Journal of Lie Theory
Volume **27** (2017) 297–314

Multivariate Hilbert Series of Lattice Cones

Wayne A. Johnson

Communicated by M. Moskowitz

Abstract. We consider the dimensions of irreducible representations whose highest weights lie on a given lattice cone. We present a simple closed form for the multivariate formal power series which generates these dimensions. This closed form is a direct generalization of a formula for the Hilbert series of an equivariant embedding of a homogeneous variety, obtained by Gross and Wallach. We use this generalization to study multivariate and single variable Hilbert series for many varieties of interest in representation theory, including the Kostant cone and various determinantal varieties. We show how the classical Hilbert series of determinantal varieties may be obtained from the multivariate series by a simple recursive relationship. We also prove some combinatorial properties of the multivariate series.
Mathematics Subject Classification 2010: 17B10, 20G05.
Key Words and Phrases: Hilbert series, highest weight theory, equivariant embeddings, Weyl Dimension Formula.

1. Introduction

Let G be a semisimple, simply connected linear algebraic group over $\mathbb{C}$. The dimension of any finite dimensional irreducible representation of G is given by the Weyl Dimension Formula. We approach the problem of finding a suitable closed form for the multivariate power series which generates the dimensions of irreducible representations whose highest weights lie in a finitely generated lattice cone in $P_+(G)$.

When the cone is generated by a single dominant integral weight, the generating function is the Hilbert series for an equivariant embedding of G/P into a projective space for some parabolic subgroup P. In fact, this case describes the Hilbert series for all such equivariant embeddings. A closed form for these Hilbert series was established by Gross and Wallach in [7]. Note that these embeddings include the most important projective embeddings of homogeneous projective varieties, including the Segre embedding of products of projective spaces, the Veronese embeddings of projective space, and the Plücker embeddings of Grassmannians. Another important variety whose coordinate ring behaves much like that of G/P is the closure of the G-orbit of a highest weight vector in a finite

ISSN 0949–5932 / $2.50

dimensional irreducible representation of G. The coodinate ring was studied by Vinberg and Popov in [12], and has a decomposition into irreducible highest weight representations similar to that of G/P.

We study the case of a general finitely generated lattice cone, and show that there is a multivariate extension of the results in [7] and [12]. This extension describes multivariate Hilbert series on many of the most interesting homogeneous varieties. As a special case, we recover the closed form for the Hilbert series of an equivariant embedding of G/P given in [7]. We also obtain multivariate Hilbert series on any variety whose coordinate ring decomposes as a G-representation over a finitely generated lattice cone, such as the symmetric, antisymmetric, and standard determinantal varieties.

In addition to these varieties, we obtain a closed form for a multivariate Hilbert series on the Kostant cones, which generalize the orbit of a single highest weight vector. These varieties go back to Kostant, who proved that their ideals are generated by quadratic elements. See [6], [10], [11], and the upcoming book by Nolan R. Wallach on the subject, [13]. We give a proof that the coordinate ring of a Kostant cone decomposes as a G-representation over a finitely generated cone in $P_+(G)$.

This paper consists of five sections. In this first section, we set up background material and notation necessary to prove the main theorem, which is established in Section 2, and recall the existing results on the single variable case. In Section 3, we discuss the generating function as a multivariate Hilbert series on the Kostant cones. In Section 4, we present further examples, including a recursive relation on multivariate Hilbert series on the symmetric and antisymmetric determinantal varieties. Finally, we close in Section 5 with a discussion of combinatorial properties of the generating function.

1.1. Notation and the Weyl Dimension Formula.

Throughout this paper, G will denote a semisimple, simply connected linear algebraic group over $\mathbb{C}$. We fix a choice $T \subset B \subset G$ of maximal torus and Borel subgroup. Let U be a maximal unipotent subgroup of B such that $B = T \cdot U$. We denote by $\mathfrak{g}, \mathfrak{h}$, and $\mathfrak{b}$ the Lie algebras of G, T, and B, respectively. Let Φ be the root system given by the pair $(\mathfrak{g}, \mathfrak{h})$, and let Φ^+ denote the set of positive roots corresponding to $\mathfrak{b}$. We set $d := |\Phi^+|$.

Let $P_+(G)$ denote the set of dominant integral weights of G. To each weight $\lambda \in P_+(G)$, let $L(\lambda)$ denote the irreducible representation of G with highest weight λ, and denote by $(\cdot,\cdot)$ the non-degenerate bilinear form on $\mathfrak{h}^*$ induced by the Killing form.

The following is well known (see p.336 in [5]).

Weyl Dimension Formula 1. *Let $\lambda \in P_+(\mathfrak{g})$. Then*

$$\dim(L(\lambda)) = \prod_{\alpha\in\Phi^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)},$$

where $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$.

Following the notation in [7], let $c_\lambda(\alpha) := \dfrac{(\lambda,\alpha)}{(\rho,\alpha)}$. Then the above formula can be written as

$$\dim(L(\lambda)) = \prod_{\alpha\in\Phi^+} (1 + c_\lambda(\alpha)).$$

Let L be a lattice in $\mathbb{R}^n$, i.e. a subset isomorphic to $\mathbb{N}^k$ for some $k \in \mathbb{N}$. If $C \subset \mathbb{R}^n$ is the closure of a non-empty cone, we call the intersection $L \cap C$ a *lattice cone*. For dominant integral weights $\lambda_1, \ldots, \lambda_k$, we use the notation $\langle\lambda_1, \ldots, \lambda_k\rangle$ to denote the lattice cone $\mathbb{N}\lambda_1 \oplus \cdots \oplus \mathbb{N}\lambda_k$ in the dominant weight lattice $P_+(G)$. In particular, $\langle\lambda\rangle$ denotes the ray $\mathbb{N}\lambda$ generated by the single dominant weight λ. We define $HS_{\mathbf{q}}\langle\lambda_1, \ldots, \lambda_k\rangle$ to be the formal power series generating the dimensions of the irreducible highest weight representations whose weights lie in $\langle\lambda_1, \ldots, \lambda_k\rangle$, i.e.

$$HS_{\mathbf{q}}\langle\lambda_1, \ldots, \lambda_k\rangle := \sum_{\mathbf{a}\in\mathbb{N}^k} \dim(L(a_1\lambda_1 + \cdots + a_k\lambda_k))\mathbf{q}^{\mathbf{a}},$$

where $\mathbf{q} := (q_1, q_2, \ldots, q_k)$ is a k-tuple of indeterminates, and $\mathbf{q}^{\mathbf{a}} := q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$.

Our goal is to prove the following.

Main Theorem 1. *Let $\lambda_1, \ldots, \lambda_k$ be dominant integral weights, and define $c_\lambda(\alpha) := \dfrac{(\lambda,\alpha)}{(\rho,\alpha)}$. Then*

$$HS_{\mathbf{q}}\langle\lambda_1, \ldots, \lambda_k\rangle = \prod_{\alpha\in\Phi^+} \left(1 + c_{\lambda_1}(\alpha) q_1 \frac{\partial}{\partial q_1} + \cdots + c_{\lambda_k}(\alpha) q_k \frac{\partial}{\partial q_k}\right) \prod_{i=1}^{k} \frac{1}{1-q_i}.$$

Note that the formula above should be read as applying the partial differential operator

$$\prod_{\alpha\in\Phi^+} \left(1 + c_{\lambda_1}(\alpha) q_1 \frac{\partial}{\partial q_1} + \cdots + c_{\lambda_k}(\alpha) q_k \frac{\partial}{\partial q_k}\right)$$

to the rational function

$$\prod_{i=1}^{k} \frac{1}{1-q_i}.$$

1.2. Hilbert series.

Given a graded $\mathbb{C}$-algebra A with ith homogeneous component A_i, define its *Hilbert function* to be the map $HF_A : \mathbb{N} \to \mathbb{N}$, given by $HF_A(i) = \dim(A_i)$. Then the *Hilbert series* of A is the formal power series

$$HS_q(A) := \sum_{n\in\mathbb{N}} HF_A(n) q^n.$$

We give some basic properties of the Hilbert function and series for a graded $\mathbb{C}$-algebra A. For further information, see, for example, [1],[8]. If A is generated by A_1, then the Hilbert series of A must represent a rational function of the form

$$\frac{p(q)}{(1-q)^l},$$

where $p(q) \in \mathbb{Z}[q]$ is a polynomial in q with integer coefficients. Further, if we consider the variety given by the spectrum of A, then the dimension of this variety is l.

As a generalization of the above, if we have an $\mathbb{N}^k$-graded $\mathbb{C}$-algebra A with homogeneous component $A_{(a_1,\ldots,a_k)}$ corresponding to the element $(a_1,\ldots,a_k) \in \mathbb{N}^k$, we can define its $\mathbb{N}^k$*-graded Hilbert series* as the formal powers series

$$\sum_{(a_1,\ldots,a_k)\in\mathbb{N}^k} \dim(A_{(a_1,\ldots,a_k)})q_1^{a_1}\ldots q_k^{a_k}.$$

This multivariate series can be restricted via a substitution to a single grading on A. For example, we could make the substitution $q_i \mapsto q$ to get a Hilbert series for A. Note that different restrictions correspond to different gradations of A, and these may give different Hilbert series.

1.3. Results of Gross and Wallach.

In [7], the authors are interested in computing the Hilbert series of the homogeneous coordinate ring of an equivariant embedding of G/P into a projective space. We recall the details. Given any irreducible highest weight representation $L(\lambda)$ of G, we can consider the parabolic subgroup given by the stabilizer of the unique hyperplane H in $L(\lambda)$ fixed by the Borel subgroup B. If we denote by $\mathbb{P}(L(\lambda))$ the projective space of all hyperplanes in $L(\lambda)$, then we have an embedding

$$\pi_\lambda : G/P \to \mathbb{P}(L(\lambda)),$$

given by the formula $\pi_\lambda(gP) := g(H)$.

It is a consequence of the Borel-Weil theorem that the homogeneous coordinate ring $A_\lambda(G/P)$ is a sum of highest weight representations. Namely,

$$A_\lambda(G/P) = \bigoplus_{n\in\mathbb{N}} L(n\lambda).$$

We consider this decomposition as happening over the cone in $P_+(G)$ generated by the single dominant weight λ. Then the Hilbert series of the embedding is given by

$$\sum_{n\in\mathbb{N}} \dim(L(n\lambda))q^n.$$

The authors prove that this series has the following closed form.

Theorem (Gross and Wallach) 1. *The Hilbert series of the embedding π_λ of G/P is*

$$\prod_{\alpha\in\Phi^+} \left(c_\lambda(\alpha)q\frac{d}{dq}+1\right)\frac{1}{1-q}.$$

In the next section, we will give a proof of the above theorem, and the rest of the paper will be devoted to examples of its applicability.

2. Proof of the main theorem

We extend the theorem from the previous subsection to the case of a finitely generated lattice cone. Consider the multivariate series given by

$$HS_{\mathbf{q}}\langle\lambda_1,\ldots,\lambda_k\rangle := \sum_{\mathbf{a}\in\mathbb{N}^k} \dim(L(a_1\lambda_1+\cdots+a_k\lambda_k))\mathbf{q}^{\mathbf{a}}.$$

We wish to find a closed form for $HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle$ that directly generalizes the form given in [7] for a single dominant integral weight.

Main Theorem 2. *Let $\lambda_1, \dots, \lambda_k$ be dominant integral weights. Then*

$$HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle = \prod_{\alpha\in\Phi^+}\left(1 + c_{\lambda_1}(\alpha)q_1\frac{\partial}{\partial q_1} + \cdots + c_{\lambda_k}(\alpha)q_k\frac{\partial}{\partial q_k}\right)\prod_{i=1}^{k}\frac{1}{1-q_i},$$

where $c_\lambda(\alpha) := \dfrac{(\lambda, \alpha)}{(\rho, \alpha)}$.

Proof. Consider $L(a_1\lambda_1 + \cdots + a_k\lambda_k)$. By the Weyl Dimension Formula, we have

$$\dim(L(a_1\lambda_1 + \cdots + a_k\lambda_k)) = \prod_{\alpha\in\Phi^+}\frac{(a_1\lambda_1 + \cdots + a_k\lambda_k + \rho, \alpha)}{(\rho, \alpha)}.$$

Since the Killing form $(\cdot, \cdot)$ is bilinear, the above product can be rewritten as

$$\dim(L(a_1\lambda_1 + \cdots + a_k\lambda_k)) = \prod_{\alpha\in\Phi^+}(1 + a_1c_{\lambda_1}(\alpha) + \cdots + a_kc_{\lambda_k}(\alpha)).$$

Therefore, we can rewrite the series as

$$HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle = \sum_{\mathbf{a}\in\mathbb{N}^k}\prod_{\alpha\in\Phi^+}(1 + a_1c_{\lambda_1}(\alpha) + \cdots + a_kc_{\lambda_k}(\alpha))\mathbf{q}^{\mathbf{a}}, \tag{1}$$

where $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$, and $\mathbf{q}^{\mathbf{a}} := q_1^{a_1} \dots q_k^{a_k}$. Consider the product

$$\prod_{\alpha\in\Phi^+}(1 + a_1c_{\lambda_1}(\alpha) + \cdots + a_kc_{\lambda_k}(\alpha)).$$

Note that the above is a polynomial in the a_i for $1 \le i \le k$. Let $d := |\Phi^+|$ and $|\mathbf{i}| := i_1 + \cdots + i_k$, for any $\mathbf{i} \in \mathbb{N}^k$. Then, expanding the product of sums into a sum of products, we have

$$\prod_{\alpha\in\Phi^+}(1 + a_1c_{\lambda_1}(\alpha) + \cdots + a_kc_{\lambda_k}(\alpha)) = \sum_{|\mathbf{i}|\le d} b_{\mathbf{i}}\mathbf{a}^{\mathbf{i}}, \tag{2}$$

where $b_{\mathbf{i}}$ is the coefficient of the monomial $\mathbf{a}^{\mathbf{i}}$ for each $\mathbf{i}$ with $|\mathbf{i}| \le d$. Combining (1) and (2), we have

$$HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle = \sum_{\mathbf{a}\in\mathbb{N}^k}\sum_{|\mathbf{i}|\le d} b_{\mathbf{i}}\mathbf{a}^{\mathbf{i}}\mathbf{q}^{\mathbf{a}}.$$

The coefficients of the polynomial (2) do not depend on $\mathbf{a}$. Therfore, we may rearrange the order of summation as follows:

$$HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle = \sum_{|\mathbf{i}|\le d} b_{\mathbf{i}}\sum_{\mathbf{a}\in\mathbb{N}^k}\mathbf{a}^{\mathbf{i}}\mathbf{q}^{\mathbf{a}}. \tag{3}$$

We now find a rational function representing the series $\sum_{\mathbf{a}\in\mathbb{N}^k}\mathbf{a}^{\mathbf{i}}\mathbf{q}^{\mathbf{a}}$. Fix a k-tuple $(i_1, \dots, i_k) \in \mathbb{N}^k$, and define $f_{(i_1,\dots,i_k)}(\mathbf{q})$ to be the formal power series

$$f_{(i_1,\ldots,i_k)}(\mathbf{q}) := \sum_{\mathbf{a}\in\mathbb{N}^k} \mathbf{a}^{\mathbf{i}}\mathbf{q}^{\mathbf{a}}.$$

Then applying the partial differential operator $q_j\dfrac{\partial}{\partial q_j}$ to $f_{(i_1,\ldots,i_k)}(\mathbf{q})$ increases the integer in the jth coordinate by one. To see this, note that for each summand $\mathbf{a}^{\mathbf{i}}\mathbf{q}^{\mathbf{a}}$, we have

$$\frac{\partial}{\partial q_j}(a_1^{i_1}\ldots a_j^{i_j}\ldots a_k^{i_k}q_1^{a_1}\ldots q_j^{a_j}\ldots q_k^{a_k}) = a_1^{i_1}\ldots a_j^{i_j+1}\ldots a_k^{i_k}q_1^{a_1}\ldots q_j^{a_j-1}\ldots q_k^{a_k}.$$

Multiplying both sides by q_j, we have

$$q_j\frac{\partial}{\partial q_j}(f_{(i_1,\ldots,i_k)}(\mathbf{q})) = f_{(i_1,\ldots,i_j+1,\ldots,i_k)}(\mathbf{q}).$$

Define $f_{(0,\ldots,0)}(\mathbf{q}) := \displaystyle\prod_{j=1}^{k}\frac{1}{1-q_k}$. Because the differential operators $q_j\dfrac{\partial}{\partial q_j}$ commute for all j, we have

$$f_{(i_1,\ldots,i_k)}(\mathbf{q}) = \left(q_1\frac{\partial}{\partial q_1}\right)^{i_1}\cdots\left(q_k\frac{\partial}{\partial q_k}\right)^{i_k}\prod_{j=1}^{k}\frac{1}{1-q_j}$$

Consider the k-tuple $\left(q_1\dfrac{\partial}{\partial q_1},\ldots,q_k\dfrac{\partial}{\partial q_k}\right)$, and define

$$\left(\mathbf{q}\frac{\partial}{\partial\mathbf{q}}\right)^{\mathbf{i}} := \left(q_1\frac{\partial}{\partial q_1}\right)^{i_1}\cdots\left(q_k\frac{\partial}{\partial q_k}\right)^{i_k}.$$

Then, we have

$$f_{(i_1,\ldots,i_k)}(\mathbf{q}) = \left(\mathbf{q}\frac{\partial}{\partial\mathbf{q}}\right)^{\mathbf{i}}\prod_{j=1}^{k}\frac{1}{1-q_j}.$$

Therefore, (3) becomes

$$\left[\sum_{|\mathbf{i}|\le d} b_{\mathbf{i}}\left(\mathbf{q}\frac{\partial}{\partial\mathbf{q}}\right)^{\mathbf{i}}\right]\prod_{j=1}^{k}\frac{1}{1-q_j}. \tag{4}$$

Note that the sum

$$\sum_{|\mathbf{i}|\le d} b_{\mathbf{i}}\left(\mathbf{q}\frac{\partial}{\partial\mathbf{q}}\right)^{\mathbf{i}} \tag{5}$$

is the same polynomial as in (2), after making the substitution $a_i \mapsto q_i\dfrac{\partial}{\partial q_i}$, for each $i=1,\ldots,k$. Therefore, the polynomial in (5) factors in the same way. Namely,

$$\sum_{|\mathbf{i}|\le d} b_{\mathbf{i}}\left(\mathbf{q}\frac{\partial}{\partial\mathbf{q}}\right)^{\mathbf{i}} = \prod_{\alpha\in\Phi^+}\left(1+c_{\lambda_1}(\alpha)q_1\frac{\partial}{\partial q_1}+\cdots+c_{\lambda_k}(\alpha)q_k\frac{\partial}{\partial q_k}\right).$$

Combining this fact with (4), we have shown that

$$HS_{\mathbf{q}}\langle\lambda_1,\ldots,\lambda_k\rangle = \prod_{\alpha\in\Phi^+}\left(1 + c_{\lambda_1}(\alpha)q_1\frac{\partial}{\partial q_1} + \cdots + c_{\lambda_k}(\alpha)q_k\frac{\partial}{\partial q_k}\right)\prod_{j=1}^{k}\frac{1}{1-q_j} \quad \blacksquare$$

3. The Kostant cone

This section is split into three subsections. The first of which presents without proof the basic definitions and theorems about the Kostant cone, as given in the upcoming book "Geometric invariant theory over the real and complex numbers" by Nolan R. Wallach. The term *Kostant cone* first appeared in a book by Kumar (see [9]), where he presents the single variate case. The definitions and results in the first the first subsection below concern the multivariate case. These results, attributed to Kostant, will be published in detail in [13]. The next subsection will present a construction of the coordinate ring of a Kostant cone as an infinite dimensional G-module that decomposes into finite dimensional irreducible representations whose highest weights trace out a lattice cone in $P_+(G)$. We have not found this construction anywhere previously in the literature. The final subsection makes explicit the relationship between the Kostant cone and the multivariate generating function given in the main theorem.

3.1. Basic definitions and theorems.

We begin by defining the Kostant cones given by a k-tuple of highest weight representations of G, as given in [13]. Let $L(\lambda_1),\ldots,L(\lambda_k)$ be highest weight representations of G of highest weight λ_i, and choose a highest weight vector $v_i \in L(\lambda_i)$ for each i. Let V be the direct sum $L(\lambda_1)\oplus\cdots\oplus L(\lambda_k)$. Then G acts on V diagonally. Let $S(V)$ be the symmetric algebra on V. Then $S(V)$ is $\mathbb{N}^k$-graded. Let $S^{n_1,\ldots,n_k}(V) \cong S^{n_1}(V)\otimes\cdots\otimes S^{n_k}(V)$ be a multi-homogeneous component of $S(V)$. The multiplicity of the irreducible representation with highest weight $\sum_i n_i\lambda_i$ in $S^{n_1,\ldots,n_k}(V)$ is one. Denote this subrepresentation $V^{n_1,\ldots,n_k}$. Then we have the following definition.

Definition 1. The *Kostant cone* X of V is the set of $v \in V$ such that

$$v^n \in \sum_{n_1+\cdots+n_k=n} V^{n_1,\ldots,n_k}.$$

There is a nice, concrete way of describing X in terms of an 'augmented' G-action on V. We let the group $G\times(\mathbb{C}^\times)^k$ act on V via

$$(g,z_1,\ldots,z_k).(v_1,\ldots,v_k) = (z_1g.v_1,\ldots,z_kg.v_k).$$

This is a quasi-affine variety, and the Zariski closure of this variety is equal to the Kostant cone X on V. Note that, if the highest weights λ_i are all linearly independent, then the action of $G\times(\mathbb{C}^\times)^k$ on V is the same as the diagonal action of G on V. In general, we are required to augment the action with an additional torus action in the case where the weights are dependent.

Note that the variety X is a direct generalization of the orbit of a highest weight vector. In the case of a highest weight λ, with highest weight vector v, $G\times\mathbb{C}^\times.v = G.v$, and the closure of the orbit of the highest weight vector v is the Kostant cone of $V = L(\lambda)$.

One of the interesting properties of the Kostant cone, originally due to Kostant, is the fact that the ideal of polynomials vanishing on X is always generated by quadratic polynomials. This result is referred to as Kostant's quadratic generation theorem, and proofs can be found in [6], [10], and [11] for the case of a single weight, and [13] for the general case.

3.2. The coordinate ring of X.

Note that, in the case that $k = 1$, the Kostant cone is the closure $G.v_\lambda$ in $L(\lambda)$. These varieties have been intensely studied, and their coordinate rings are computed in [12]. Here, we generalize the results in [12] to the multivariate case. Let $L(\lambda_1), \dots, L(\lambda_k)$ be finite dimensional irreducible representations of G with highest weights $\lambda_1, \dots, \lambda_k$, and choose a highest weight vector v_i from each $L(\lambda_i)$. Define $V := L(\lambda_1) \oplus \cdots \oplus L(\lambda_k)$ as before. Let $X = \overline{G \times (\mathbb{C}^\times)^k.(v_1, \dots, v_k)}$ be the Kostant cone corresponding to $\lambda_1, \dots, \lambda_k$.

We consider G and its subgroups as subgroups of $G \times (\mathbb{C}^\times)^k$ via the map $g \mapsto (g, 1, \dots, 1)$. We will abuse notation and write $g.(v_1, \dots, v_k)$ when we mean $(g, 1, \dots, 1).(v_1, \dots, v_k)$. We then have

$$b.(v_1, \dots, v_k) = (\lambda_1(b)v_1, \dots, \lambda_k(b)v_k), \forall b \in B. \tag{6}$$

Let O be the orbit of $(v_1, \dots, v_k)$, π the canonical mapping of $V/\{0\}$ onto $\mathbb{P}(V)$, and P the isotropy subgroup of $\pi(v_1, \dots, v_k)$.

Proposition 1. *P is a parabolic subgroup, containing B.*

Proof. Let π_i be the canonical mapping of $L(\lambda_i)/\{0\}$ onto $\mathbb{P}(L(\lambda_i))$, and let P_i be the isotropy subgroup of $\pi_i v_i$. Then it follows from (6) that P_i is a parabolic subgroup containing B. We will show that $P = P_1 \cap \cdots \cap P_k$.

To this end, assume $p \in P$. Then p, acting diagonally, stabilizes the line through the origin in V containing $(v_1, \dots, v_k)$. In particular, p stabilizes the line through the origin in $L(\lambda_i)$ containing v_i. Thus, $P \subset P_1 \cap \cdots \cap P_k$.

Now assume $p \in P_1 \cap \cdots \cap P_k$. Then p stabilizes the line through the origin in each $L(\lambda_i)$ that contains v_i. Since P acts diagonally on $\pi(v_1, \dots, v_k)$, this gives $p \in P$. ∎

Note that, as characters of B, the weights $\lambda_1, \dots, \lambda_k$ extend uniquely to characters of P. Let H be the isotropy subgroup of $(v_1, \dots, v_k)$.

Proposition 2. *$H = \{p \in P \mid \lambda_i(p) = 1, i = 1, \dots, k\}$.*

Proof. Note that if $\lambda_i(p) = 1$, for all $i = 1, \dots, k$, we have

$$p.(v_1, \dots, v_k) = (\lambda(p)v_1, \dots, \lambda(p)v_k) = (v_1, \dots, v_k),$$

so H contains $\{p \in P \mid \lambda_i(p) = 1, i = 1, \dots, k\}$.

For the opposite inclusion, let $h \in H$. Then, since h fixes $(v_1, \dots, v_k)$ and acts linearly, it must fix the line $\pi(v_1, \dots, v_k)$. So $H \subset P$. Thus $h.(v_1, \dots, v_k) = (\lambda_1(h)v_1, \dots, \lambda_k(h)v_k)$. This implies that $\lambda_i(h) = 1$, for all $i = 1, \dots, k$, since H is the isotropy group of $(v_1, \dots, v_k)$. ∎

We have the following further characterization of X.

Theorem 1. $X = O \cup \{0\}$.

Proof. O is invariant under multiplication by any element of $\mathbb{C}^\times$. To see this, let $z \in \mathbb{C}^\times$. Then $(g, z_1, \dots, z_k).(zv_1, \dots, zv_k) = (zz_1 g.v_1, \dots, zz_r g.v_k)$. But this is just $(g, zz_1, \dots, zz_k).(v_1, \dots, v_k)$, which is in O. Thus, the claim follows from the fact that πO is closed in $\mathbb{P}(V)$. This is true, since $\pi O \cong G/P$, and P is a parabolic subgroup of G. ■

The maps $G \xrightarrow{\tau} O \xrightarrow{\iota} X$, where $\tau(g) = g.(v_1, \dots, v_k)$ and ι is the canonical inclusion generate inclusions

$$\mathbb{C}[X] \hookrightarrow \mathbb{C}[O] \hookrightarrow \mathbb{C}[G]$$

on the level of homogeneous coordinate rings. Further, the maps ι and τ commute with left translations. To see that τ commutes with left translations, consider $\tau(gh)$. Then

$$\tau(gh) = gh.(v_1, \dots, v_k) = g.(hv_1, \dots, hv_k) = g.\tau(h).$$

Since τ and ι commute with left translations, $\mathbb{C}[X]$ and $\mathbb{C}[O]$ are left-invariant subalgebras of $\mathbb{C}[G]$. Thus, $\mathbb{C}[O] = \mathbb{C}[G/H] = \mathbb{C}[G]^H$, where H is considered to act on the right. H is normal in P. Thus, $\mathbb{C}[O]$ is right-invariant with respect to P. The action of P by right translation then reduces to the action of the torus P/H. Therefore, we have a decomposition into weight spaces

$$\mathbb{C}[O] = \bigoplus_{\lambda \in \mathfrak{X}(P)} \mathbb{C}[O]_\lambda,$$

where $\mathbb{C}[O]_\lambda := \{f \in \mathbb{C}[O] \mid f(gp) = \lambda(p) f(g), g \in G, p \in P\}$.

We define $S(\lambda) := \{f \in \mathbb{C}[G] \mid f(gb) = \lambda(b) f(g), g \in G, b \in B\}$. Note that this is a finite dimensional, left-invariant subspace of $\mathbb{C}[G]$. The set

$$\mathfrak{X}(B)^+ := \{\lambda \in \mathfrak{X}(B) \mid S(\lambda) \neq 0\}$$

is the set of all highest weights of irreducible representations of G. The representation $S(\lambda)$, when $\lambda \in \mathfrak{X}(B)^+$, is dual to the irreducible representation of G of highest weight λ. The duality can be expressed explicitly by

$$\langle f, g.v_\lambda \rangle = f(g) \tag{7}$$

for all $f \in S(\lambda)$, $g \in G$, and v_λ a highest weight vector of weight λ.

It is clear that $\mathbb{C}[O]_\lambda \subset S(\lambda)$. Let $\langle \lambda_1, \dots, \lambda_k \rangle$ be the set of all non-negative integer combinations of $\lambda_1, \dots, \lambda_k$.

Proposition 3. *$\mathbb{C}[O]_\lambda \neq 0$ implies λ is an integer combination of $\lambda_1, \dots, \lambda_k$.*

Proof. Assume that $\mathbb{C}[O]_\lambda \neq 0$. Take a nonzero $f \in \mathbb{C}[O]$ such that $f(gp) = \lambda(p) f(g)$ for all $g \in G$, $p \in P$. In particular, $f(gh) = \lambda(h) f(g)$ for all $g \in G$, $h \in H$. By the way we identify $\mathbb{C}[O]$ as a subalgebra of $\mathbb{C}[G]$, we have

$$f(gh) = f \circ \tau(gh) = f(gh.(v_1, \dots, v_k)) = f(g.(\lambda_1(h)v_1, \dots, \lambda_k(h)v_k)).$$

By Proposition 2,

$$H = \{p \in P \mid \lambda_i(p) = 1, i = 1, \dots, k\}.$$

Thus, $f(gh) = f(g.(v_1, \dots, v_k)) = f \circ \tau(g) = f(g)$. Summarizing, we have

$$f(g) = f(gh) = \lambda(h) f(g).$$

Then $f \neq 0$, implies $\lambda(h) = 1$. Therefore, we have

$$Ker(\lambda) \supset Ker(\lambda_1) \cap \cdots \cap Ker(\lambda_k).$$

There exists a bijective correspondence between subgroups of T and subgroups of $\mathfrak{X}(T)$, given by $(\Gamma \leq \mathfrak{X}(T))$

$$\Gamma \mapsto T^\Gamma := \{t \in T \mid \chi(t) = 1 \text{ for all } \chi \in \Gamma\}.$$

This correspondence reverses inclusions: if $T^{\Gamma_1} \subset T^{\Gamma_2}$, then $\Gamma_1 \supset \Gamma_2$. To see this, note that Γ consists of ALL characters whose value on T^Γ is one. If $T^{\Gamma_1} \subset T^{\Gamma_2}$, then any element of Γ_2 has value one on T^{Γ_1}. Thus, $\Gamma_2 \subset \Gamma_1$. Now, let Γ_1 be the subgroup generated by $\lambda_1, \dots, \lambda_r$, and let Γ_2 be the subgroup generated by λ.

We have $T^{\Gamma_1} = Ker(\lambda_1) \cap \cdots \cap Ker(\lambda_k)$ and $T^{\Gamma_2} = Ker(\lambda)$. We have already shown that $T^{\Gamma_1} \subset T^{\Gamma_2}$. Thus, $\Gamma_2 \subset \Gamma_1$, and we have proven the claim. ■

Note that Proposition 3 implies that

$$\mathbb{C}[O] \subset \bigoplus_{\lambda} S(\lambda), \tag{8}$$

where λ runs through all integer combinations of $\lambda_1, \dots, \lambda_k$.

Proposition 4. *For each* $\lambda \in \langle \lambda_1, \dots, \lambda_k \rangle$, $S(\lambda) \subset \mathbb{C}[X]$.

Proof. We need to show that $S(\lambda_i) \subset \mathbb{C}[X]$ for all $i = 1, \dots, k$. Then the claim follows from the fact that $S(\lambda) \cdot S(\mu) = S(\lambda + \mu)$ for all $\lambda, \mu \in \mathfrak{X}^+(B)$. Take $f \in S(\lambda_i)$. Then $f(gb) = \lambda_i(b) f(g)$, for all $g \in G, b \in B$. Let v_i be a highest weight vector for $L(\lambda_i)$ as before. Then by (7), $f(g) = \langle f, gb.v_i \rangle = \langle f, g.v_i \rangle$. Let $O_i := G.v_i$ for each $i = 1, \dots, k$, and define a map $O_i \to X$ by

$$g.v_i \mapsto g.(v_1, \dots, v_i, \dots, v_k),$$

for each i. We can think of these maps as inclusions of the O_i into X. In this way, we consider $g.v_i$ to be an element of X. Then, by (7), f is a function on X, and $S(\lambda_i) \subset \mathbb{C}[X]$. ■

Note that Proposition 4 implies that

$$\bigoplus_{\lambda \in \langle \lambda_1, \dots, \lambda_r \rangle} S(\lambda) \subset \mathbb{C}[X]. \tag{9}$$

Then, by the chain of inclusions $\mathbb{C}[X] \hookrightarrow \mathbb{C}[O] \hookrightarrow \mathbb{C}[G]$ and the fact that λ cannot be a highest weight if the coefficients on any λ_i are negative, (8) and (9) imply the following theorem.

Theorem 2. *The homogeneous coordinate ring* $\mathbb{C}[X]$ *of the Kostant cone* X *given by the weights* $\lambda_1, \dots, \lambda_k$ *is*

$$\bigoplus_{\lambda \in \langle \lambda_1, \ldots, \lambda_k \rangle} S(\lambda).$$

3.3. Multivariate Hilbert series on X.

Note that Kostant's quadratic generation theorem gives a method of computing multivariate Hilbert series on the Kostant cone X. The multivariate Hilbert series of a variety whose ideal I has a quadratic generating set can be computed using methods from commutative algebra by looking at the graph of relations given by the leading quadratic monomials in I. However, this method rarely leads to simple direct computations of Hilbert series.

On the otherhand, the main theorem gives a direct way of computing the multivariate Hilbert series of X. Let $\mathbb{C}[X]$ be the homogeneous coordinate ring of X. By Theorem 3, we have

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \langle \lambda_1, \ldots, \lambda_k \rangle} S(\lambda).$$

The representations $S(\lambda)$ are dual to the irreducible highest weight representations $L(\lambda)$. In particular, they have the same dimension. So we can use the formula for the main theorem to compute the multivariate Hilbert series on X given by the decomposition into weight spaces in Theorem 3. Let $HS(X)$ be the multivariate Hilbert series of X. Then, explicitly,

$$HS(X) = \prod_{\alpha \in \Phi^+} \left(1 + c_{\lambda_1}(\alpha) q_1 \frac{\partial}{\partial q_1} + \cdots + c_{\lambda_k}(\alpha) q_k \frac{\partial}{\partial q_k}\right) \prod_{i=1}^{k} \frac{1}{1 - q_i}.$$

This formula holds for any Kostant cone X, regardless of the number of dominant weights involved. Also, this formula is explicit, and can be computed for specific examples with the help of a computer algebra system.

4. Other examples

By setting $k = 1$, we obtain the Hilbert series of an equivariant embedding of a projective variety, as in the case of [7]. If G has rank k, and we look at the formal power series given by the fundamental dominant weights $\langle \omega_1, \ldots, \omega_k \rangle$, we obtain a generating function for the dimensions of all finite dimensional irreducible representations of G, as in [2].

We can interpret the multivariate series $HS_{\mathbf{q}}\langle \lambda_1, \ldots, \lambda_k \rangle$ geometrically via Hilbert series. Inside our given Borel subgroup $B \subset G$, we have a maximal unipotent subgroup U such that $B = T \cdot U$. The quotient G/U has a natural structure as an affine variety. This variety has coordinate ring

$$\mathbb{C}[G/U] \cong \bigoplus_{\lambda \in P_+(G)} \mathbb{C}_\lambda \otimes L(\lambda), \tag{10}$$

(see §3.3 in [14]). If we set $V(\lambda) := \mathbb{C}_\lambda \otimes L(\lambda)$, we have a gradation on $\mathbb{C}[G/U]$ given by $V(\lambda)V(\mu) = V(\lambda + \mu)$. Then replacing $P_+(G)$ with a lattice cone $\langle \lambda_1, \ldots, \lambda_k \rangle$ gives a subalgebra of $\mathbb{C}[G/U]$. The spectrum of this subalgebra is a variety, and this variety has an $\mathbb{N}^k$-graded Hilbert series given by $HS_{\mathbf{q}}\langle \lambda_1, \ldots, \lambda_k \rangle$.

In this section, our main interest in examples is going to be using the formula from the main theorem to find a series in k variables and then specializing that series to a single variable Hilbert series on the underlying variety given by subalgebras of (10) corresponding to a given lattice cone $\langle \lambda_1, \ldots, \lambda_k \rangle$.

Some of the most interesting examples are those given by looking at the homogeneous coordinate ring of the three determinantal varieties $\mathcal{D}_{m,n}^{\leq k}, \mathcal{SD}_n^{\leq k}$, and $\mathcal{AD}_n^{\leq 2k}$. We begin with the symmetric determinantal varieties $\mathcal{SD}_n^{\leq k}$. Note that finding the Hilbert series for these varieties is in general a very difficult thing to do (see, for example, [3], [4]).

The Second Fundamental Theorem of Invariant Theory for $O(n)$ (see [5], p.561), states that the homogeneous coordinate ring $\mathcal{SD}_n^{\leq k}$ decomposes as an $SL(n, \mathbb{C})$-module in the following way:

$$\mathbb{C}[\mathcal{SD}_n^{\leq k}] \cong \bigoplus_{\lambda} L(\lambda),$$

where λ runs over all even dominant integral weights of depth at most k. Here, an even weight of depth at most k is one that lies in the lattice cone $\langle 2\omega_1, \ldots, 2\omega_k \rangle$, where $\omega_1, \ldots, \omega_{n-1}$ are the fundamental dominant weights of $SL(n, \mathbb{C})$, and we are using the standard Borel subgroup of upper triangular matrices in $SL(n, \mathbb{C})$.

So we can compute the series $HS_{\mathbf{q}}\langle 2\omega_1, \ldots, 2\omega_k \rangle$ and specialize the variables in an appropriate way to recover the Hilbert series of the standard embedding of the symmetric determinantal variety. The standard Hilbert series on $\mathcal{SD}_n^{\leq k}$ is given by

$$\sum_{\lambda} \dim(L(\lambda)) q^{|\lambda|},$$

where again, λ runs over all even dominant integral weights of depth at most k. After computing the series $HS_{\mathbf{q}}\langle 2\omega_1, \ldots, 2\omega_k \rangle$, we specialize to the standard Hilbert series by making the substitution $q_i \mapsto q^i$ for $i = 1, \ldots, k$.

We now compute some examples. We consider the variety $\mathcal{SD}_4^{\leq 2}$. We will compute the series $HS_{\mathbf{q}}\langle 2\omega_1, 2\omega_2 \rangle$, where ω_1 and ω_2 are the first two fundamental dominant weights of $SL(4, \mathbb{C})$. The main theorem gives us the following closed form for $HS_{\mathbf{q}}\langle 2\omega_1, 2\omega_2 \rangle$:

$$\prod_{1 \leq i < k \leq 4} \left(1 + 2c_{\omega_1}(\epsilon_i - \epsilon_j) q_1 \frac{\partial}{\partial q_2} + 2c_{\omega_2}(\epsilon_i - \epsilon_j) q_2 \frac{\partial}{\partial q_2}\right) \frac{1}{(1-q_1)(1-q_2)},$$

where $\Phi^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq 4\}$, and ϵ_i is the functional that gives the ith diagonal element of a matrix in $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$. Then computing $c_{\omega_1}(\epsilon_i - \epsilon_j)$ and $c_{\omega_2}(\epsilon_i - \epsilon_j)$ for $1 \leq i < j \leq 4$ gives us

$$(1+2q_1\tfrac{\partial}{\partial q_1})(1+2q_2\tfrac{\partial}{\partial q_2})(1+q_1\tfrac{\partial}{\partial q_1}+q_2\tfrac{\partial}{\partial q_2})(1+q_2\tfrac{\partial}{\partial q_2})(1+\tfrac{2}{3}q_1\tfrac{\partial}{\partial q_1}+\tfrac{2}{3}q_2\tfrac{\partial}{\partial q_2})\tfrac{1}{(1-q_1)(1-q_2)}.$$

Applying the differential operators then yields

$$\frac{1+6q_1+15q_2+q_1^2+16q_1q_2+15q_2^2+q_2^3-50q_1q_2^2-29q_1^2q_2-4q_1q_2^3-25q_1^2q_2^2+6q_1^3q_2+21q_1^2q_2^3+20q_1^3q_2^2+6q_1^3q_2^3}{(1-q_1)^4(1-q_2)^5}.$$

This formula seems unmanagable, but is easy to compute with Mathematica or Maple, and after we make the substitution $q_i \mapsto q^i$, we get

$$\frac{1+3q+6q^2}{(1-q)^7},$$

which is the Hilbert series for the standard embedding of $\mathcal{SD}_4^{\leq 2}$.

We can then increase the size of the matrices to recursively find the Hilbert series of $\mathcal{SD}_n^{\leq 2}$. Let $\{\alpha_1, \ldots, \alpha_{n-1}\}$ be the simple roots of $SL(n, \mathbb{C})$. The only positive roots of $SL(n,\mathbb{C})$ that contribute to the product in $HS_{\mathbf{q}}\langle\omega_1,\omega_2\rangle$, are those which can be written as a sum of consecutive simple roots $\sum\alpha_i$ beginning at either α_1 or α_2. So, as we go from $n-1$ to n, we add two differential operators, namely, those which correspond to the positive roots $\alpha_2+\cdots+\alpha_{n-1}$ and $\alpha_1+\cdots+\alpha_{n-1}$. If we define $HS^n_{\mathbf{q}}\langle\omega_1,\omega_2\rangle$ to be the series given by the first two fundamental dominant weights of $SL(n,\mathbb{C})$, we have the following recursive formula. Note that

$$HS^3_{\mathbf{q}}\langle 2\omega_1,2\omega_2\rangle=\frac{1+3q_1+3q_2-3q_1^2q_2-3q_1q_2^2-q_1^2q_2^2}{(1-q_1)^3(1-q_2)^3}.$$

Lemma 1. *For $n>3$,*

$$HS^n_{\boldsymbol{q}}\langle 2\omega_1,2\omega_2\rangle=(1+\tfrac{2}{n-2}q_2\tfrac{\partial}{\partial q_2})(1+\tfrac{2}{n-1}q_1\tfrac{\partial}{\partial q_1}+\tfrac{2}{n-1}q_2\tfrac{\partial}{\partial q_2})HS^{n-1}_{\boldsymbol{q}}\langle 2\omega_1,2\omega_2\rangle.$$

We obtain the recursion by simply computing the coefficients for the two new weights. Note that this recursion is on the multivariate series, but it does not pass to a recursion on the single variable Hilbert series for the varieties $\mathcal{SD}^{\le 2}_n$. The recursion is linear in the number of differential operators that contribute to the series when we go from $SL(n-1,\mathbb{C})$ to $SL(n,\mathbb{C})$. In this way, the multivariate series behaves more nicely than the single variable Hilbert series. This multi-variate series then allows us to more easily compute the Hilbert series of $\mathcal{SD}^{\le k}_n$.

This method generalizes to the rank k symmetric determinantal variety. Again, we write $HS^n_{\mathbf{q}}\langle 2\omega_1,\ldots,2\omega_k\rangle$ when considering the weights as weights of $SL(n,\mathbb{C})$. We have the following.

Proposition 5. *Let $n>k+1$. Then*

$$HS^n_{\boldsymbol{q}}\langle 2\omega_1,\ldots,2\omega_k\rangle=\prod_{i=1}^{k}\left(1+\frac{2}{n-i}\sum_{j=i}^{k}q_j\frac{\partial}{\partial q_j}\right)HS^{n-1}_{\boldsymbol{q}}\langle 2\omega_1,\ldots,2\omega_k\rangle.$$

Proof. We have a labelling of the fundamental dominant weights and the simple roots for $SL(n,\mathbb{C})$ as $\omega_1,\ldots,\omega_{n-1}$ and $\alpha_1,\ldots,\alpha_{n-1}$, resp., such that

$$\Phi^+=\{\alpha_i+\alpha_{i+1}+\cdots+\alpha_j\mid 1\le i\le j\le n-1\},$$

and

$$c_{\omega_k}(\alpha_i+\cdots+\alpha_j)=\begin{cases}\dfrac{1}{j-i+1} & : i\le k\le j\\ 0 & : \textit{otherwise}\end{cases}$$

independent of n. Under this labelling, the positive roots of $SL(n,\mathbb{C})$ which contribute to the product but did not contribute at the $(n-1)$-st step are precisely those roots whose sum begins with α_i for some $i\le k$ and ends with α_{n-1}. There are k of these roots, namely $\alpha_1+\cdots+\alpha_{n-1},\alpha_2+\cdots+\alpha_{n-1},\ldots,\alpha_k+\cdots+\alpha_{n-1}$. Then we have

$$2c_{\omega_i}(\alpha_1+\cdots+\alpha_{n-1})=\frac{2}{n-1},\text{ for }1\le i\le k,$$
$$2c_{\omega_i}(\alpha_2+\cdots+\alpha_{n-1})=\frac{2}{n-2},\text{ for }2\le i\le k,$$
$$\cdots$$

$$2c_{\omega_k}(\alpha_k + \cdots + \alpha_{n-1}) = \frac{2}{n-k}.$$

This proves the Proposition. ■

There is a similar story for the Hilbert series of the standard embedding of $\mathcal{AD}_n^{\leq 2k}$. The Second Fundamental Theorem of Invariant Theory for $Sp(2n, \mathbb{C})$ (see p. 562 in [5]) states that the homogeneous coordinate ring of $\mathcal{AD}_n^{\leq 2k}$ decomposes as an $SL(2n, \mathbb{C})$-module as

$$\mathbb{C}[\mathcal{AD}_n^{\leq 2k}] \cong \sum_{\lambda} L(\lambda),$$

where λ runs over the lattice cone $\langle \omega_2, \omega_4, \ldots, \omega_{2k} \rangle$. Then $HS_{\mathbf{q}}\langle \omega_2, \omega_4, \ldots, \omega_{2k} \rangle$ can again be specialized to the standard Hilbert series given by

$$\sum_{\lambda} \dim(L(\lambda)) q^{|\lambda|},$$

where λ runs over $\langle \omega_2, \omega_4, \ldots, \omega_{2k} \rangle$, by making the substitution $q_i \mapsto q^i$. We compute the series $HS_{\mathbf{q}}\langle \omega_2, \ldots, \omega_{2k} \rangle$ and specialize the variables in the appropriate way to recover the Hilbert series of the standard embedding of the antisymmetric determinantal variety.

As with the symmetric determinantal variety, we start with the simplest case with a multi-gradation. Consider $\mathcal{AD}_n^4$, where $n \geq 6$. In this case, we wish to compute $HS_{\mathbf{q}}^n\langle \omega_2, \omega_4 \rangle$, where again, the superscript refers to the fact that we are considering ω_2 and ω_4 as fundamental dominant weights of the group SL_n. Let $n = 6$. There are twelve differential operators in the formula in Theorem 4.2.1 which are nonzero in this case. These differential operators correspond to positive root strings $\alpha_i + \cdots + \alpha_j$, where α_2 and/or α_4 show up somewhere in the string.

As we increase n, we add four new differential operators at each step. To see this, note that as n increases to $n+1$, the new root strings which contribute at least one nonzero differential operator are those which contain a copy of α_2 or α_4 and end in n. These new root strings are of the form $\alpha_i + \cdots + \alpha_n$ for $i = 1, 2, 3, 4$. There are obviously four such root strings. We then have the following recursive relation.

Lemma 2. *Let $n > 6$. Then*

$$HS_{\boldsymbol{q}}^n\langle \omega_2, \omega_4 \rangle = D.HS_{\boldsymbol{q}}^{n-1}\langle \omega_2, \omega_4 \rangle,$$

where D is the partial differential operator

$$(1 + \tfrac{1}{n-1} q_1 \tfrac{\partial}{\partial q_1} + \tfrac{1}{n-1} q_2 \tfrac{\partial}{\partial q_2})(1 + \tfrac{1}{n-2} q_1 \tfrac{\partial}{\partial q_1} + \tfrac{1}{n-2} q_2 \tfrac{\partial}{\partial q_2})(1 + \tfrac{1}{n-3} q_2 \tfrac{\partial}{\partial q_2})(1 + \tfrac{1}{n-4} q_2 \tfrac{\partial}{\partial q_2}).$$

This recursion is obtained by computing the coefficients of the four new differential operators obtained by increasing the size of the matrices from $n-1 \times n-1$ to $n \times n$. Note that even though the recursion is significantly more complicated than in the symmetric case, we still add a fixed amount of differential operators at each step, and the actual computational complexity does not really increase. In a similar fashion, we can obtain a recursive relationship for the multivariate Hilbert series of $\mathcal{AD}_n^{\leq 2k}$.

5. Some combinatorial results on $HS_{\mathbf{q}}\langle\lambda_1,\ldots,\lambda_k\rangle$.

We conclude this paper with a brief study on the behavior of the closed form presented in the main theorem. Our goal is to prove the following. To simplify terminology, we equate the multivariate series $HS_{\mathbf{q}}\langle\lambda_1,\ldots,\lambda_k\rangle$ with the closed form presented in the main theorem.

Theorem 3. *Let $\lambda_1,\ldots,\lambda_k$ be dominant integral weights for a semisimple, simply connected linear algebraic group G. Then, for $k>1$, the numerator of the multivariate series*

$$HS_{\boldsymbol{q}}\langle\lambda_1,\ldots,\lambda_k\rangle$$

has coefficient sum zero.

Since the numerator of $HS_{\mathbf{q}}\langle\lambda_1,\ldots,\lambda_k\rangle$ is never zero, this theorem implies that the multivariate Hilbert series converges to a rational function whose numerator contains both positive and negative values. As long as $k>1$, then the numerator will never be a positive integer polynomial.

Note that this is not true when $k=1$. For example, let $G=SL(3,\mathbb{C})$, and consider $HS_q\langle\omega_1\rangle$. Using the formula in the main theorem (or even the formula in [7]), we find that

$$HS_q\langle\omega_1\rangle=\frac{1}{(1-q)^2},$$

whose numerator does not have coefficient sum zero.

We prove the theorem in steps. We begin by proving something general.

Lemma 3. *Let $n_1,n_2,\ldots,n_k>0$ be integers. Assume*

$$H(\boldsymbol{q})=\frac{N(\boldsymbol{q})}{(1-q_1)^{n_1}(1-q_2)^{n_2}\ldots(1-q_k)^{n_k}},$$

where $N(q)$ is a polynomial with coefficient sum equal to zero and $H(\boldsymbol{q})$ is as reduced as possible. Then the numerator of $q_i\dfrac{\partial}{\partial q_i}H(\boldsymbol{q})$ also has coefficient sum zero.

Proof. By the quotient rule,

$$\begin{aligned}q_i\frac{\partial}{\partial q_i}H(\mathbf{q})&=\frac{q_i[\prod(1-q_j)^{n_j}\frac{\partial N}{\partial q_i}+Nn_i(1-q_1)^{n_1}\ldots(1-q_i)^{n_i-1}\ldots(1-q_k)^{n_k}]}{\prod(1-q_j)^{2n_j}}\\&=\frac{q_i(1-q_1)^{n_1}\ldots(1-q_i)^{n_i-1}\ldots(1-q_k)^{n_k}[\frac{\partial N}{\partial q_i}(1-q_i)+n_iN]}{\prod(1-q_j)^{n_j}}\\&=\frac{q_i[(1-q_i)\frac{\partial N}{\partial q_i}+n_iN]}{(1-q_1)^{n_1}\ldots(1-q)_i^{n_i+1}\ldots(1-q_k)^{n_k}}.\end{aligned}$$

Note that, since $(1-q_i)$ does not divide N, the final copy of $(1-q_i)$ in the numerator cannot be cancelled. Then, since both N and $(1-q_i)$ have coefficient sum zero, the sum of the coefficients in the numerator is zero. ■

What we have shown here is that the partial differential operators $q_i\dfrac{\partial}{\partial q_i}$ preserve the property that a rational function will have a numerator with coefficient sum equal to zero. If the numerator of a rational function has coefficient sum zero, we will simply say that the rational function itself has coefficient sum zero. Note that this implies immediately that the operators

$$\left(1 + c_1 q_1 \frac{\partial}{\partial q_1} + \cdots + c_k q_k \frac{\partial}{\partial q_k}\right),$$

where the c_i are constants, also preserve the property that the coefficient sum of the numerator is zero.

Lemma 4. *Assume* $H(q,r) = \dfrac{1}{(1-q)(1-r)}$*, and let* a, b *be nonzero constants. Then*

$$\left(1 + aq\frac{\partial}{\partial q} + br\frac{\partial}{\partial r}\right) H(q,r)$$

has coefficient sum zero.

Proof. A simple computation yields

$$\left(1 + aq\frac{\partial}{\partial q} + br\frac{\partial}{\partial r}\right) H(q,r) = \frac{(1-q)(1-r) + aq(1-r) + br(1-q)}{(1-q)^2(1-r)^2},$$

which has coefficient sum zero. ■

Note that the above Lemma 4 can be used as the base case for an induction argument to prove the following.

Lemma 5. *Assume* $H(\boldsymbol{q}) = \dfrac{1}{(1-q_1)\dots(1-q_k)}$*, for* $k > 1$*. Let* $c_1, \dots, c_k$ *be nonzero constants. Then the rational function*

$$\left(1 + c_1 q_1 \frac{\partial}{\partial q_1} + \cdots + c_k q_k \frac{\partial}{\partial q_k}\right) H(\boldsymbol{q})$$

has coefficient sum zero.

We can apply Lemma 5 to $HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle$ in the following way.

Proposition 6. *Let* G *be a semisimple, simply connected linear algebraic group over* $\mathbb{C}$ *of rank* l*. Let* $\alpha = \alpha_1 + \cdots + \alpha_l$*, where the* α_i *are a choice of simple roots for* $\mathfrak{g}$*. Assume* $\lambda_1, \dots, \lambda_k$ *be dominant integral weights of* $\mathfrak{g}$*. Then, for* $k > 1$*,*

$$\left(1 + c_{\lambda_1}(\alpha) q_1 \frac{\partial}{\partial q_1} + \cdots + c_{\lambda_k}(\alpha) q_k \frac{\partial}{\partial q_k}\right) \frac{1}{(1-q_1)\dots(1-q_k)}$$

is a rational function with coefficient sum zero.

Proof. Since $\lambda_1, \dots, \lambda_k$ are dominant integral and $\alpha = \alpha_1 + \cdots + \alpha_l$, the coefficients $c_{\lambda_i}(\alpha)$ are all nonzero constants. Then the claim follows immediately from Lemma 5. ■

We can now prove the Theorem in this section.

Proof. The main theorem states that $HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle$ has the following rational form:

$$\prod_{\alpha\in\Phi^+}\left(1+c_{\lambda_1}(\alpha)q_1\frac{\partial}{\partial q_1}+\cdots+c_{\lambda_k}(\alpha)q_k\frac{\partial}{\partial q_k}\right)\frac{1}{(1-q_1)\dots(1-q_k)}.$$

Proposition 6 provides us with a positive root α such that each of the $c_{\lambda_i}(\alpha)$ is nonzero. Therefore, after applying the operator corresponding to this root, we have a rational function with coefficient sum zero. Note that the operators in the product commute. Therefore, we may as well apply this differential operator first.

By the discussion after Lemma 3, all of the other operators preserve the property that the coefficient sum of the numerator is zero. Therefore, the coefficient sum of $HS_{\mathbf{q}}\langle\lambda_1,\dots,\lambda_k\rangle$ is zero. ∎

References

[1] Bump, D., "Algebraic Geometry," World Scientific Publishing Co. Inc., 1998.

[2] Cohen, A., and G. C. M. Ruitenberg, "Generating functions and Lie groups," CWI Tract, 1990.

[3] Enright, T., and M. Hunziker, *Resolutions and Hilbert series of determinantal varieties and unitary highest weight modules*, J. of Algebra **273** (2004), 608–639.

[4] Enright, T., M. Hunziker, and W. A. Pruett, *Diagrams of Hermitian type, highest weight modules, and syzygies of determinantal varieties*, in: "Symmetry: Representation Theory and Its Applications: In Honor of Nolan R. Wallach", Birkhäuser, Boston, 2015, 125–188.

[5] Goodman, R., and N. R. Wallach, "Symmetry, representations, and invariants", Springer, Berlin etc., 2009.

[6] Gorodentsev, A. L., A. S. Khoroshkin, and A. N. Rudakov, *On syzygies of highest weight orbits*, "Moscow Seminar on Mathematical Physics II", Amer. Math. Soc., Providence, RI, 2007.

[7] Gross, B. H., and N. R. Wallach, *On the Hilbert polynomials and Hilbert series of homogeneous projective varieties*, in: "Arithmetic geometry and automorphic forms", Int. Press, 2011, 253–263.

[8] Harris, J., "Algebraic Geometry," Springer, Berlin etc., 1995.

[9] Kumar, S., "Kac-Moody groups, their flag varieties and representation theory", Birkhäuser, Boston, 2002.

[10] Lancaster, G., and J. Towber, *Representation-functors and flag-algebras for the classical groups. I*, J. of Algebra **59** (1979), 16–38.

[11] Lichtenstein, W., *A system of quadrics describing the orbit of the highest weight vector*, Proc. Amer. Math, Soc. **84** (1982), 605–608.

[12] Vinberg, È. B., and V. L. Popov, *A certain class of quasihomogeneous affine varieties*, Izv. Akad. Nauk SSSR Ser. Mat. (Russian) **36** (1972), 749–764.

[13] Wallach, N. R., "Geometric invariant theory over the real and complex numbers," forthcoming.

[14] Willenbring, J. F., and G. J. Zuckerman, *Small semisimple subalgebras of semisimple Lie algebras*, in: "Harmonic analysis, group representations, automorphic forms and invariant theory," World Sci. Publ., 2007, 403–429.

Wayne A. Johnson
Department of Mathematics
Truman State University
100 East Normal Avenue
Kirksville, MO 63501, USA
wjohnson@truman.edu

Received July 22, 2016
and in final form August 4, 2016