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Tulczyjew's Triplet for Lie Groups II: Dynamics

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Abstract. Taking configuration space as a Lie group, the trivialized Euler-Lagrange and Hamilton's equations are obtained and presented as Lagrangian submanifolds of the trivialized Tulczyjew's symplectic space. Euler-Poincaré and Lie-Poisson equations are presented as Lagrangian submanifolds of the reduced Tulczyjew's symplectic space. Tulczyjew's generalized Legendre transformations for trivialized and reduced dynamics are constructed.
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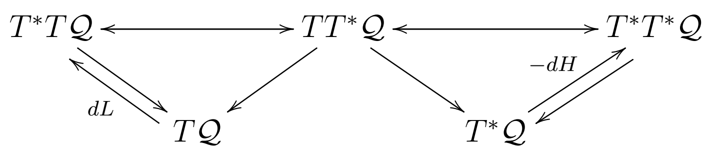
1. Introduction

In 1970's, W.Tulczyjew introduced a geometric framework, nowadays called Tulczyjew's triplet, where Hamiltonian and Lagrangian dynamics can be presented as Lagrangian submanifolds of a certain symplectic manifold, [44, 45, 46, 47, 48]. For a mechanical system having the configuration space $\mathcal{Q}$, the two wings of the Tulczyjew triplet



(1)

defines two different special symplectic structures for Tulczyjew's symplectic space $TT^*\mathcal{Q}$ symplectomorphic to $T^*T\mathcal{Q}$ and $T^*T^*\mathcal{Q}$. A Lagrangian L on $T\mathcal{Q}$ (or a Hamiltonian H on $T^*\mathcal{Q}$) generates a Lagrangian submanifold $\mathcal{S}_{TT^*\mathcal{Q}}$ of $TT^*\mathcal{Q}$. In this geometrization, Tulczyjew introduced a generalization of the Legendre transformation, applicable also for degenerate cases, as a transformation between realizations of the same Lagrangian submanifold with two different functions.

In the literature, various analysis and generalizations of Tulczyjew's triplet can be found. To cite some examples, we refer to [9, 33, 19, 20, 21] for construction

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for field theories using jet bundles, to [8, 22, 23] for constructions in the framework of Lie algebroids, to [42] and [25] for $k-$cosymplectic and presymplectic structures, to [24] for double groups, and to a recent preprint [18] for the Marsden-Weinstein reduction of Tulczyjew triplets. We also refer [51].

The present work is the second of a series of papers. In the first paper [16], the global trivializations of Tulczyjew's triplet (1) were adapted for Lie groups

$$\begin{array}{ccccc} {}^1T^*TG & \longleftrightarrow & {}^1TT^*G & \longleftrightarrow & {}^1T^*T^*G \\ & G \circledS \mathfrak{g} & & G \circledS \mathfrak{g}^* & \end{array} \tag{2}$$

where $\mathfrak{g}$ is Lie algebra of the group G, $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$, the superscript 1 denotes the global trivialization of the first kind that lifts the Lie group action to iterated bundles. The global trivialization of the first kind is the one that preserves the group structures [16]. The reason to choose the under the global trivializations is to see the symmetries clearly, and to simplify the possible reduction procedures. The reason to choose right trivializations is to apply this geometry to our motivational question, to find a proper generalized (inverse) Legendre transformation of Hamiltonian formulation of Poisson-Vlasov equations governing the non-relativistic motion of the plasma [17].

In [16], after the trivialization of the triplet, we constructed the reduced Tulczyjew's triplet

$$\begin{array}{ccccc} \mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^* & \longleftrightarrow & \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} & \longleftrightarrow & \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \\ & \mathfrak{g} & & \mathfrak{g}^* & \end{array} \tag{3}$$

where the first order bundles were reduced by the action of base group G. For the tangent bundle, we arrived Lie algebra $\mathfrak{g}$ of G. For the cotangent bundle T^*G, we preferred to apply Poisson reduction to the symplectic manifold T^*G and arrived to the dual space $\mathfrak{g}^*$. The reason of choosing the Poisson reduction is to increase the application area of the triplet since Hamiltonian formulations of many systems, such as rigid bodies, fluids and plasmas, are defined on the dual space $\mathfrak{g}^*$. The second order bundles are reduced by the application of the Marsden-Weinstein symplectic reduction theorem. For example, for TT^*G, the reduced space is the symplectic manifold $\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}^*$ where $\mathcal{O}_\lambda$ is the coadjoint orbit through the point λ in $\mathfrak{g}^*$. In the forthcoming sections all the details about the mappings and projections in the triplets (2) and (3) will be clearly given.

Aim and content of the paper: The outmost goal of this paper is to recast (trivialized/reduced) Lagrangian and Hamiltonian dynamics on Lie groups as Lagrangian submanifolds of (trivialized/reduced) Tulczyjew's symplectic manifolds in order to define generalized Legendre transformation, working also for degeneracy cases, of the trivialized Euler-Lagrange and Hamilton's equations as well as Euler-Poincaré and Lie-Poisson equations.

To this end, in the next section, we shall derive trivialized Euler-Lagrange, trivialized Hamilton's, Euler-Poincaré and Lie-Poisson equations. In section 3, geometry of the trivialized Tulczyjew's triplet in diagram (2) will be summarized. We shall represent trivialized Euler-Lagrange and trivialized Hamilton's equations as Lagrangian submanifolds of trivialized Tulczyjew's symplectic space ${}^{1}TT^{*}G$. We shall then use special symplectic structures associated with the fibrations of $G \circledS \mathfrak{g}$ and $G \circledS \mathfrak{g}^{*}$ in order to define Legendre and inverse Legendre transformations, and arrive at Morse families on the trivialized Pontryagin bundle $G \circledS (\mathfrak{g} \times \mathfrak{g}^{*})$ with fibrations over $\mathfrak{g}^{*}$ and $\mathfrak{g}$, respectively. In section 4, we shall start with the reduced Tulczyjew's triplet in diagram (3) and present Euler-Poincaré and Lie-Poisson dynamics as Lagrangian submanifolds of the reduced Tulczyjew's symplectic space $\mathcal{O}_{\lambda} \times \mathfrak{g}^{*} \times \mathfrak{g}$. We shall then establish the Legendre transformations of reduced dynamics. In the last section, we shall present the example where G is the group of diffeomorphisms on a manifold.

The main role of the present construction will be played by the two (trivialized/reduced) special symplectic structures (including two potential one-forms) underlying the (trivialized/reduced) Tulczyjew's symplectic manifold. In the present paper, in order to maintain the characteristic properties (existance of two potential forms and two special symplectic structures) of the Tulczyjew's triplet, we write all the trivializations/reductions of the symplectic manifolds in canonical ways. The reduced Tulczyjew's symplectic two-form on $\mathcal{O}_{\lambda} \times \mathfrak{g}^{*} \times \mathfrak{g}$ having two different potential one forms result with two different special symplectic structures. One of the special symplectic structure of $\mathcal{O}_{\lambda} \times \mathfrak{g}^{*} \times \mathfrak{g}$ is for the Euler-Poincaré dynamics on the Lie algebra $\mathfrak{g}$ and the other is for the Lie-Poisson dynamics on the dual space $\mathfrak{g}^{*}$.

The main orientation of this present paper and [18] are almost the same except that the latter is studying on a general manifold Q instead of focusing on Lie groups. In [18], Marsden-Weinstein reduction are achieved by taking the isotropy group G_{λ} is equal to the whole Lie group G. This results with the reduced Tulczyjew's space $\mathfrak{g}^{*} \times \mathfrak{g}$ which is lack of two potential one-forms hence lack of two special symplectic structures. In the present paper, no such preferences is done so that the result is the reduced Tulczyjew space $\mathcal{O}_{\lambda} \times \mathfrak{g}^{*} \times \mathfrak{g}$ having two special symplectic structures. We also point out that, in [18], the direct product trivializations are presented. We remark also that, in [8, 22, 23], the Tulczyjew's triplet is written for the case of Lie algebroids. If one starts with this, the reduced Tulczyjew's space will be $\mathfrak{g}^{*} \times \mathfrak{g}$ for the particular choice of the Lie algebras.

2. Dynamics on Lie Groups

2.1. Notations.

G is a Lie group. Its Lie algebra $\mathfrak{g} \simeq T_{e}G$ is assumed to be reflexive. The dual of $\mathfrak{g}$ is $\mathfrak{g}^{*} = Lie^{*}(G) \simeq T_{e}^{*}G$. Throughout the work, we shall designate

$$g, h \in G, \quad \xi, \eta, \zeta \in \mathfrak{g}, \quad \mu, \nu, \lambda \in \mathfrak{g}^{*}. \tag{4}$$

For a tensor field which is either right or left invariant, we shall use $V_{g} \in T_{g}G$ or $\alpha_{g} \in T_{g}^{*}G$ etc... For an arbitrary manifold $\mathcal{M}$, we shall use

$$u, v \in \mathcal{M}, \quad V_{u}, U_{u} \in T_{u}\mathcal{M}, \quad \alpha_{u}, \beta_{u}, \gamma_{u} \in T_{u}^{*}\mathcal{M} \tag{5}$$

to denote vectors and one-forms over specific points. We shall denote left and right multiplications on G by L_g and R_g, respectively. The right inner automorphism

$$I_g = L_{g^{-1}} \circ R_g \tag{6}$$

will be a right action of G on G satisfying $I_g \circ I_h = I_{hg}$. The *right* adjoint action $Ad_g = T_e I_g$ of G on $\mathfrak{g}$ is defined as the tangent map of I_g at the identity $e \in G$. The infinitesimal *right* adjoint representation $ad_\xi \eta$ is $[\xi, \eta]_{\mathfrak{g}}$ and it is defined as the derivative of Ad_g at the identity. A right invariant vector field X_ξ^G on G can be obtained by right translation

$$X_\xi^G(g) = T_e R_g \xi \tag{7}$$

of $\xi \in \mathfrak{g}$ for each $g \in G$. The identity

$$[\xi, \eta] = \left[X_\xi^G, X_\eta^G\right]_{JL} \tag{8}$$

gives the isomorphism between $\mathfrak{g}$ and the space $\mathfrak{X}^R(G)$ of right invariant vector fields endowed with the Jacobi-Lie bracket. The coadjoint action Ad_g^* of G on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ is a right representation and is the linear algebraic dual of $Ad_{g^{-1}}$, namely,

$$\left\langle Ad_g^* \mu, \xi \right\rangle = \left\langle \mu, Ad_{g^{-1}} \xi \right\rangle \tag{9}$$

holds for all $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. The infinitesimal coadjoint action ad_ξ^* of $\mathfrak{g}$ on $\mathfrak{g}^*$ is the linear algebraic dual of ad_ξ. Note that, the infinitesimal generator of the coadjoint action Ad_g^* is minus the infinitesimal coadjoint action ad_ξ^*, that is, if $g^t \subset G$ is a curve passing through the identity in the direction of $\xi \in \mathfrak{g}$, then

$$\left.\frac{d}{dt}\right|_{t=0} Ad_{g^t}^* \mu = -ad_\xi^* \mu. \tag{10}$$

The *right* trivialization maps on TG and T^*G are defined to be

$$tr_{TG}^R : TG \to G \circledS \mathfrak{g} : U_g \to \left(g, T_g R_{g^{-1}} U_g\right), \tag{11}$$

$$tr_{T^*G}^R : T^*G \to G \circledS \mathfrak{g}^* : \alpha_g \to \left(g, T_e^* R_g \alpha_g\right). \tag{12}$$

We refer to [16] for further details about the right actions and representations.

2.2. Lagrangian dynamics.

For a Lagrangian density $L : TG \to \mathbb{R}$, define the unique function $\bar{L}$ on $G \circledS \mathfrak{g}$ by

$$\bar{L}(g, \xi) = \bar{L} \circ tr_{TG}^R(V_g) = L(V_g), \tag{13}$$

where $\xi = T_g R_{g^{-1}} V_g$.

Proposition 2.1. *A Lagrangian density $\bar{L}$ on $G \circledS \mathfrak{g}$ defines the trivialized Euler-Lagrange dynamics*

$$\frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi} = T_e^* R_g \frac{\delta \bar{L}}{\delta g} + ad_\xi^* \frac{\delta \bar{L}}{\delta \xi}. \tag{14}$$

Proof. Using reduced variational principle, one computes

$$\delta\int_b^a \bar{L}(g,\xi)\,dt = \int_b^a \left(\left\langle \frac{\delta \bar{L}}{\delta g},\delta g\right\rangle_g + \left\langle \frac{\delta \bar{L}}{\delta \xi},\delta \xi\right\rangle_e\right)dt$$

$$= \int_b^a \left(\left\langle \frac{\delta \bar{L}}{\delta g},\delta g\right\rangle_g + \left\langle \frac{\delta \bar{L}}{\delta \xi},\dot{\eta}+[\xi,\eta]\right\rangle_e\right)dt$$

$$= -\left\langle \frac{\delta \bar{L}}{\delta \xi},\eta\right\rangle_e\Big|_b^a + \int_b^a \left(\left\langle \frac{\delta \bar{L}}{\delta g},\delta g\right\rangle_g + \left\langle -\frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi}+ad_\xi^*\frac{\delta \bar{L}}{\delta \xi},\eta\right\rangle_e\right)dt$$

$$= -\left\langle \frac{\delta \bar{L}}{\delta \xi},T_gR_{g^{-1}}\delta g\right\rangle_e\Big|_b^a + \int_b^a \left\langle \frac{\delta \bar{L}}{\delta g},\delta g\right\rangle_g + \left\langle ad_\xi^*\frac{\delta \bar{L}}{\delta \xi}-\frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi},T_gR_{g^{-1}}\delta g\right\rangle_e dt$$

$$= -\left\langle T_g^*R_{g^{-1}}\frac{\delta \bar{L}}{\delta \xi},\delta g\right\rangle_g\Big|_a^b + \int_b^a \left\langle \frac{\delta \bar{L}}{\delta g}+T_g^*R_{g^{-1}}\left(ad_\xi^*\frac{\delta \bar{L}}{\delta \xi}-\frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi}\right),\delta g\right\rangle_g dt.$$

and the conclusion follows if δg vanishes at boundaries. Here, the variation of the fiber (Lie algebra) variable ξ is taken by the reduced variational principle [10, 6, 16, 28, 29, 35, 38]

$$\delta\xi = \dot{\eta} + [\xi,\eta]\,. \tag{15}$$

■

The trivialized Euler-Lagrange equation (14) is defined over the identity $e \in G$. Because $\delta\bar{L}/\delta g \in T_g^*G$ and the functor $T_e^*R_g$ takes this to the dual space $\mathfrak{g}^* = T_e^*G$. Eq.(14) appeared in [13] with the missing operator $T_e^*R_g$. It also appeared in some recent works [11, 12] on the higher order Lagrangian and Hamiltonian dynamics on trivialized iterated bundles of Lie groups. See also [7].

When the Lagrangian $\bar{L}$ is independent of g, that is, $\bar{L}(g,\xi) = l(\xi)$ and L on TG is right invariant, then Eq.(14) reduces to the Euler-Poincaré equation

$$ad_\xi^*\frac{\delta l}{\delta \xi} - \frac{d}{dt}\frac{\delta l}{\delta \xi} = 0. \tag{16}$$

2.3. Hamiltonian dynamics.

By pushing forward the canonical one-form θ_{T^*G} and symplectic two-form Ω_{T^*G} on T^*G with the trivialization map $tr_{T^*G}^R$, $G\circledS\mathfrak{g}^*$ can be endowed with an exact symplectic two-form $\Omega_{G\circledS\mathfrak{g}^*} = d\theta_{G\circledS\mathfrak{g}^*}$. If

$$X_{(\xi,\nu)}^{G\circledS\mathfrak{g}^*}(g,\mu) = \left(T_eR_g\xi, \nu + ad_\xi^{R*}\mu\right),$$

is a right invariant vector field at the point $(g,\mu) \in G\circledS\mathfrak{g}^*$ generated by the Lie algebra element $(\xi,\nu) \in Lie\,(G\circledS\mathfrak{g}^*) \simeq \mathfrak{g}\circledS\mathfrak{g}^*$ then, the values of canonical forms $\theta_{G\circledS\mathfrak{g}^*}$ and $\Omega_{G\circledS\mathfrak{g}^*}$ on $X_{(\xi,\nu)}^{G\circledS\mathfrak{g}^*}(g,\mu)$ are [1, 16]

$$\left\langle \theta_{G\circledS\mathfrak{g}^*}, X_{(\xi,\nu)}^{G\circledS\mathfrak{g}^*}\right\rangle(g,\mu) = \langle \mu,\xi\rangle \tag{17}$$

$$\left\langle \Omega_{G\circledS\mathfrak{g}^*};\left(X_{(\xi,\nu)}^{G\circledS_R\mathfrak{g}^*}, X_{(\eta,\lambda)}^{G\circledS\mathfrak{g}^*}\right)\right\rangle(g,\mu) = \langle \nu,\eta\rangle - \langle \lambda,\xi\rangle + \left\langle \mu,[\xi,\eta]_{\mathfrak{g}}\right\rangle \tag{18}$$

which are considered for linearizations of Hamiltonian systems in [38] and, for higher order dynamics in [11]. Let H be a function on T^*G and define $\bar{H} : G \circledS \mathfrak{g}^* \to \mathbb{R}$ by $\bar{H} \circ tr^R_{T^*G} = H$, that is, for $\alpha_g = T^*_g R_{g^{-1}} \mu$, we have $\bar{H}(g, \mu) = H(\alpha_g)$ and Hamilton's equations on $(G \circledS \mathfrak{g}^*, \Omega_{G \circledS \mathfrak{g}^*})$ are

$$i_{X^{G \circledS \mathfrak{g}^*}_{\bar{H}}} \Omega_{G \circledS \mathfrak{g}^*} = -d\bar{H}. \tag{19}$$

Proposition 2.2. *The Hamiltonian vector field $X^{G \circledS_R \mathfrak{g}^*}_{\bar{H}}$, defined in Eq.(19), is generated by the element*

$$\left(\frac{\delta \bar{H}}{\delta \mu}, -T^*_e R_g \left(\frac{\delta \bar{H}}{\delta g}\right)\right)$$

of the Lie algebra $\mathfrak{g} \circledS \mathfrak{g}^*$ of $G \circledS \mathfrak{g}^*$ and components of $X^{G \circledS_R \mathfrak{g}^*}_{\bar{H}}$ are given by the trivialized Hamilton's equations

$$\frac{dg}{dt} = T_e R_g \left(\frac{\delta \bar{H}}{\delta \mu}\right), \quad \frac{d\mu}{dt} = ad^*_{\frac{\delta \bar{H}}{\delta \mu}} \mu - T^*_e R_g \frac{\delta \bar{H}}{\delta g}. \tag{20}$$

Note that the second term on the right hand side of the second equation in Eq.(20) is a consequence of the semidirect product structure on $G \circledS \mathfrak{g}^*$. Accordingly, if we let $\bar{H}$ to be independent of the group element, that is, $\bar{H}(g, \mu) = h(\mu)$ and H on T^*G is right invariant, then the trivialized Hamilton's equations (20) reduce to

$$\frac{dg}{dt} = T_e R_g \left(\frac{\delta h}{\delta \mu}\right), \quad \frac{d\mu}{dt} = ad^*_{\frac{\delta h}{\delta \mu}} \mu. \tag{21}$$

The canonical Poisson bracket on $G \circledS \mathfrak{g}^*$ is

$$\begin{aligned}
\{\bar{F}, \bar{K}\}_{G \circledS \mathfrak{g}^*}(g, \mu) &= \Omega_{G \circledS \mathfrak{g}^*}\left(X^{G \circledS \mathfrak{g}^*}_{\bar{F}}, X^{G \circledS \mathfrak{g}^*}_{\bar{K}}\right)(g, \mu) \\
&= \Omega_{G \circledS \mathfrak{g}^*}\left(X^{G \circledS \mathfrak{g}^*}_{\left(\frac{\delta \bar{F}}{\delta \mu}, -T^* R_g \frac{\delta \bar{F}}{\delta g}\right)}, X^{G \circledS \mathfrak{g}^*}_{\left(\frac{\delta \bar{K}}{\delta \mu}, -T^* R_g \frac{\delta \bar{K}}{\delta g}\right)}\right)(g, \mu) \\
&= \left\langle T^*_e R_g \frac{\delta \bar{K}}{\delta g}, \frac{\delta \bar{F}}{\delta \mu}\right\rangle - \left\langle T^*_e R_g \frac{\delta \bar{F}}{\delta g}, \frac{\delta \bar{K}}{\delta \mu}\right\rangle + \left\langle \mu, \left[\frac{\delta \bar{F}}{\delta \mu}, \frac{\delta \bar{K}}{\delta \mu}\right]_{\mathfrak{g}}\right\rangle,
\end{aligned}$$

for two function(al)s $\bar{F}$ and $\bar{K}$ defined on $G \circledS \mathfrak{g}^*$. The Poisson bracket $\{\ ,\ \}_{G \circledS \mathfrak{g}^*}$ is non-degenerate. When $\bar{F}$ and $\bar{K}$ are independent of the group variable $g \in G$, that is, $\bar{F} = f(\mu)$ and $\bar{K} = k(\mu)$, we have the Lie-Poisson bracket

$$\{f, k\}_{\mathfrak{g}^*}(\mu) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu}\right]_{\mathfrak{g}}\right\rangle \tag{22}$$

on the dual space $\mathfrak{g}^*$ [3, 29, 35, 30]. This is a manifestation of the fact that the projection $G \circledS \mathfrak{g}^* \to \mathfrak{g}^*$ is the momentum map for the cotangent lifted left action of G on $G \circledS \mathfrak{g}^*$. In this case, the dynamics is driven by the Hamiltonian vector field $X^{\mathfrak{g}^*}_h$ satisfying

$$\{f, h\}_{\mathfrak{g}^*} = -\left\langle df, X^{\mathfrak{g}^*}_h\right\rangle$$

for a given Hamiltonian function(al) h on $\mathfrak{g}^*$. More explicitly, the value of Hamiltonian vector field $X_h^{\mathfrak{g}^*}$ at $\mu \in \mathfrak{g}^*$ is defined by the Lie-Poisson equations

$$\frac{d\mu}{dt} = ad^*_{\frac{\delta h}{\delta \mu}} \mu. \tag{23}$$

We shall refer to both of the equations in (21) and (23) as Lie-Poisson equations [38].

3. The Trivialized Dynamics

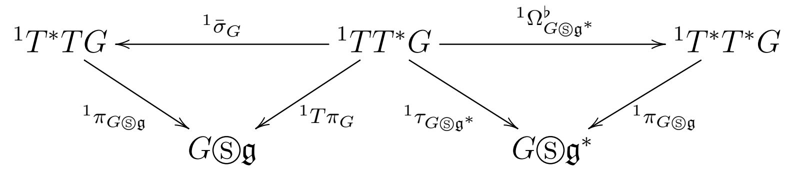
3.1. Trivialization of the Tulczyjew's Triplet.

The tangent and cotangent lifts of the group structure on G define group structures on TG and T^*G, respectively. The trivialization maps tr^R_{TG} and $tr^R_{T^*G}$ on TG and T^*G are defined in such a way that they are not only diffeomorphisms but also group isomorphisms [27, 31, 36, 37, 38, 39, 40]. For iterated bundles, with the same understanding of trivializations we obtained [16]

$$\begin{aligned}
tr^1_{T^*(G\circledS\mathfrak{g})} &: T^*\left(G\circledS\mathfrak{g}\right) \to \left(G\circledS\mathfrak{g}\right)\circledS\left(\mathfrak{g}^*\times\mathfrak{g}^*\right) = {}^1T^*TG & (24)\\
&: \left(\alpha_g,\alpha_\xi\right) \to \left(g,\xi,T_e^*R_g\left(\alpha_g\right)+ad^*_\xi\alpha_\xi,\alpha_\xi\right), \\
tr^1_{T^*(G\circledS\mathfrak{g}^*)} &: T^*T^*G \to \left(G\circledS\mathfrak{g}^*\right)\circledS\left(\mathfrak{g}^*\times\mathfrak{g}\right) = {}^1T^*T^*G & (25)\\
&: \left(\alpha_g,\alpha_\mu\right) \to \left(g,\mu,T_e^*R_g\left(\alpha_g\right)-ad^*_{\alpha_\mu}\mu,\alpha_\mu\right), \\
tr^1_{T(G\circledS\mathfrak{g}^*)} &: TT^*G \to \left(G\circledS\mathfrak{g}^*\right)\circledS\left(\mathfrak{g}\circledS\mathfrak{g}^*\right) = {}^1TT^*G & (26)\\
&: \left(V_g,V_\mu\right) \to \left(g,\mu,TR_{g^{-1}}V_g,V_\mu-ad^*_{TR_{g^{-1}}V_g}\mu\right).
\end{aligned}$$

Although, not unique, this way of trivializing iterated bundles enables us to perform reductions of Tulczyjew's triplet.

The symplectic two-forms on the trivialized bundles ${}^1T^*TG$, ${}^1T^*T^*G$ and ${}^1TT^*G$ have been constructed based on the fact that the trivialization maps $tr^1_{T^*(G\circledS\mathfrak{g})}$, $tr^1_{T^*(G\circledS\mathfrak{g}^*)}$ and $tr^1_{T(G\circledS\mathfrak{g}^*)}$ are symplectic diffeomorphisms. The trivialized Tulczyjew's triplet



(27)

consists of trivialized symplectic diffeomorphisms ${}^1\bar{\sigma}_G$ and ${}^1\Omega^\flat_{G\circledS\mathfrak{g}^*}$, and projections whose local expressions are

$$\begin{aligned}
{}^1\bar{\sigma}_G &: {}^1TT^*G \to {}^1T^*TG : (g,\mu,\xi,\nu) \to \left(g,\xi,\nu+ad^*_\xi\mu,\mu\right), & (28)\\
{}^1\Omega^\flat_{G\circledS\mathfrak{g}^*} &: {}^1TT^*G \to {}^1T^*T^*G : (g,\mu,\xi,\nu) \to \left(g,\mu,\nu+ad^*_\xi\mu,-\xi\right), & (29)\\
{}^1T\pi_G &: {}^1TT^*G \to G\circledS\mathfrak{g} : (g,\mu,\xi,\nu) \to (g,\xi), & (30)\\
{}^1\pi_{G\circledS\mathfrak{g}^*} &: {}^1T^*T^*G \to G\circledS\mathfrak{g}^* : (g,\mu,\nu,\xi) \to (g,\mu), & (31)\\
{}^1\pi_{G\circledS\mathfrak{g}} &: {}^1T^*TG \to G\circledS\mathfrak{g} : (g,\xi,\mu,\nu) \to (g,\xi). & (32)
\end{aligned}$$

3.2. Trivialized Tulczyjew's Symplectic Space.

Lie algebra of the group ${}^1TT^*G$ is the semi-direct product $(\mathfrak{g}Ⓢ\mathfrak{g}^*)Ⓢ(\mathfrak{g}Ⓢ\mathfrak{g}^*)$. A Lie algebra element

$$(\xi_2,\nu_2,\xi_3,\nu_3)\in(\mathfrak{g}Ⓢ\mathfrak{g}^*)Ⓢ(\mathfrak{g}Ⓢ\mathfrak{g}^*)$$

defines a right invariant vector field on ${}^1TT^*G$ by the tangent lift of right translation in ${}^1TT^*G$. At a point (g,μ,ξ,ν), a right invariant vector field takes the value

$$X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}=\left(TR_g\xi_2,\nu_2+ad^*_{\xi_2}\mu,\xi_3+[\xi,\xi_2]_{\mathfrak{g}},\nu_3+ad^*_{\xi_2}\nu-ad^*_{\xi}\nu_2\right). \tag{33}$$

By requiring the trivialization $tr^1_{TT^*G}$ be a symplectic mapping, we obtain an exact symplectic structure $\Omega_{{}^1TT^*G}$ with two potential one-forms θ_1 and θ_2 on ${}^1TT^*G$. At a point $(g,\mu,\xi,\nu)\in{}^1TT^*G$, the values of the potential one-forms θ_1 and θ_2 on the right invariant vector field of the form of Eq.(33) are

$$\left\langle\theta_1,X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle=\langle\nu,\xi_2\rangle-\langle\nu_2,\xi\rangle+\left\langle\mu,[\xi,\eta]_{\mathfrak{g}}\right\rangle, \tag{34}$$

$$\left\langle\theta_2,X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle=\langle\mu,\xi_3\rangle+\langle\nu,\xi_2\rangle+\left\langle\mu,[\xi,\xi_2]_{\mathfrak{g}}\right\rangle, \tag{35}$$

respectively. At the same point, the value of symplectic two-form $\Omega_{{}^1TT^*G}$ on two right invariant vector fields is

$$\begin{aligned}\left\langle\Omega_{{}^1TT^*G};\left(X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)},X^{{}^1TT^*G}_{(\bar\xi_2,\bar\nu_2,\bar\xi_3,\bar\nu_3)}\right)\right\rangle&=\langle\nu_3,\bar\xi_2\rangle+\langle\nu_2,\bar\xi_3\rangle-\langle\bar\nu_2,\xi_3\rangle-\langle\bar\nu_3,\xi_2\rangle\\&+\left\langle\nu,[\xi_2,\bar\xi_2]_{\mathfrak{g}}\right\rangle+\left\langle\mu,[\xi_3,\bar\xi_2]_{\mathfrak{g}}+[\xi_2,\bar\xi_3]_{\mathfrak{g}}+\left[\xi,[\xi_2,\bar\xi_2]_{\mathfrak{g}}\right]\right\rangle.\end{aligned} \tag{36}$$

Existence of potential one-forms in Eqs.(34) and (35) leads us to define two special symplectic structures

$$\left({}^1TT^*G,\ {}^1\tau_{GⓈ\mathfrak{g}^*},\ {}^1T^*T^*G,\ \theta_1,\ {}^1\Omega^\flat_{GⓈ\mathfrak{g}^*}\right) \tag{37}$$

$$\left({}^1TT^*G,\ {}^1T\pi_G,\ {}^1T^*TG,\ \theta_2,\ {}^1\bar\sigma_G\right), \tag{38}$$

on the trivialized Tulczyjew's symplectic manifold $({}^1TT^*G,\Omega_{{}^1TT^*G})$. The structures in Eqs.(37) and (38) are the right and left wings of the trivialized Tulczyjew's triplet (27), respectively. We refer to [16] for details.

3.3. Trivialized Lagrangian Dynamics as a Lagrangian Submanifold.

Proposition 3.1. *Let $\bar L$ be a Lagrangian on $GⓈ\mathfrak{g}$, then the Lagrangian submanifold $S_{{}^1TT^*G}$ defined by the equation*

$$\left({}^1T\pi_G\right)^*d\bar L=\theta_2, \tag{39}$$

gives the trivialized Euler-Lagrange equations (14). Here, the projection ${}^1T\pi_G$ is given by Eq.(30) and θ_2 is the one-form in Eq.(35).

Proof. Under the global trivialization ${}^{1}TT^{*}G$ of $T\left(G\circledS\mathfrak{g}^{*}\right)$, given in Eq.(26), the Lagrangian submanifold described by Eq.(39) becomes

$$S_{\ ^{1}TT^{*}G}=\left\{\left(g,\frac{\delta\bar{L}}{\delta\xi},\xi,T^{*}R_{g}\frac{\delta\bar{L}}{\delta g}\right)\in\ {}^{1}TT^{*}G:(g,\xi)\in G\circledS\mathfrak{g}\right\}. \tag{40}$$

To relate this to the trivialized Euler-Lagrange equations (14), we recall the reconstruction mapping

$$\left(tr^{1}_{T(G\circledS\mathfrak{g}^{*})}\right)^{-1}:\ {}^{1}TT^{*}G\rightarrow T\left(G\circledS\mathfrak{g}^{*}\right):(g,\mu,\xi,\nu)\rightarrow\left(g,\mu,TR_{g}\xi,\nu+ad^{*}_{\xi}\mu\right), \tag{41}$$

computed from Eq.(26). $\left(tr^{1}_{T(G\circledS\mathfrak{g}^{*})}\right)^{-1}$ maps $S_{\ ^{1}TT^{*}G}$ to the Lagrangian submanifold

$$S_{T(G\circledS\mathfrak{g}^{*})}=\left\{\left(g,\frac{\delta\bar{L}}{\delta\xi};TR_{g}\xi,T^{*}R_{g}\frac{\delta\bar{L}}{\delta g}+ad^{*}_{\xi}\frac{\delta\bar{L}}{\delta\xi}\right)\in S_{\ ^{1}TT^{*}G}:(g,\xi)\in G\circledS\mathfrak{g}\right\} \tag{42}$$

of $T\left(G\circledS\mathfrak{g}^{*}\right)$ and this determines the trivialized Euler-Lagrange equations (14).

An alternative way to obtain $S_{\ ^{1}TT^{*}G}$ is to consider a function $\bar{L}$ on $G\circledS\mathfrak{g}$ together with the special symplectic structure (38). This time using the trivialization $tr^{1}_{T^{*}(G\circledS\mathfrak{g})}$ in Eq.(24), we obtain the trivialization of exterior derivative ${}^{1}d\bar{L}:=tr^{R}_{T^{*}(G\circledS\mathfrak{g})}\circ d\bar{L}$ which defines, through $im\left({}^{1}d\bar{L}\right)$, the Lagrangian submanifold

$$S_{\ ^{1}T^{*}TG}=\left\{\left(g,\xi,T^{*}_{e}R_{g}\frac{\delta\bar{L}}{\delta g}+ad^{*}_{\xi}\frac{\delta\bar{L}}{\delta\xi},\frac{\delta\bar{L}}{\delta\xi}\right)\in\ {}^{1}T^{*}TG:(g,\xi)\in G\circledS\mathfrak{g}\right\} \tag{43}$$

of $\left({}^{1}T^{*}TG,\ {}^{1}\Omega_{T^{*}(G\circledS\mathfrak{g})}\right)$. The inverse ${}^{1}\bar{\sigma}_{G}^{-1}$ of the diffeomorphism ${}^{1}\bar{\sigma}_{G}$ in Eq.(28) takes the Lagrangian submanifold $im\left({}^{1}d\bar{L}\right)$ in Eq.(43) to the Lagrangian submanifold $S_{\ ^{1}TT^{*}G}$ in Eq.(40).

■

3.4. Trivialized Hamiltonian Dynamics as a Lagrangian Submanifold.

Proposition 3.2. *The Lagrangian submanifold defined by the equation*

$$-\left({}^{1}\tau_{G\circledS\mathfrak{g}^{*}}\right)^{*}d\bar{H}=\theta_{1} \tag{44}$$

determines the trivialized Hamilton's equations (20). Here, ${}^{1}\tau_{G\circledS\mathfrak{g}^{*}}$ is the tangent bundle projection and θ_{1} is the one-form in Eq.(34).

Proof. Under the global trivialization ${}^{1}TT^{*}G$ of $TT^{*}G$ given in Eq.(26), the Lagrangian submanifold (44) can be described as

$$S'_{\ ^{1}TT^{*}G}=\left\{\left(g,\mu,\frac{\delta\bar{H}}{\delta\mu},-T^{*}R_{g}\frac{\delta\bar{H}}{\delta g}\right)\in\ {}^{1}TT^{*}G:(g,\mu)\in G\circledS\mathfrak{g}^{*}\right\}. \tag{45}$$

The reconstruction mapping $\left(tr^1_{T(G\circledS\mathfrak{g}^*)}\right)^{-1}$ in Eq.(41) maps $S'{}_{{}^1TT^*G}$ to the Lagrangian submanifold

$$S'_{T(G\circledS\mathfrak{g}^*)}=\left\{\left(TR_g\left(\frac{\delta\bar{H}}{\delta\mu}\right),ad^*_{\frac{\delta\bar{H}}{\delta\mu}}\mu-TR^*_g\frac{\delta\bar{H}}{\delta g}\right)\in T\left(G\circledS\mathfrak{g}^*\right):(g,\mu)\in G\circledS\mathfrak{g}^*\right\} \tag{46}$$

which is the image of Hamiltonian vector field $X^{G\circledS\mathfrak{g}^*}_{\bar{H}}$ defined in Eq.(20).

Alternatively, using the trivialization of the exterior derivative

$$-\ {}^1d\bar{H}=-tr^R_{T^*(G\circledS\mathfrak{g}^*)}\circ d\left(\bar{H}\right)$$

we obtain the Lagrangian submanifold

$$S'_{{}^1T^*T^*G}=\left\{\left(g,\mu,ad^*_{\frac{\delta\bar{H}}{\delta\mu}}\mu-T^*_eR_g\frac{\delta\bar{H}}{\delta g},-\frac{\delta\bar{H}}{\delta\mu}\right)\in\ {}^1T^*T^*G:(g,\mu)\in G\circledS\mathfrak{g}^*\right\} \tag{47}$$

of ${}^1T^*T^*G$. The inverse ${}^1\Omega^\sharp_{G\circledS\mathfrak{g}^*}$ of the isomorphism ${}^1\Omega^\flat_{G\circledS\mathfrak{g}^*}$ maps $S'_{{}^1T^*T^*G}$ to the Lagrangian submanifold $S'{}_{{}^1TT^*G}$. This description of $S'_{{}^1TT^*G}$ is the usual form of Hamilton's equation with respect to the symplectic two-form ${}^1\Omega_{G\circledS\mathfrak{g}^*}$.

■

3.5. Legendre Transformation for Trivialized Dynamics.

In the previous section, the trivialized Euler-Lagrange equations (14) have been reformulated as the Lagrangian submanifold $S_{{}^1TT^*G}$ described in Eq.(40). We are now ready to perform the Legendre transformation, that is to describe $S_{{}^1TT^*G}$ from Hamiltonian side (bundles over $G\circledS\mathfrak{g}^*$) of the trivialized Tulczyjew's triplet (27).

Proposition 3.3. *The Lagrangian dynamics determined by the Lagrangian submanifold $S_{{}^1TT^*G}$ in Eq.(40) is generated by the Morse family*

$$E^{\bar{L}\to\bar{H}}=\left(\bar{L}\circ\ {}^1T\pi_G\right)+\Delta=\bar{L}\left(g,\xi\right)-\left\langle\mu,\xi\right\rangle \tag{48}$$

defined on the (right) trivialized Pontryagin bundle ${}^1PG=G\circledS\left(\mathfrak{g}\times\mathfrak{g}^*\right)$ over $G\circledS\mathfrak{g}^*$. Here, the function $\Delta=\langle\mu,\xi\rangle$ is defined as to satisfy

$$d\Delta=\theta_1-\theta_2=-\left\langle\mu,\xi_3\right\rangle-\left\langle\nu_2,\xi\right\rangle.$$

Remark 3.4. The right trivialization of the Pontryagin bundle $PG=TG\times_G T^*G$ is

$$\begin{aligned} tr^1_{PG}&:TG\times_G T^*G\to G\circledS\left(\mathfrak{g}\times\mathfrak{g}^*\right)=:\ {}^1PG\\ &:\left(V_g,\alpha_g\right)\to\left(g,T_gR_{g^{-1}}V_g,T^*_eR_g\alpha_g\right).\end{aligned}$$

Remark 3.5. The potential function Δ is the value of canonical one-form $\theta_{G\circledS\mathfrak{g}^*}$ on the right invariant vector field $X^{G\circledS\mathfrak{g}^*}_{(\xi,\nu)}$ as given in Eq.(17).

Proof. The Morse family $E^{\bar{L}\to\bar{H}}$, in Eq.(48), determines a Lagrangian submanifold $S_{T^*(G Ⓢ\mathfrak{g}^*)}$ which can be described by the equations

$$\alpha_g = \frac{\delta E^{\bar{L}\to\bar{H}}}{\delta g} = \frac{\delta \bar{L}}{\delta g}, \quad \alpha_\mu = \frac{\delta E^{\bar{L}\to\bar{H}}}{\delta \mu} = -\xi, \quad 0 = \frac{\delta E^{\bar{L}\to\bar{H}}}{\delta \xi} = \frac{\delta \bar{L}}{\delta \xi} - \mu$$

defined on the coordinates (α_g, α_μ) of $T^*_{(g,\mu)}(G Ⓢ \mathfrak{g}^*)$. The trivialization $tr^1_{T^*(G Ⓢ\mathfrak{g}^*)}$ maps $S_{T^*(G Ⓢ\mathfrak{g}^*)}$ to the Lagrangian submanifold

$$S_{\,{}^1T^*T^*G} = \left(g, \frac{\delta \bar{L}}{\delta \xi}, T^*R_g \frac{\delta \bar{L}}{\delta g} - ad^*_\xi \frac{\delta \bar{L}}{\delta \xi}, -\xi\right)$$

of ${}^1T^*T^*G$. The musical isomorphism ${}^1\Omega^\sharp_{G Ⓢ\mathfrak{g}^*}$, in turn, maps $S_{\,{}^1T^*T^*G}$ to the Lagrangian submanifold $S_{\,{}^1TT^*G}$ in Eq.(40). ■

Remark 3.6. If we have $\bar{L} = l(\xi)$, the trivialized Euler-Lagrange equations reduce to Euler-Poincaré equations. In this case, the Legendre transformation is generated by the Morse family

$$E^{\bar{L}\to\bar{H}} = l(\xi) - \langle \mu, \xi \rangle . \tag{49}$$

The inverse Legendre transformation defines a Lagrangian formulation for the trivialized Hamilton's Eq.(20) which is represented by the Lagrangian submanifold $S'_{\,{}^1TT^*G}$ described in Eq.(45). The following proposition shows how to find an alternative generating family for $S'_{\,{}^1TT^*G}$ that will lead to its representation on the Lagrangian side of the triplet (27).

Proposition 3.7. *The Morse family*

$$E^{\bar{H}\to\bar{L}} = \left(-\bar{H} \circ^1 T\pi_G\right) - \Delta = \langle \mu, \xi \rangle - \bar{H}(g, \mu) \tag{50}$$

defined on the trivialized Pontryagin bundle ${}^1PG = G Ⓢ (\mathfrak{g} \times \mathfrak{g}^*)$ *over* $G Ⓢ \mathfrak{g}$ *determines the Lagrangian submanifold* $S'_{{}^1TT^*G}$ *in Eq.(45).*

Proof. The Lagrangian submanifold $S_{T^*(G Ⓢ\mathfrak{g})}$ of $T^*(G Ⓢ \mathfrak{g})$ defined by the Morse family (50) is given by

$$\alpha_g = \frac{\delta E^{\bar{H}\to\bar{L}}}{\delta g} = -\frac{\delta \bar{H}}{\delta g}, \quad \alpha_\xi = \frac{\delta E^{\bar{H}\to\bar{L}}}{\delta \xi} = \mu, \quad 0 = \frac{\delta E^{\bar{H}\to\bar{L}}}{\delta \mu} = -\frac{\delta \bar{H}}{\delta \mu} + \xi,$$

where (α_g, α_ξ) are coordinates on $T^*_{(g,\xi)}(G Ⓢ \mathfrak{g})$. The trivialization $tr^1_{T^*(G Ⓢ\mathfrak{g})}$ in Eq.(24) maps $S_{T^*(G Ⓢ\mathfrak{g})}$ to the Lagrangian submanifold

$$S_{\,{}^1T^*TG} = \left(g, \xi, -T^*R_g \frac{\delta \bar{H}}{\delta g} + ad^*_{\frac{\delta \bar{H}}{\delta \mu}} \mu, \mu\right)$$

of ${}^1T^*TG$. The inverse of the isomorphism ${}^1\bar{\sigma}_G$ in Eq.(28) takes $S_{\,{}^1T^*TG}$ to the Lagrangian submanifold $S'_{{}^1TT^*G}$ in Eq.(45). ■

Remark 3.8. When $\bar{H} = h(\mu)$, the resulting Morse family

$$E^{\bar{H}\to\bar{L}} = \langle \mu, \xi \rangle - h(\mu) \tag{51}$$

generates the Lie-Poisson dynamics.

4. The Reduced Dynamics

4.1. Reduction of Tulczyjew's symplectic space.

The left action

$$(g;(h,\lambda,\xi_2,\nu_2)) \to \left(gh, Ad^*_{g^{-1}}\lambda, Ad_{g^{-1}}\xi_2, Ad^*_{g^{-1}}\nu_2\right) \tag{52}$$

of G on ${}^1TT^*G$ is symplecitc which has the momentum mapping $\mathbf{J}_{\,{}^1TT^*G}$ defined by

$$\left\langle \mathbf{J}_{\,{}^1TT^*G}(g,\mu,\xi,\nu),\eta\right\rangle = \theta_1\left(X^{{}^1TT^*G}_{(\eta,0,0,0)}\right) = \theta_2\left(X^{{}^1TT^*G}_{(\eta,0,0,0)}\right). \tag{53}$$

Here, $X^{{}^1TT^*G}_{(\eta,0,0,0)}$ is the infinitesimal generators of the action and derived from Eq.(33). θ_1 and θ_2 are potential one-forms on ${}^1TT^*G$ given in Eqs.(34) and (35), respectively. Explicitly, we have that

$$\mathbf{J}_{\,{}^1TT^*G} : {}^1TT^*G \to \mathfrak{g}^* : (g,\mu,\xi,\nu) \to \nu + ad^*_\xi\mu$$

and the inverse image $\mathbf{J}^{-1}_{\,{}^1TT^*G}(\lambda)$ for a regular value $\lambda = \nu + ad^*_\xi\mu \in \mathfrak{g}^*$ is a submanifold of ${}^1TT^*G$. Note that the restriction to what defines an equivalence relation, will be denoted by $\sim_\lambda$, only on the fiber components, that is

$$\begin{aligned}\mathbf{J}^{-1}_{\,{}^1TT^*G}(\lambda) &= {}^1TT^*G/\sim_\lambda = ((G \circledS \mathfrak{g}^*) \circledS (\mathfrak{g} \circledS \mathfrak{g}^*))/\sim_\lambda \\ &= G \circledS (\mathfrak{g}^* \times \mathfrak{g} \times \mathfrak{g}^*)/\sim_\lambda .\end{aligned}$$

We may consider $\mathbf{J}^{-1}_{\,{}^1TT^*G}(\lambda)$ as an embedded submanifold of ${}^1TT^*G$ by the image of inclusion

$$\imath_{\,{}^1TT^*G} : G \times (\mathfrak{g}^* \times \mathfrak{g}) \to \mathbf{J}^{-1}_{\,{}^1TT^*G}(\lambda) = (g,\mu,\xi) \to \left(g,\mu,\xi,\lambda - ad^*_\xi\mu\right). \tag{54}$$

The isotropy group G_λ of $\lambda \in \mathfrak{g}^*$ acts on $\mathbf{J}^{-1}_{\,{}^1TT^*G}(\lambda)$ and the quotient is

$$\begin{aligned}\left({}^1TT^*G\right)^{/G} &:= \mathbf{J}^{-1}_{\,{}^1TT^*G}(\lambda)/G_\lambda \simeq G \times (\mathfrak{g}^* \times \mathfrak{g})/G_\lambda \\ &\simeq G/G_\lambda \times (\mathfrak{g}^* \times \mathfrak{g}) \simeq \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g},\end{aligned} \tag{55}$$

where $\mathcal{O}_\lambda$ is the coadjoint orbit throught the point $\lambda \in \mathfrak{g}^*$. The reduction can be summarized by the following diagram

$$\begin{array}{ccc}\mathbf{J}^{-1}_{\,{}^1TT^*G}(\lambda) = G \circledS (\mathfrak{g}^* \times \mathfrak{g}) & \xrightarrow{\ \imath_{{}^1TT^*G}\ } & {}^1TT^*G \\ \downarrow \chi_{{}^1TT^*G} & \swarrow p_{{}^1TT^*G} & \\ \mathbf{J}^{-1}_{\,{}^1TT^*G}(\lambda)/G_\lambda = \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} & & \end{array} \tag{56}$$

where the projections are given by

$$p_{\,{}^1TT^*G} : {}^1T^*T^*G \to \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} : (g,\mu,\xi,\nu) \to \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) \tag{57}$$

$$\chi_{\,{}^1TT^*G} : G \circledS (\mathfrak{g}^* \times \mathfrak{g}) \to \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} : (g,\mu,\xi) \to \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right). \tag{58}$$

We will use the abbreviation $\mathfrak{z}_d$ to denote the triple $\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}$.

We now compute the vector fields and one-forms on the reduced Tulczyjew's symplectic space $\mathfrak{z}_d$, by pushing the tensor fields on ${}^1TT^*G$ forward by the projection $p_{\ {}^1TT^*G}$. At the point (g,μ,ξ,ν), the tangent mapping of $p_{\ {}^1TT^*G}$ is

$$\begin{aligned} T_{(g,\mu,\xi,\nu)}\left(p_{\ {}^1TT^*G}\right) : T_{(g,\mu,\xi,\nu)}\left({}^1TT^*G\right) &\to T_{\left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right)}\left(\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}\right) \\ : (V_g, V_\mu, V_\xi, V_\nu) &\to \left(ad^*_{TR_{g^{-1}}V_g} \circ Ad^*_{g^{-1}}\lambda, V_\mu, V_\xi\right). \end{aligned}$$

To obtain vector fields on the reduced space, we push a right invariant vector field $X^{{}^1TT^*G}_{(\eta,\upsilon,\zeta,\bar{\upsilon})}$, in the form given by Eq.(33), forward by $p_{\ {}^1TT^*G}$. To do this, we derive the constraint $\lambda = \nu + ad^*_\xi\mu$ and obtain

$$\nu_3 + ad^*_{\xi_2}\nu + ad^*_{\xi_2} \circ ad^*_\xi\mu + ad^*_{\xi_3}\mu = 0,$$

and use this in the definition of $X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ to define elements of the tangent space of $\mathbf{J}^{-1}_{{}^1TT^*G}(\lambda)$. Hence we arrive at the vector field

$$X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) = \left(ad^*_\eta \circ Ad^*_{g^{-1}}\lambda, \upsilon + ad^*_\eta\mu, \zeta + [\xi,\eta]\right) \tag{59}$$

on $\mathfrak{z}_d$. The Jacobi-Lie bracket of two such vector fields is

$$\left[X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}, X^{\mathfrak{z}_d}_{(\bar{\eta},\bar{\upsilon},\bar{\zeta})}\right] = X^{\mathfrak{z}_d}_{\left([\eta,\bar{\eta}], ad^*_{\bar{\eta}}\upsilon - ad^*_\eta\bar{\upsilon}, \left[\eta,\bar{\zeta}\right] - [\bar{\eta},\zeta]\right)}. \tag{60}$$

Acoording to the Marsden-Weinstein symplectic reduction theorem, the quotient $\mathfrak{z}_d$ is a symplectic manifold with symplectic two form $\Omega_{\mathfrak{z}_d}$ satisfying

$$\left(\chi_{\ {}^1T^*T^*G}\right)^* \Omega_{\mathfrak{z}_d} = \left(\imath_{\ {}^1T^*T^*G}\right)^* \Omega_{\ {}^1TT^*G}, \tag{61}$$

where $\Omega_{\ {}^1TT^*G}$ is the symplectic two-form in Eq.(36).

Proposition 4.1. *The reduced Tulczyjew's manifold $\mathfrak{z}_d = \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}$ is an exact symplectic manifold with symplectic two-form $\Omega_{\mathfrak{z}_d}$, potential one-forms χ_1 and χ_2 whose values on vector fields of the form of Eq.(59) at a point $\left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) \in \mathfrak{z}_d$ are*

$$\left\langle \Omega_{\mathfrak{z}_d}, \left(X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}, X^{\mathfrak{z}_d}_{(\bar{\eta},\bar{\upsilon},\bar{\zeta})}\right)\right\rangle = \left\langle \upsilon, \bar{\zeta}\right\rangle - \left\langle \bar{\upsilon}, \zeta\right\rangle - \left\langle \lambda, [\eta,\bar{\eta}]\right\rangle, \tag{62}$$

$$\left\langle \chi_1, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\right\rangle \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) = \left\langle \lambda, \eta\right\rangle - \left\langle \upsilon, \xi\right\rangle, \tag{63}$$

$$\left\langle \chi_2, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\right\rangle \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) = \left\langle \lambda, \eta\right\rangle + \left\langle \mu, \zeta\right\rangle, \tag{64}$$

respectively.

Proof. Using the defining relation in Eq.(61), a direct calculation shows that the reduced symplectic two-form is in form (62). We recall definitions of the potential one-forms θ_1 and θ_2 in Eqs.(34) and (35). Define one-forms χ_1 and χ_2 on $\mathfrak{z}_d$ by the equations

$$\begin{aligned}\left\langle \theta_1, X^{\,{}^1TT^*G}_{(\eta,\upsilon,\zeta,\bar{\upsilon})}\right\rangle (g,\mu,\xi,\nu) &= \left\langle \chi_1, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\right\rangle \left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right),\\ \left\langle \theta_2, X^{\,{}^1TT^*G}_{(\eta,\upsilon,\zeta,\bar{\upsilon})}\right\rangle (g,\mu,\xi,\nu) &= \left\langle \chi_2, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\right\rangle \left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right).\end{aligned}$$

The exterior derivative of χ_1 in Eq.(63) gives the symplectic two-form $\Omega_{\mathfrak{z}_d}$. Using the invariant definition of exterior derivative we obtain

$$\begin{aligned}\left\langle \Omega_{\mathfrak{z}_d};\left(X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)},X^{\mathfrak{z}_d}_{(\bar{\eta},\bar{\upsilon},\bar{\zeta})}\right)\right\rangle &= X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\left\langle \chi_1, X^{\mathfrak{z}_d}_{(\bar{\eta},\bar{\upsilon},\bar{\zeta})}\right\rangle - X^{\mathfrak{z}_d}_{(\bar{\eta},\bar{\upsilon},\bar{\zeta})}\left\langle \chi_1, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\right\rangle\\ &\quad -\left\langle \chi_1,\left[X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)},X^{\mathfrak{z}_d}_{(\bar{\eta},\bar{\upsilon},\bar{\zeta})}\right]\right\rangle\\ &= -\left\langle \bar{\upsilon},\zeta+[\xi,\eta]\right\rangle - \left(-\left\langle \upsilon,\bar{\zeta}+[\xi,\bar{\eta}]\right\rangle\right)\\ &\quad -\left\langle \lambda,[\eta,\bar{\eta}]\right\rangle - \left\langle ad^*_{\bar{\eta}}\upsilon - ad^*_{\eta}\bar{\upsilon},\xi\right\rangle\\ &= \left\langle \upsilon,\bar{\zeta}\right\rangle - \left\langle \bar{\upsilon},\zeta\right\rangle - \left\langle \lambda,[\eta,\bar{\eta}]\right\rangle, \end{aligned} \tag{65}$$

where we used the fact that $\langle\lambda,\eta\rangle$ is a constant for a fixed λ, and the Jacobi Lie bracket in Eq.(60). Similarly, we can show $\Omega_{\mathfrak{z}_d} = d\chi_2$. It follows from Eqs.(63) and (64) that the difference

$$\chi_2-\chi_1 = d\langle\mu,\xi\rangle = d\Delta \tag{66}$$

is an exact one-form on $\mathfrak{z}_d$. ■

Remark 4.2. The reduced symplectic two-form $\Omega_{\mathfrak{z}_d}$ is an example of the reduced product dynamics defined in the proposition 5.4 of [49].

4.2. Reduction of Tulczyjew's triplet.

Application of the Marsden-Weinstein reduction for the left action of G to the iterated bundles in the trivialized Tulczyjew's triplet results in symplectic projections

$$\begin{aligned} p_{\,{}^1T^*TG} &: \left({}^1T^*TG,\Omega_{\,{}^1T^*TG}\right)\to\left(\mathfrak{z}_l=\mathcal{O}_\lambda\times\mathfrak{g}\times\mathfrak{g}^*,\Omega_{\mathfrak{z}_l}\right) &(67)\\ &: (g,\xi,\lambda,\nu)\to\left(Ad^*_{g^{-1}}\lambda,\xi,\nu\right),\\ p_{\,{}^1T^*T^*G} &: \left({}^1T^*T^*G,\Omega_{\,{}^1T^*T^*G}\right)\to\left(\mathfrak{z}_d=\mathcal{O}_\lambda\times\mathfrak{g}^*\times\mathfrak{g},\Omega_{\mathfrak{z}_d}\right) &(68)\\ &: (g,\mu,\lambda,\xi)\to\left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right),\\ p_{\,{}^1TT^*G} &: \left({}^1TT^*G,\Omega_{\,{}^1TT^*G}\right)\to\left(\mathfrak{z}_h=\mathcal{O}_\lambda\times\mathfrak{g}^*\times\mathfrak{g},\Omega_{\mathfrak{z}_h}\right) &(69)\\ &: (g,\mu,\xi,\nu)\to\left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right), \end{aligned}$$

into reduced spaces, where $\mathcal{O}_\lambda$ is the coadjoint orbit through $\lambda\in\mathfrak{g}^*$. In the last line, we take the fiber coordinate $\upsilon=\lambda-ad^*_{\xi}\mu$ [16] in order to have convenience in projected coordinates as described by Eqs.(71) and (72) below, as well as in projections in Eqs.(74)-(77).

Remark 4.3. In [16], it is shown that, the left action of G on iterated bundles can be trivialized to act on the fiber variables ξ, λ and ν. That means, while performing symplectic quotients, one should consider, literally, the orbits $G_{\lambda}\backslash(G \times \mathfrak{g} \times \mathfrak{g}^{*})$. However, to have a more clear notation, we prefer to take $\mathcal{O}_{\lambda} \times \mathfrak{g} \times \mathfrak{g}^{*}$ which is, indeed, diffeomorphic to the correct reduced space [2].

Following [16], we rewrite the reduced Tulczyjew's triplet (3) with a more clear notation as follows:

$$
\begin{array}{ccccc}
\mathfrak{z}_l = \mathcal{O}_{\lambda} \times \mathfrak{g} \times \mathfrak{g}^{*} & \xleftarrow{\ \varkappa\ } & \mathfrak{z}_d = \mathcal{O}_{\lambda} \times \mathfrak{g}^{*} \times \mathfrak{g} & \xrightarrow{\ \omega^{\flat}\ } & \mathfrak{z}_h = \mathcal{O}_{\lambda} \times \mathfrak{g}^{*} \times \mathfrak{g} \\
\tau_{\mathfrak{z}_l} \searrow & & \swarrow \tau_{\mathfrak{z}_d} \quad \pi_{\mathfrak{z}_d} \searrow & & \swarrow \pi_{\mathfrak{z}_h} \\
& \mathfrak{g} & & \mathfrak{g}^{*} &
\end{array}
\tag{70}
$$

consisting of the symplectic diffeomorphisms

$$
\varkappa : \mathfrak{z}_d \to \mathfrak{z}_l : \left(Ad^{*}_{g^{-1}}\lambda, \mu, \xi\right) \to \left(Ad^{*}_{g^{-1}}\lambda, \xi, \mu\right), \tag{71}
$$

$$
\omega^{\flat} : \mathfrak{z}_d \to \mathfrak{z}_h : \left(Ad^{*}_{g^{-1}}\lambda, \mu, \xi\right) \to \left(Ad^{*}_{g^{-1}}\lambda, \mu, -\xi\right) \tag{72}
$$

obtained from the trivialized symplectic diffeomorphisms ${}^{1}\bar{\sigma}_{G}$ and ${}^{1}\Omega^{\flat}_{G\circledS\mathfrak{g}^{*}}$ in Eqs.(28) and (29) by the equations

$$
\varkappa \circ p_{\ {}^{1}TT^{*}G} = p_{\ {}^{1}T^{*}TG} \circ {}^{1}\bar{\sigma}_{G}, \quad \text{and} \quad \omega^{\flat} \circ p_{\ {}^{1}TT^{*}G} = p_{\ {}^{1}T^{*}TG} \circ {}^{1}\Omega^{\flat}_{G\circledS\mathfrak{g}^{*}}. \tag{73}
$$

The projections $\tau_{\mathfrak{z}_l}$, $\tau_{\mathfrak{z}_d}$, $\pi_{\mathfrak{z}_d}$ and $\pi_{\mathfrak{z}_h}$ are trivial

$$
\tau_{\mathfrak{z}_l} : \mathfrak{z}_l \to \mathfrak{g} : \left(Ad^{*}_{g^{-1}}\lambda, \xi, \mu\right) \to \xi, \tag{74}
$$

$$
\tau_{\mathfrak{z}_d} : \mathfrak{z}_d \to \mathfrak{g} : \left(Ad^{*}_{g^{-1}}\lambda, \mu, \xi, \right) \to \xi, \tag{75}
$$

$$
\pi_{\mathfrak{z}_d} : \mathfrak{z}_d \to \mathfrak{g}^{*} : \left(Ad^{*}_{g^{-1}}\lambda, \mu, \xi\right) \to \mu, \tag{76}
$$

$$
\pi_{\mathfrak{z}_h} : \mathfrak{z}_h \to \mathfrak{g}^{*} : \left(Ad^{*}_{g^{-1}}\lambda, \mu, \xi\right) \to \mu. \tag{77}
$$

The explicit expressions of the reduced symplectic two-forms $\Omega_{\mathfrak{z}_l}$ and $\Omega_{\mathfrak{z}_h}$ on the product bundles $\mathfrak{z}_l$ and $\mathfrak{z}_h$ can be obtained by the pull-back of $\Omega_{\mathfrak{z}_d}$ in Eq.(62) with the symplectic diffeomorphisms $\varkappa$ and $\omega^{\flat}$ in Eqs.(71) and (72), respectively.

In [2], a detailed study on the Hamiltonian dynamics on $\mathfrak{z}_l$, in connection with those on $T^{*}TG$, can be found.

4.3. Euler-Poincaré dynamics as a Lagrangian submanifold.

When $\bar{L} = l(\xi)$, the trivialized exterior derivative ${}^{1}d\bar{L}$ obtained in Eq.(43) becomes

$$
{}^{1}dl : \mathfrak{g} \to {}^{1}T^{*}TG : \xi \to \left(g, \xi, ad^{*}_{\xi}\frac{\delta l}{\delta \xi}, \frac{\delta l}{\delta \xi}\right). \tag{78}
$$

The image of trivialized exterior derivative ${}^{1}dl$ can be reduced to $\mathfrak{z}_l$ by composition with the projection map $p_{\ {}^{1}T^{*}TG}$ in Eq.(67). That is, we define

$$
d^{G\backslash}l = p_{\ {}^{1}T^{*}TG} \circ {}^{1}dl : \mathfrak{g} \to \mathfrak{z}_l : \xi \to \left(ad^{*}_{\xi}\frac{\delta l}{\delta \xi}, \xi, \frac{\delta l}{\delta \xi}\right),
$$

where we choose $g=e$ without loss of generality. Applying the inverse $\varkappa^{-1}$ of the symplectic diffeomorphism $\varkappa:\mathfrak{z}_d\to\mathfrak{z}_l$ in Eq.(71), we define Lagrange-Dirac derivative

$$\mathfrak{d}l=\varkappa^{-1}\circ d^{G\backslash}l=\varkappa^{-1}\circ p_{\ ^1T^*TG}\circ\ ^1dl:\mathfrak{g}\to\mathfrak{z}_d:\xi\to\left(ad_\xi^*\frac{\delta l}{\delta\xi},\frac{\delta l}{\delta\xi},\xi\right). \tag{79}$$

Proposition 4.4. *The image of Lagrange-Dirac derivative* $\mathfrak{d}l$*, in Eq.(79), is a Lagrangian submanifold* $s_{\mathfrak{z}_d}$ *of* $(\mathfrak{z}_d,\Omega_{\mathfrak{z}_d})$ *defining the Euler-Poincaré equations (16).*

Proof. Since, the trivialization map $tr^1_{T^*T^*G}$ is symplectic, the image of ^1dl is a Lagrangian submanifold of $^1T^*TG$. The projection $p_{\ ^1T^*TG}$ is symplectic, and hence the image of $d^{G\backslash}l$ is a Lagrangian submanifold $s_{\mathfrak{z}_l}$ of $\mathfrak{z}_l$. The inverse symplectic diffeomorphism $\varkappa^{-1}$ maps this Lagrangian submanifold $s_{\mathfrak{z}_l}$ to a Lagrangian submanifold $s_{\mathfrak{z}_d}$ of $\mathfrak{z}_d$. So, the image $s_{\mathfrak{z}_d}$ of $\mathfrak{d}l$ is a Lagrangian submanifold of $(\mathfrak{z}_d,\Omega_{\mathfrak{z}_d})$. Under the global trivialization, $s_{\mathfrak{z}_d}$ is obtained to be

$$s_{\mathfrak{z}_d}=\left\{\left(ad_\xi^*\frac{\delta l}{\delta\xi},\frac{\delta l}{\delta\xi},\xi\right)\in\mathfrak{z}_d:\xi\in\mathfrak{g}\right\}. \tag{80}$$

When $\bar{L}=l(\xi)$, the Lagrangian submanifold $S_{\ ^1TT^*G}$ in Eq.(40) reduces to

$$s_{\ ^1TT^*G}=\left\{\left(e,\frac{\delta l}{\delta\xi},\xi,0\right)\in\ ^1TT^*G:\xi\in\mathfrak{g}\right\},$$

and the first definition in Eq.(73) shows that the projection of $s_{\ ^1TT^*G}$ by $p_{\ ^1TT^*G}$ is $s_{\mathfrak{z}_d}$. The reconstruction mapping $^1TT^*G\to T(G\circledS\mathfrak{g}^*)$ in Eq.(41) takes $s_{\ ^1TT^*G}$ to the Lagrangian submanifold

$$s_{T(G\circledS\mathfrak{g}^*)}=\left\{\left(e,\frac{\delta l}{\delta\xi};\xi,ad_\xi^*\frac{\delta l}{\delta\xi}\right)\in T(G\circledS\mathfrak{g}^*):\xi\in\mathfrak{g}\right\}$$

of $T(G\circledS\mathfrak{g}^*)$, and this defines Euler-Poincaré equations (16). ■

Alternatively, the formulation that uses the de Rham exterior derivative and the potential one-form χ_2 in Eq.(64) goes as follows.

Proposition 4.5. *The identity*

$$\tau_{\mathfrak{z}_d}^*dl=\chi_2$$

defines the Lagrangian submanifold $s_{\mathfrak{z}_d}$ *in Eq.(80), hence the Euler-Poincaré equations (16). Here,* $\tau_{\mathfrak{z}_d}$ *is the projection* $\mathfrak{z}_d\to\mathfrak{g}$*,* dl *is the (de Rham) exterior derivative of* l *on* $\mathfrak{g}$*, and* χ_2 *is the potential one-form in Eq.(64).*

Proof. We compute the value of exact one-from $\tau_{\mathfrak{z}_d}^*dl=d(l\circ\tau_{\mathfrak{z}_d})$ on a vector field $X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}$ in Eq.(59). At a point $\left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right)$, we have

$$\begin{aligned}\left\langle \tau^*_{\mathfrak{z}_d} dl, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\right\rangle \left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right) &= \left\langle dl, \left(\tau_{\mathfrak{z}_d}\right)_* X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\right\rangle \\ &= \left\langle \frac{\delta l}{\delta \xi}, \zeta+[\xi,\eta]\right\rangle \\ &= \left\langle \frac{\delta l}{\delta \xi}, \zeta\right\rangle + \left\langle ad^*_{\xi}\frac{\delta l}{\delta \xi}, \eta\right\rangle ,\end{aligned}$$

where $\left(\tau_{\mathfrak{z}_d}\right)_* X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}$ is the push forward of the vector field $X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}$ by the projection $\tau_{\mathfrak{z}_d}$ from $\mathfrak{z}_d$ to its third factor $\mathfrak{g}$, that is, to the vector $\zeta+[\xi,\eta]$ in $T_\xi\mathfrak{g}\simeq\mathfrak{g}$. Equating this to $\left\langle \chi_2, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}\right\rangle$ in Eq.(64) gives the Lagrangian submanifold $s_{\mathfrak{z}_d}=im\left(\mathfrak{d}l\right)$ in Eq.(80) via

$$\lambda = ad^*_{\xi}\frac{\delta l}{\delta \xi} \quad \text{and} \quad \mu=\frac{\delta l}{\delta \xi}$$

in coordinates (λ,μ,ξ) of $\mathfrak{z}_d$. ■

4.4. Lie-Poisson dynamics as a Lagrangian submanifold.

Consider a Hamiltonian function $\bar{H}$ on GⓈ$\mathfrak{g}^*$ and define $h:\mathfrak{g}^*\to\mathbb{R}$ by $\bar{H}=h(\mu)$. With the trivialized exterior derivative

$$-\,{}^1dh:\mathfrak{g}^*\to\ {}^1T^*T^*G:(g,\mu)\to\left(g,\mu,ad^*_{\frac{\delta h}{\delta\mu}}\mu,-\frac{\delta h}{\delta \mu}\right),$$

and the projection $p_{\ {}^1T^*T^*G}:{}^1T^*T^*G\to\mathfrak{z}_h$ in Eq.(68), we define

$$-d^{G\backslash}h=p_{\ {}^1T^*T^*G}\circ\ {}^1d\left(-h\right):\mathfrak{g}^*\to\mathfrak{z}_h:\mu\to\left(ad^*_{\frac{\delta h}{\delta\mu}}\mu,\mu,-\frac{\delta h}{\delta \mu}\right)$$

by choosing $g=e$. Applying the inverse $\omega^\sharp$ of the symplectic diffeomorphism $\omega^\flat$ in Eq.(72), we obtain the Hamilton-Dirac derivative

$$-\mathfrak{d}h=\omega^\sharp\circ d^{G\backslash}\left(-h\right):\mathfrak{g}^*\to\mathfrak{z}_d:\mu\to\left(ad^*_{\frac{\delta h}{\delta\mu}}\mu,\mu,\frac{\delta h}{\delta \mu}\right). \tag{81}$$

Proposition 4.6. *The image of the Hamilton-Dirac derivative $-\mathfrak{d}h$ is a Lagrangian submanifold $s'_{\mathfrak{z}_d}$ of $(\mathfrak{z}_d,\Omega_{\mathfrak{z}_d})$ and it defines the Lie-Poisson equations (23).*

Proof. The image of $-\,{}^1dh$ is a Lagrangian submanifold of ${}^1T^*T^*G$ and $p_{\ {}^1T^*T^*G}$ maps this Lagrangian submanifold to a Lagrangian submanifold $s'_{\mathfrak{z}_h}$ of $\mathfrak{z}_h$. The musical isomorphism $\omega^\sharp$ takes $s'_{\mathfrak{z}_h}$ to the Lagrangian submanifold $s'_{\mathfrak{z}_d}$ of $(\mathfrak{z}_d,\Omega_{\mathfrak{z}_d})$. Thus,

$$s'_{\mathfrak{z}_d}=im\left(-\mathfrak{d}h\right)=\omega^\sharp\circ im\left(d^{G\backslash}\left(-h\right)\right)=\omega^\sharp\circ p_{\ {}^1T^*T^*G}\circ im\left(-\,{}^1dh\right). \tag{82}$$

From Eqs. (68) and (69) we have

$$\omega^\sharp\circ p_{\ {}^1T^*T^*G}=p_{\ {}^1TT^*G}\circ\ {}^1\Omega^\sharp_{G Ⓢ\mathfrak{g}^*},$$

where ${}^{1}\Omega^{\sharp}_{G\circledS\mathfrak{g}^*}$ is the inverse of the isomorphism ${}^{1}\Omega^{\flat}_{G\circledS\mathfrak{g}^*}$ in Eq.(29). This implies that $s'_{\mathfrak{z}_d}$ is the projection $p_{\,{}^1TT^*G}\left(s'_{\,{}^1TT^*G}\right)$ of the Lagrangian submanifold

$$s'_{\,{}^1TT^*G} = \left\{ \left(g, \mu; \frac{\delta h}{\delta \mu}, 0 \right) \in {}^1TT^*G : \mu \in \mathfrak{g}^* \right\}$$

obtained from $S'_{\,{}^1TT^*G}$ in Eq.(45) by substituting $\bar{H} = h(\mu)$. The reconstruction mapping ${}^1TT^*G \to T(G\circledS\mathfrak{g}^*)$ in the Eq.(41) takes $s'_{\,{}^1TT^*G}$ to the Lagrangian submanifold

$$s'_{T(G\circledS\mathfrak{g}^*)} = \left\{ \left(g, \mu; TR_g \frac{\delta h}{\delta \mu}, ad^*_{\frac{\delta h}{\delta \mu}} \mu \right) \in T(G\circledS\mathfrak{g}^*) : \mu \in \mathfrak{g}^* \right\}$$

of $T(G\circledS\mathfrak{g}^*)$ which is the Lie-Poisson equation (21) equivalent to (23). ■

Alternatively, with exterior derivative and the potential one-form χ_1 in Eq.(63), we have

Proposition 4.7. *The equation*

$$-\pi^*_{\mathfrak{z}_d} dh = \chi_1$$

defines the Lagrangian submanifold $s'_{\mathfrak{z}_d}$ in Eq.(82) and gives the Lie-Poisson equations (23). Here, $\pi_{\mathfrak{z}_d}$ is the projection $\mathfrak{z}_d \to \mathfrak{g}^*$, dh is the (de Rham) exterior derivative of h on $\mathfrak{g}^*$, and χ_1 is the potential one-form in Eq.(63).

Proof. To prove this identity, we compute the value of exact one-from $\pi^*_{\mathfrak{z}_d} dh = d(h \circ \pi_{\mathfrak{z}_d})$ on a vector field $X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}$ over $\mathfrak{z}_d$. At a point $\left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right)$, we have

$$\begin{aligned}
\left\langle -\pi^*_{\mathfrak{z}_d} dh, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)} \right\rangle \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) &= -\left\langle dh, (\pi_{\mathfrak{z}_d})_* X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)} \right\rangle \\
&= -\left\langle \frac{\delta h}{\delta \mu}, \upsilon + ad^*_\eta \mu \right\rangle \\
&= -\left\langle \frac{\delta h}{\delta \mu}, \upsilon \right\rangle + \left\langle ad^*_{\frac{\delta h}{\delta \mu}} \mu, \eta \right\rangle,
\end{aligned}$$

where $(\pi_{\mathfrak{z}_d})_* X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}$ is the push forward of the vector field $X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)}$ by the projection $\pi_{\mathfrak{z}_d}$ from $\mathfrak{z}_d$ to its second factor $\mathfrak{g}^*$, that is to the dual vector $\upsilon + ad^*_\eta \mu$ in $T_\mu \mathfrak{g}^* \simeq \mathfrak{g}^*$. Equating this to $\left\langle \chi_1, X^{\mathfrak{z}_d}_{(\eta,\upsilon,\zeta)} \right\rangle$ in Eq.(63) defines the Lagrangian submanifold $s'_{\mathfrak{z}_d} = im(\mathfrak{d}h)$ in Eq.(82) given in coordinates (λ, μ, ξ) of $\mathfrak{z}_d$ by

$$\lambda = ad^*_{\frac{\delta h}{\delta \mu}} \mu \quad \text{and} \quad \xi = \frac{\delta h}{\delta \mu}.$$ ■

4.5. Legendre Transformation for Reduced Dynamics.

Being cotangent bundles, $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^*$ and $T^*\mathfrak{g}^* = \mathfrak{g}^* \times \mathfrak{g}$ are canonically symplectic. It is possible to embed $T^*\mathfrak{g}$ and $T^*\mathfrak{g}^*$ symplectically into the total space $\mathfrak{z}_d$

$$\hat{\varkappa} : T^*\mathfrak{g} \to \mathfrak{z}_d : (\xi, \mu) \to \left(ad^*_\xi \mu, \mu, \xi\right), \tag{83}$$

$$\hat{\omega} : T^*\mathfrak{g}^* \to \mathfrak{z}_d : (\mu, \xi) \to \left(ad^*_\xi \mu, \mu, \xi\right). \tag{84}$$

The following proposition shows how to define Lagrangian submanifold $im\,(\mathfrak{d}l) = s_{\mathfrak{z}_d}$ in Eq.(80) from the right wing (that is from the Hamiltonian side) of the reduced Tulczyjew's triplet (70).

Proposition 4.8. *The Lagrangian dynamics determined by the Lagrangian submanifold $s_{\mathfrak{z}_d}$ in Eq.(80) is generated by the Morse family*

$$E^{l\to h} = (l \circ \tau_{\mathfrak{z}_d}) + \Delta = l\,(\xi) - \langle \mu, \xi \rangle \tag{85}$$

on the bundle $\mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$. Here, Δ is a real valued function on $\mathfrak{g} \times \mathfrak{g}^*$ obtained from the equation

$$\chi_2 - \chi_1 = d\Delta = d\,\langle \mu, \xi \rangle$$

where χ_1 and χ_2 are given in Eqs.(63) and (64), respectively.

Proof. In the trivialized cases, the Legendre transformations have been achieved by Morse families on the trivialized Pontryagin bundle $G \circledS (\mathfrak{g} \times \mathfrak{g}^*)$. For the reduced dynamics, due to the invariance under the group action G, the Morse families will be defined on $G \backslash (G \circledS (\mathfrak{g} \times \mathfrak{g}^*)) \simeq \mathfrak{g} \times \mathfrak{g}^*$. Recall the generating object

$$\begin{array}{ccc} \mathfrak{z}_d & \xleftarrow{\quad\hat{\omega}\quad} & \mathfrak{g}^* \times \mathfrak{g} \\ & \searrow^{\pi_{\mathfrak{z}_d}} & \downarrow \\ & & \mathfrak{g}^* \end{array}$$

where $\hat{\omega}$ is the embedding in Eq.(84). The Morse family $E^{l\to h}$ generates a Lagrangian submanifold $s_{T^*\mathfrak{g}^*}$ of $T^*\mathfrak{g}^*$ given by

$$s_{T^*\mathfrak{g}^*} = \left\{ (\mu, \xi) \in T^*\mathfrak{g}^* : T^*\pi_{\mathfrak{z}_d}\,(\mu, \xi) = dE^{l\to h}\,(\mu, \xi) \right\}. \tag{86}$$

Explicitly, the Lagrangian submanifold $s_{T^*\mathfrak{g}^*}$ consists of two-tuples $(\delta l/\delta\xi, \xi)$. Hence, the image of $s_{T^*\mathfrak{g}^*}$ under the map $\hat{\omega}$ is $s_{\mathfrak{z}_d}$. When the Lagrangian l is not regular then it is not possible to define a function h on $\mathfrak{g}^*$ generating $s_{\mathfrak{z}_d}$. In this case, we only have Morse family $E^{l\to h}$. ■

Remark 4.9. Recall that, the Morse family $E^{\bar{L}\to\bar{H}}$, defined in Eq.(49), is also a generating family for Euler-Poincaré equations. Here, the function $E^{\bar{L}\to\bar{H}}$ is defined on the Pontryagin bundle ${}^1PG$ with base manifold $G \circledS \mathfrak{g}^*$ whereas $E^{l\to h}$, in Eq.(85), is defined on $T^*\mathfrak{g}^*$ with cotangent bundle projection.

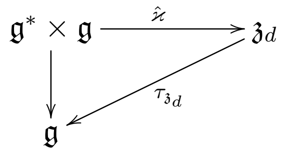
Now, we will establish the inverse Legendre transformation. The Hamiltonian dynamics is defined by a Hamiltonian function h on $\mathfrak{g}^*$. A Hamiltonian functional on $\mathfrak{g}^*$ defines the Lagrangian submanifold $s'_{\mathfrak{z}_d}$, in Eq.(82), of $(\mathfrak{z}_d, \Omega_{\mathfrak{z}_d})$. The following proposition shows how to generate $s'_{\mathfrak{z}_d}$ using the left wing (that is the Lagrangian side) of the reduced Tulczyjew's triplet (70).

Proposition 4.10. *The Morse family*

$$E^{h\to l} = \Delta - h\,(\mu) = \langle \mu, \xi \rangle - h\,(\mu) \tag{87}$$

on the bundle $\mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}$ generates the Lagrangian submanifold $s'_{\mathfrak{z}_d}$ in Eq.(82).

Proof. In this case, diagram is



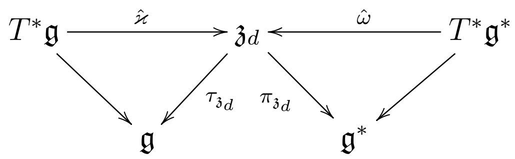
where $\hat{\varkappa}$ is the embedding, in Eq.(83), of $T^*\mathfrak{g}$ into $\mathfrak{z}_d$. The Morse family $E^{h\to l}$ in Eq.(87) generates a Lagrangian submanifold

$$s_{T^*\mathfrak{g}} = \left\{(\xi,\mu) \in T^*\mathfrak{g} : T^*\tau_{\mathfrak{z}_d}(\xi,\mu) = dE^{h\to l}(\xi,\mu)\right\}, \tag{88}$$

of $T^*\mathfrak{g}$. This Lagrangian submanifold consists of two-tuples $(\delta h/\delta\mu, \mu)$. $\hat{\varkappa}$ maps $s_{T^*\mathfrak{g}}$ to $s'_{\mathfrak{z}_d}$. When the Hamiltonian h is not regular then it is not possible to find a Lagrangian function l on $\mathfrak{g}$. In this case, we only have Morse family $E^{h\to l}$. ■

Remark 4.11. Recall that, the Morse family $E^{\bar{H}\to\bar{L}}$, defined in Eq.(51), is another generating family for Lie-Poisson dynamics. $E^{\bar{H}\to\bar{L}}$ is defined on the Pontryagin bundle ^1PG with base manifold GⓈ$\mathfrak{g}$ whereas $E^{l\to h}$, in Eq.(87), is defined on $T^*\mathfrak{g}$ with tangent bundle projection.

To summarize, in order to use the classical formulations of generating objects, we employ the following reduced form of Tulczyjew's triplet



where we replace the total spaces $\mathfrak{z}_l$ and $\mathfrak{z}_h$ by $T^*\mathfrak{g}$ and $T^*\mathfrak{g}^*$, respectively. The payoff is that the mappings $\hat{\varkappa}$ and $\hat{\omega}$ are symplectic embeddings but not isomorphisms.

Note that, after the Legendre transformation, we arrive dynamics on the Pontryagin bundle $PG = TG \times_G T^*G$ or its reduction $\mathfrak{g} \oplus \mathfrak{g}^*$. In the following section we study the dynamics on the Pontryagin bundle.

5. Example: Diffeomorphism Groups

5.1. Group Structure.

Group $\mathcal{D}$ of diffeomorphisms on $\mathcal{Q}$ is a Lie group [4, 15, 41]. Lie algebra of $\mathcal{D}$ is the space $\mathfrak{X}$ of vector fields on $\mathcal{Q}$. The (right) adjoint action Ad of $\mathcal{D}$ on $\mathfrak{X}$ is given by the pull-back operation φ^*X, for $\varphi \in G$ and $X \in \mathfrak{X}$. The infinitesimal adjoint action of an element $Y \in \mathfrak{X}$ on $X \in \mathfrak{X}$ is the Lie derivative of X in the direction of Y, that is $\mathcal{L}_Y X$. The tangent space

$$T_\varphi\mathcal{D} = \{X_\varphi : \mathcal{Q} \to T\mathcal{Q} : X_\varphi = X \circ \varphi \text{ for some } X \in \mathfrak{X}\}$$

at $\varphi \in \mathcal{D}$ consists of material velocity fields. The lifted group multiplication on the tangent bundle $T\mathcal{D}$ is

$$\varpi_{T\mathcal{D}}(X_\varphi, Y_\psi) = X_{\varphi\circ\psi} + T\varphi \circ Y_\psi. \tag{89}$$

The right and the left trivializations of $T\mathcal{D}$ are

$$\begin{aligned} tr^{R}_{T\mathcal{D}} &: T\mathcal{D} \to \mathcal{D} \times \mathfrak{X} : X_{\varphi} \to (\varphi, X) \\ tr^{L}_{T\mathcal{D}} &: T\mathcal{D} \to \mathcal{D} \times \mathfrak{X} : X_{\varphi} \to (\varphi, \varphi^{*} X) . \end{aligned} \tag{90}$$

After choosing the right trivialization $tr^{R}_{T\mathcal{D}}$, we arrive at the semidirect product group multiplication

$$\varpi_{\mathcal{D}\circledS\mathfrak{X}} \left((\varphi, X) , (\psi, Y) \right) = (\varphi\psi, X + \varphi_{*} Y) \tag{91}$$

on $\mathcal{D}\circledS\mathfrak{X}$. The Lie algebra of $\mathcal{D}\circledS\mathfrak{X}$ is $\mathfrak{X}\circledS\mathfrak{X}$ with semi-direct product

$$\left[(X_1, X_2) , (Y_1, Y_2) \right]_{\mathfrak{X}\circledS\mathfrak{X}} = \left([X_1, Y_1] , [X_1, Y_2] - [Y_1, X_2,] \right) .$$

The dual space $\mathfrak{X}^{*}$ of the Lie algebra $\mathfrak{X}$ is the space $\Lambda^{1}(\mathcal{Q}) \otimes Den(\mathcal{Q})$ of one-form densities on $\mathcal{Q}$. The pairing between $\mu \otimes d^{n}q \in \mathfrak{X}^{*}$ and $X \in \mathfrak{X}$ is given by the integration

$$\langle \mu \otimes d^{n}q, X \rangle = \int_{\mathcal{M}} \langle \mu, X \rangle_{\mathcal{Q}} \, d^{n}q, \tag{92}$$

where $d^{n}q$ is the top-form on $\mathcal{Q}$. The pairing inside the integral is the natural pairing of finite dimensional spaces $T_q\mathcal{Q}$ and $T^{*}_{q}\mathcal{Q}$. The coadjoint action Ad^{*} of $\mathcal{D}$ on $\mathfrak{X}^{*}$ is the pull-back operation $\varphi^{*}(\mu \otimes d^{n}q)$ for $\varphi \in \mathcal{D}$ and $\mu \otimes d^{n}q \in \mathfrak{X}^{*}$. The infinitesimal coadjoint action ad^{*} of an element $X \in \mathfrak{X}$ on $\mu \otimes d^{n}q \in \mathfrak{X}^{*}$ is minus the Lie derivative of $\mu \otimes d^{n}q$ by X, that is

$$ad^{*}_{X} : \mathfrak{X}^{*} \to \mathfrak{X}^{*} : \mu \otimes d^{n}q \to -\left(\mathcal{L}_{X}\mu + (div_{d^{n}q}X)\, \mu \right) \otimes d^{n}q. \tag{93}$$

Here, $div_{d^{n}q}X$ denotes divergence of the vector field X with respect to the top-form $d^{n}q$. The cotangent space at φ is

$$T^{*}_{\varphi}\mathcal{D} = \left\{ (\mu_{\varphi} : \mathcal{Q} \to T^{*}\mathcal{Q}) \otimes d^{n}q : \mu_{\varphi} = \mu \circ \varphi, \, \mu \in \Lambda^{1}(\mathcal{Q}) \right\} .$$

The pairing between $T^{*}_{\varphi}\mathcal{D}$ and $T_{\varphi}\mathcal{D}$ is taken to be the right invariant $L^{2}-$integral. Cotangent lifts of right and left actions of $\mathcal{D}$ on $T^{*}\mathcal{D}$ can be computed using

$$T^{*}_{\varphi\circ\psi}R_{\psi^{-1}}(\mu_{\varphi}) = \mu_{\varphi\circ\psi}, \quad T^{*}_{\psi\circ\varphi}L_{\psi^{-1}}\mu_{\varphi} = T^{*}\psi^{-1} \circ \mu_{\varphi},$$

respectively. The cotangent bundle $T^{*}\mathcal{D}$ is a Lie group with the group multiplication

$$(\mu_{\varphi}, \nu_{\psi}) = \mu_{\varphi\circ\psi} + T^{*}\varphi^{-1} \circ \nu_{\psi}.$$

The right and left trivializations of $T^{*}\mathcal{D}$ are

$$\begin{aligned} tr^{R}_{T^{*}\mathcal{D}} &: T^{*}\mathcal{D} \to \mathcal{D} \times \mathfrak{X}^{*} : \mu_{\varphi} \otimes d^{n}q \to (\varphi, \mu \otimes d^{n}q) \\ tr^{L}_{T^{*}\mathcal{D}} &: T^{*}\mathcal{D} \to \mathcal{D} \times \mathfrak{X}^{*} : \mu_{\varphi} \otimes d^{n}q \to (\varphi, \varphi^{*}\mu \otimes \varphi^{*}d^{n}q) . \end{aligned} \tag{94}$$

We choose the right trivialization to arrive at the semi-direct product group structure with multiplication

$$\varpi_{\mathcal{D}\circledS\mathfrak{X}^{*}} \left((\varphi, \mu \otimes d^{n}q) , (\psi, \nu \otimes d^{n}q) \right) = \left(\varphi \circ \psi, (\mu + \varphi_{*}\nu) \otimes \varphi_{*}d^{n}q \right)$$

on the trivialization $\mathcal{D}Ⓢ\mathfrak{X}^*$.

5.2. The Trivialized Dynamics.

Let $\bar{L} = \bar{L}(\varphi, X)$ be a Lagrangian density on $\mathcal{D}Ⓢ\mathfrak{X}$, the trivialized Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\delta \bar{L}}{\delta X} = \frac{\delta \bar{L}}{\delta \varphi} \circ \varphi^{-1} - \mathcal{L}_X \frac{\delta \bar{L}}{\delta X} - (div_{d^n q} X) \frac{\delta \bar{L}}{\delta X}. \tag{95}$$

$\bar{L}$ generates a Lagrangian submanifold

$$S_{^1TT^*\mathcal{D}} = \left(\varphi, \frac{\delta \bar{L}}{\delta X}, X, \frac{\delta \bar{L}}{\delta \varphi} \circ \varphi^{-1}\right) \tag{96}$$

of the trivialized Tulczyjew's symplectic space $^1TT^*\mathcal{D}$ defined by the semi-direct product $(\mathcal{D}Ⓢ\mathfrak{X}^*)\,Ⓢ\,(\mathfrak{X}Ⓢ\mathfrak{X}^*)$. Here, the trivialization map

$$T(\mathcal{D}Ⓢ\mathfrak{X}^*) \to {}^1TT^*\mathcal{D} : (X_\varphi, Y_\mu) \to (\varphi, \mu \otimes d^n q, X, Y_\mu + \mathcal{L}_X \mu + (div_{d^n q} X)\, \mu \otimes d^n q) \tag{97}$$

realizes the relation between the Lagrangian submanifold $S_{^1TT^*\mathcal{D}}$ and the trivialized Euler-Lagrange equation (95). The Legendre transformation of trivialized Euler-Lagrange equation can be achieved by the Morse family

$$E(\varphi, \mu \otimes d^n q, X) = \bar{L}(\varphi, X) - \int_{\mathcal{Q}} \langle \mu, X \rangle_{\mathcal{Q}}\, d^n q$$

on the Pontryagin bundle $\mathcal{D}Ⓢ(\mathfrak{X}^* \oplus \mathfrak{X})$ over $\mathcal{D}Ⓢ\mathfrak{X}^*$.

A right invariant vector field on $\mathcal{D}Ⓢ\mathfrak{X}^*$ is given by

$$X^{\mathcal{D}Ⓢ\mathfrak{X}^*}_{(X,\nu)}(\varphi, \mu) = (X_\varphi, \nu - \mathcal{L}_X \mu - (div_{d^n q} X)\, \mu \otimes d^n q).$$

At $(\varphi, \mu \otimes d^n q)$, the values of canonical one-form $\theta_{\mathcal{D}Ⓢ\mathfrak{X}^*}$ and the symplectic two-form $\Omega_{\mathcal{D}Ⓢ\mathfrak{X}^*}$ on the right invariant vector fields are

$$\left\langle \theta_{\mathcal{D}Ⓢ\mathfrak{X}^*}, X^{\mathcal{D}Ⓢ\mathfrak{X}^*}_{(X,\nu)} \right\rangle = \int_{\mathcal{Q}} \langle \mu, X \rangle_{\mathcal{Q}}\, d^n q,$$

$$\left\langle \Omega_{\mathcal{D}Ⓢ\mathfrak{X}^*}; \left(X^{\mathcal{D}Ⓢ\mathfrak{X}^*}_{(X,\nu)}, X^{\mathcal{D}Ⓢ\mathfrak{X}^*}_{(Y,\lambda)}\right) \right\rangle = \int_{\mathcal{Q}} \langle \nu, Y \rangle_{\mathcal{Q}} - \langle \lambda, X \rangle_{\mathcal{Q}} + \langle \mu, [X, Y] \rangle_{\mathcal{Q}}\, d^n q.$$

For a Hamiltonian function $\bar{H}$ on $\mathcal{D}Ⓢ\mathfrak{X}^*$, the trivialized Hamilton's equations are

$$\frac{d\varphi}{dt} = \left(\frac{\delta \bar{H}}{\delta \mu}\right)_\varphi, \quad \frac{d\mu}{dt} = -\mathcal{L}_{\frac{\delta \bar{H}}{\delta \mu}} \mu - \left(div_{d^n q} \frac{\delta \bar{H}}{\delta \mu}\right) \mu - \left(\frac{\delta \bar{H}}{\delta \varphi}\right) \circ \varphi^{-1}, \tag{98}$$

where, due to the reflexivity assumption, $\delta \bar{H}/\delta \mu$ is assumed to be a vector field in the Lie algebra, and $\left(\delta \bar{H}/\delta \mu\right)_\varphi$ is the material velocity field. The Lagrangian submanifold generated by the Hamiltonian function $\bar{H}$ is

$$S'_{^1TT^*\mathcal{D}} = \left(\varphi, \mu \otimes d^n q, \frac{\delta \bar{H}}{\delta \mu}, -\frac{\delta \bar{H}}{\delta \varphi} \circ \varphi^{-1} \otimes d^n q\right).$$

To establish the link between $S'_{{}^1TT^*\mathcal{D}}$ and Eq.(98), we refer to the trivialization map (97). The Legendre transformation of the Hamiltonian dynamics described by Eqs.(98) results from the Morse family

$$E(\varphi, \mu \otimes d^n q, X) = \int_{\mathcal{Q}} \langle \mu, X \rangle_{\mathcal{Q}} d^n q - \bar{H}(\varphi, \mu)$$

on the Pontryagin bundle $\mathcal{D}Ⓢ(\mathfrak{X}^* \oplus \mathfrak{X})$ over $\mathcal{D}Ⓢ\mathfrak{X}$.

5.3. The Reduced Dynamics.

When the Lagrangian $\bar{L}$ is free of the group variable, we have $\bar{L} = l(X)$ and the trivialized Euler-Lagrange equations (95) reduces to the Euler-Poincaré equations

$$\frac{d}{dt}\frac{\delta l}{\delta X} = -\mathcal{L}_X \frac{\delta l}{\delta X} - (div_{d^n q} X)\frac{\delta l}{\delta X}. \tag{99}$$

Similarly, when the Hamiltonian depends on the fiber variable μ only, $\bar{H}(g, \mu) = h(\mu)$, the trivialized Hamilton's equation (98) gives the Lie-Poisson equation

$$\frac{d\mu}{dt} = -\mathcal{L}_{\frac{\delta h}{\delta \mu}} \mu - \left(div_{d^n q}\frac{\delta h}{\delta \mu}\right)\mu. \tag{100}$$

In order to perform the Legendre transformations of the reduced dynamics, we present them as Lagrangian submanifolds of the reduced Tulczyjew's symplectic space $\mathcal{O}_\lambda \times \mathfrak{X}^* \times \mathfrak{X}$. Here, $\mathcal{O}_\lambda \times \mathfrak{X}^* \times \mathfrak{X}$ is obtained by application of the Marsden-Weinstein reduction to the trivialized Tulczyjew's symplectic space ${}^1TT^*\mathcal{D}$, that is

$$ {}^1TT^*\mathcal{D} \to \mathcal{O}_\lambda \times \mathfrak{X}^* \times \mathfrak{X} : (\varphi, \mu \otimes d^n q, X, \nu \otimes d^n q) \to (\varphi_* (\lambda \otimes d^n q), \mu \otimes d^n q, X), \tag{101}$$

where $\lambda = \nu - \mathcal{L}_X \mu - (div_{d^n q} X)\mu$. For a Lagrangian l on $\mathfrak{X}$, image of the Lagrange-Dirac derivative

$$\mathfrak{d}l : \mathfrak{X} \to \mathcal{O}_\lambda \times \mathfrak{X}^* \times \mathfrak{X} : X \to \left(-\mathcal{L}_X \frac{\delta l}{\delta X} - (div_{d^n q} X)\frac{\delta l}{\delta X} \otimes d^n q, \frac{\delta l}{\delta X} \otimes d^n q, X\right)$$

is a Lagrangian submanifold of $\mathcal{O}_\lambda \times \mathfrak{X}^* \times \mathfrak{X}$. The image $im(\mathfrak{d}l)$ determines Euler-Poincaré equations. The Legendre transformation is generated by the Morse family

$$E^{l \to h}(\mu \otimes d^n q, X) = l(X) - \int_{\mathcal{Q}} \langle \mu, X \rangle_{\mathcal{Q}} d^n q$$

on the bundle $\mathfrak{X}^* \times \mathfrak{X} \to \mathfrak{X}^*$. Similarly, for a Hamiltonian h on $\mathfrak{X}^*$, the image of Hamilton-Dirac derivative

$$-\mathfrak{d}h : \mathfrak{X}^* \to \mathcal{O}_\lambda \times \mathfrak{X}^* \times \mathfrak{X} : \mu \to \left(-\mathcal{L}_{\frac{\delta h}{\delta \mu}} \mu - \left(div_{d^n q}\frac{\delta h}{\delta \mu}\right)\mu \otimes d^n q, \mu \otimes d^n q, \frac{\delta h}{\delta \mu}\right)$$

is a Lagrangian submanifold of $\mathcal{O}_\lambda \times \mathfrak{X}^* \times \mathfrak{X}$ and determines Lie-Poisson equations. The inverse Legendre transformation of the Euler-Poincaré dynamics is generated by the Morse family

$$E^{h \to l}(\mu \otimes d^n q, X) = \int_{\mathcal{Q}} \langle \mu, X \rangle_{\mathcal{Q}} d^n q - h(\mu) \tag{102}$$

on the bundle $\mathfrak{X}^* \times \mathfrak{X} \to \mathfrak{X}$.

6. Summary, Discussions and Prospectives

We obtain trivialized and reduced dynamics for Hamiltonian and Lagrangian formulations of systems with configuration space G. Following diagram summarizes these equations and their representations by Lagrangian submanifolds.

	Dynamics	Lagrangian Submanifold
Trivialized Euler-Lagrange Equations on $G Ⓢ \mathfrak{g}$	$\frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi} = T_e^* R_g \frac{\delta \bar{L}}{\delta g} + ad_\xi^* \frac{\delta \bar{L}}{\delta \xi}$	$\left\{ g, \frac{\delta \bar{L}}{\delta \xi}, \xi, T^* R_g \frac{\delta \bar{L}}{\delta g} \right\} \subset {}^1TT^*G$
Trivialized Hamilton's Equations on $G Ⓢ \mathfrak{g}^*$	$\frac{dg}{dt} = T_e R_g \left(\frac{\delta \bar{H}}{\delta \mu} \right)$, $\frac{d\mu}{dt} = ad^*_{\frac{\delta \bar{H}}{\delta \mu}} \mu - T_e^* R_g \frac{\delta \bar{H}}{\delta g}$	$\left\{ g, \mu, \frac{\delta \bar{H}}{\delta \mu}, -T^* R_g \frac{\delta \bar{H}}{\delta g} \right\} \subset {}^1TT^*G$
Euler-Poincaré Equations on $\mathfrak{g}$	$ad_\xi^* \frac{\delta l}{\delta \xi} - \frac{d}{dt}\frac{\delta l}{\delta \xi} = 0$	$\left\{ ad_\xi^* \frac{\delta l}{\delta \xi}, \frac{\delta l}{\delta \xi}, \xi \right\} \subset \mathfrak{z}_d$
Lie-Poisson Equations on $\mathfrak{g}^*$	$\frac{d\mu}{dt} = ad^*_{\frac{\delta h}{\delta \mu}} \mu$	$\left\{ ad^*_{\frac{\delta h}{\delta \mu}} \mu, \mu, \frac{\delta h}{\delta \mu} \right\} \subset \mathfrak{z}_d$

The Legendre transformation is a change of representation of Lagrangian submanifolds generated by certain Morse families viewed as functions defined on fibered manifolds. We identify the following Morse families for trivialized and reduced dynamics.

	Morse family	Bundle
Trivialized Euler-Lagrange Equations on $G Ⓢ \mathfrak{g}$	$E^{\bar{L} \to \bar{H}} (g, \xi, \mu) = \bar{L}(g, \xi) - \langle \mu, \xi \rangle$	$G Ⓢ (\mathfrak{g} \times \mathfrak{g}^*) \to G Ⓢ \mathfrak{g}^*$
Trivialized Hamilton's Equations on $G Ⓢ \mathfrak{g}^*$	$E^{\bar{H} \to \bar{L}} (g, \xi, \mu) = \langle \mu, \xi \rangle - \bar{H}(g, \mu)$	$G Ⓢ (\mathfrak{g} \times \mathfrak{g}^*) \to G Ⓢ \mathfrak{g}$
Euler-Poincaré Equations on $\mathfrak{g}$	$E^{l \to h} (\xi, \mu) = l(\xi) - \langle \mu, \xi \rangle$	$\mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$
Lie-Poisson Equations on $\mathfrak{g}^*$	$E^{h \to l} (\xi, \mu) = \langle \mu, \xi \rangle - h(\mu)$	$\mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}.$

Obviously, the form of dynamical equations obtained in this work depends on the trivialization we employed. What we refer to trivialization of the first kind carries the group operations to iterated bundles and contributes additional term due to semi-direct product structures. Higher order dynamics on Lie groups with adapted trivializations of higher order and iterated bundles as well as their symplectic and Poisson reductions are available in [14].

The reduction of Tulczyjew's symplectic space can be generalized to symplectic reduction of tangent bundle of a symplectic manifold with the lifted symplectic structure. This could be the first step toward more general studies on the reduction of the special symplectic structures and the reduction of Tulczyjew's triplet with configuration manifold $\mathcal{Q}$.

Finally, we want to mention that the foremost example of degenerate system that falls into application area of present formulation is the Vlasov-Poisson equation of plasma dynamics which was, indeed, the motivation for this work.

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