On the Schur Multiplier of n-Lie Algebras

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Abstract. In this article, we give the structure of all covers of n-Lie algebras with finite dimensional Schur multipliers, which generalizes an earlier work of Salemkar et. al. Also, For an n-Lie algebra A of dimension d, we find the upper bound $\dim \mathcal{M}(A) \leq \binom{d}{n}$, where $\mathcal{M}(A)$ denotes the Schur multiplier of A and that the equality holds if and only if A is abelian. Finally, we give a formula for the dimension of the Schur multiplier of the direct sum of two n-Lie algebras.
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1. Notations and preparatory results

In 1985, Filippov [9] introduced the concept of n-*Lie algebras*, as an n-ary multilinear and skew-symmetric operation $[x_1, \ldots, x_n]$, which satisfies the following generalized Jacobi identity

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n].$$

Clearly, such an algebra becomes a ordinary Lie algebra when $n = 2$. Beside presenting many examples of n-Lie algebras, he also extended the notions of simplicity and nilpotency and determined all $(n+1)$-dimensional n-Lie algebras over an algebraically closed field of characteristic zero. There are great deal of difference between n-Lie algebras and the ordinary Lie algebras. For example, Williams [17] shows that the non-nilpotent n-Lie algebras can admit a regular automorphism of order p, and the sum of nilpotent ideals need not be nilpotent.

Let $A_1, A_2, \ldots, A_n$ be subalgebras of n-Lie algebra A. The subalgebra of A is generated by all vectors $[x_1, x_2, \ldots, x_n]$, where $x_i \in A_i$, $i = 1, 2, \ldots, n$, denoted by $[A_1, A_2, \ldots, A_n]$. The subalgebra $[A, A, \ldots, A]$ is called the *derived algebra* of A and it is denoted by A^2. If $A^2 = 0$ then A is called an abelian algebra. An *ideal I* of an n-Lie algebra A is a subspace of A such that $[I, A, \ldots, A] \subseteq I$. If $[I, I, A, \ldots, A] = 0$, then I is called an *abelian ideal*.

The *center* of A is defined by

$$Z(A) = \{x \in A : [x, A, \ldots, A] = 0\}.$$

Let $Z_0(A) = \langle 0 \rangle$. Then the ith center of A is defined inductively by

$$\frac{Z_i(A)}{Z_{i-1}(A)} = Z\left(\frac{A}{Z_{i-1}(A)}\right)$$

for all $i \geq 1$. Clearly, $Z_1(A) = Z(A)$, is an abelian ideal of A.

Let A be an n-Lie algebra over a field Λ with a free presentation

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0,$$

in which F is a free n-Lie algebra. The *Schur multiplier* of A, denoted by $\mathcal{M}(A)$, is defines as

$$\mathcal{M}(A) = \frac{R \cap F^2}{[R, F, \ldots, F]}.$$

Notice that the Schur multiplier of an n-Lie algebra A is always abelian and that every two Schur multipliers of A are isomorphic (see [1], [2], [3], [6], [10], [13], [14], [15] and [18] for more information on the Schur multiplier of Lie algebras). In 2003, Casas [7] showed that the Schur multiplier of A is isomorphic to $HL_1(A)$ the second cohomology of A.

The second author and Veisi [16] introduced the isoclinism for n-Lie algebras as follows: Let A_1 and A_2 be two n-Lie algebras. Then A_1 and A_2 are said to be *isoclinic*, denoted by $A_1 \sim A_2$, if there exist two isomorphisms $\alpha : A_1/Z(A_1) \to A_2/Z(A_2)$ and $\beta : A_1^2 \to A_2^2$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\frac{A_1}{Z(A_1)} \times \cdots \times \frac{A_1}{Z(A_1)} & \longrightarrow & A_1^2 \\
{\scriptstyle \alpha \times \cdots \times \alpha} \downarrow & & \downarrow {\scriptstyle \beta} \\
\frac{A_2}{Z(A_2)} \times \cdots \times \frac{A_2}{Z(A_2)} & \longrightarrow & A_2^2
\end{array}
$$

where the horizontal maps are defined by $(x_1 + Z(A_i), \ldots, x_n + Z(A_i)) \mapsto [x_1, \ldots, x_n]$.

Note that isoclinism among n-Lie algebras is an equivalence relation. The following lemma is straightforward.

Lemma 1.1. *Let A be an n-Lie algebra, and H and I be a subalgebra and an ideal of A, respectively. Then the following statements hold:*

(i) *$H \sim H + Z(A)$. In particular, if $A = H + Z(A)$, then $A \sim H$. Conversely, if $A/Z(A)$ satisfies the descending chain condition on subalgebras and $A \sim H$, then $A = H + Z(A)$.*

(ii) *$A/I \sim A/(I \cap A^2)$. In particular, if $I \cap A^2 = 0$, then $A \sim A/I$. Conversely, if A^2 satisfies the ascending chain condition on ideals and $A \sim A/I$, then $I \cap A^2 = 0$.*

The following corollary is useful in the next section.

Corollary 1.2. *Let H be a subalgebra of an n-Lie algebra A. If β is an epimorphism from A onto H, then β induces an isoclinism between A and H if and only if $\ker \beta \cap A^2 = 0$.*

Proof. If β induces an isoclinism between A and H, then $\beta|_{A^2} : A^2 \longrightarrow H^2$ is an isomorphism. Hence, $\ker \beta \cap A^2 = 0$. The 'if' part follows from Lemma 1.1(ii). ∎

This paper is organized as follows. Section 2 studies the central extension of n-Lie algebras and section 3 introduces dimension of the Schur multiplier of a direct sum of two n-Lie algebras.

2. Covering n-Lie algebras

Central extension theory is studied in many contexts such as the central extension of Virasoro algebras, the central extension of Kac-Moody algebras and the central extension of finite generated graded Lie algebras (see [4],[5],[8],[11] and [12]). In this section, we study the central extension of n-Lie algebras.

An exact sequence $0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$ of n-Lie algebras is called a *central extension* of A, if $C \subseteq Z(B)$. The central extension is said to be a *stem extension* of A if $C \subseteq Z(B) \cap B^2$. The stem extension is *maximal* if every epimorphism of any other stem extension of A onto $0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$ is necessarily an isomorphism. Finally, we call the stem extension a *stem cover* if $C \cong \mathcal{M}(A)$. In this case, B is said to be a cover (or covering n-Lie algebras).

In what follows, we show that every n-Lie algebra A has a stem cover and also prove that the extension $0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$ of the finite dimensional n-Lie algebra A is maximal (or equivalently, B has maximal dimension) if and only if it is a stem cover of A.

The following result shows the existence of covers for arbitrary n-Lie algebras.

Lemma 2.1. *Every n-Lie algebra A has a stem cover.*

Proof. Let $F/R \cong A$ be a free presentation of A and $S/[R, F, \dots, F]$ be a complement to $\mathcal{M}(A)$ in $R/[R, F, \dots, F]$. Clearly, S is an ideal of F. Now put $B = F/S$ and $C = R/S$. Then

$$\frac{B}{C} \cong \frac{F}{R} \cong A, \quad C \cong \mathcal{M}(A) \quad \text{and} \quad R = R \cap F^2 + S \subseteq F^2 + S.$$

Hence

$$C = \frac{R}{S} \subseteq \frac{F^2 + S}{S} = \left(\frac{F}{S}\right)^2 \cong B^2$$

and

$$C = \frac{R}{S} \subseteq Z\left(\frac{F}{S}\right) = Z(B),$$

which is the desired result. ∎

The following lemma plays an essential role in our investigation.

Lemma 2.2. Let $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} A \longrightarrow 0$ be a free presentation of an n-Lie algebra A and let $0 \longrightarrow C \longrightarrow B \xrightarrow{\theta} \overline{A} \longrightarrow 0$ be a central extension of another n-lie algebra $\overline{A}$. Then for each homomorphism $\alpha : A \longrightarrow \overline{A}$, there exists a homomorphism $\beta : F/[R, F, \ldots, F] \longrightarrow B$ such that $\beta(R/[R, F, \ldots, F]) \subseteq C$ and the following diagram is commutative:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{R}{[R,F,\ldots,F]} & \longrightarrow & \frac{F}{[R,F,\ldots,F]} & \xrightarrow{\overline{\pi}} & A & \longrightarrow & 0 \\
& & \downarrow{\beta_1} & & \downarrow{\beta} & & \downarrow{\alpha} & & \\
0 & \longrightarrow & C & \longrightarrow & B & \xrightarrow{\theta} & \overline{A} & \longrightarrow & 0
\end{array}
\tag{1}
$$

Here $\overline{\pi}$ is the natural epimorphism induced by π and β_1 is the restriction of β to $R/[R, F, \ldots, F]$.

Proof. Since F is a free n-Lie algebra, there exists a homomorphism $\beta' : F \longrightarrow B$ such that the following diagram commutes:

$$
\begin{array}{ccc}
F & \xrightarrow{\pi} & A \\
\downarrow{\beta'} & & \downarrow{\alpha} \\
B & \xrightarrow{\theta} & \overline{A}
\end{array}
$$

A simple verification shows that $\beta'(R) \subseteq C$ and $\beta'([R, F \ldots, F]) = 0$. Hence β' induces homomorphisms

$$
\beta : \frac{F}{[R, F, \ldots, F]} \longrightarrow B \quad \text{and} \quad \beta_1 : \frac{R}{[R, F, \ldots, F]} \longrightarrow C
$$

in such a way that the diagram (1) is commutative. ∎

In the following, we determine the structure of covers of the n-Lie algebra A, for which $\mathcal{M}(A)$ is finite dimensional. This result is similar to the work of Salemkar and the proof is essentially the same as in [14, Theorem 2.2] and it is omitted.

Theorem 2.3. Let A be an n-Lie algebra such that its Schur multiplier is finite dimensional and let $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} A \longrightarrow 0$ be a free presentation of A. Then the extension $0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$ is a stem cover of A if and only if there exists an ideal S of F such that:

(i) $B \cong F/S$ and $C \cong R/S$;

(ii) $\frac{R}{[R,F,\ldots,F]} \cong \mathcal{M}(A) \oplus \frac{S}{[R,F,\ldots,F]}$.

Lemma 2.4. Let A be an n-Lie algebra such that the following diagram is commutative:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_1 & \longrightarrow & B_1 & \longrightarrow & A & \longrightarrow & 0 \\
& & \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\
0 & \longrightarrow & C_2 & \longrightarrow & B_2 & \longrightarrow & A & \longrightarrow & 0
\end{array}
$$

where the first row is exact and the second row is an stem extension. If the homomorphism γ is onto, then so is β.

Proof.   Clearly, $B_2 = \operatorname{Im}\beta + C_2$. Hence $(B_2)^2 = (\operatorname{Im}\beta)^2$. Thus $C_2 \subseteq (\operatorname{Im}\beta)^2$, that is β is an epimorphism.   ∎

By invoking Lemmas 2.2 and 2.4, one obtains the following results.

Corollary 2.5.   *Let $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} A \longrightarrow 0$ be a free presentation of an n-Lie algebra A. Then every cover of A is a homomorphic image of $F/[R, F, \ldots, F]$.*

From the following corollary, one can deduce that all covers of an n-Lie algebras with finite dimensional Schur multiplier are isoclinic.

Corollary 2.6.   *Let A be an n-Lie algebra with finite dimensional Schur multiplier. If $0 \longrightarrow C_i \longrightarrow B_i \longrightarrow A \longrightarrow 0$ $(i = 1, 2)$ are two stem covers of A, then*

(1) B_1 and B_2 are isoclinic.

(2) $Z(B_1)/C_1 \cong Z(B_2)/C_2$.

(3) if $\beta : B_1 \longrightarrow B_2$ is an epimorphism satisfying $\beta(C_1) = C_2$, then β is an isomorphism.

Proof.   (1) Let $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} A \longrightarrow 0$ be a free presentation of A and $0 \longrightarrow C \longrightarrow A^* \longrightarrow A \longrightarrow 0$ be a stem cover of A. By Theorem 2.3, there exists an epimorphism $\beta : F/[R, F, \ldots, F] \longrightarrow A^*$ such that

$$\frac{R}{[R, F, \ldots, F]} \cong \mathcal{M}(A) \oplus \ker \beta.$$

It is clear that, $\ker \beta \cap (F/[R, F, \ldots, F])^2 = 0$. Hence, by Corollary 1.2, A^* and $F/[R, F, \ldots, F]$ are isoclinic, as required.

(2) Let $F/R \cong A$ be a free presentation of A. By Theorem 2.3, $B_1 \cong F/S$ and $C_1 \cong R/S$ for some ideal S of F such that $R/[R, F, \ldots, F] = \mathcal{M}(A) \oplus S/[R, F, \ldots, F]$. Put $Z(F/[R, F, \ldots, F]) = T/[R, F, \ldots, F]$. It is enough to prove that $Z(B_1)/C_1 \cong T/R$. If $x + S \in Z(F/S)$, then $x + [R, F, \ldots, F] \in Z(F/[R, F, \ldots, F])$, hence $Z(F/S) = T/S$ so that

$$\frac{Z(B_1)}{C_1} \cong \frac{T/S}{R/S} \cong \frac{T}{R}.$$

(3) It is similar to the proof of [15, Theorem 3.3] and we omit it.   ∎

Theorem 2.7.   *Let $0 \longrightarrow C \longrightarrow B \xrightarrow{\theta} A \longrightarrow 0$ be a stem extension of a finite dimensional n-Lie algebra A. Then there is a cover A^* of A such that B is a homomorphic image of A^*.*

Proof.   Let $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} A \longrightarrow 0$ be a free presentation of A. by Lemma 2.2, there exists an epimorphism $\beta : F/[R, F, \ldots, F] \longrightarrow A$ such that the following diagram is commutative:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{R}{[R,F,\ldots,F]} & \longrightarrow & \dfrac{F}{[R,F,\ldots,F]} & \xrightarrow{\bar{\pi}} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta_1} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle I_A} & & \\
0 & \longrightarrow & C & \longrightarrow & B & \xrightarrow{\theta} & A & \longrightarrow & 0
\end{array}
\tag{2}
$$

Since $B = Im\beta + C$, we find that β is onto and $\beta(R/[R, F, \ldots, F]) = C$. Put $\ker \beta_1 = \ker \beta = T/[R, F, \ldots, F]$, for some ideal T in R. Then $B \cong F/T$ and $C \cong R/T$. Clearly,

$$\beta((R \cap F^2)/[R, F, \ldots, F]) \subseteq \beta(R/[R, F, \ldots, F]) \cap \beta(F^2/[R, F, \ldots, F]) = C \cap B^2 = C.$$

To prove the reverse containment, assume that $y = \beta(r + [R, F, \ldots, F]) = \beta(x + [R, F, \ldots, F])$, for some $r \in R$ and $x \in F^2$. Then $(x - r) + [R, F, \ldots, F] \in T/[R, F, \ldots, F] \subseteq R/[R, F, \ldots, F]$.

It follows that $x \in R$ and $y \in \beta((R \cap F^2)/[R, F, \ldots, F])$. Therefore, β restrictions to an epimorphism from $(R \cap F^2)/[R, F, \ldots, F]$ to C. Hence

$$\frac{R}{T} \cong \frac{R/[R, F, \ldots, F]}{T/[R, F, \ldots, F]} \cong C \cong \frac{(R \cap F^2)/[R, F, \ldots, F]}{(T \cap F^2)/[R, F, \ldots, F]} \cong \frac{(R \cap F^2) + T}{T}$$

and since C is finite dimensional, we have $(R \cap F^2) + T = R$. Now suppose that $S/[R, F, \ldots, F]$ is a complement of $(T \cap F^2)/[R, F, \ldots, F]$ in $T/[R, F, \ldots, F]$, then $S \cap (R \cap F^2) = [R, F, \ldots, F]$ and $(R \cap F^2) + S = R$, which imply that

$$\frac{R}{[R, F, \ldots, F]} \cong \mathcal{M}(A) \oplus \frac{S}{[R, F, \ldots, F]}.$$

Then by Theorem 2.3, F/S is a cover of A, which completes the proof.   ∎

The following corollary is an immediate consequence of the above theorem.

Corollary 2.8.   *The maximal stem extension of a finite dimensional n-Lie algebra are precisely the same as its stem covers.*

Corollary 2.9.   *Let $0 \longrightarrow C_i \longrightarrow B_i \longrightarrow A \longrightarrow 0$ $(i = 1, 2)$ be two maximal stem extensions of a finite dimensional n-Lie algebra A. Then $\dim B_1 = \dim B_2$.*

3.   Schur multiplier of n-Lie algebras

Relations between the center and derived subgroups of a group G back to 1904 when Schur proved that if G is a group such that the order of $G/Z(G)$ is finite, then so is the derived subgroup of G. Let L be a Lie algebra. Similar to Schur's theorem, Maneyhun [13] in 1994 proved that if $\dim L/Z(L) = n$, then $\dim L^2 \leq n(n-1)/2$.

The following Theorem is a generalization of the above result in n-Lie algebras.

Theorem 3.1. *Let A be an n-Lie algebra such that $\dim A/Z(A) = d$. Then*

$$\dim A^2 \le \binom{d}{n}.$$

Proof. Let $\{\overline{x}_1, \ldots, \overline{x}_d\}$ be a basis for $A/Z(A)$. Clearly, A^2 generates by

$$\{[x_{i_1}, \ldots, x_{i_n}] : 1 \le i_1 < \cdots < i_n \le d\},$$

from which the result follows. ∎

Corollary 3.2. *Let A be a d-dimensional n-Lie algebra and $0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$ be a stem extension of A. Then $\dim B \le \binom{d}{n} + d$.*

Proof. Since $C \subseteq Z(B)$, we have $\dim B/Z(B) \le \dim B/C = d$ and hence $\dim B^2 \le \binom{d}{n}$. As $C \subseteq B^2$, it follows that

$$\dim B = \dim A + \dim C$$
$$\le d + \dim B^2$$
$$\le d + \binom{d}{n},$$

as required. ∎

As a consequence of Theorem 3.1, we derive

Theorem 3.3. *Let A be a d-dimensional n-Lie algebra. Then*

$$\dim \mathcal{M}(A) \le \binom{d}{n}.$$

Proof. Let $0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$ be a stem cover of A. As $C \subseteq Z(B)$, we have $\dim B/Z(B) \le \dim B/C = d$. Hence, by Theorem 3.1, $\dim B^2 \le \binom{d}{n}$. Again, since $C \subseteq B^2$ and $C \cong \mathcal{M}(A)$, it follows that

$$\dim \mathcal{M}(A) = \dim C \le \dim B^2 \le \binom{d}{n},$$

as required. ∎

Theorem 3.4. *Let A be a d-dimensional n-Lie algebra. Then $\dim \mathcal{M}(A) = \binom{d}{n}$ if and only if A is abelian.*

Proof. First suppose that A is abelian. Clearly, every stem extension of A is of the form $0 \longrightarrow B^2 \longrightarrow B \longrightarrow A \longrightarrow 0$. Let C and D be vector spaces with bases $\{x_1, \ldots, x_d\}$ and $\{y_{i_1,\ldots,i_n} : 1 \le i_1 < \cdots < i_n \le d\}$, respectively. Put $B = C + D$. Then B turns into an n-Lie algebra via the following brackets:

$$[x_{i_1}, \ldots, x_{i_n}] = y_{i_1,\ldots,i_n} \quad \text{for } 1 \le i_1 < \cdots < i_n \le d,$$
$$[x_{i_1}, \ldots, x_{i_r}, \ldots, x_{i_s}, \ldots, x_{i_n}] = 0 \quad \text{for } i_r = i_s,$$
$$[y_{i_1,\ldots,i_n}, b_2, \ldots, b_n] = 0 \quad \text{for all } b_i \in B.$$

Now, B/B^2 is an n-Lie algebra of dimension d and $B^2 \subseteq Z(B)$. Hence $0 \longrightarrow B^2 \longrightarrow B \longrightarrow A \longrightarrow 0$ is a maximal stem extension so that, by Corollary 2.8, it is a stem cover. Hence $\dim \mathcal{M}(A) = \binom{d}{n}$.

Conversely, suppose that $\dim \mathcal{M}(A) = \binom{d}{n}$ and $0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$ be a stem cover of A. Since $C \subseteq Z(B)$, we have $\dim B/Z(B) \leq d$ and hence $\dim B^2 \leq \binom{d}{n}$. Now,

$$\binom{d}{n} = \dim \mathcal{M}(A) = \dim C \leq \dim B^2 \leq \binom{d}{n},$$

which implies that $C = B^2$, that is, $A \cong B/C$ is abelian. ∎

In what follows, we shall compute the dimension of Schur multiplier of a direct sum of two n-Lie algebras. First we give a key lemma for our approach.

Lemma 3.5.   *Let A be a finite dimensional n-Lie algebra and I be an ideal of A. Let $H = A/I$. Then there exists a finite dimensional n-Lie algebra L with ideal M such that*

(i) $I \cap A^2 \cong L/M$,

(ii) $M \cong \mathcal{M}(A)$,

(iii) $\mathcal{M}(H)$ is an epimorphic image of L, and

(iv) if $I \subseteq Z(A)$, then $I \cap A^2$ is an epimorphic image of $\mathcal{M}(H)$.

Proof.   Let $F/R \cong A$ be a free presentation of A and $T/R \cong I$ be a free presentation for I, where T is an ideal of F. Then

$$A^2 \cap I \cong \left(\frac{F}{R}\right)^2 \cap \frac{T}{R} \cong \frac{F^2 + R}{R} \cap \frac{T}{R}$$
$$= \frac{F^2 \cap T}{F^2 \cap R} \cong \frac{(F^2 \cap T)/[R, F, \ldots, F]}{(F^2 \cap R)/[R, F, \ldots, F]}.$$

By putting $L = (T \cap F^2)/[R, F, \ldots, F]$ and $M = (R \cap F^2)/[R, F, \ldots, F]$, the result follows immediately. ∎

Now, we are ready to obtain our formula.

Theorem 3.6.   *Let A and B be finite dimensional n-Lie algebras. Then*

$$\dim \mathcal{M}(A \oplus B) = \dim \mathcal{M}(A) + \dim \mathcal{M}(B) + \binom{a+b}{n} - \binom{a}{n} - \binom{b}{n},$$

in which $a = \dim A/A^2$ and $b = \dim B/B^2$.

Proof. Let $0 \longrightarrow N \longrightarrow H \longrightarrow A \oplus B \longrightarrow 0$ be a stem cover for $A \oplus B$ and let X and Y be ideals of H such that $X/N \cong A$ and $Y/N \cong B$, respectively. Clearly, $H = X + Y$,

$$H^2 = X^2 + Y^2 + \sum_{r=1}^{n-1} [\underbrace{X, \ldots, X}_{r}, \underbrace{Y, \ldots, Y}_{n-r}]$$

and

$$[X, \ldots, X, Y, \ldots, Y] \subseteq X \cap Y = N.$$

Since $N \subseteq H^2$, one gets

$$N = (X^2 \cap N) + (Y^2 \cap N) + \sum_{r=1}^{n-1} [\underbrace{X, \ldots, X}_{r}, \underbrace{Y, \ldots, Y}_{n-r}].$$

By Lemma 3.5, $X^2 \cap N$ and $Y^2 \cap N$ are epimorphic images of $\mathcal{M}(A)$ and $\mathcal{M}(B)$, respectively. Thus

$$\dim \mathcal{M}(A \oplus B) = \dim N \leq \dim \mathcal{M}(A) + \dim \mathcal{M}(B) + \sum_{r=1}^{n-1} \dim([\underbrace{X, \ldots, X}_{r}, \underbrace{Y, \ldots, Y}_{n-r}]).$$

Now, consider the following multilinear map

$$f : \frac{A}{A^2} \times \cdots \times \frac{A}{A^2} \times \frac{B}{B^2} \times \cdots \times \frac{B}{B^2} \longrightarrow [X, \ldots, X, Y, \ldots, Y]$$
$$(\bar{a}_1, \ldots, \bar{a}_r, \bar{b}_1, \ldots, \bar{b}_{n-r}) \longmapsto [x_{a_1}, \ldots, x_{a_r}, y_{b_1}, \ldots, y_{b_{n-r}}],$$

in which $x_{a_i} + N \longrightarrow a_i$ and $y_{b_i} + N \longrightarrow b_i$ comes from the isomorphisms $X/N \cong A$ and $Y/N \cong B$. So that

$$\dim \mathcal{M}(A \oplus B) \leq \dim \mathcal{M}(A) + \dim \mathcal{M}(B) + \sum_{r=1}^{n-1} \binom{a}{r} \binom{b}{n-r}.$$

Let $0 \longrightarrow L \longrightarrow U \longrightarrow A \longrightarrow 0$ and $0 \longrightarrow M \longrightarrow V \longrightarrow B \longrightarrow 0$ be the stem covers of A and B, respectively, P and Q be complements to U^2 in U and V^2 in V, respectively, and $\{u_i\}$ and $\{v_i\}$ be bases for U and V, respectively. Let W be a vector space with a basis containing all notations $[u_{i_1}, \ldots, u_{i_r}, v_{i_{r+1}}, \ldots, v_{i_n}]$. Then

$$W = [U, \underbrace{V, \ldots, V}_{n-1}] + [U, U, \underbrace{V, \ldots, V}_{n-2}] + \cdots + [\underbrace{U, \ldots, U}_{n-1}, V]$$

and

$$\dim W = \sum_{r=1}^{n-1} \binom{\dim \frac{U}{U^2}}{r} \binom{\dim \frac{V}{V^2}}{n-r}.$$

Let $S = U \oplus V \oplus W$ and extent the bracket on U and V in such a way that $[[\underbrace{U, \ldots, U}_{r}, \underbrace{V, \ldots, V}_{n-r}], \underbrace{U, \ldots, U}_{s}, \underbrace{V, \ldots, V}_{n-s-1}] = 0$ for all $0 \leq r \leq n, 0 \leq s \leq n-1, (r,s) \neq (0,0), (r,s) \neq (n, n-1)$ and $W \subseteq Z(S)$. Then

$$W = [P, Q, \ldots, Q] + [P, P, Q, \ldots, Q] + \cdots + [P, \ldots, P, Q].$$

One can easily see that S is an n-Lie algebra. Now, consider $T = L + M + W$. Since $L \subseteq U^2 \cap Z(U)$ and $M \subseteq V^2 \cap Z(V)$, we have $T \subseteq S^2$ and $\frac{S}{T} = \frac{U+V+W}{L+M+W}$.

The map $g : U + V + W \longrightarrow U/L \oplus V/M$ given by $g(u, v, w) = (\overline{u}, \overline{v})$ is an epimorphism whose kernel is $L + M + W$. Thus,

$$\frac{S}{T} \cong \frac{U}{L} \oplus \frac{V}{M} \cong A \oplus B.$$

Hence, $0 \longrightarrow T \longrightarrow S \longrightarrow A \oplus B \longrightarrow 0$ is a stem extension of $A \oplus B$. On the other hand,

$$\dim T = \dim L + \dim M + \dim W$$

$$= \dim \mathcal{M}(A) + \dim \mathcal{M}(B) + \sum_{r=1}^{n-1} \binom{\dim \frac{U}{U^2}}{r} \binom{\dim \frac{V}{V^2}}{n-r}$$

$$= \dim \mathcal{M}(A) + \dim \mathcal{M}(B) + \sum_{r=1}^{n-1} \binom{\dim \frac{A}{A^2}}{r} \binom{\dim \frac{B}{B^2}}{n-r}$$

$$= \dim \mathcal{M}(A) + \dim \mathcal{M}(B) + \binom{a+b}{n} - \binom{a}{n} - \binom{b}{n}.$$

Therefore, $0 \longrightarrow T \longrightarrow S \longrightarrow A \oplus B \longrightarrow 0$ is a stem cover of $A \oplus B$, which completes the proof. ∎

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