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Cohomological Rigidity of the Schrödinger Algebra $S(N)$ and its Central Extension $\widehat{S}(N)$

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Abstract. It is shown that for any $N \neq 2$, the Schrödinger algebra $S(N)$ and its central extension $\widehat{S}(N)$ are cohomologically rigid Lie algebras, i.e., have a vanishing second Chevalley cohomology group with values in the adjoint representation. Further, it is shown that the main cohomological difference between these algebras lies in the structure of the third cohomology space.
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1. Introduction

The deformation theory of algebraic structures, initiated by Gerstenhaber in the mid 1960s [11], soon appeared to be a powerful tool not only in the algebraic sense, but also a valuable technique to analyze the geometrical and topological properties of algebras [19]. In the context of Lie algebras, this approach turned out to be deeply connected with cohomology and representation theory of (semisimple) Lie algebras [13, 20], as well as with the notion of contractions of Lie algebras introduced earlier by Inönü and Wigner [14]. For physical applications in particular, the concept of structural stability served as a measure to test a model with respect to some small changes of the fundamental parameters. The first symmetry algebras to be studied from this point of view, the classical kinematical algebras [16], constituted a strong motivation to study the stability properties of (symmetry) algebras in various geometric applications (see e.g. [3, 10, 27] and references therein), both from the perspective of deformations and contractions of Lie algebras [9, 7, 17].

In this work, the Chevalley cohomology of the (centrally extended) Schrödinger algebras $\widehat{S}(N) = (\mathfrak{so}\,(N) \oplus \mathfrak{sl}\,(2,\mathbb{R}))\,\overrightarrow{\oplus}_{(D_{\frac{1}{2}}\otimes\Lambda)\oplus D_0}\mathfrak{h}_N$ in $(N+1)$ dimensions with values in the adjoint representation are determined, proving that for $N \neq 2$, the $\widehat{S}(N)$ algebras are rigid, i.e., satisfy the Nijenhuis-Richardson rigidity con-

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dition $H^2\left(\widehat{S}\left(N\right),\widehat{S}\left(N\right)\right)=0$. It is further shown that the third cohomology group $H^3\left(\widehat{S}\left(N\right),\widehat{S}\left(N\right)\right)$ does not vanish and is completely determined by the cohomology space $H^3\left(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(N),\mathbb{R}\right)$. The computation of the cohomology can be adapted easily to the case of the usual Schrödinger algebra $S(N)=\widehat{S}(N)/Z\left(\widehat{S}(N)\right)$, resulting in the cohomological rigididity of the latter for $N\neq 2$. The main cohomological difference between these Lie algebras is shown to be in the characterization of the third cohomology $H^3\left(S\left(N\right),S\left(N\right)\right)$.

2. Cohomology and deformation of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra and V a representation of $\mathfrak{g}$. Let $C^p\left(\mathfrak{g},V\right)=Hom(\bigwedge^p\mathfrak{g},V)$ denote the soace of p-cochains and $d_p:C^p\left(\mathfrak{g},V\right)\longrightarrow C^{p+1}\left(\mathfrak{g},V\right)$ the coboundary operator, where

$$d_p\Phi\left(X_1,...,X_{p+1}\right)=\sum_{1\leq s\leq p+1}\left(-1\right)^{s+1}\left(X_s.\Phi\right)\left(X_1,...,\widehat{X}_s,...,X_{p+1}\right)+ \\ +\sum_{1\leq s\leq t\leq p+1}\left(-1\right)^{s+t}\Phi\left(\left[X_s,X_t\right],X_1,...,\widehat{X}_s,...,\widehat{X}_{t,...},X_{p+1}\right) \quad (1)$$

If $Z^p\left(\mathfrak{g},V\right)=\ker d_p$ is the space of p-cocycles and $B^p\left(\mathfrak{g},V\right)=\mathrm{Im}d_p$ that of coboundaries, the p^{th}-cohomology space with values in V is given by

$$H^p\left(\mathfrak{g},V\right)=Z^p\left(\mathfrak{g},V\right)/B^p\left(\mathfrak{g},V\right). \quad (2)$$

For the case of Lie algebras having a non-trivial Levi decomposition, there exists a useful reduction, called the Hochschild-Serre factorization theorem [13]. If $\mathfrak{g}$ has the Levi decomposition $\mathfrak{g}=\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$, where $\mathfrak{s}$ denotes the Levi subalgebra and $\mathfrak{r}$ the radical of $\mathfrak{g}$, then the adjoint cohomology $H^p(\mathfrak{g},\mathfrak{g})$ admits the following decomposition:

$$H^p\left(\mathfrak{g},\mathfrak{g}\right)\simeq\sum_{i+j=p}H^i\left(\mathfrak{g},\mathbb{F}\right)\otimes H^j\left(\mathfrak{r},\mathfrak{g}\right)^{\mathfrak{g}}, \quad (3)$$

where $\mathbb{F}$ denotes the base field and $H^j\left(\mathfrak{r},\mathfrak{g}\right)^{\mathfrak{g}}$ is the space of $\mathfrak{g}$-invariant cocycles. A direct proof of the Hochschild-Serre factorization theorem, as well as some additional properties, can be found [13, 15].

2.1. Deformations of Lie algebras.

If $\mathcal{L}^n$ denotes the (algebraic) set consisting of all Lie algebras of dimension n over the vector space V, the general linear group $GL(n,V)$ acts naturally on $\mathcal{L}^n$ by

$$f\star\mu=f^{-1}\left(\mu\left(f,f\right)\right),\quad f\in GL(n,V). \quad (4)$$

It is clear from this action that the orbit $\mathcal{O}(\mu)$ corresponding to the structure tensor μ of a Lie algebra $\mathfrak{g}$ is nothing but the isomorphism class of $\mathfrak{g}$:

$$\mathcal{O}(\mu)=\{\mu'\mid\mathfrak{g}=(V,\mu)\simeq\mathfrak{g}'=(V,\mu')\} \quad (5)$$

Definition 2.1. A Lie algebra $\mathfrak{g}=(V,\mu)$ is rigid (or stable) if the orbit $\mathcal{O}(\mu)$ is an open set (with respect to the Euclidean topology).

The analysis of the topological definition of rigidity or stability led to the famous algebraic rigidity criterion of Nijenhuis and Richardson [19]:

Theorem 2.2. *If a Lie algebra* $\mathfrak{g} = (V, \mu)$ *satisfies* $H^2(\mathfrak{g}, \mathfrak{g}) = 0$*, then it is rigid.*

As pointed out in [24], the vanishing of the second cohomology group is a sufficient, but not necessary condition for the stability of a Lie algebra. Rigid algebras with vanishing second cohomology are sometimes called cohomologically rigid (or stable). The usual method to analyze rigidity is based on deformation theory, as developed by Gerstenhaber [11]. In this context, we only recall that for a formal one-parameter deformation $\mathfrak{g}_t$ of a Lie algebra $\mathfrak{g} = (V, \mu)$

$$[X, Y]_t := [X, Y] + \psi_m(X, Y)t^m, \tag{6}$$

the quadratic Rim map $\mathrm{Sq} : H^2(\mathfrak{g}, \mathfrak{g}) \to H^3(\mathfrak{g}, \mathfrak{g})$ defined by (see e.g. [8, 23])

$$\mathrm{Sq}(\psi_1)(X_i, X_j, X_k) := \psi_1(\psi_1(X_i, X_j), X_k) + \psi_1(\psi_1(X_j, X_k), X_i) + \psi_1(\psi_1(X_k, X_i), X_j)$$

provides information on the integrability of the deformation.[1] A Lie algebra $\mathfrak{g}$ is rigid if any infinitesimal deformation of type (6) is isomorphic to $\mathfrak{g}$, encompassing both the cohomological and topological rigidity.

The cohomology of semidirect sums provides us with some useful criteria to study their deformations. The central result concerning the structure of such algebras is the Page-Richardson stability theorem [20]. This result essentially establishes that if the Lie algebra $\mathfrak{g}$ possesses a semisimple subalgebra $\mathfrak{s}$, then its deformations will always have some subalgebra isomorphic to $\mathfrak{s}$, and the action of $\mathfrak{s}$ on the remaining generators is preserved. Combining this with the Hochschild-Serre factorization theorem, it follows that deformations are determined by alterations of the commutators in the radical $\mathfrak{r}$ of $\mathfrak{g}$, corresponding to one of the following possibilities:

1. $\mathfrak{s}$ is a maximal semisimple subalgebra of $\mathfrak{s}'$, and either $\mathfrak{g}_t$ is isomorphic to $\mathfrak{s}'$, or there exists a solvable Lie algebra $\mathfrak{r}'$ such that $\mathfrak{g}_t \simeq \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}'$.
2. $\mathfrak{s}$ is not maximal semisimple subalgebra of $\mathfrak{s}'$. Then, a non-semisimple deformation $\mathfrak{g}_t$ is either isomorphic to $\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}'$ with $\mathfrak{r}'$ solvable, or there exists a semisimple subalgebra $\mathfrak{s}_1$ of $\mathfrak{s}'$ and a representation R_1 of $\mathfrak{s}_1$ such that $\mathfrak{g}_t \simeq \mathfrak{s}_1 \overrightarrow{\oplus}_{R_1} \mathfrak{r}'$ for some solvable Lie algebra $\mathfrak{r}'$.

We finally observe that, to a certain extent, deformations can be seen as an inverse procedure to that of contractions [16]. Let $\mathfrak{g}$ be a Lie algebra and $\Phi_t \in End(\mathfrak{g})$ a family of non-singular linear maps of $\mathfrak{g}$, where $t \in [1, \infty)$. For any $X, Y \in \mathfrak{g}$, the bracket over the transformed basis has the form

$$[X, Y]_{\Phi_t} := \Phi_t^{-1} [\Phi_t(X), \Phi_t(Y)], \tag{7}$$

[1] It should be mentioned that in the case of solvable Lie algebras, the latter does not necessarily provide information about obstructions to integrability.

If the limit

$$[X,Y]_\infty := \lim_{t\to\infty} \Phi_t^{-1}\left[\Phi_t(X),\Phi_t(Y)\right] \tag{8}$$

exists for any $X, Y \in \mathfrak{g}$, then equation (8) defines a Lie algebra $\mathfrak{g}'$ called the contraction of $\mathfrak{g}$. If there exists a basis $\{Y_1, .., Y_n\}$ such that the contraction matrix A_Φ has the form

$$(A_\Phi)_{ij} = \delta_{ij} t^{n_j}, \quad n_j \in \mathbb{R}, t > 0,$$

then we speak of a generalized Inönü-Wigner contraction [28].

3. The extended Schrödinger algebra

The extended Schrödinger algebra $\widehat{S}(N)$ was first considered in [6, 12, 18] as the invariance algebra of the Schrödinger equation in (N+1)-dimensional space time, and since then it has attracted considerable interest in applications (see e.g. [5]). Formally, the centrally extended Schrödinger algebra $\widehat{S}(N)$ in $(N+1)$-dimensions has dimension $\frac{1}{2}\left(N^2+3N+8\right)$ and non-trivial commutators

$$\begin{array}{ll}
[J_{\mu\nu}, J_{\lambda\sigma}] = \delta_{\mu\lambda}J_{\nu\sigma} + \delta_{\nu\sigma}J_{\mu\lambda} - \delta_{\mu\sigma}J_{\nu\lambda} - \delta_{\nu\lambda}J_{\mu\sigma}, & \\
{[J_{\mu\nu}, P_\lambda]} = \delta_{\mu\lambda}P_\nu - \delta_{\nu\lambda}P_\mu, & [J_{\mu\nu}, G_\lambda] = \delta_{\mu\lambda}G_\nu - \delta_{\nu\lambda}G_\mu, \\
{[P_t, G_\mu]} = P_\mu; & [K, P_\mu] = -G_\mu, \\
{[D, G_\mu]} = G_\mu, & [D, P_\mu] = -P_\mu, \\
{[D, K]} = 2K, & [D, P_t] = -2P_t, \\
{[K, P_t]} = -D. & [P_\mu, G_\nu] = \delta_{\mu\nu}M
\end{array} \tag{9}$$

over the basis $\{J_{ij}, P_k, G_k, K, D, P_t, M\}$, where $J_{\mu\nu} + J_{\nu\mu} = 0$ are rotations, P_μ spatial translation generators, P_t the time translation, G_μ special Galilei transformations, D the generator of scale transformations, K the generator of Galilean conformal transformations and M is seen as a central "charge" [12, 18]. From (9) it is not difficult to see that the Levi decomposition of $\widehat{S}(N)$ for $N \geq 3$ is given by

$$\widehat{S}(N) = \left(\mathfrak{so}(N) \oplus \mathfrak{sl}(2,\mathbb{R})\right) \overrightarrow{\oplus}_{\left(D_{\frac{1}{2}}\otimes\Lambda\right)\oplus D_0} \mathfrak{h}_N, \tag{10}$$

where $\mathfrak{h}_N$ is the $(2N+1)$-dimensional Heisenberg algebra, $D_{\frac{1}{2}}$ is the standard representation of $\mathfrak{sl}(2,\mathbb{R})$ and Λ the standard representation of $\mathfrak{so}(N)$, respectively [5]. D_0 denotes the trivial representation. For the special value $N = 1$ the Levi decomposition reduces to

$$\widehat{S}(1) = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{D_{\frac{1}{2}}\oplus D_0} \mathfrak{h}_1, \tag{11}$$

while for $N = 2$ we have[2]

$$\widehat{S}(2) = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{D_{\frac{1}{2}}\oplus 2D_0} \mathfrak{g}_{6,82}. \tag{12}$$

3.1. The special case $\widehat{S}(2)$.

[2] $\widehat{S}(2)$ coincides with the Lie algebra $L_{9,52}$ as listed in the classification of [25].

As said before, the case $N = 2$ in dimension 9 plays a special role among the centrally extended Schrödinger algebras, as the radical is not isomorphic to the Heisenberg algebra, but to a solvable Lie algebra containing $\mathfrak{h}_2$ as maximal nilpotent ideal.

Lemma 3.1. *The Lie algebra $\widehat{S}(2)$ is not rigid. For $k = 1, 2, 3$, the following identity holds*

$$\dim H^k\left(\widehat{S}(2), \widehat{S}(2)\right) = 2. \tag{13}$$

Moreover, the quadratic Rim map $Sq{:}H^2\left(\widehat{S}(2), \widehat{S}(2)\right) \to H^3\left(\widehat{S}(2), \widehat{S}(2)\right)$ is zero.

Proof. The result is proved by direct computation. For $k = 1$, it can be easily shown that the only outer derivations are given by

$$\begin{array}{l} f_1(J_{12}) = M, \\ f_2(P_i) = P_i, \quad f_2(G_i) = G_i, \quad 1 \leq i \leq 2; \quad f_2(M) = 2M \end{array} \tag{14}$$

The cohomology space $H^2\left(\widehat{S}(2), \widehat{S}(2)\right)$ is generated by the cocycles

$$\begin{array}{llll} \varphi(P_1, G_2) = -D, & \varphi(G_1, P_2) = D & \varphi(P_1, P_2) = 2P_t, & \varphi(G_1, G_2) = 2K, \\ \varphi(P_1, G_1) = -3J_{12} & \varphi(P_2, G_2) = -3J_{12}. & & \\ \psi(J_{12}, P_1) = P_1, & \psi(J_{12}, P_2) = P_2, & \psi(J_{12}, G_1) = G_1, & \psi(J_{12}, G_2) = G_2, \\ \psi(J_{12}, M) = 2M. & & & \end{array} \tag{15}$$

In both cases it is immediate to verify that

$$\mathrm{Sq}(\varphi) = \mathrm{Sq}(\psi) = 0,$$

thus there is no obstruction to integrability and $\widehat{S}(2)$ is not a rigid Lie algebra. Finally, after some computation, a basis for $H^3\left(\widehat{S}(2), \widehat{S}(2)\right)$ can be chosen with the following generators

$$\begin{array}{lll} \xi(J_{12}, P_1, G_2) = -D, & \xi(J_{12}, G_1, P_2) = D & \xi(J_{12}, P_1, P_2) = 2P_t, \\ \xi(J_{12}, G_1, G_2) = 2K, & \xi(J_{12}, P_1, G_1) = -3J_{12} & \xi(J_{12}, P_2, G_2) = -3J_{12}. \\ \chi(J_{12}, K, P_t) = M. & & \end{array}$$

■

Since $\widehat{S}(2)$ is not rigid, we determine the Lie algebras that arise as its deformation.

Proposition 3.2. *For $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$, the deformation $\widehat{S}(2) + \varepsilon_1\varphi + \varepsilon_2\psi$ satisfies the Jacobi identity if and only if*

$$\varepsilon_1\varepsilon_2 = 0.$$

Moreover,

1. $\widehat{S}(2) + \varepsilon_1\varphi \simeq \mathfrak{su}(2,1) \oplus \mathbb{R}$ *for any* ε_1,
2. $\widehat{S}(2) + \varepsilon_2\psi \simeq \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{2D_{\frac{1}{2}} \oplus 2D_0} \mathfrak{g}_{6,13} = L^1_{9,46}$.

Proof. It is trivial to verify that one of the deformation parameters must vanish if $\widehat{S}\,(2)+\varepsilon_1\varphi+\varepsilon_2\psi$ satisfies the Jacobi identity. Taking $\varepsilon_1=0$ and $\varepsilon_2=1$, it follows at once that the radical of $\widehat{S}\,(2)+\psi$ is a solvable Lie algebra isomorphic to $\mathfrak{g}_{6,13}$ as classified in [26]. The semidirect product coincides with the nine dimensional Lie algebra $L^1_{9,46}$ listed in [25].
On the other hand, for $\varepsilon_1\neq 0$ and $\varepsilon_2=0$, the Killing form κ of the deformation $\widehat{S}\,(2)+\varepsilon_1\varphi$ has eigenvalues

$$\mathrm{Spec}\,(\kappa)=\{0,-4,12,\pm 2,\pm 6\varepsilon_1,\pm 12\varepsilon_1\}$$

and thus signature $\sigma=0$. According to the classification of real forms of simple Lie algebras [21], the Levi subalgebra $\mathfrak{s}$ of $\widehat{S}\,(2)+\varepsilon_1\varphi$ is isomorphic to the non-compact algebra $\mathfrak{su}\,(2,1)$, and we obtain the reductive deformation $\widehat{S}\,(2)+\varepsilon_1\varphi\simeq\mathfrak{su}\,(2,1)\oplus\mathbb{R}$. ■

Remark 3.3. The reason for the non-rigidity of $\widehat{S}\,(2)$ lies ultimately in the fact that the rotation generator J_{12} does not belong to the Levi subalgebra of $\widehat{S}\,(2)$, thus preventing invariant $\mathfrak{sl}\,(2,\mathbb{R})$-cocycles from appearing as a coboundary.

4. Cohomology of $\widehat{S}(N)$ for $N\geq 3$

In this section, using the Page-Richardson and the Hochschild-Serre factorization theorems, we prove that the extended Schrödinger algebras $\widehat{S}(N)$ are cohomologically rigid, and that the third cohomology group with coefficients in the adjoint module is always non-zero. We can restrict ourselves to compute the spaces of cocycles of $\mathfrak{h}_N$ with values in $\widehat{S}(N)$ that are invariant by the Levi subalgebra of the latter. In order to simplify notations, in the following, we denote the Levi subalgebra $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(N)$ of $\widehat{S}(N)$ by $\mathfrak{s}$.

4.1. Invariant 1-cochains.

A generic 1-cochain of $\mathfrak{h}_N$ with values in $\widehat{S}(N)$ has the shape

$$\begin{aligned}
\psi\,(P_\mu) &= a^{\lambda\sigma}_{1\mu}\,J_{\lambda\sigma}+b^1_{1\mu}D+b^2_{1\mu}K+b^3_{1\mu}P_t+c^\nu_{1\mu}P_v+d^\nu_{1\mu}G_\nu+f_1M,\\
\psi\,(G_\mu) &= a^{\lambda\sigma}_{2\mu}\,J_{\lambda\sigma}+b^1_{2\mu}D+b^2_{2\mu}K+b^3_{2\mu}P_t+c^\nu_{2\mu}P_v+d^\nu_{2\mu}G_\nu+f_2M,\\
\psi\,(M) &= a^{\lambda\sigma}_{3}\,J_{\lambda\sigma}+b^1_{3}D+b^2_{3}K+b^3_{3}P_t+c^\nu_{3}P_v+d^\nu_{3}G_\nu+f_3M.
\end{aligned}\tag{16}$$

For any $Z\in\mathfrak{h}_N$, the invariance of ψ with respect to D leads to the identity

$$(D.\psi)\,(Z)=[D,\,\psi\,(Z)]-\psi\,([D,Z])=0.\tag{17}$$

As D acts diagonally on the radical, $[D,Z]=\lambda_Z\,Z$, we deduce from (17) that the only terms of (16) that can have nonzero coefficients are those having the same eigenvalue λ_Z as Z with respect to D. Hence a D-invariant cocycle ψ must have the shape

$$\begin{aligned}
\psi\,(P_\mu)=c^\nu_{1\mu}P_v,\quad \psi\,(G_\mu)=d^\nu_{2\mu}G_\nu,\\
\psi\,(M)=a^{\lambda\sigma}_{3}\,J_{\lambda\sigma}+b^1_3D+f_3M.
\end{aligned}\tag{18}$$

Imposing now invariance with respect to K, we get the constraints

$$\begin{aligned}(K.\psi)(P_\mu) &= [K,\psi(P_\mu)] + \psi(G_\mu) = \left(-c_{1\mu}^{\nu} + d_{2\mu}^{\nu}\right) G_\nu = 0, &(19)\\ (K.\psi)(M) &= [K,\psi(M)] = -2b_3^1 K = 0, &(20)\end{aligned}$$

thus $c_{1\mu}^{\nu} = d_{2\mu}^{\nu}$ for all $1 \leq \mu,\nu \leq N$ and $b_3^1 = 0$. Checking the invariance with respect to P_t gives no new relations. It remains to determine the restrictions imposed by the invariance with respect to the orthogonal subalgebra $\mathfrak{so}(N)$. To this extent, let $1 \leq \alpha < \beta \leq N$ and consider

$$(J_{\alpha\beta}.\psi)(P_\alpha) = [J_{\alpha\beta},\psi(P_\alpha)] - \psi(P_\beta) = c_{1\alpha}^{\nu}\left(\delta_\alpha^\nu P_\beta - \delta_\beta^\nu P_\alpha\right) - c_{1\beta}^{\nu} P_\nu = 0. \quad (21)$$

This implies that for any $\nu \neq \alpha,\beta$ we have $c_{1\beta}^{\nu} = 0$, while $c_{1\alpha}^{\alpha} = c_{1\beta}^{\beta}$ and $c_{1\alpha}^{\beta} = -c_{1\beta}^{\alpha}$. However, if $\mu \neq \alpha,\beta$, the identity

$$(J_{\alpha\beta}.\psi)(P_\mu) = [J_{\alpha\beta},\psi(P_\mu)] = 0 \quad (22)$$

shows that $c_{1\mu}^{\alpha} = c_{1\mu}^{\beta} = 0$ and $c_{1\mu}^{\mu} = c_0$ for all μ. As the indices α,β,μ go through $\{1,\cdots,N\}$, we conclude that

$$\psi(P_\mu) = c_0 P_\mu, \quad \psi(G_\mu) = c_0 G_\mu, \quad 1 \leq \mu \leq N. \quad (23)$$

We finally check the condition

$$(J_{\alpha\beta}.\psi)(M) = [J_{\alpha\beta},\psi(M)] = a_3^{\lambda\sigma}[J_{\alpha\beta},J_{\lambda\sigma}] = 0. \quad (24)$$

Since $\mathfrak{so}(N)$ is a simple Lie algebra, for any pair (λ,σ) there exists at least one another pair (α,β) such that $[J_{\alpha\beta},J_{\lambda\sigma}] \neq 0$ [21]. Going through all indices $1 \leq \alpha,\beta,\lambda,\sigma \leq N$, it follows at once that $a_3^{\lambda\sigma} = 0$ for all pairs (λ,σ). As a consequence, an invariant 1-cochain has the form

$$\psi(P_\mu) = c_0 P_\mu, \quad \psi(G_\mu) = c_0 G_\mu, \quad \psi(M) = f_3 M, \quad 1 \leq \mu \leq N, \quad (25)$$

and we conclude that $\dim C^1\left(\mathfrak{h}_N, \widehat{S}(N)\right)^{\mathfrak{s}} = 2$. Computing the coboundary operator $d\psi$, it is straightforward to verify that for such a cochain we get

$$\begin{aligned}d\psi(P_\mu,P_\nu) &= d\psi(G_\mu,G_\nu) = d\psi(P_\mu,M) = d\psi(G_\mu,M) = 0,\\ d\psi(P_\mu,G_\nu) &= \delta_\mu^\nu\left(c_{1\mu}^{\mu} + c_{1\nu}^{\nu} - f_3\right) M. &(26)\end{aligned}$$

This equation shows that the space $Z^1\left(\mathfrak{h}_N, \widehat{S}(N)\right)^{\mathfrak{s}}$ has dimension 1, generated by the cocycle ψ for the values $c_0 = 1, f_3 = 2$. In particular, the cochain ψ_0 defined by $c_0 = 1, f_3 = 0$ and given by

$$\psi_0(P_\mu) = P_\mu,\ \psi_0(G_\mu) = G_\mu,\ 1 \leq \mu \leq N \quad (27)$$

gives rise to the 2-coboundary[3]

$$d\psi_0(P_\mu,G_\nu) = \delta_\mu^\nu M. \quad (28)$$

[3]Thus the cochain defined by $c_0 = 0, f_3 = 1$ is of the same cohomology class as ψ_0.

It follows from these computations that for $N\geq 3$ $\dim H^{1}\left(\mathfrak{h}_N,\widehat{S}(N)\right)^{\mathfrak{s}}=1$.

4.2. Invariant 2-cochains.

The computation of $H^{2}\left(\mathfrak{h}_N,\widehat{S}(N)\right)^{\mathfrak{s}}$ is formally very similar to the previous case. Let $\varphi\in C^{2}\left(\mathfrak{h}_N,\widehat{S}(N)\right)^{\mathfrak{s}}$ be an $\mathfrak{s}$-invariant cochain. We first consider the invariance with respect to $D\in\mathfrak{sl}(2,\mathbb{R})$. As the latter acts diagonally on the elements of the Heisenberg algebra, taking two arbitrary elements $X_1,X_2\in\mathfrak{h}_N$ we arrive at the expression

$$(D.\varphi)(X_1,X_2)=[D,\varphi(X_1,X_2)]-\varphi([D,X_1],X_2)-\varphi(X_1,[D,X_2])=0. \quad (29)$$

As before, the identity implies that only the terms having eigenvalue the sum of the eigenvalues of X_1 and X_2 with respect to D can survive. From the commutators (9) it follows at once that an invariant 2-cochain reduces to

$$\begin{gathered}\varphi(P_\mu,P_\nu)=a_{\mu\nu}P_t,\quad \varphi(G_\mu,G_\nu)=b_{\mu\nu}K,\\ \varphi(P_\mu,G_\nu)=c_{\mu\nu}^{\lambda\sigma}J_{\lambda\sigma}+d_{\mu\nu}D+k_{\mu\nu}M,\\ \varphi(P_\mu,M)=f_\mu^\nu P_v,\quad \varphi(G_\mu,M)=g_\mu^\nu G_\nu.\end{gathered} \quad (30)$$

Further relations among the coefficients are obtained from the invariance with respect to K. Displaying only those that give nontrivial relations, we have that

$$\begin{aligned}(K.\varphi)(P_\mu,P_\nu)=(d_{\mu\nu}-d_{\nu\mu}-a_{\mu\nu})D+\left(c_{\mu\nu}^{\lambda\sigma}-c_{\nu\mu}^{\lambda\sigma}\right)J_{\lambda\sigma}+(k_{\mu\nu}-k_{\nu\mu})M=0,\\ (K.\varphi)(P_\mu,P_\nu)=(b_{\mu\nu}-2d_{\mu\nu})K=0,\\ (K.\varphi)(P_\mu,M)=\left(g_\mu^\nu-f_\mu^\nu\right)G_\nu=0.\end{aligned}$$

Again, invariance by P_t does not add new relations. A $\mathfrak{sl}(2,\mathbb{R})$-invariant 2-cochain has thus the form

$$\begin{gathered}\varphi(P_\mu,P_\nu)=(d_{\mu\nu}-d_{\nu\mu})P_t,\quad \varphi(G_\mu,G_\nu)\ =2d_{\mu\nu}K,\\ \varphi(P_\mu,G_\nu)=c_{\mu\nu}^{\lambda\sigma}J_{\lambda\sigma}+d_{\mu\nu}D+k_{\mu\nu}M,\\ \varphi(P_\mu,M)=g_\mu^\nu P_v,\quad \varphi(G_\mu,M)=g_\mu^\nu G_\nu,\end{gathered} \quad (31)$$

where $c_{\mu\nu}^{\lambda\sigma}=c_{\nu\mu}^{\lambda\sigma}$ and $k_{\mu\nu}=k_{\nu\mu}$. It remains to check the invariance by $\mathfrak{so}(N)$ generators. To this extent, let $1\leq\alpha<\beta\leq N$ be a fixed pair of indices. Take $s\neq\alpha,\beta$. Then

$$(J_{\alpha\beta}.\varphi)(P_\alpha,P_s)=[J_{\alpha\beta},\varphi(P_\alpha,P_s)]-\varphi(P_\alpha,P_s)=-\varphi(P_\alpha,P_s)=0. \quad (32)$$

From this equation we obtain that for all $1\leq\alpha,s\leq N$

$$d_{\alpha s}=d_{s\alpha}. \quad (33)$$

However, since φ is a skew-symmetric map, we have that $\varphi(G_\alpha,G_s)=-\varphi(G_s,G_\alpha)$, meaning that $d_{\alpha s}=-d_{s\alpha}$. The condition (33) therefore implies that

$$d_{\alpha s}=0,\ 1\leq\alpha,s\leq N. \quad (34)$$

Since α, s cover all indices, we conclude from (34) that $\varphi(P_\mu, P_\nu) = \varphi(G_\mu, G_\nu) = 0$. Now let μ, ν be indices such that $\alpha \neq \mu, \nu$ and $\beta \neq \mu, \nu$. The invariance condition $(J_{\alpha\beta}.\varphi)(P_\mu, G_\nu)$ then simplifies to

$$(J_{\alpha\beta}.\varphi)(P_\mu, G_\nu) = [J_{\alpha\beta}, \varphi(P_\mu, G_\nu)] = c_{\mu\nu}^{\lambda\sigma}[J_{\alpha\beta}, J_{\lambda\sigma}] = 0. \tag{35}$$

Using once more the simplicity of $\mathfrak{so}(N)$, for any pair (λ, σ) there exists at least one pair (α, β) such that the commutator $[J_{\alpha\beta}, J_{\lambda\sigma}]$ does not vanish. Letting the indices run through $\{1, \cdots, N\}$, a routine computation shows that

$$c_{\mu\nu}^{\lambda\sigma} = 0,\ 1 \leq \mu < \nu \leq N,\ 1 \leq \lambda < \sigma \leq N. \tag{36}$$

Taking now an index $\nu \neq \alpha, \beta$ further implies that

$$(J_{\alpha\beta}.\varphi)(P_\alpha, G_\nu) = [J_{\alpha\beta}, \varphi(P_\alpha, G_\nu)] - \varphi(P_\beta, G_\nu) = -k_{\beta\nu}M = 0, \tag{37}$$

so that $k_{\beta\nu} = 0$ for $\beta \neq \nu$. Similarly,

$$(J_{\alpha\beta}.\varphi)(P_\alpha, G_\beta) = [J_{\alpha\beta}, \varphi(P_\alpha, G_\nu)] - \varphi(P_\beta, G_\nu) + \varphi(P_\alpha, G_\alpha) = 0 \tag{38}$$

shows that $k_{11} = \cdots = k_{NN} = k_0$. Finally, evaluating the condition

$$(J_{\alpha\beta}.\varphi)(P_\mu, M) = [J_{\alpha\beta}, \varphi(P_\mu, M)] - \varphi([J_{\alpha\beta}, P_\mu], M) = 0 \tag{39}$$

we get the generic expression

$$(J_{\alpha\beta}.\varphi)(P_\mu, M) = \delta_\nu^\alpha g_\mu^\nu P_\beta - \delta_\nu^\beta g_\mu^\nu P_\alpha - \delta_\mu^\alpha g_\beta^\sigma P_\sigma + \delta_\mu^\beta g_\alpha^\omega P_\omega = 0. \tag{40}$$

Taking for example $\mu = \alpha$ (the case $\mu = \beta$ is completely analogous) the identity reduces to

$$g_\alpha^\alpha P_\beta - g_\alpha^\beta P_\alpha - g_\beta^\sigma P_\sigma = 0. \tag{41}$$

Expanding the latter we deduce that $g_\alpha^\alpha = g_\beta^\beta$, $g_\alpha^\beta + g_\beta^\alpha = 0$ and that $g_\beta^\sigma = 0$ for all $\sigma \neq \alpha, \beta$. As the indices run through the set $\{1, \cdots, N\}$, all coefficients g_μ^ν vanish, up to the diagonal terms g_μ^μ. Writing $g_0 = g_\mu^\mu$, $1 \leq \mu \leq N$, a 2-cochain invariant by $\mathfrak{s} = \mathfrak{so}(N) \oplus \mathfrak{sl}(2, \mathbb{R})$ has the form

$$\begin{aligned} \varphi(P_\mu, P_\nu) = 0, \quad \varphi(G_\mu, M) = g_0 G_\mu \\ \varphi(P_\mu, G_\nu) = \delta_\mu^\nu k_0 M, \\ \varphi(G_\mu, G_\nu) = 0, \quad \varphi(P_\mu, M) = g_0 P_\mu, \end{aligned} \tag{42}$$

and hence $\dim C^2\left(\mathfrak{h}_N, \widehat{S}(N)\right)^{\mathfrak{s}} = 2$. In order to deduce the invariant cocycles, we must still compute the coboundary operator $d\varphi$. Taking for example the triple $\{P_\mu, G_\mu, M\}$ we obtain

$$d\varphi(P_\mu, G_\mu, M) = [P_\mu, \varphi(G_\mu, M)] - [G_\mu, \varphi(P_\mu, M)] = 2g_0\,[P_\mu, G_\mu] = 2g_0\,M. \tag{43}$$

If $d\varphi(P_\mu, G_\mu, M) = 0$ holds, then necessarily $g_0 = 0$ must be satisfied. The other triples do not provide additional information. It results that the space $Z^2\left(\mathfrak{h}_N, \widehat{S}(N)\right)^{\mathfrak{s}}$ is generated by the cocycle

$$\varphi_0(P_\mu, G_\nu) = \delta_\mu^\nu M. \tag{44}$$

Now, from equation (27) we immediately see that

$$\varphi_0 = d\,\psi_0. \tag{45}$$

This proves the following

Theorem 4.1. *For any $N \geq 3$, the extended Schrödinger algebra $\widehat{S}(N)$ satisfies the condition*

$$H^2\left(\widehat{S}(N), \widehat{S}(N)\right) = 0. \tag{46}$$

The consequence is the (cohomological) rigidity of $\widehat{S}(N)$ for $N \neq 2$. In particular, for these values $\widehat{S}(N)$ cannot be obtained as a contraction of another Lie algebra.

Remark 4.2. Since for $N \neq 2$ the Lie algebra $\widehat{S}(N)$ has only one outer derivation, it results in particular that $\dim Der\left(\widehat{S}(N)\right) = \frac{1}{2}(N^2+3N)+5$. From the classical cohomology theory [13] we know that $\dim B^2\left(\widehat{S}(N), \widehat{S}(N)\right) =$

$$\left(\dim \widehat{S}(N)\right)^2 - \dim Der\left(\widehat{S}(N)\right) = \frac{(N^2+3N+8)(N^2+3N+6)}{4}. \tag{47}$$

As the second cohomology vanishes,

$$\dim Z^2\left(\widehat{S}(N), \widehat{S}(N)\right) = \dim B^2\left(\widehat{S}(N), \widehat{S}(N)\right).$$

Albeit the nullity of the second cohomology group prevents us from the necessity of further inspecting the quadratic Rim map, we observe that the preceding argumentation also serves to obtain information about $H^3\left(\widehat{S}(N), \widehat{S}(N)\right)$.

Proposition 4.3. *For any $N \geq 3$, the following isomorphism holds:*

$$H^3\left(\widehat{S}(N), \widehat{S}(N)\right) \simeq H^3(\mathfrak{s}, \mathbb{R}).M. \tag{48}$$

Up to one difference in the factorization, the proof is completely analogous to the previous ones, for which reason we omit the details. According to the Hochschild-Serre factorization theorem, and taking into account that for a semisimple Lie algebra $\mathfrak{s}$ the spaces $H^1(\mathfrak{s}, \mathbb{R})$ and $H^2(\mathfrak{s}, \mathbb{R})$ are zero [13], we have the isomorphism $H^3\left(\widehat{S}(N), \widehat{S}(N)\right) \simeq$

$$\left(H^3\left(\mathfrak{h}_N, \widehat{S}(N)\right)^{\mathfrak{s}} \otimes H^0(\mathfrak{s}, \mathbb{R})\right) \oplus \left(H^0\left(\mathfrak{h}_N, \widehat{S}(N)\right)^{\mathfrak{s}} \otimes H^3(\mathfrak{s}, \mathbb{R})\right). \tag{49}$$

Repeating the arguments used for $i = 1, 2$, it can be routinely shown that the space $H^3\left(\mathfrak{h}_N, \widehat{S}(N)\right)^{\mathfrak{s}}$ of invariant 3-cocycle classes is zero, and the first term of (49) disappears. On the other hand, $H^0\left(\mathfrak{h}_N, \widehat{S}(N)\right)^{\mathfrak{s}} = M$, this implies that

$$H^3\left(\widehat{S}(N), \widehat{S}(N)\right) = H^3(\mathfrak{s}, \mathbb{R}).M. \tag{50}$$

For a semisimple Lie algebra $\mathfrak{s}$ it constitutes a classical result that $H^3(\mathfrak{s}, \mathbb{R}) \neq 0$ (see e.g. [13, 21]), from which we conclude that the third cohomology group of the extended Schrödinger algebra is not zero.

Remark 4.4. For the case $N = 1$ the assertion still holds. Here

$$H^3\left(\widehat{S}(1), \widehat{S}(1)\right) = H^3(\mathfrak{sl}(2,\mathbb{R}), \mathbb{R}).M$$

and a basis of $H^3(\mathfrak{sl}(2,\mathbb{R}), \mathbb{R})$ is given by the 3-cocycle

$$\varphi(D, K, P_t) = 1.$$

4.3. The Schrödinger algebra $S(N) = \widehat{S}(N)/Z\left(\widehat{S}(N)\right)$.

The previous proofs for the first and second cohomology spaces with values in the adjoint representation can be easily adapted to the case of the "usual" Schrödinger algebra $S(N)$, i.e., the algebra obtained from $\widehat{S}(N)$ by quotient by its centre, by simply removing the parts making reference to the central charge M. Essentially the same conclusions arise, namely, that $S(N)$ possesses exactly one outer derivation if $N \neq 2$, given by

$$f(P_\mu) = P_\mu, \quad F(G_\mu) = G_\mu, \quad 1 \leq \mu \leq N. \tag{51}$$

For $N \neq 2$, the radical of $S(N)$ is Abelian, while for $N = 2$ it is solvable.

Theorem 4.5. *For any $N \neq 2$ the Schrödinger algebra $S(N)$ is cohomologically rigid:*

$$H^2(S(N), S(N)) = 0. \tag{52}$$

Moreover,

$$H^3(S(N), S(N)) = H^3\left(\mathbb{R}^{2N}, S(N)\right)^{\mathfrak{s}} \otimes \mathbb{R}, \tag{53}$$

and $H^3(S(N), S(N)) \neq 0$ for $N \geq 3$.

The essential difference between $S(N)$ and its central extension $\widehat{S}(N)$ lies in the third cohomology space. As the centre of $S(N)$ is zero, it follows at once that $H^0\left(\mathbb{R}^{2N}, \widehat{S}(N)\right)^{\mathfrak{s}} = 0$, where $\mathfrak{s} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(N)$, and therefore formula (49) reduces to

$$H^3(S(N), S(N)) \simeq H^3\left(\mathbb{R}^{2N}, S(N)\right)^{\mathfrak{s}} \otimes H^0(\mathfrak{s}, \mathbb{R}), \tag{54}$$

where $H^0(\mathfrak{s}, \mathbb{R}) = \mathbb{R}$. Clearly, for $N = 1$ the space (54) reduces to zero. For $N \geq 3$ it is straightforward to verify that the 3-cochain defined by

$$\begin{aligned} \varphi(P_\mu, P_\nu, G_\lambda) &= \delta_\mu^\lambda P_\nu - \delta_\nu^\lambda P_\mu, \\ \varphi(P_\mu, G_\nu, G_\lambda) &= \delta_\mu^\lambda G_\nu - \delta_\nu^\lambda G_\mu \end{aligned} \tag{55}$$

is a cocycle $d\varphi = 0$ and cannot appear as a coboundary. This proves that $H^3(S(N), S(N)) \neq 0$.

We finally observe that the centrally extended Schrödinger algebra $\widehat{S}(N)$ can be seen as a deformation of the Lie algebra $S(N) \oplus \mathbb{R}$, or, in alternative form, the latter Lie algebra as a generalized Inönü-Wigner contraction of $\widehat{S}(N)$.

Remark 4.6. For $N=2,\ S(2)$ is isomorphic to the Lie algebra $L_{8,17}^{-1}$ of [26]. In [2] it was shown that it satisfies

$$\dim H^1\left(S\left(2\right),S\left(2\right)\right)=2,\ \dim H^2\left(S\left(2\right),S\left(2\right)\right)=2,\ \dim H^3\left(S\left(2\right),S\left(2\right)\right)=2,$$

and the corresponding deformations were computed.

Final remarks

By means of the Page-Richardson stability theorem and the Hochschild-Serre factorization theorem for the Chevalley cohomology of Lie algebras, it has been shown that for values $N\neq 2$, the centrally extended Schrödinger algebra $\widehat{S}(N)$ in $(N+1)$ dimensions is a cohomologically rigid Lie algebra, while the third cohomology group $H^3\left(\widehat{S}(N),\widehat{S}(N)\right)$ is always non-zero and determined by $H^3\left(\mathfrak{sl}(2,\mathbb{R}\oplus\mathfrak{so}(N),\mathbb{R})\right)$. This confirms the stability of the model as inferred in its applications to covariant Field Theory [12, 18]. The method of cocycles invariant with respect to the Levi subalgebra of $\widehat{S}(N)$ can be easily adapted to the case of the non-extended Schrödinger algebra $S(N)$, which enables us to determine also its cohomological rigidity for $N\neq 2$ and that $H^3\left(S(N),S(N)\right)\neq 0$. However, as $S(N)$ has no centre, the latter space is characterized not by the third cohomology group of the Levi subalgebra with values in $\mathbb{R}$, but in contrast by the subspace of invariant 3-cocycles.

The computation method of invariant cocycles can be enlarged in straightforward manner to determine the second cohomology space and deformations of other types of semidirect sums of semisimple Lie algebras and Heisenberg algebras, as well as their corresponding central quotient. In this situation, it is also expected that the third cohomology space differs structurally depending whether we are considering the Lie algebra with centre or not. We merely mention that this type of semidirect sums are of interest for various applications, as the Lie algebras $\mathfrak{wsp}(N)$ with Levi subalgebra isomorphic to the symplectic Lie algebra $\mathfrak{sp}(N)$ [1, 22], or the 190-dimensional Lie algebra $E_{7\frac{1}{2}}$ that appeared in the context of the Deligne conjecture.

References

[1] Campoamor-Stursberg, R., *Explizite Formeln für die Casimiroperatoren des semidirekten Produktes einer Heisenberg Lie-Algebra $\mathfrak{h}$ mit der einfachen Lie-Algebra $\mathfrak{sl}\left(2,\mathbb{C}\right)$*, Acta Phys. Polonica B **35** (2004), 2059–2069.

[2] —, *A new matrix method for the Casimir operators of the Lie algebras $w\mathfrak{sp}\left(N,\mathbb{R}\right)$ and $I\mathfrak{sp}\left(2N,\mathbb{R}\right)$*, J. Phys. A: Math. Gen. **38** (2005), 4187–4208.

[3] —, *A comment concerning cohomology and invariants of Lie algebras with respect to contractions and deformations*, Phys. Lett. **A362** (2007), 360–367.

[4] Carles, R., *Sur la structure des algèbres de Lie rigides*, Ann. Inst. Fourier **34** (1984), 65–84.

[5] Dobrev, V. K., H. D. Doebner, and Ch. Mrugalla, *Lowest weight representations of the Schrödinger algebra and generalized heat Schrödinger equations*, Rep. Math. Phys. **39** (1997), 201–218.

[6] Feinsilver, P., J. Kocik, and B. Schott, *Representations of the Schrödinger algebra and Appell systems*, Fortschritte der Physik **52** (2004), 343–359.

[7] Fialowski, A., *Deformations and contractions of algebraic structures*, Proc. Steklov Inst. Math. **286** (2014), 240–252.

[8] Fialowski, A., and D. Fuchs, *Construction of miniversal deformations of Lie algebras*, J. Funct. Anal. **161** (1999), 76–110.

[9] Fialowski, A., and J. O'Halloran, *A comparison of deformations and orbit closure*, Commun. Algebra **18** (1990), 4121–4140.

[10] Finkelstein, D. R., *General quantization*, Int. J. Theor. Phys. **45** (2006), 1397–1427.

[11] Gerstenhaber, M., *On the deformation of rings and algebras*, Ann. Math. **79** (1964), 59–103.

[12] Hagen, C. R., *Scale and conformal transformations in Galilean-covariant field theory*, Phys. Rev. **D5** (1972), 377–388.

[13] Hochschild, G., and J. P. Serre, *Cohomology of Lie algebras*, Ann. Math. **57** (1953), 591–603.

[14] Inönü, E., and E. P. Wigner, *On the contraction of groups and their representations*, Proc. Natl. Acad. Sci. U.S. **39** (1953), 510–524.

[15] Leger, G. F., *On cohomology of Lie algebras*, Proc. Amer. Math. Soc. **8** (1957), 1010–1020.

[16] Levy-Nahas, M., *Deformation and contraction of Lie algebras*, J. Math. Phys. **8** (1967), 1211–1222.

[17] Moylan, P., *Models of contractions*, J. Phys. Conf. Ser. **343** 012083 (11 p)

[18] Niederer, U., *The maximal kinematical invariance group of the free Schrödinger equation*, Helv. Phys. Acta **45** (1972), 802–810.

[19] Nijenhuis, A., and R. W. Richardson, *Deformations of homomorphisms of Lie groups and Lie algebras*, Bull. Amer. Math. Soc. **73** (1967), 175–179.

[20] Page, S., and R. W. Richardson, *Stable subalgebras of Lie algebras and associative algebras*, Trans. Amer. Math. Soc. **127** (1967), 302–312.

[21] Postnikov, M. M., "Lie Groups and Lie Algebras," Nauka, 1982.

[22] Quesne, Ch., *Casimir operators of semidirect sum Lie algebras*, J. Phys. A: Math. Gen. **21** (1988), L321–L324.

[23] Rauch, G., *Effaçement et déformation*, Ann. Inst. Fourier **22**(1972), 239–269.

[24] Richardson, R. W., *On the rigidity of semi-direct products of Lie algebras*, Pacific J. Math. **22** (1967), 339–344.

[25] Turkowski, P., *Structure of real Lie algebras*, Linear Alg. Appl. **171** (1992), 197–212.

[26] —, *Low dimensional real Lie algebras*, J. Math. Phys. **29** (1988), 2139–2144.

[27] Vilela Mendes, R., *Deformations, stable theories and fundamental constants*, J. Phys. A: Math. Gen. **27** (1994), 8091–8104.

[28] Weimar-Woods, E., *Contractions, generalized Inönü-Wigner contractions and deformations of finite-dimensional Lie algebras*, Rev. Math. Phys. **12** (2000), 1505–1529.

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