that of $B_r(2n)$.

Theorem 2.4 (Grood [11]). *The map $\Phi : D_r(2n) \to \mathrm{End}_{\mathbb{C}}(V^{\otimes r})$, defined above by Brauer, has its image inside $\mathrm{End}_{\mathbb{C}SO(V,q)}(V^{\otimes r})$ and is, in fact, an epimorphism.*

2.3. $\mathbb{Z}$-forms of Brauer and even Brauer algebras. We see in this subsection that there exist forms of the Brauer and the even Brauer algebras over $\mathbb{Z}$ for the quadratic form q and the maps Φ and Ψ also make sense over $\mathbb{Z}$.

Over $\mathbb{Z}$, we work with a free $\mathbb{Z}$-module M of rank $2n$ and consider the bilinear form on M given by the matrix $[\delta_{ij}]$ with respect to some basis of M over $\mathbb{Z}$, say $\{w_1, \dots, w_{2n}\}$. We denote the corresponding quadratic form by q_M. Then it is clear that the quadratic module (M, q_M) over $\mathbb{Z}$ is a $\mathbb{Z}$-form of the quadratic module (V, q) over $\mathbb{C}$. In fact, after identifying the basis $\{v_i\}$ of V with the basis $\{w_i \otimes 1\}$ of $M \otimes \mathbb{C}$ the module M can be considered as a $\mathbb{Z}$-submodule of V and then q_M is just the restriction of q to M.

Continuing with this identification of $M \otimes \mathbb{C}$ and V we get that the group $O(M, q_M)$ is a subgroup of $O(V, q)$ and $SO(M, q_M)$ is a subgroup of $SO(V, q)$. Recall that the group $O(M, q_M)$ is the group of $\mathbb{Z}$-linear automorphisms of M which preserve the form q_M and $SO(M, q_M)$ is the subgroup of $O(M, q_M)$ consisting of elements of determinant one. We now denote by $D_r(2n)_\mathbb{Z}$ the $\mathbb{Z}$-submodule of $D_r(2n)$ generated by the r-diagrams and the $r\backslash n$-diagrams and by $\Phi_\mathbb{Z}$ the restriction of Φ to $D_r(2n)_\mathbb{Z}$. Further, it is clear by the definition of Φ that the $\mathbb{Z}$-submodule of $V^{\otimes r}$ generated by $v_{j_1} \otimes \cdots \otimes v_{j_r}$ is left invariant under the action of $D_r(2n)_\mathbb{Z}$. By our identification of basis vectors, we have that the action of $D_r(2n)_\mathbb{Z}$ on $(M \otimes \mathbb{C})^{\otimes r}$ leaves $M^{\otimes r}$ invariant.

Thus we have an algebra homomorphism $\Phi_\mathbb{Z} : D_r(2n)_\mathbb{Z} \to \mathrm{End}_\mathbb{Z}(M^{\otimes r})$. Since every element of the image $\Phi_\mathbb{Z}(D_r(2n)_\mathbb{Z})$ commutes with every element of the image $\Psi(SO(M, q_M)) \subseteq \mathrm{End}_\mathbb{Z}(M^{\otimes r})$ we get that $\Phi_\mathbb{Z}(D_r(2n)_\mathbb{Z})$ is contained in $\mathrm{End}_{\mathbb{Z}SO(M,q_M)}(M^{\otimes r})$ and $\Psi(SO(M, q_M)) \subseteq \mathrm{End}_{D_r(2n)_\mathbb{Z}}(M^{\otimes r})$.

Thus we have obtained $\mathbb{Z}$-forms of the even Brauer algebra $D_r(2n)$ as well as of the map Φ.

2.4. Brauer and even Brauer algebras over an infinite field. Let K be an infinite field of characteristic not two and consider a vector space V over K of dimension $2n$. We consider the quadratic form q given by the bilinear form $[\delta_{ij}]$ with respect to some basis of V. We then identify the quadratic space (V, q) over K with $(M, q_M) \otimes_\mathbb{Z} K$ by identifying the above basis of V with $\{w_i \otimes 1\}$. It also follows that we have an algebra homomorphism $\Phi_K : D_r(2n)_\mathbb{Z} \otimes K \to \mathrm{End}_K(V^{\otimes r})$.

For a positive integer m, we denote by $B_r(m)_K$ the vector space generated by the r-diagrams. This is identified with the algebra $B_r(m)_\mathbb{Z} \otimes K$ and hence is an algebra in a natural way. We denote by $D_r(2n)_K$ the K-vector space generated by the r-diagrams and the $r\backslash n$-diagrams. This is also identified with the algebra $D_r(2n)_\mathbb{Z} \otimes K$ and hence is an algebra in a natural way. As an algebra, $D_r(2n)_K$ is generated by the subalgebra $B_r(2n)_K$ and the element t_0. Further, the definition of Φ_K is the same as the one given by Brauer, except that we take into consideration the characteristic of the field K while multiplying by the constants $C_{I,J}$ and $a!(2n)^c$. Note also that there is always the map $\Psi_K : KSO(V, q) \to \mathrm{End}_K(V^{\otimes r})$ given by the natural action of $SO(V, q)$ on $V^{\otimes r}$.

Remark 2.5. *As $B_r(2n)_K$ is a subalgebra of $D_r(2n)_K$ generated by the r-diagrams, the above construction of this subsection can be carried out for $B_r(2n)_K$ and the orthogonal group $O(V, q)$ to get a map $\Phi_K : B_r(2n)_K \to \mathrm{End}_{KO(V,q)}(V^{\otimes r})$*

over K. Note that the map Φ_K defined on $B_r(2n)_K$ is the restriction of Φ_K defined on $D_r(2n)_K$ to $B_r(2n)_K$.

By [8, Theorem 1.2], we see that $\Phi_K(B_r(2n)_K) \subseteq \mathrm{End}_{KO(V,q)}(V^{\otimes r}) \subseteq \mathrm{End}_{KSO(V,q)}(V^{\otimes r})$. We further recall that the transformation T_0 corresponding to the diagram t_0 is in $\mathrm{End}_{KSO(V,q)}(V^{\otimes r})$, it then follows that $\Phi_K(D_r(2n)_K) \subseteq \mathrm{End}_{KSO(V,q)}(V^{\otimes r})$.

Doty and Hu ([8, Theorem 1.2]) extended Theorem 2.2 to general infinite fields of odd characteristic. However, they considered the quadratic form q' on V, a vector space of dimension m, whose bilinear form is given by the matrix $[\delta_{ij'}]$, with respect to some basis, where $j' = m + 1 - j$. So, naturally, the map Φ on $B_r(m)$ is defined in a different way than the definition given by Brauer, while the map Ψ on $KO(V, q')$ is the map given by the natural action of $KO(V, q')$ on V.

Theorem 2.6 (Doty, Hu). *Let K be an infinite field of odd characteristic and let V be an m-dimensional vector space over K. Let q', Φ and Ψ be as in the previous paragraph. Then*

1. $\Psi(KO(V, q')) = \mathrm{End}_{B_r(m)}(V^{\otimes r})$,
2. $\Phi(B_r(m)) = \mathrm{End}_{KO(V,q')}(V^{\otimes r})$, *and,*
3. *if $m \geq r$, then Φ is injective map and hence is an isomorphism onto $\mathrm{End}_{KO(V,q')}(V^{\otimes r})$.*

Over an algebraically closed field $\overline{K}$, any two non-degenerate quadratic forms of dimension m are isometric. It then follows that the corresponding orthogonal and the special orthogonal groups are conjugate inside $GL_m(\overline{K})$. And then the commutants of these groups will also be conjugate in the corresponding endomorphism ring of the underlying vector space as well as the endomorphism ring of any tensor power of it.

We have defined the quadratic form q and Doty-Hu defined the quadratic form q' on the vector space V. These two forms are isometric over $\overline{K}$ and hence as remarked above the subalgebras $\mathrm{End}_{\overline{K}O(V_{\overline{K}},q_{\overline{K}})}(V_{\overline{K}}^{\otimes r})$ and $\mathrm{End}_{\overline{K}O(V_{\overline{K}},q'_{\overline{K}})}(V_{\overline{K}}^{\otimes r})$ are conjugate in $\mathrm{End}_{\overline{K}}(V_{\overline{K}}^{\otimes r})$. Here the suffix $\overline{K}$ means that the algebraic structure defined over K, either of a vector space or of a quadratic space, is base-changed to $\overline{K}$.

The actions of the Brauer algebras are also different in our definition and in that of Doty and Hu. Endomorphism E_i of $V_{\overline{K}}^{\otimes r}$ is defined in Section 2.1. The endomorphism $E_{i'}$ defined by Doty and Hu is given as follows:
For the quadratic space (V, q'), there is an ordered basis $\{w_1, \ldots, w_m\}$ such that $q'(w_i, w_j) = \delta_{i,j'}$ for all $1 \leq i, j \leq m$.

$$E'_i(w_{j_1} \otimes w_{j_2} \otimes \cdots \otimes w_{j_r}) = \delta_{j_i, j'_{i+1}} \sum_{l=1}^{m} w_{j_1} \otimes \cdots \otimes w_{j_{i-1}} \otimes (w_l \otimes w_{l'}) \otimes w_{j_{i+2}} \otimes \cdots \otimes w_{j_r}.$$

This general definition of E'_i can be found in [9, Page 430].

Lemma 2.7. *The two transformations E_i and E_i' of $V_{\overline{K}}^{\otimes r}$ are conjugate.*

Proof. To show that the two actions are conjugate, we produce an element $P \in GL(V_{\overline{K}})$ such that $P^{-1^{\otimes r}} E_i' P^{\otimes r} = E_i$. The bilinear form corresponding to q is given by the identity matrix I and the bilinear form corresponding to q' is given by the matrix J where $J = [\delta_{i,j'}]$ with $j' = m + 1 - j$. Since the forms q, q' are isometric over $\overline{K}$ there exists an automorphism P of $V_{\overline{K}}$ such that $P^t J P = I$. It can then be checked that $P^{-1^{\otimes r}} E_i' P^{\otimes r} = E_i$. ■

Proposition 2.8. *Over an algebraically closed field $\overline{K}$ of characteristic not two we have:*

1. $\Psi(\overline{K}O(V_{\overline{K}}, q_{\overline{K}})) = \mathrm{End}_{B_r(m)_{\overline{K}}}(V_{\overline{K}}^{\otimes r})$,
2. $\Phi(B_r(m)_{\overline{K}}) = \mathrm{End}_{\overline{K}O(V_{\overline{K}}, q_{\overline{K}})}(V_{\overline{K}}^{\otimes r})$

   where q is any non-degenerate quadratic form defined on V.

Proof. All the spaces in the statement are finite dimensional, so to prove equality of two such spaces it is enough to prove that one space is contained in the other and that their dimensions are the same.

Since the elements in the image of the Brauer algebra commute with the elements in the image of the orthogonal group, it follows that the spaces on the left hand side of the equality are contained in those in the right hand side. Further, we have equality of the spaces for one (non-degenerate) quadratic form q' as proved by Doty and Hu. Then we use above lemma to conclude that the spaces on both sides of the equalities have the same dimension and hence they are equal. ■

3. Proof of the main theorem

This section is devoted to proving the main theorem, Theorem 1.2. The notations are as in the last section. The base field is denoted by K, this is an infinite field of characteristic not equal to two, and we work with the quadratic form q defined over the vector space V of dimension $2n$ over K. The group $SO(V, q)$ denotes the (K-rational points of the) special orthogonal group preserving the form q and $D_r(2n)$ denotes the even Brauer algebra. This is a K-algebra generated by the r-diagrams and the $r \backslash n$ diagrams. The maps $\Phi : D_r(2n) \to \mathrm{End}_{KSO(V,q)}(V^{\otimes r})$ and $\Psi : KSO(V, q) \to \mathrm{End}_{D_r(2n)}(V^{\otimes r})$ are defined in the previous section. In the notations of the previous section $D_r(2n)$ is the algebra $D_r(2n)_K$, and the maps Φ, Ψ are the respective maps Φ_K, Ψ_K.

We summarise our strategy of the proofs. We first show that the dimensions of the four subalgebras of $\mathrm{End}_K(V^{\otimes r})$, namely, $\Psi(KSO(V, q))$, $\mathrm{End}_{D_r(2n)}(V^{\otimes r})$, $\Phi(D_r(2n))$ and $\mathrm{End}_{KSO(V,q)}(V^{\otimes r})$ are invariant under a base change. Further, since the maps Φ and Ψ are well-behaved with base change, we work out the proof of the main theorem first for an algebraically closed field $\overline{K}$, and conclude the result for an arbitrary infinite field by dimension comparison. More precisely, assume that $\Phi(D_r(2n)_{\overline{K}}) = \mathrm{End}_{\overline{K}SO(V,q)}(V^{\otimes r})$, now $\Phi(D_r(2n)_K) \subseteq$

$\mathrm{End}_{KSO(V,q)}(V^{\otimes r})$ as vector spaces. But, both the spaces have equal dimensions since $\dim_K(\Phi(D_r(2n)_K))=\dim_{\overline{K}}(\Phi(D_r(2n)_{\overline{K}})) = \dim_{\overline{K}}(\mathrm{End}_{\overline{K}SO(V,q)}(V^{\otimes r})) = \dim_K(\mathrm{End}_{KSO(V,q)}(V^{\otimes r}))$. Hence, $\Phi(D_r(2n)_K) = \mathrm{End}_{KSO(V,q)}(V^{\otimes r})$. A similar argument works for the map Ψ. This strategy of dimension comparison is due to S. Koenig as mentioned in [7].

Lemma 3.1. *Let L be an extension of the field K. Then we have:*

1. $\dim_K \Phi(D_r(2n)) = \dim_L \Phi_L(D_r(2n)_L)$.
2. $\dim_K \mathrm{End}_{\Phi(D_r(2n))}(V^{\otimes r}) = \dim_L \mathrm{End}_{\Phi_L(D_r(2n)_L)}(V_L^{\otimes r})$

Proof. It is immediate that (1) holds. After all, $\Phi(D_r(2n))$ is a finite dimensional K-vector space and its dimension does not change after a base change as $\Phi(D_r(2n)) \otimes_K L$ is nothing but $\Phi_L(D_r(2n)_L)$.

For proving (2), we choose a basis of $\Phi(D_r(2n))$, say $\{y_1, y_2, \ldots, y_s\}$. Then the set of simple tensors $\{y_1 \otimes 1, \ldots, y_s \otimes 1\}$ is a basis of $\Phi_L(D_r(2n)_L)$. We denote by $Z_{\mathrm{End}_K(V^{\otimes r})}(y)$ the centralizer of the endomorphism y in the endomorphism ring. Then

$$\mathrm{End}_{D_r(2n)}(V^{\otimes r}) = \cap_i Z_{\mathrm{End}_K(V^{\otimes r})}(y_i) \text{ and } \mathrm{End}_{D_r(2n)_L}(V_L^{\otimes r}) = \cap_i Z_{\mathrm{End}_L(V_L^{\otimes r})}(y_i \otimes 1).$$

The centralizer of each y_i in $\mathrm{End}_K(V^{\otimes r})$ is a certain vector subspace of $\mathrm{End}_K(V^{\otimes r})$ given by solving some linear equations. The dimensions of these subspaces do not change under a base change and hence the same holds for $\mathrm{End}_{D_r(2n)}(V^{\otimes r})$ as well.

This proves (2). ∎

Lemma 3.2. *Let L be an extension of the field K. The dimension of the K-vector space $\Psi(KSO(V,q))$ is the same as the dimension of the L-vector space $\Psi_L(LSO(V_L, q_L))$.*

Proof. We observe that we are working over an infinite field K and that $KSO(V,q)$ is the group of K-rational points of a connected linear algebraic group defined over K, say H. The group H is nothing but a K-form of the algebraic group SO_{2n}. By Grothendieck's theorem [1, Corollary 7.12], the group $KSO(V,q)$ is Zariski dense in the algebraic group H and hence in $LSO(V_L, q_L)$ which are the L-rational points of H. Then it follows that $\Psi(KSO(V,q))$ is dense in $\Psi_L(LSO(V_L, q_L))$. Since these are finite dimensional vector spaces, it follows that the dimension of $\Psi_L(LSO(V_L, q_L))$ is the same as the dimension of $\Psi(KSO(V,q)) \otimes_K L$ which is the same as that of $\Psi(KSO(V,q))$. ∎

Lemma 3.3. *Let L be an extension of the field K. Then the dimension of the K-vector space $\mathrm{End}_{KSO(V,q)}(V^{\otimes r})$ is the same as the dimension of the L-vector space $\mathrm{End}_{LSO(V_L,q_L)}(V_L^{\otimes r})$.*

Proof. We use the following sequence of $KSO(V,q)$-equivariant isomorphisms:

$$V^{\otimes 2r} \xrightarrow{\sim} V^{\otimes r} \otimes V^{\otimes r} \xrightarrow{\sim} V^{\otimes r} \otimes V^{*\otimes r} \xrightarrow{\sim} \mathrm{End}_K(V^{\otimes r})$$

where the middle isomorphism is obtained by the non-degeneracy of q. Note that the $KSO(V,q)$-action on $\mathrm{End}_K(V^{\otimes r})$ is conjugation action of $\Psi(KSO(V,q))$. The space $\mathrm{End}_{KSO(V,q)}(V^{\otimes r})$ is simply the fixed points under the action of $KSO(V,q)$ in $\mathrm{End}_K(V^{\otimes r})$ and hence is isomorphic to the space of fixed points in $V^{\otimes 2r}$ under the natural $KSO(V,q)$-action. It follows from De Concini-Procesi [6, Theorem 5.6] that the dimension of the fixed points in $V^{\otimes 2r}$ under natural $KSO(V,q)$-action is invariant under a base change, and hence we get that the dimension of $\mathrm{End}_{KSO(V,q)}(V^{\otimes r})$ is also invariant under a base change. ■

We now start proving the main theorem. By above lemmas, it is enough to prove the first two parts of Theorem 1.2 where the base field is algebraically closed.

Theorem 3.4. *We have $\Phi(D_r(2n)_{\overline{K}}) = \mathrm{End}_{\overline{K}SO(V_{\overline{K}},q_{\overline{K}})}(V_{\overline{K}}^{\otimes r})$, where the notations are as above.*

Proof. We know that the map $\Phi_{\overline{K}}$ defined on the Brauer algebra $B_r(2n)_{\overline{K}}$ is just the restriction of the map $\Phi_{\overline{K}}$ defined on $D_r(2n)_{\overline{K}}$. We also know that $\Phi_{\overline{K}}(B_r(2n)_{\overline{K}}) = \mathrm{End}_{\overline{K}O(V_{\overline{K}},q_{\overline{K}})}(V_{\overline{K}}^{\otimes r})$ [8, Theorem 1.2(b)]. Moreover, the algebra $\mathrm{End}_{\overline{K}SO(V_{\overline{K}},q_{\overline{K}})}(V_{\overline{K}}^{\otimes r})$ is generated by $\mathrm{End}_{\overline{K}O(V_{\overline{K}},q_{\overline{K}})}(V_{\overline{K}}^{\otimes r})$ and an element $T_0 \in \mathrm{End}_{\overline{K}}(V^{\otimes r})$ which corresponds to the determinant [2, Page 870] and [6, Theorem 5.6]. We also know that $D_r(2n)_{\overline{K}}$ is generated by $B_r(2n)_{\overline{K}}$ and the $r\backslash n$-diagram t_0. It can be checked that $\Phi_{\overline{K}}(t_0) = T_0$ [12, Page 687] and so $\Phi_{\overline{K}}(D_r(2n)_{\overline{K}}) = \mathrm{End}_{\overline{K}SO(V_{\overline{K}},q_{\overline{K}})}(V_{\overline{K}}^{\otimes r})$. ■

Theorem 3.5. *With the notations as above, we have $\Psi_{\overline{K}}(\overline{K}SO(V_{\overline{K}},q_{\overline{K}})) = \mathrm{End}_{D_r(2n)_{\overline{K}}}(V_{\overline{K}}^{\otimes r})$.*

Proof. We observe that the map $\Psi_{\overline{K}} : \overline{K}SO(V_{\overline{K}},q_{\overline{K}}) \to \mathrm{End}_{\overline{K}}(V_{\overline{K}}^{\otimes r})$ is obtained by the natural action of $SO(V_{\overline{K}},q_{\overline{K}})$ on $V_{\overline{K}}^{\otimes r}$. Hence $\Psi_{\overline{K}}$ is the restriction of the same map defined on $\overline{K}O(V_{\overline{K}},q_{\overline{K}})$, or on $\overline{K}GL(V_{\overline{K}})$, for that matter. We know that $\Psi_{\overline{K}}(\overline{K}O(V_{\overline{K}},q_{\overline{K}})) = \mathrm{End}_{B_r(2n)_{\overline{K}}}(V_{\overline{K}}^{\otimes r})$ by [8, Theorem 1.2(a)] and Proposition 2.8. Since $D_r(2n)_{\overline{K}}$ is generated by $B_r(2n)_{\overline{K}}$ and the element t_0, it is clear that $\mathrm{End}_{D_r(2n)_{\overline{K}}}(V_{\overline{K}}^{\otimes r})$ consists of those elements of $\mathrm{End}_{B_r(2n)_{\overline{K}}}(V_{\overline{K}}^{\otimes r})$ which commute with $\Phi_{\overline{K}}(t_0) = T_0$.

Choose a basis of $\Psi_{\overline{K}}(\overline{K}SO(V_{\overline{K}},q_{\overline{K}}))$ consisting of elements of the group $\Psi_{\overline{K}}(SO(V_{\overline{K}},q_{\overline{K}}))$, say $\{g_1, g_2, \ldots, g_a\}$, and extend it to a basis of the algebra $\Psi_{\overline{K}}(\overline{K}O(V_{\overline{K}},q_{\overline{K}}))$ consisting of the elements of the group $\Psi_{\overline{K}}(O(V_{\overline{K}},q_{\overline{K}}))$, say $\{g_1, g_2, \ldots, g_a, g_{a+1}, \ldots, g_{a+b}\}$.

For any $X \in \mathrm{End}_{D_r(2n)_{\overline{K}}}(V_{\overline{K}}^{\otimes r})$, we want to show that there is an element $\sum_{i=1}^{a} x_i g_i$ in the image $\Psi_{\overline{K}}(\overline{K}SO(V_{\overline{K}},q_{\overline{K}}))$, with $x_i \in \overline{K}$, such that $\sum_{i=1}^{a} x_i g_i = X$.

Since $X \in \mathrm{End}_{B_r(2n)_{\overline{K}}}(V_{\overline{K}}^{\otimes r})$, we get that there is an element $\sum_{i=1}^{a+b} x_i g_i$ in $\Psi_{\overline{K}}(\overline{K}O(V_{\overline{K}},q_{\overline{K}}))$ such that $\sum_{i=1}^{a+b} x_i g_i = X$. We conjugate this last equation by the element T_0 to get

$$\sum_{i=1}^{a+b} x_i(T_0^{-1}g_iT_0) = T_0^{-1}XT_0 = X = \sum_{i=1}^{a+b} x_ig_i.$$

For $1 \leq i \leq a$, we have that $g_i \in \Psi_{\overline{K}}(\overline{K}SO(V_{\overline{K}}, q_{\overline{K}})) \subseteq \mathrm{End}_{D_r(2n)_{\overline{K}}}(V_{\overline{K}}^{\otimes r})$ and hence $T_0^{-1}g_iT_0 = g_i$ for $1 \leq i \leq a$. Moreover, since T_0 corresponds to the determinant multilinear map, we have $T_0^{-1}gT_0 = -g$ for every $g \in \Psi_{\overline{K}}(O(V_{\overline{K}}, q_{\overline{K}}))\backslash\Psi_{\overline{K}}(SO(V_{\overline{K}}, q_{\overline{K}}))$. Thus the last equation becomes

$$\sum_{i=1}^{a} x_ig_i - \sum_{i=a+1}^{a+b} x_ig_i = \sum_{i=1}^{a} x_ig_i + \sum_{i=a+1}^{a+b} x_ig_i.$$

Since the elements $\{g_i\}$ are all linearly independent, we get that $x_i = 0$ for $i \geq a+1$. This proves the result. ∎

Lemma 3.6. *If $r \leq 2n$, then the kernel of the restriction map $\Phi_K|_{B_r(2n)_K} : B_r(2n)_K \to \mathrm{End}_{KO(V,q)}(V_K^{\otimes r})$ is trivial.*

Proof. Let $d \in B_r(2n)$ be such that $\Phi_K(d) = 0$. Note that $d \in B_r(2n)_{\overline{K}}$ as $d\otimes 1$ and that $\Phi_{\overline{K}}(d \otimes 1) = \Phi_K(d) \otimes 1 \in \mathrm{End}_{KO(V,q)}(V^{\otimes r}) \otimes_K \overline{K} = \mathrm{End}_{\overline{K}O(V_{\overline{K}}, q_{\overline{K}})}(V_{\overline{K}}^{\otimes r})$. Hence, $d \otimes 1 \in \ker(\Phi_{\overline{K}}|_{B_r(2n)_{\overline{K}}})$. Over the algebraic closure $\overline{K}$, all the bilinear forms are equivalent and Doty-Hu's result [8, Theorem 1.2(b)] applied to $\Phi_{\overline{K}}|_{B_r(2n)_{\overline{K}}}$ implies that $d \otimes 1 = 0$. Hence, $d = 0$ and the proof is complete. ∎

We now prove the remaining part of Theorem 1.2.

Theorem 3.7. *If $n \leq r \leq 2n$, and if $V^{\otimes r}$ is a semisimple module for the group $SO(V,q)$ then the largest associative semisimple quotient of $D_r(2n)$ is isomorphic to $\mathrm{End}_{KSO(V,q)}(V^{\otimes r})$.*

Let $\overline{D_r(2n)}$ denote the largest associative quotient of $D_r(2n)$ and let $\overline{\Phi}$ be the induced map from $\overline{D_r(2n)}$ to $\mathrm{End}_{KSO(V,q)}(V^{\otimes r})$. Clearly, $\overline{\Phi}$ is surjective.

We now see the proof of the Theorem 3.7.

Proof. We prove that kernel of the map $\overline{\Phi}$ is equal to the Jacobson radical of $\overline{D_r(2n)}$, J. Under the algebra homomorphism $\overline{\Phi}$, $\overline{\Phi}(J)$ is contained in the Jacobson radical of $\mathrm{End}_{KSO(V,q)}(V^{\otimes r})$. Since $V^{\otimes r}$ is semi-simple $SO(V,q)$-module, $J(\mathrm{End}_{KSO(V,q)}(V^{\otimes r})) = 0$, $\overline{\Phi}(J) = 0$ and $J \subseteq \ker(\overline{\Phi})$.

For the reverse inclusion, we first observe that $B_r(2n) \subseteq \overline{D_r(2n)}$ and due to Theorem 2.6, we have $\ker(\overline{\Phi}) \cap B_r(2n) = 0$. Further, it follows from the definition of the product in $D_r(2n)$ that the product of any two $r\backslash n$-diagrams is in $B_r(2n)$. Hence we conclude that $\ker(\overline{\Phi})^2 = 0$. Therefore, $\ker(\overline{\Phi})$ is nilpotent and hence contained in J. ∎

Serre, in his paper [14, Theorem 1], had proved that if $r(m-1) < p$ where p is the characteristic of the field K then $V^{\otimes r}$ is a semi-simple $SO(V,q)$ module. We then have:

Corollary 3.8. *If dim $V = m$ is such that $r \leq m$ and $r(m-1) < p$ where p is the characteristic of the field K, then $\overline{D_r(m)}/J \simeq \mathrm{End}_{SO(V,q)}(V^{\otimes r})$.*

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