Springer Correspondence for Exceptional Lie Algebras and their Duals in Small Characteristic

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Communicated by E. B. Vinberg

Abstract. We describe the Springer correspondence explicitly for exceptional Lie algebras of type G_2 and F_4 and their duals in bad characteristics, i.e. in characteristics 2 and 3.
Mathematics Subject Classification 2010: 20G99.
Key Words and Phrases: Springer correspondence, exceptional groups, bad characteristic.

1. Introduction

In this paper we describe the Springer correspondence explicitly for exceptional Lie algebras of type G_2 and F_4 and their duals in bad characteristics. In what follows we describe the set-up of the paper in more detail.

Let G be an almost-simple algebraic group defined over an algebraically closed field $\mathbf{k}$ of prime characteristic and $\mathfrak{g}$ the Lie algebra of G. Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. Denote by $\mathcal{U}_G$, $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}^*}$ the varieties of unipotent elements in G, nilpotent elements in $\mathfrak{g}$, and nilpotent elements in $\mathfrak{g}^*$, respectively. We recall that an element in $\mathfrak{g}^*$ is called nilpotent if it annihilates a Borel subalgebra of $\mathfrak{g}$ (see [KW]). The group G acts on $\mathfrak{g}$ and $\mathfrak{g}^*$ by adjoint and coadjoint action, respectively. It is known that the number of G-orbits in $\mathcal{U}_G$, $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}^*}$, respectively, is finite (for $\mathcal{N}_{\mathfrak{g}^*}$ see [X4] and references there). We fix a prime $l \neq \operatorname{char} \mathbf{k}$. Let $\mathcal{A}_G$ (resp. $\mathcal{A}_{\mathfrak{g}}$, $\mathcal{A}_{\mathfrak{g}^*}$) denote the set of all pairs $(\mathcal{O}, \mathcal{E})$, where $\mathcal{O} \subset \mathcal{U}_G$ (resp. $\mathcal{O} \subset \mathcal{N}_{\mathfrak{g}}$, $\mathcal{O} \subset \mathcal{N}_{\mathfrak{g}^*}$) is a G-orbit and $\mathcal{E}$ is an irreducible G-equivariant $\bar{\mathbb{Q}}_l$-local system on $\mathcal{O}$ (up to isomorphism). Let W be the Weyl group of G and let $\operatorname{Irr}(W)$ denote the set of irreducible characters (over $\bar{\mathbb{Q}}_l$) of the Weyl group W. The Springer correspondence for G (resp. $\mathfrak{g}$, $\mathfrak{g}^*$) maps the set $\operatorname{Irr}(W)$ injectively into the set $\mathcal{A}_G$ (resp. $\mathcal{A}_{\mathfrak{g}}$, $\mathcal{A}_{\mathfrak{g}^*}$); we denote the map by γ_G (resp. $\gamma_{\mathfrak{g}}$, $\gamma_{\mathfrak{g}^*}$). The construction was originally given for $\mathfrak{g}$ by Springer in large characteristic [Sp] and generalized to arbitrary characteristic for G by Lusztig [L1]. The same construction as Lusztig's works for $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) in any characteristic assuming

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that G is adjoint (resp. simply connected), see for example [X1, X2]; this in turn gives rise to Springer correspondences for all $\mathfrak{g}$ and $\mathfrak{g}^*$ in any characteristic.

If the characteristic p of $\mathbf{k}$ is very good for G (namely p is good, and p does not divide $n+1$ if G is of type A_n), there exists a G-equivariant isomorphism between $\mathcal{U}_G$ and $\mathcal{N}_{\mathfrak{g}}$ by a theorem of Springer, moreover, there exists a G-invariant non-degenerate bilinear form on $\mathfrak{g}$ which we can use to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Thus in such cases the sets $\mathcal{A}_G$, $\mathcal{A}_{\mathfrak{g}}$ and $\mathcal{A}_{\mathfrak{g}^*}$ can be naturally identified. The Springer correspondence for G, i.e. the map $\gamma_G : \mathrm{Irr}(W) \to \mathcal{A}_G$ has been described explicitly in all characteristics [L1, LS2, Sp, Sh, AL, S2]. When G is of type A_n and p divides $n+1$, the maps γ_G, $\gamma_{\mathfrak{g}}$, $\gamma_{\mathfrak{g}^*}$ coincide with the corresponding maps for $GL(n+1)$.

Assume now that the characteristic of $\mathbf{k}$ is bad for G. In general we can no longer identify the sets $\mathcal{A}_G$, $\mathcal{A}_{\mathfrak{g}}$ and $\mathcal{A}_{\mathfrak{g}^*}$. When G is a classical group, the Springer correspondence for $\mathfrak{g}$ and $\mathfrak{g}^*$ in bad characteristic has been determined explicitly in [X3]. When G is of type F_4 and char $\mathbf{k} = 2$, the Springer correspondence for $\mathfrak{g}$ has been described by Spaltenstein [S4]. When G is of type E_6, E_7 or E_8 in bad characteristic, the Springer correspondence for $\mathfrak{g}$ was partially described by the work of Spaltenstein [S3] and Holt-Spaltenstein [HS]. In this paper we describe the Springer correspondence for $\mathfrak{g}$ and $\mathfrak{g}^*$ in the remaining cases when G is of type G_2 or F_4. We will focus on the case of $\mathfrak{g}^*$ with details given and treat the case of $\mathfrak{g}$ briefly. The Springer correspondence for $\mathfrak{g}$ and $\mathfrak{g}^*$ turns out to be the same as in characteristic 0 when G is of type G_2 and char $\mathbf{k} = 2$, or when G is of type F_4 and char $\mathbf{k} = 3$. The same holds for $\mathfrak{g}^*$ when G is of type G_2 and char $\mathbf{k} = 3$.

The paper is organized as follows. In §2 we discuss some preliminaries. In particular, we determine the existence of G-invariant non-degenerate bilinear forms on $\mathfrak{g}$ (see Lemma 2.1). In §3 we describe the component groups of centralizers for nilpotent elements in $\mathfrak{g}^*$ and we compute the dimensions of the Springer fibers. In §4 we describe the Springer correspondence $\gamma_{\mathfrak{g}^*} : \mathrm{Irr}\, W \to \mathcal{A}_{\mathfrak{g}^*}$ for $\mathfrak{g}^*$ explicitly. In particular, following [L3], we give an a priori description of the set $\gamma_{\mathfrak{g}^*}^{-1}\{(\mathcal{O}, \bar{\mathbb{Q}}_l) \,|\, (\mathcal{O}, \bar{\mathbb{Q}}_l) \in \mathcal{A}_{\mathfrak{g}*}\}$, i.e. a set of Weyl group representations that parametrize nilpotent coadjoint orbits in $\mathfrak{g}^*$. Finally, in §5 we describe the maps $\gamma_{\mathfrak{g}} : \mathrm{Irr}\, W \to \mathcal{A}_{\mathfrak{g}}$ for $\mathfrak{g}$ explicitly.

Acknowledgement. The author thanks the referee for carefully reading the manuscript and for many helpful comments.

2.   Preliminaries

In this section let G be an exceptional group of type G_2 or F_4 defined over an algebraically closed field $\mathbf{k}$.

2.1. Structural constants and Bruhat decomposition.   As in [X3], we make use of the generators and relations in G, $\mathfrak{g}$, $\mathfrak{g}^*$, and the Bruhat decomposition for G. We recall some notations from [X3, §7.3-§7.5], for details, see *loc. cit.* For $g \in G$ and $\xi \in \mathfrak{g}^*$, we write $g.\xi$ for the coadjoint action, i.e., $g.\xi(x) = \xi(\mathrm{Ad}(g^{-1})x)$ for all $x \in \mathfrak{g}$, where Ad denotes the adjoint action of G on $\mathfrak{g}$.

Fix a maximal torus T of G and $B \supset T$ a Borel subgroup of G. Let R, R^+ and Π be the set of roots, positive roots, and simple roots, determined by

(G, T, B), respectively. For each $\alpha \in R$, let $x_\alpha : \mathbb{G}_a \to U_\alpha \subset G$ be an isomorphism such that

$$s x_\alpha(t) s^{-1} = x_\alpha(\alpha(s)t) \text{ for all } s \in T \text{ and } t \in \mathbb{G}_a,$$

$$n_\alpha(t) := x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t) \in N_G(T) \text{ and } n_\alpha(1) := n_\alpha \text{ has image } w_\alpha \text{ in } W,$$

where w_α denotes the reflection with respect to α. Define

$$h_\alpha(t) = n_\alpha(t) n_\alpha(-1) \in T.$$

By Bruhat decomposition, each $g \in G$ can be written uniquely in the form

$$g = b n_w u_w \text{ for some } b \in B, \ u_w \in U_w := \{ \prod_{\alpha > 0, w(\alpha) < 0} x_\alpha(t_\alpha) \,|\, t_\alpha \in \mathbb{G}_a \}, \tag{1}$$
$$\text{and } n_w \in N_G(T) \text{ a representative of } w \in W.$$

Let $\mathfrak{t}$, $\mathfrak{b}$, $\mathfrak{n}$ be the Lie algebra of T, B, and the unipotent radical U of B respectively. Define $\mathfrak{n}^* = \{\xi \in \mathfrak{g}^* \,|\, \xi(\mathfrak{b}) = 0\}$. Then for any $\xi \in \mathcal{N}_{\mathfrak{g}^*}$, there exists $g \in G$ such that $g.\xi \in \mathfrak{n}^*$. We make use of the Chevalley basis $\{h_\alpha, \ \alpha \in \Pi, \ e_\alpha, \ \alpha \in R\}$ of $\mathfrak{g}$, and the elements $\{e'_\alpha \in \mathfrak{g}^*, \ \alpha \in R\}$ (in particular, $\{e_\alpha, \ \alpha \in R^+\}$ is a basis of $\mathfrak{n}$ and $\{e'_\alpha, \alpha \in R^+\}$ is a basis of $\mathfrak{n}^*$), chosen in [X3, §7.3], where the adjoint action of G is given as follows

$$\mathrm{Ad}(x_\alpha(t)) \, e_\beta = \sum_i t^i M_{\alpha,\beta,i} \, e_{i\alpha+\beta}, \ \alpha, \beta \in R^+$$
$$\mathrm{Ad}(h_\alpha(t)) \, e_\beta = t^{A_{\alpha\beta}} e_\beta, \ \mathrm{Ad}(n_\alpha) \, e_\beta = \eta_{\alpha,\beta} \, e_{w_\alpha(\beta)}, \ \alpha \in \Pi, \tag{2}$$

and the coadjoint action of G is given as follows

$$x_\alpha(t).e'_\beta = \sum_i (-1)^i t^i M_{\alpha,-i\alpha-\beta,i} \, e'_{i\alpha+\beta}, \ \alpha, \beta \in R^+$$
$$h_\alpha(t).e'_\beta = t^{A_{\alpha\beta}} e'_\beta, \ n_\alpha.e'_\beta = \eta_{\alpha,\beta} \, e'_{w_\alpha(\beta)}, \ \alpha \in \Pi. \tag{3}$$

We will also make use of the following relations in G (see for example [C, §5.2, §7.2]), for $\alpha, \beta \in R$,

$$x_\alpha(s)^{-1} x_\beta(t)^{-1} x_\alpha(s) x_\beta(t) = \prod_{i,j>0} x_{i\beta+j\alpha}(c_{ij\beta\alpha} (-t)^i s^j), \ \alpha \neq \pm\beta$$
$$n_\alpha^2 = h_\alpha(-1), \ n_\alpha x_\beta(t) n_\alpha^{-1} = x_{w_\alpha(\beta)}(\eta_{\alpha,\beta} t), \ n_\alpha n_\beta n_\alpha^{-1} = h_{w_\alpha(\beta)}(\eta_{\alpha,\beta}) n_{w_\alpha(\beta)}. \tag{4}$$

Here $A_{\alpha,\beta}$, $M_{\alpha,\beta,i}$, $c_{ij\beta\alpha}$ and $\eta_{\alpha,\beta}(= \pm 1)$ are structural constants (for their determination see [X3, §7] and [C, pages 77 and 94]), in particular, they are integers.

2.2. Invariant non-degenerate bilinear forms on $\mathfrak{g}$. The following lemma was explained to the author by G. Lusztig. We include a proof here for completeness.

Lemma 2.1. *Assume that G is of type G_2 (resp. F_4). Then there exists a G-invariant non-degenerate bilinear form on $\mathfrak{g}$ if and only if $\mathrm{char}\, \boldsymbol{k} \neq 3$ (resp. $\mathrm{char}\, \boldsymbol{k} \neq 2$).*

Thus in the above cases, we have a G-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$. In turn the Springer correspondence maps $\gamma_\mathfrak{g}$ and $\gamma_{\mathfrak{g}^*}$ can be identified.

To prove Lemma 2.1, we first note that the Lie algebra $\mathfrak{g}$ is simple in the above cases [H]. Assume that G is of type G_2 and char $\mathbf{k} \neq 3$. Let α be the short simple root and β the long simple root. Define a symmetric bilinear form $(,)$ on $\mathfrak{g}$ as follows

$$
\begin{aligned}
(h_\alpha, h_\alpha) = 6, \ (h_\alpha, h_\beta) = -3, \ (h_\beta, h_\beta) = 2 \\
(e_\lambda, e_\mu) = 0 \text{ if } \lambda + \mu \neq 0, \\
(e_\lambda, e_{-\lambda}) = 3 \text{ (resp. 1) if } \lambda \text{ is a short (resp. long) root.}
\end{aligned}
\tag{5}
$$

Using [K, Theorem 2.2] one checks that (5) defines an invariant non-degenerate bilinear form on $\mathfrak{g}$ over $\mathbb{C}$, thus it is a scalar multiple of the Killing form, which is G-invariant. As all the structure constants are in $\mathbb{Z}$, reducing mod char $\mathbf{k}$, we see that the above bilinear form remains non-degenerate and G-invariant.

Assume that G is of type F_4 and char $\mathbf{k} \neq 2$. Let p, q be the long simple roots and r, s the short simple roots such that $(q, r) \neq 0$. Define a symmetric bilinear form $(,)$ on $\mathfrak{g}$ as follows

$$
\begin{aligned}
(h_p, h_p) = (h_q, h_q) = 2, \ (h_p, h_q) = -1, \ (h_p, h_r) = (h_p, h_s) = (h_q, h_s) = 0, \\
(h_q, h_r) = (h_r, h_s) = -2, \ (h_r, h_r) = (h_s, h_s) = 4, \\
(e_\lambda, e_\mu) = 0 \text{ if } \lambda + \mu \neq 0, \\
(e_\lambda, e_{-\lambda}) = 2 \text{ (resp. 1) if } \lambda \text{ is a short (resp. long) root.}
\end{aligned}
\tag{6}
$$

The same argument as in the case of G_2 shows that the bilinear form $(,)$ of (6) is non-degenerate and G-invariant.

Now assume that char $\mathbf{k} = 3$ and G is of type G_2, there are 6 nilpotent orbits in $\mathfrak{g}$ [St] and 5 nilpotent coadjoint orbits in $\mathfrak{g}^*$ [X4]. Similarly, when char $\mathbf{k} = 2$ and G is of type F_4, there are 22 nilpotent orbits in $\mathfrak{g}$ [S4] and 18 nilpotent coadjoint orbits in $\mathfrak{g}^*$ [X4]. Thus in these cases there can not exist a G-invariant non-degenerate bilinear form on $\mathfrak{g}$. This completes the proof of Lemma 2.1.

Remark 2.2.   Alternatively, as pointed out by the referee, it is known that $\mathfrak{g}$ and $\mathfrak{g}^*$ are not isomorphic as G-modules in these cases. Then Lemma 2.1 follows.

2.3. Weyl group representations and truncated induction.   Let W be a Weyl group and let $\operatorname{Irr} W$ denote the set of irreducible characters of W. For $\rho \in \operatorname{Irr} W$,

$$
\begin{aligned}
&\text{let } b_\rho \text{ denote the minimal integer } d \text{ such that } \rho \text{ occurs} \\
&\text{in the } W\text{-module } \mathfrak{S}_d(V) \text{ of all homogeneous polynomials} \\
&\text{of degree } d \text{ on the reflection space } V.
\end{aligned}
\tag{7}
$$

For a subgroup $W' \subset W$, let $j_{W'}^W : \operatorname{Irr} W' \to \operatorname{Irr} W$ denote the truncated induction, i.e., for $\rho' \in \operatorname{Irr} W'$, $j_{W'}^W \rho'$ is the unique $\rho \in \operatorname{Irr} W$ such that ρ occurs in $\operatorname{Ind}_{W'}^W \rho'$, written as $\langle \rho, \operatorname{Ind}_{W'}^W \rho' \rangle \neq 0$, and $b_\rho = b_{\rho'}$.

We label $\mathrm{Irr}\,W$ using the notations of [A]. In particular, $\mathrm{Irr}\,W(A_{n-1}) = \mathrm{Irr}\,S_n$ is labelled by partitions of n, where (n) denotes the trivial character and (1^n) denotes the sign character, and $\mathrm{Irr}\,W(B_n) = \mathrm{Irr}\,W(C_n)$ is labelled by a pair of partition $[\lambda : \mu]$. For W of type G_2 or F_4, $\chi_{i,j} \in \mathrm{Irr}\,W$ has degree i. For the reader's convenience, we list the values b_ρ (see (7)) for $\rho \in \mathrm{Irr}\,W(G_2)$ and $\rho \in \mathrm{Irr}\,W(F_4)$ in the following tables (see for example [GP]).

ρ	$\chi_{1,1}$	$\chi_{2,1}$	$\chi_{2,2}$	$\chi_{1,3}$	$\chi_{1,4}$	$\chi_{1,2}$
b_ρ	0	1	2	3	3	6

Table 1: b-invariants for $W(G_2)$

ρ	$\chi_{1,1}$	$\chi_{4,2}$	$\chi_{9,1}$	$\chi_{8,1}$	$\chi_{8,3}$	$\chi_{2,1}$	$\chi_{2,3}$	$\chi_{12,1}$	$\chi_{16,1}$	$\chi_{6,1}$	$\chi_{6,2}$	$\chi_{9,2}$	$\chi_{9,3}$
b_ρ	0	1	2	3	3	4	4	4	5	6	6	6	6
ρ	$\chi_{4,3}$	$\chi_{4,4}$	$\chi_{4,1}$	$\chi_{8,2}$	$\chi_{8,4}$	$\chi_{9,4}$	$\chi_{1,2}$	$\chi_{1,3}$	$\chi_{4,5}$	$\chi_{2,2}$	$\chi_{2,4}$	$\chi_{1,4}$	
b_ρ	7	7	8	9	9	10	12	12	13	16	16	24	

Table 2: b-invariants for $W(F_4)$

3.   Nilpotent coadjoint orbits

Let G continue to be an exceptional group of type G_2 or F_4 over an algebraically closed field $\mathbf{k}$. In this section, we describe the component groups $A_G(\xi) := Z_G(\xi)/Z_G(\xi)^0$ for $\xi \in \mathcal{N}_{\mathfrak{g}^*}$. Moreover, we compute the dimensions of Springer fibers. More precisely, let B be a fixed Borel subgroup of G with unipotent radical U. We write $\mathfrak{b} = \mathrm{Lie}\,B$ and $\mathfrak{n} = \mathrm{Lie}\,U$. Define

$$\mathfrak{b}^* = \{\zeta \in \mathfrak{g}^* \,|\, \zeta(\mathfrak{n}) = 0\} \text{ and } \mathcal{B}_\xi^G = \{gB \in G/B \,|\, g^{-1}.\xi \in \mathfrak{b}^*\},\ \xi \in \mathfrak{g}^*.$$

We show in §3 that

Proposition 3.1.   *For $\xi \in \mathcal{N}_{\mathfrak{g}^*}$,*

$$\dim \mathcal{B}_\xi^G = \frac{\dim Z_G(\xi) - \mathrm{rank}\,G}{2}. \tag{8}$$

The component groups $A_G(x)$ for $x \in \mathcal{N}_{\mathfrak{g}}$ have been determined in all characteristics, as well as the dimensions of Springer fibers, see §5 for exposition. In view of Lemma 2.1, it suffices to consider the cases when G is an exceptional group of type G_2 (resp. F_4) over an algebraically closed field $\mathbf{k}$ of characteristic 3 (resp. 2), which we will assume in the remainder of this section.

The nilpotent coadjoint orbits in $\mathfrak{g}^*$ have been classified in [X4, §7]. We use the same notation and representatives for the orbits as in *loc.cit.* In particular, the orbits are labeled as in the Bala-Carter classification (while the two new orbits in

bad characteristic are labeled as $(B_3)_2$ and $(A_2)_2$). The tilde in $\widetilde{A}_1$ etc. indicates short roots.

The results are listed in Table 3 when G is of type G_2 and char $\mathbf{k} = 3$, and in Table 4 when G is of type F_4 and char $\mathbf{k} = 2$.

Orbit	representative ξ	dim $Z_G(\xi)$	$A_G(\xi)$	dim $\mathcal{B}_\xi$
G_2	$e'_\alpha + e'_\beta$	2	1	0
$G_2(a_1)$	$e'_\beta + e'_{2\alpha+\beta}$	4	S_3	1
$\widetilde{A}_1$	e'_α	6	1	2
A_1	e'_β	8	1	3
$\emptyset$	0	14	1	6

Table 3: Type G_2, nilpotent coadjoint orbits, char $\mathbf{k} = 3$

Orbit	representative ξ	dim $Z_G(\xi)$	$A_G(\xi)$	dim $\mathcal{B}_\xi$
F_4	$e'_p + e'_q + e'_r + e'_s$	4	1	0
$F_4(a_1)$	$e'_p + e'_{qr} + e'_{q2r} + e'_s$	6	S_2	1
$F_4(a_2)$	$e'_p + e'_{qr} + e'_{rs} + e'_{q2r2s}$	8	1	2
B_3	$e'_p + e'_{qrs} + e'_{q2r} + e'_{pq2rs}$	10	1	3
C_3	$e'_s + e'_{q2r} + e'_{pqr}$	10	1	3
$F_4(a_3)$	$e'_{pqr} + e'_{qrs} + e'_{pq2r} + e'_{q2r2s}$	12	S_4	4
$(B_3)_2$	$e'_p + e'_{qr} + e'_{q2r2s}$	12	1	4
$C_3(a_1)$	$e'_{pqr} + e'_{q2rs} + e'_{q2r2s}$	14	S_2	5
B_2	$e'_{pqr} + e'_{q2r2s}$	16	S_2	6
$\widetilde{A}_2 + A_1$	$e'_{pqr} + e'_{q2rs} + e'_{p2q2r2s}$	16	1	6
$A_2 + \widetilde{A}_1$	$e'_{p2q2r} + e'_{q2r2s} + e'_{pq2rs}$	18	1	7
$\widetilde{A}_2$	$e'_{pqrs} + e'_{q2rs}$	22	1	9
A_2	$e'_{p2q2r} + e'_{pq2r2s} + e'_{p2q3r2s}$	22	1	9
$A_1 + \widetilde{A}_1$	$e'_{p2q2r2s} + e'_{p2q3rs}$	24	1	10
$(A_2)_2$	$e'_{p2q2r} + e'_{pq2r2s}$	28	1	12
$\widetilde{A}_1$	$e'_{p2q3r2s}$	30	S_2	13
A_1	$e'_{2p3q4r2s}$	36	1	16
$\emptyset$	0	52	1	24

Table 4: Type F_4, nilpotent coadjoint orbits, char $\mathbf{k} = 2$

3.1. Component groups of centralizers. To describe the structure of the component groups $A_G(\xi)$, $\xi \in \mathcal{N}_{\mathfrak{g}^*}$, we make use of the following lemma.

Lemma 3.2 ([B, 3.4, p176]).   *Let $\sigma : G \to G$ be the Frobenius endomorphism induced by $\mathbf{k} \to \mathbf{k} : a \mapsto a^q$. Let $\mathcal{O} \subset \mathfrak{g}^*$ be a G-orbit. Then the set of G^σ-orbits*

in $\mathcal{O}^\sigma$ is in bijection with $H^1(\sigma, A_G(\xi))$, where $\xi \in \mathcal{O}$.

Note that for q large enough, σ acts trivially on $A_G(\xi)$. Thus $|H^1(\sigma, A_G(\xi))|$ equals the number of conjugacy classes in $A_G(\xi)$. It then follows from [X4, §7, Table 1 and Table 2] that we only need to consider the representative $\xi = e'_\beta + e'_{2\alpha+\beta}$ for the orbit $G_2(a_1)$ and the representative $\xi_5 = e'_{pqr} + e'_{qrs} + e'_{pq2r} + e'_{q2r2s}$ for the orbit $F_4(a_3)$. Here we have used the fact that $S_2 \cong \mathbb{Z}/2\mathbb{Z}$ is the only finite group with exactly 2 conjugacy classes.

Assume that G is of type G_2 and char $\mathbf{k} = 3$. We determine $A_G(\xi)$ for $\xi = e'_\beta + e'_{2\alpha+\beta}$. Using (1) and (3), one can check that $Z_G(\xi) \subset B$ and

$$A_G(\xi) \cong \langle \gamma_1, \gamma_2 \rangle \cong S_3, \tag{9}$$

where $\gamma_1 = h_\beta(-1)$ and $\gamma_2 = h_\beta(-1)x_\alpha(\eta)$ $(\eta^2 = -1)$. In particular, we have used the following formula

$$u(\mathbf{t}).\xi = \xi + (t_1^3 + t_1)e'_{3\alpha+\beta} + (t_5 + t_3)e'_{3\alpha+2\beta}.$$

where $u(\mathbf{t}) = x_\alpha(t_1)x_\beta(t_2)x_{\alpha+\beta}(t_3)x_{2\alpha+\beta}(t_4)x_{3\alpha+\beta}(t_5)x_{3\alpha+2\beta}(t_6)$, $t_i \in \mathbb{G}_a$.

Assume that G is of type F_4 and char $\mathbf{k} = 2$. We determine $A_G(\xi_5)$ for $\xi_5 = e'_{pqr} + e'_{qrs} + e'_{pq2r} + e'_{q2r2s}$. Let P be the standard parabolic group P_J, where $J = \{p, r, s\}$. Then using [CP, Theorem 7.3] and the description of nilpotent pieces in $\mathfrak{g}^*$ in [X4, §7.6] we see that $Z_G(\xi_5) \subset P$. Using (1) and (3), one checks that

$$Z_G^0(\xi_5) = Z_{U_P}(\xi_5), \quad A_G(\xi_5) \cong \langle \gamma_1, \gamma_2, \gamma_3 \rangle \cong S_4, \tag{10}$$

where

$$\gamma_1 = x_p(1)\,x_s(1), \quad \gamma_2 = n_p\,n_s, \quad \gamma_3 = x_p(1)\,x_s(1)\,x_r(1)\,n_r\,x_r(1)$$

with

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = 1, \ (\gamma_1\gamma_2)^3 = (\gamma_2\gamma_3)^3 = 1, \ (\gamma_1\gamma_3)^2 = 1.$$

In particular we have used the following formula (here the assumption char $\mathbf{k} = 2$ is used)

$$u(\mathbf{t}).\xi_5 = e'_{pqr} + e'_{qrs} + (1 + t_2)e'_{pq2r} + t_2\,e'_{q2rs} + (t_1 + t_3)\,e'_{pqrs} + (1 + t_4)e'_{q2r2s}$$
$$+(t_1t_2 + t_2t_3 + t_4)e'_{pq2rs} + (t_1 + t_3^2 + t_4(t_1 + t_3))e'_{pq2r2s}$$

where $u(\mathbf{t}) = x_p(t_1)\,x_r(t_2)\,x_s(t_3)\,x_{rs}(t_4)$, $t_i \in \mathbb{G}_a$.

Remark 3.3. For the element ξ_5 above, we have another interesting way to deduce that $A_G(\xi_5) \cong S_4$ as follows. Taking q big enough, we can choose a representative ξ of the orbit of ξ_5 such that the Frobenius morphism σ fixes ξ and acts trivially on $A_G(\xi)$. This implies that $|Z_G(\xi)^\sigma| = |A_G(\xi)||(Z_G(\xi)^0)^\sigma|$. In view of [X4, Table 2], we conclude that $A_G(\xi)$ is a finite group of order 24 with exactly 5 conjugacy classes. It turns out that the finite groups with exactly 5 conjugacy classes have been classified by Miller [M] in 1919; the list is the following: the alternating group A_5, S_4, the non-cyclic group of order 21, the metacylic group of order 20, the dihedral group of order 14, the dihedral group of order 8, the quaternion group, and $\mathbb{Z}/5\mathbb{Z}$. We can then easily identify our group from the list.

3.2. Springer fibers. Let L denote a Levi subgroup of a proper parabolic subgroup P of G. We write $\mathfrak{l} = \operatorname{Lie} L$, $\mathfrak{n}_P = \operatorname{Lie} U_P$, where U_P is the unipotent radical of P. Let

$$\mathfrak{l}^* = \{\zeta \in \mathfrak{g}^* \mid \zeta(\mathfrak{n}_P \oplus \mathfrak{n}_P^-) = 0\} \text{ and } \mathfrak{n}_P^* = \{\zeta \in \mathfrak{g}^* \mid \zeta(\mathfrak{l} \oplus \mathfrak{n}_P) = 0\},$$

where $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{n}_P \oplus \mathfrak{n}_P^-$.

Let $\xi \in \mathcal{N}_{\mathfrak{g}^*}$. We prove Proposition 3.1 using the following statements.

$$\begin{gathered} \text{Assume that } \xi \text{ lies in } \mathfrak{l}^* \text{ and } \dim Z_L(\xi) = 2\dim \mathcal{B}_\xi^L + \operatorname{rank} L. \\ \text{Then } \dim Z_G(\xi) = 2\dim \mathcal{B}_\xi^G + \operatorname{rank} G. \end{gathered} \quad (11)$$

$$\begin{gathered} \text{If } \xi \text{ lies in the orbit obtained by inducing the orbit of } \xi' \in \mathfrak{l}^* {}^1, \text{ and} \\ \dim Z_L(\xi') = 2\dim \mathcal{B}_{\xi'}^L + \operatorname{rank} L, \text{ then } \dim Z_G(\xi) = 2\dim \mathcal{B}_\xi^G + \operatorname{rank} G. \end{gathered} \quad (12)$$

The statement (11) is proved using the same argument as in [S1, II 3.14]. In particular, one uses that the equality in (8) holds if there exists $w \in W$ such that $\mathcal{O}_\xi \cap \mathfrak{n}^* \cap \mathfrak{n}_w^*$ is dense in $\mathfrak{n}^* \cap \mathfrak{n}_w^*$, where $\mathfrak{n}_w^* = \{\zeta \in \mathfrak{g}^* \mid \zeta(w\mathfrak{b}w^{-1}) = 0\}$. The statement (12) is proved using the same argument as in [LS1, Theorem 1.3]. In particular, the proof shows if $\mathcal{O}_\xi = \operatorname{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*} \mathcal{O}_{\xi'}^L$, then

$$\dim \mathcal{O}_\xi = \dim \mathcal{O}_{\xi'}^L + 2\dim U_P. \quad (13)$$

Note that by [X4, Propositon 3.1], the equality (8) holds for any $\xi' \in \mathcal{N}_{\mathfrak{l}^*}$ (replacing G by L). Thus it suffices to show that

$$\begin{gathered} \text{every } \xi \in \mathcal{N}_{\mathfrak{g}^*} \text{ either lies in } \mathfrak{l}^* \text{ or lies in an orbit induced from a} \\ \text{nilpotent orbit in } \mathfrak{l}^* \text{ for a proper Levi subgroup } L. \end{gathered} \quad (14)$$

For G_2, one readily checks (14). For example, the orbit G_2 is induced from the zero orbit in a maximal torus, the orbit $G_2(a_1)$ is induced from the zero orbit in a Levi subgroup of type $\widetilde{A}_1$. For F_4, we make use of the following table for induced orbits. The first row contains orbits $\widetilde{\mathcal{O}}$ in $\mathcal{N}_{\mathfrak{g}^*}$, the second row indicates the type of the Levi subgroup L, and the third row contains orbits $\mathcal{O}$ in $\mathcal{N}_{\mathfrak{l}^*}$ such that $\widetilde{\mathcal{O}} = \operatorname{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*} \mathcal{O}$, where 0 denotes the zero orbit and we use the notation in [X2] for orbits in a Levi of type B_3 or C_3. The last row of the table is included for use in §4 and will be explained there. We remark that we did not attempt to identify all induced orbits here, rather only the cases that are needed for our purpose. As an example, we explain the case of the orbit $F_4(a_3)$ being induced from the orbit $(1; 1_1^4)$ in $\mathfrak{l}^*$ for L of type B_3; the other cases can be checked similarly. Let $\xi = e'_{pqr} + e'_{qrs} + e'_{pq2r} + e_{q2r2s}$, which is in the orbit $F_4(a_3)$, and let $\xi' = e'_{pqr} + e'_{pq2r}$. Let P be the standard parabolic subgroup P_J, $J = \{p, q, r\}$ and L the standard Levi subgroup of P. Then L is of type B_3. Moreover, $\xi' \in \mathfrak{l}^*$ and $\xi \in \xi' + \mathfrak{n}_P^*$. Using [X2], one checks that ξ' lies in the orbit $(1; 1_1^4)$ of $\mathfrak{l}^*$, moreover, $\dim \mathcal{O}_{\xi'}^L = 10$. Note that $\dim \mathcal{O}_\xi = 40 = \dim \mathcal{O}_{\xi'} + 2\dim U_P = \dim \operatorname{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*} \mathcal{O}_{\xi'}^L$

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[1]see [X4, §4], i.e., $\mathcal{O}_\xi \cap (\mathcal{O}_{\xi'}^L + \mathfrak{n}_P^*)$ is dense in $\mathcal{O}_{\xi'}^L + \mathfrak{n}_P^*$, where $\mathcal{O}_{\xi'}^L$ denotes the L-orbit of ξ'; we write $\mathcal{O}_\xi = \operatorname{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*} \mathcal{O}_{\xi'}^L$.

$\widetilde{\mathcal{O}}$	$\widetilde{A}_2$	A_2	B_2	$F_4(a_3)$	$F_4(a_3)$	B_3	$F_4(a_1)$	$F_4(a_2)$
type of L	B_3	C_3	C_3	$\widetilde{A}_2 A_1$	B_3	B_3	B_3	$A_1 \widetilde{A}_1$
$\mathcal{O}$	0	0	1_1^6	0	$(1; 1_1^4)$	$(0; 3_3^2)$	$(2; 1_1^2)$	0
$\rho_{\xi',1}^L$	$[-:1^3]$	$[-:1^3]$	$[1^3:-]$	$[1^3] \times [1^2]$	$[1:1^2]$	$[-:3]$	$[2:1]$	$[2] \times [2]$

Table 5: Induced nilpotent coadjoint orbits, type F_4, char $\mathbf{k} = 2$

(in the last equality we use (13)). Since $\mathrm{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*} \mathcal{O}_{\xi'}^L$ is the unique dense orbit in $G.(\mathcal{O}_{\xi'} + \mathfrak{n}_P^*)$, we obtain that

$$\mathcal{O}_\xi = \mathrm{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*} \mathcal{O}_{\xi'}^L, \ \xi = e'_{pqr} + e'_{qrs} + e'_{pq2r} + e_{q2r2s}, \ \xi' = e'_{pqr} + e'_{pq2r}. \quad (15)$$

Now one checks readily that (14) holds for F_4. This completes the proof of Proposition 3.1.

4. The Springer correspondence for $\mathfrak{g}^*$

Let G be a connected simply connected algebraic group defined over an algebraically closed field $\mathbf{k}$ of prime characteristic. Fix a prime $l \neq \mathrm{char}\,\mathbf{k}$. Let

$$\gamma_{\mathfrak{g}^*} : \mathrm{Irr}\,W \to \mathcal{A}_{\mathfrak{g}^*} \quad (16)$$

be the injective Springer correspondence map constructed for $\mathfrak{g}^*$ as in [X2, §5], where $\mathrm{Irr}\,W$ and $\mathcal{A}_{\mathfrak{g}^*}$ are the sets defined in the introduction. Namely, $\mathrm{Irr}\,W$ denotes the set of irreducible characters of the Weyl group W of G and $\mathcal{A}_{\mathfrak{g}^*}$ is the set of all pairs $(\mathcal{O}, \mathcal{E})$ where $\mathcal{O} \subset \mathcal{N}_{\mathfrak{g}^*}$ is a G-orbit and $\mathcal{E}$ is an irreducible G-equivariant $\bar{\mathbb{Q}}_l$-local system on $\mathcal{O}$ (up to isomorphism). The constructions and proofs in [X2, §5] apply for any simply connected G. In what follows we briefly recall the construction. Consider the following proper maps

$$\varphi_0 : G \times^B \mathfrak{n}^* \to \mathcal{N}_{\mathfrak{g}^*}, \ \varphi : G \times^B \mathfrak{b}^* \to \mathfrak{g}^*.$$

The map φ_0 is semismall and the map φ is small. Moreover,

$$\begin{aligned}
\varphi_{0*}\bar{\mathbb{Q}}_l[-] &\cong \bigoplus_{(\mathcal{O}, \mathcal{E}) \in \mathcal{A}_{\mathfrak{g}^*}} \mathrm{IC}(\bar{\mathcal{O}}, \mathcal{E}) \otimes V_{\mathcal{O}, \mathcal{E}} \\
&\cong \varphi_* \bar{\mathbb{Q}}_l|_{\mathcal{N}_{\mathfrak{g}^*}}[-] \cong \bigoplus_{\rho \in \mathrm{Irr}\,W} \rho \otimes (\varphi_* \bar{\mathbb{Q}}_l)_\rho|_{\mathcal{N}_{\mathfrak{g}^*}}[-]
\end{aligned} \quad (17)$$

where $[-]$ denotes shift by $\dim \mathcal{N}_{\mathfrak{g}^*}$, $\mathrm{IC}(\bar{\mathcal{O}}, \mathcal{E})$ is the perverse IC-extension of the local system $\mathcal{E}$ on $\mathcal{O}$, and $(\varphi_* \bar{\mathbb{Q}}_l)_\rho = \mathrm{Hom}_{\bar{\mathbb{Q}}_l[W]}(\rho, \varphi_* \bar{\mathbb{Q}}_l)$. The map $\gamma_{\mathfrak{g}^*}$ maps $\rho \in \mathrm{Irr}\,W$ to the unique pair $(\mathcal{O}, \mathcal{E}) \in \mathcal{A}_{\mathfrak{g}^*}$ such that $\mathrm{IC}(\bar{\mathcal{O}}, \mathcal{E}) \cong (\varphi_* \bar{\mathbb{Q}}_l)_\rho|_{\mathcal{N}_{\mathfrak{g}^*}}[-]$. Note that this implies that $\rho \cong V_{\mathcal{O}, \mathcal{E}}$ under W-action.

In this section we describe the map $\gamma_{\mathfrak{g}^*}$ of (16) explicitly assuming that G is of type G_2 (resp. F_4) and char $\mathbf{k} = 3$ (resp. char $\mathbf{k} = 2$). The results are given in Table 6 (resp. Table 7). For a pair $(\mathcal{O}, \mathcal{E}) \in \mathcal{A}_{\mathfrak{g}^*}$, we write $\rho_{\xi,\phi}^G$ (or simply $\rho_{\xi,\phi}$)

for the inverse image $\gamma_{\mathfrak{g}^*}^{-1}(\mathcal{O}, \mathcal{E})$, where $\xi \in \mathcal{O}$ and $\phi \in \operatorname{Irr} A_G(\xi)$ corresponds to the local system $\mathcal{E}$. In particular, as mentioned in the introduction, the Springer correspondence for $\mathfrak{g}^*$ turns out to be the same as in characteristic 0 when G is of type G_2, or when G is of type F_4 and char $\mathbf{k} = 3$.

Orbit of ξ	ϕ	$\rho_{\xi,\phi}$	Orbit of ξ	ϕ	$\rho_{\xi,\phi}$
G_2	(1)	$\chi_{1,1}$	$\widetilde{A_1}$	(1)	$\chi_{2,2}$
$G_2(a_1)$	(3)	$\chi_{2,1}$	A_1	(1)	$\chi_{1,4}$
$G_2(a_1)$	$(2,1)$	$\chi_{1,3}$	$\emptyset$	(1)	$\chi_{1,2}$
$G_2(a_1)$	(1^3)	$-$			

Table 6: Springer correspondence for $\mathfrak{g}^*$, type G_2 and char $\mathbf{k} = 3$

Orbit of ξ	ϕ	$\rho_{\xi,\phi}$	Orbit of ξ	ϕ	$\rho_{\xi,\phi}$	Orbit of ξ	ϕ	$\rho_{\xi,\phi}$
F_4	(1)	$\chi_{1,1}$	$F_4(a_3)$	$(2,1,1)$	$\chi_{1,3}$	$\widetilde{A_2}$	(1)	$\chi_{8,2}$
$F_4(a_1)$	(2)	$\chi_{4,2}$	$F_4(a_3)$	(1^4)	$-$	A_2	(1)	$\chi_{8,4}$
$F_4(a_1)$	(1^2)	$\chi_{2,3}$	$(B_3)_2$	(1)	$\chi_{2,1}$	$A_1 + \widetilde{A_1}$	(1)	$\chi_{9,4}$
$F_4(a_2)$	(1)	$\chi_{9,1}$	$C_3(a_1)$	(2)	$\chi_{16,1}$	$(A_2)_2$	(1)	$\chi_{1,2}$
B_3	(1)	$\chi_{8,1}$	$C_3(a_1)$	(1^2)	$\chi_{4,4}$	$\widetilde{A_1}$	(2)	$\chi_{4,5}$
C_3	(1)	$\chi_{8,3}$	B_2	(2)	$\chi_{9,2}$	$\widetilde{A_1}$	(1^2)	$\chi_{2,2}$
$F_4(a_3)$	(4)	$\chi_{12,1}$	B_2	(1^2)	$\chi_{4,1}$	A_1	(1)	$\chi_{2,4}$
$F_4(a_3)$	$(3,1)$	$\chi_{9,3}$	$\widetilde{A_2} + A_1$	(1)	$\chi_{6,1}$	$\emptyset$	(1)	$\chi_{1,4}$
$F_4(a_3)$	$(2,2)$	$\chi_{6,2}$	$A_2 + \widetilde{A_1}$	(1)	$\chi_{4,3}$			

Table 7: Springer correspondence for $\mathfrak{g}^*$, type F_4 and char $\mathbf{k} = 2$

We remark that in view of the results in [X3], Tables 6 and 7, the pair $(G_2(a_1), \mathcal{E}_{(1^3)})$ for G_2 and the pair $(F_4(a_3), \mathcal{E}_{(1^4)})$ for F_4 are the cuspidal pairs in the sense of [L1], where $\mathcal{E}_\phi$ denotes the local system corresponding to $\phi \in \operatorname{Irr} A_G(\xi)$.

4.1. The methods.

We describe the map $\gamma_{\mathfrak{g}^*}$ of (16) following the methods in [AL, S2, S5]. In particular, we use extensively the tables of induce/restrict matrix for Weyl group representations given by Alvis in [A]. More specifically, we make use of the following statements, for which the proofs follow the same arguments as in [AL, S2, S5] and the references there. Thus we only briefly comment on the proofs. Throughout this subsection $\xi \in \mathcal{N}_{\mathfrak{g}^*}$, L is a Levi subgroup of a proper parabolic subgroup P of G, $\mathfrak{l} = \operatorname{Lie} L$, U_P is the unipotent radical of P, and W_L is the Weyl group of L, regarded naturally as a subgroup of W.

To begin with, let b_ρ be defined as in (7) for $\rho \in \operatorname{Irr} W$. Using the same argument as in, for example, [J, §13.12], i.e. taking stalks at 0 in (17), using

Proposition 3.1 and the fact that $[\rho : H^{2i}(\mathcal{B}, \bar{\mathbb{Q}}_l)] = [\rho : \mathfrak{S}_i(V)]$ (this follows from the facts that $H^\bullet(\mathcal{B}, \bar{\mathbb{Q}}_l)$ is isomorphic to the coinvariant algebra C of $\mathfrak{S}(V)$ and that $C \otimes \mathfrak{S}(V)^W \cong \mathfrak{S}(V)$), we see that

$$b_{\rho^G_{\xi,1}} = \dim \mathcal{B}^G_\xi. \tag{18}$$

Next let $\xi' \in \mathcal{N}_{\mathfrak{l}^*}$. Consider the permutation representation $\varepsilon_{\xi,\xi'}$ of $A_G(\xi) \times A_L(\xi')$ afforded by the finite set $S_{\xi,\xi'}$ of irreducible components of dimension $d_{\xi,\xi'}$ of $Y_{\xi,\xi'} := \{g \in G \,|\, g^{-1}.\xi \in \xi' + \mathfrak{n}^*_P\}$ [2]. The same proof as in [L1] (see also [X3]) shows that for $\phi \in \operatorname{Irr} A_G(\xi)$ and $\phi' \in \operatorname{Irr} A_L(\xi')$, we have

$$\langle \phi \otimes \phi', \varepsilon_{\xi,\xi'} \rangle = \langle \operatorname{Res}^W_{W_L} \rho^G_{\xi,\phi}, \rho^L_{\xi',\phi'} \rangle_{W_L} := n_{\xi,\xi',\phi,\phi'}. \tag{19}$$

In particular, we make use of the following special cases of (19).

$$\text{If } n_{\xi,\xi',1,1} = 0, \text{ then } S_{\xi,\xi'} = \emptyset \text{ and all } n_{\xi,\xi',\phi,\phi'} = 0. \tag{20}$$

If the orbit of ξ is induced from that of $\xi' \in \mathfrak{l}^*$, then we can assume that $\xi \in \xi' + \mathfrak{n}^*_P$. Using the same argument as in [LS1, 1.3, 1.5] and [S5], we see that in this case

$$S_{\xi,\xi'} \text{ is isomorphic to } A_G(\xi)/N \text{ as sets with } A_G(\xi) \times A_L(\xi')\text{-actions}, \tag{21}$$

where

$$\begin{aligned} N = Z_{Z^0_L(\xi')U_P}(\xi)/Z^0_G(\xi) \text{ is a normal subgroup of} \\ H = Z_P(\xi)/Z^0_G(\xi), \ H/N \cong A_L(\xi'), \end{aligned} \tag{22}$$

and the group $A_G(\xi) \times H/N \cong A_G(\xi) \times A_L(\xi')$ acts on $A_G(\xi)/N$ by $(a, hN).(xN) = axh^{-1}N$. Thus it follows from (19) and (21) that for $\phi \in \operatorname{Irr} A_G(\xi)$ and $\phi' \in \operatorname{Irr} A_L(\xi')$,

$$n_{\xi,\xi',\phi,\phi'} = \langle \tilde{\phi}', \operatorname{Res}^{A_G(\xi)}_H \phi \rangle, \text{ where } \tilde{\phi}' \text{ is the lift of } \phi' \text{ to } H. \tag{23}$$

Moreover, the same argument as in [LS1, 1.3] shows that $\dim \mathcal{B}^G_\xi = \dim \mathcal{B}^L_{\xi'}$. It then follows from (18), the definition of the truncated induction operator j (see §2) and the above discussion that

$$\text{If the orbit of } \xi \text{ is induced from that of } \xi' \in \mathcal{N}_{\mathfrak{l}^*}, \text{ then } \rho^G_{\xi,1} = j^W_{W_L}(\rho^L_{\xi',1}). \tag{24}$$

Finally, the same argument as in [AL] shows that

$$\text{If } \xi \in \mathfrak{l}^*, \text{ then } \langle \rho^G_{\xi,1}, \operatorname{Ind}^W_{W_L} \hat{\rho}^L_{\xi,1} \rangle \neq 0, \text{ where } \hat{\rho}^L_{\xi,1} = \sum (-1)^i H^i(\mathcal{B}^L_\xi, \bar{\mathbb{Q}}_l). \tag{25}$$

In addition, we remark that the decomposition (17) implies that we have a decomposition

$$H^{2d_\xi}(\mathcal{B}^G_\xi, \bar{\mathbb{Q}}_l) \cong \bigoplus_{\phi \in \operatorname{Irr} A_G(\xi)} \phi \otimes \rho^G_{\xi,\phi}$$

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[2] Here $d_{\xi,\xi'} = (\dim Z_G(\xi) + \dim Z_L(\xi'))/2 + \dim U_P$.

as $A_G(\xi) \times W$ representations, where $d_\xi = \dim \mathcal{B}_\xi^G$ and $A_G(\xi)$ acts via the permutation representation ϵ_ξ afforded by the set of irreducible components of $\mathcal{B}_\xi^G$. In particular, we see that

$$\rho_{\xi,\phi}^G \neq 0 \Leftrightarrow \phi \text{ occurs in the above permutation representation } \epsilon_\xi. \qquad (26)$$

4.2. Type G_2 in characteristic 3.

We verify in this subsection the description of the map $\gamma_{\mathfrak{g}^*}$ given in Table 6. First, using (18), Table 1 and Table 3, one determines easily the images of $\chi_{1,1}$, $\chi_{1,2}$, $\chi_{2,1}$ and $\chi_{2,2}$ under $\gamma_{\mathfrak{g}^*}$.

For $\xi_4 = e'_\beta$ in the orbit A_1, note that ξ_4 is a regular nilpotent element in $\mathfrak{l}^*$, where L is a Levi subgroup of type A_1. Using (25), we see that $\langle \rho_{\xi_4,1}, \mathrm{Ind}_{W(A_1)}^W[2] \rangle \neq 0$. Thus it follows from [A, Table 64] that $\rho_{\xi_4,1} = \chi_{1,4}$.

It remains to determine $\gamma_{\mathfrak{g}^*}(\chi_{1,3})$. To this end, it is enough to determine the pair (ξ_2, ϕ), $\xi_2 = e'_\beta + e'_{2\alpha+\beta}$ in the orbit $G_2(a_1)$ and $\phi \neq 1$, that appears in the image of $\gamma_{\mathfrak{g}^*}$. In view of (26) we study the permutation representation ϵ_{ξ_2} of $A_G(\xi_2) \cong S_3$ afforded by the set of irreducible components of $\mathcal{B}_{\xi_2}$. Using the Bruhat decomposition (1) and (3), one can check that $g^{-1}.\xi_2 \in \mathfrak{b}^*$ if and only if $g^{-1} \in B$, $g^{-1} \in Bs_\alpha x_\alpha(t_1)$, or $g^{-1} \in Bs_\beta s_\alpha x_\alpha(\varpi) x_{3\alpha+\beta}(t_5)$, where $\varpi^3 + \varpi = 0$. Thus $\mathcal{B}_{\xi_2}$ has 4 irreducible components. Using the description of $A_G(\xi_2)$ in (9), one readily checks that the permutation representation ϵ_{ξ_2} decomposes as $(3) \oplus (3) \oplus (2,1)$. Thus $\gamma_{\mathfrak{g}^*}(\chi_{1,3}) = (\xi_2, (2,1))$. This completes the verification of Table 6.

4.3. Type F_4 in characteristic 2.

We verify in this subsection the description of the map $\gamma_{\mathfrak{g}^*}$ given in Table 7. First, using (18), Table 2 and Table 4, one determines easily the image of $\chi_{1,1}$, $\chi_{4,2}$, $\chi_{9,1}$, $\chi_{16,1}$, $\chi_{9,4}$, $\chi_{4,5}$, and $\chi_{1,4}$, under $\gamma_{\mathfrak{g}^*}$.

Using (18), (24), [A, Tables 60-61], and Table 5 (where the last row indicates $\rho_{\xi',1}^L$ for $\xi' \in \mathcal{O}$), we see that $\gamma_{\mathfrak{g}^*}(\chi_{8,2}) = (\widetilde{A}_2, \bar{\mathbb{Q}}_l)$, $\gamma_{\mathfrak{g}^*}(\chi_{8,4}) = (A_2, \bar{\mathbb{Q}}_l)$, $\gamma_{\mathfrak{g}^*}(\chi_{9,2}) = (B_2, \bar{\mathbb{Q}}_l)$, $\gamma_{\mathfrak{g}^*}(\chi_{12,1}) = (F_4(a_3), \bar{\mathbb{Q}}_l)$ and $\gamma_{\mathfrak{g}^*}(\chi_{8,1}) = (B_3, \bar{\mathbb{Q}}_l)$.

Note that the representative ξ in Table 4 for the orbit $A_2 + \widetilde{A}_1$ (resp. $(B_3)_2$, C_3, $(A_2)_2$, A_1) is a regular nilpotent element in $\mathfrak{l}^*$ for a Levi L of type $A_2\widetilde{A}_1$ (resp. B_3, C_3, A_2, A_1). Thus using (18), (25) and again [A, Tables 60-61], we see that $\gamma_{\mathfrak{g}^*}^{-1}(\mathcal{O}_\xi, \bar{\mathbb{Q}}_l) = \chi_{4,3}$ (resp. $\chi_{2,1}$, $\chi_{8,3}$, $\chi_{1,2}$, $\chi_{2,4}$). For the orbit $(A_2)_2$, we have used that $\mathrm{Ind}_{W(A_2)}^W(3) = \mathrm{Ind}_{W(B_3)}^W([3:-] \oplus [2:1] \oplus [1:2] \oplus [-:3])$, and for the orbit (A_1), we have used that $\mathrm{Ind}_{W(A_1)}^W(2) = \mathrm{Ind}_{W(B_3)}^W([3:-] \oplus [2:1] \oplus [1:2] \oplus [-:3] \oplus [21:-] \oplus [2:1] \oplus [1:2] \oplus [-:21] \oplus [1^2:1] \oplus [1:1^2])$.

For the orbit $\widetilde{A}_2 + A_1$, again the representative ξ is a regular nilpotent element in $\mathfrak{l}^*$ for a Levi L of type $\widetilde{A}_2 A_1$. Thus $\gamma_{\mathfrak{g}^*}^{-1}(\mathcal{O}_\xi, \bar{\mathbb{Q}}_l) \in \{\chi_{6,1}, \chi_{9,3}\}$. Note that $\langle [21:-], \mathrm{Res}_{W(B_3)}^W \chi_{9,3} \rangle \neq 0$ while $\langle [2:1], \mathrm{Res}_{W(B_3)}^W \chi_{9,3} \rangle = 0$. This contradicts with (20) as $[2:1] = (\gamma_{\mathfrak{g}^*}^{B_3})^{-1}((2; 1_1^2), \bar{\mathbb{Q}}_l)$ and $[21:-] = (\gamma_{\mathfrak{g}^*}^{B_3})^{-1}((2; 1_1^2), \mathcal{E}_{(1^2)})$. Hence $\gamma_{\mathfrak{g}^*}^{-1}(\widetilde{A}_2 + A_1, \bar{\mathbb{Q}}_l) = \chi_{6,1}$.

Let ξ be in the orbit $F_4(a_1)$. Taking ξ' varying in $\mathfrak{l}^*$ for a Levi subgroup L of type B_3 and using [A, Table 60], we see that $n_{\xi,\xi',1,1} \neq 0$ if and only if ξ' satisfies $\rho_{\xi',1}^L = [3:-]$ or $\rho_{\xi',1}^L = [2:1]$. In the former case $A_L(\xi') = 1$ and in the latter case we have $A_L(\xi') = S_2$ and $\rho_{\xi',(1^2)}^L = [21:-]$ (see [X3]).

By (20), $\langle \mathrm{Res}^W_{W(B_3)} \rho_{\xi,(1^2)}, \rho' \rangle_{W(B_3)} = 0$ for any $\rho' \in \mathrm{Irr}(W(B_3)) - \{[3 : -], [2 : 1], [21 : -]\}$. Thus $\rho_{\xi,(1^2)} = \chi_{2,3}$. The same argument shows that for ξ in the orbit B_2, $\langle \mathrm{Res}^W_{W(B_3)} \rho_{\xi,(1^2)}, \rho' \rangle_{W(B_3)} = 0$ for any $\rho' \in \mathrm{Irr}(W(B_3)) - \{[2 : 1], [21 : -], [1 : 2], [- : 3], [- : 21]\}$, thus $\rho_{\xi,(1^2)} = \chi_{4,1}$.

Consider now ξ_6 in the orbit $F_4(a_3)$ and $1 \neq \phi \in \mathrm{Irr}(A_G(\xi_6))$. The same argument as above shows that $\rho_{\xi_6,\phi} \neq \chi_{2,2}$. Similarly, for ξ_8 in the orbit $C_3(a_1)$, $\rho_{\xi_8,(1^2)} \neq \chi_{2,2}$. This forces that $\chi_{2,2} = \gamma_{\mathfrak{g}^*}(\widetilde{A}_1, \mathcal{E}_{(1^2)})$.

It remains to determine the images of $\chi_{9,3}$, $\chi_{6,2}$, $\chi_{4,4}$ and $\chi_{1,3}$ under $\gamma_{\mathfrak{g}^*}$. We make use of (23). Let $\xi = e'_{pqr} + e'_{qrs} + e'_{pq2r} + e_{q2r2s}$ in the orbit $F_4(a_3)$ and $\xi' = e'_{pqr} + e'_{pq2r}$. Let P be the standard parabolic subgroup P_J, $J = \{p, q, r\}$ and L the standard Levi subgroup of P. Then L is of type B_3. Moreover, $\xi' \in \mathfrak{l}^*$ and $\xi \in \xi' + \mathfrak{n}_P^*$. It has been shown that the orbit of ξ is induced from that of ξ' (see (15)). Using the description of the component group $A_G(\xi)$ in (10), one can check that the groups defined in (22) are as follows

$$H = Z_P(\xi)/Z_G^0(\xi) \cong \langle \gamma_1, \gamma_3 \rangle \cong S_2 \times S_2 \text{ and } N = Z_{Z_L^0(\xi')U_P}(\xi)/Z_G^0(\xi) \cong \langle \gamma_1 \rangle \cong S_2.$$

One checks that

$$\mathrm{Ind}_H^{A_G(\xi)} \mathbf{1} = \mathrm{Ind}_{S_2 \times S_2}^{S_4} \mathbf{1} = (4) \oplus (3, 1) \oplus (2, 2), \quad \mathrm{Ind}_H^{A_G(\xi)} \widetilde{\phi}' = (3, 1) \oplus (2, 1, 1), \quad (27)$$

where $\mathbf{1}$ is the trivial representation and $\widetilde{\phi}'$ is the lift of the sign character ϕ' of $H/N \cong S_2$ to $H \cong S_2 \times S_2$. It is shown in [X3] that

$$\rho^L_{\xi',1} = [1 : 1^2] \text{ and } \rho^L_{\xi',\phi'} = [1^3 : -].$$

Let $\phi \in \mathrm{Irr}\, A_G(\xi)$. Using (19), (23) and (27), we conclude that

$$\langle \rho^G_{\xi,\phi}, \mathrm{Ind}_{W(B_3)}^W [1; 1^2] \rangle_W = \langle \rho^G_{\xi,\phi}, \chi_{12,1} \oplus \chi_{6,2} \oplus \chi_{9,3} \rangle_W = \langle \phi, (4) \oplus (3, 1) \oplus (2, 2) \rangle_{S_4}$$

$$\langle \rho^G_{\xi,\phi}, \mathrm{Ind}_{W(B_3)}^W [1^3; 0] \rangle_W = \langle \rho^G_{\xi,\phi}, \chi_{4,4} \oplus \chi_{9,3} \oplus \chi_{1,3} \rangle_W = \langle \phi, (3, 1) \oplus (2, 1, 1) \rangle_{S_4}$$

where in the first identity we have used [A, Table 60]. It thus follows that

$$\rho_{\xi,(3,1)} = \chi_{9,3}, \quad \rho_{\xi,(2,2)} = \chi_{6,2}, \quad \rho_{\xi,(2,1,1)} \in \{\chi_{4,4}, \chi_{1,3}\}. \quad (28)$$

Now let P be the standard parabolic subgroup P_J, where $J = \{p, q, s\}$, and let L be the standard Levi subgroup of P, which is of type $A_2\widetilde{A}_1$. Then $\xi \in \mathfrak{n}_P^*$ and ξ is induced from the zero orbit in $\mathfrak{l}^*$. Again using (10) we see that in this case

$$H = Z_P(\xi)/Z_G^0(\xi) \cong \langle \gamma_1, \gamma_2 \rangle = N \cong S_3.$$

One checks that

$$\mathrm{Ind}_H^{A_G(\xi)} \mathbf{1} = \mathrm{Ind}_{S_3}^{S_4} \mathbf{1} = (4) \oplus (3, 1).$$

The same argument as above shows that

$$\langle \rho^G_{\xi,\phi}, \chi_{4,4} \oplus \chi_{12,1} \oplus \chi_{9,3} \rangle_W = \langle \phi, (4) \oplus (3, 1) \rangle_{S_4}.$$

In view of (28), it follows from the above equation that

$$\rho_{\xi,(2,1,1)} = \chi_{1,3} \text{ and } \chi_{4,4} \neq \rho_{\xi,(1^4)}.$$

The last inequality forces that $\chi_{4,4} = \rho_{\xi_8,(1^2)}$, where ξ_8 is in the orbit $C_3(a_1)$. This completes the verification of Table 7.

4.4. A set of Weyl group representations. In this subsection we define a subset of $\text{Irr}(W)$ following [L3] which gives an a prior definition of the set $\{\gamma_{\mathfrak{g}^*}^{-1}(\mathcal{O}, \bar{\mathbb{Q}}_l) \mid (\mathcal{O}, \bar{\mathbb{Q}}_l) \in \mathcal{A}_{\mathfrak{g}^*}\}$.

Let $R^\vee$ be the set of coroots. For $\alpha \in R$, let $\alpha^\vee$ denote the corresponding coroot. Define
$$\widetilde{\Theta} = \{\beta \in R \mid \beta^\vee - \alpha^\vee \notin R^\vee, \forall \alpha \in \Pi\};$$

$$\widetilde{\Theta}_r = \{J \subset \widetilde{\Theta} \mid J \text{ linearly independent}, |\sum_{\alpha \in \Pi} \mathbb{Z}\alpha^\vee / \sum_{\beta \in J} \mathbb{Z}\beta^\vee| = r^k \text{ for some } k \in \mathbb{Z}_{>0}\}.$$

For $J \in \widetilde{\Theta}_r$, let W_J be the subgroup of W generated by the reflections $s_\alpha, \alpha \in J$.

We define a set $\mathcal{T}_W^{*,r} \subset \text{Irr } W$ by induction on $|W|$ as follows. If $W = \{1\}$, $\mathcal{T}_W^{*,r} = \text{Irr}(W)$. If $W \neq \{1\}$, then

$$\mathcal{T}_W^{*,r} = \mathcal{S}_W^1 \cup \{\rho \in \text{Irr}(W) \mid \rho = j_{W_J}^W \rho' \text{ for some } J \in \widetilde{\Theta}_r \text{ and some } \rho' \in \mathcal{T}_{W_J}^{*,r}\},$$

where $\mathcal{S}_W^1$ is defined as in [L2, 1.3], i.e. it is the set of irreducible characters of W that correspond to the pairs $(\mathcal{O}, \bar{\mathbb{Q}}_l)$ in characteristic 0.

One can verify that the set

$$\{\gamma_{\mathfrak{g}^*}^{-1}(\mathcal{O}, \bar{\mathbb{Q}}_l) \mid (\mathcal{O}, \bar{\mathbb{Q}}_l) \in \mathcal{A}_{\mathfrak{g}^*}\} \text{ coincides with the set } \mathcal{T}_W^{*,3} \text{ (resp. } \mathcal{T}_W^{*,2})$$
$$\text{for } G_2 \text{ (resp. } F_4) \text{ in characteristic 3 (resp. 2).}$$

In fact for G_2, we have $\widetilde{\Theta}_3 = \{J\}$, where $J = \{\alpha, -(2\alpha + \beta)\}$ and $W_J = W(A_2)$, and in characteristic 3

$$\{\gamma_{\mathfrak{g}^*}^{-1}(\mathcal{O}, \bar{\mathbb{Q}}_l) \mid (\mathcal{O}, \bar{\mathbb{Q}}_l) \in \mathcal{A}_{\mathfrak{g}^*}\} = \mathcal{S}_W^1 = \mathcal{T}_W^{*,3};$$

for F_4, we have $\widetilde{\Theta}_2 = \{J_1, J_2, J_3\}$, where $J_1 = \{p, q, r, -p2q3r2s\}$, $W_{J_1} = W(B_3A_1)$, $J_2 = \{p, r, s, -p2q3r2s\}$, $W_{J_2} = W(\widetilde{A}_3A_1)$, $J_3 = \{q, r, s, -p2q3r2s\}$ and $W(J_4) = W(C_4)$, and in characteristic 2

$$\{\gamma_{\mathfrak{g}^*}^{-1}(\mathcal{O}, \bar{\mathbb{Q}}_l) \mid (\mathcal{O}, \bar{\mathbb{Q}}_l) \in \mathcal{A}_{\mathfrak{g}^*}\} = \mathcal{S}_W^1 \cup \{\chi_{2,1}, \chi_{1,2}\} = \mathcal{T}_W^{*,2}.$$

Here we have used again the tables in [A] and the description of $\mathcal{T}_W^{*,2}$ in [X3] when W is of type B or C.

5. Springer correspondence for $\mathfrak{g}$

Let G be of type G_2 (resp. F_4) defined over an algebraically closed field of characteristic 2 or 3 (resp. characteristic 3). In this section, we describe the Springer correspondence maps $\gamma_{\mathfrak{g}} : \text{Irr } W \to \mathcal{A}_{\mathfrak{g}}$ explicitly, see Tables 8, 9 and 10. In particular, as mentioned in the introduction, the Springer correspondence for $\mathfrak{g}$ turns out to be the same as in characteristic 0 when G is of type G_2 and char $\mathbf{k} = 2$, or when G is of type F_4 and char $\mathbf{k} = 3$. In view of Lemma 2.1 and §4,

Orbit of x	ϕ	$\rho_{x,\phi}$	Orbit of x	ϕ	$\rho_{x,\phi}$
G_2	(1)	$\chi_{1,1}$	$\widetilde{A}_1$	(1)	$\chi_{2,2}$
$G_2(a_1)$	(3)	$\chi_{2,1}$	A_1	(1)	$\chi_{1,4}$
$G_2(a_1)$	$(2,1)$	$\chi_{1,3}$	$\emptyset$	(1)	$\chi_{1,2}$
$G_2(a_1)$	(1^3)	$-$			

Table 8: Springer correspondence for $\mathfrak{g}$, type G_2 and char $\mathbf{k} = 2$

Orbit of x	ϕ	$\rho_{x,\phi}$	Orbit of x	ϕ	$\rho_{x,\phi}$
G_2	(1)	$\chi_{1,1}$	$(\widetilde{A}_1)_2$	(1)	$\chi_{1,3}$
$G_2(a_1)$	(2)	$\chi_{2,1}$	A_1	(1)	$\chi_{1,4}$
$G_2(a_1)$	$(1,1)$	$-$	$\emptyset$	(1)	$\chi_{1,2}$
$\widetilde{A}_1$	(1)	$\chi_{2,2}$			

Table 9: Springer correspondence for $\mathfrak{g}$, type G_2 and char $\mathbf{k} = 3$

this completes the description of the Springer correspondence for $\mathfrak{g}$ and $\mathfrak{g}^*$ when G is of type G_2 and F_4 in all characteristics, as explained in the introduction.

As in §4, the orbits are labelled as in the Bala-Carter classification, and $\rho_{x,\phi}$, for $x \in \mathcal{N}_{\mathfrak{g}}$ and $\phi \in \operatorname{Irr} A_G(x)$, denotes the inverse image of $(\mathcal{O}_x, \mathcal{E})$ under $\gamma_{\mathfrak{g}}$, where $\mathcal{E}$ is the local system on the orbit $\mathcal{O}_x$ of x corresponding to ϕ.

5.1. Nilpotent orbits in $\mathfrak{g}$.

The nilpotent orbits in $\mathfrak{g}$ for G_2 have been classified by Stuhler [St] and those for F_4 by Holt and Spaltenstein [HS]. It has been shown (see [HS]) that for $x \in \mathcal{N}_{\mathfrak{g}}$,

$$\dim \mathcal{B}_x = \frac{\dim Z_G(x) - \operatorname{rank} G}{2},$$

where $\mathcal{B}_x = \{gB \mid \operatorname{Ad}(g^{-1})x \in \mathfrak{b}\}$. We list the results here for the reader's convenience, see Tables 11, 12 and 13.

Note that the nilpotent orbits in $\mathfrak{g}$ for F_4 in char $\mathbf{k} = 3$ and component groups of centralizers coincide with those in characteristic 0. In particular, there are 4 distinguished nilpotent orbits (where $x \in \mathcal{N}_{\mathfrak{g}}$ is distinguished if $Z(G)^0$ is a maximal torus in $Z_G(x)$) and every orbit contains an element that is distinguished in a Levi subgroup.

5.2. Springer correspondence.

We use the same strategy as in §4 to obtain the explicit description of the maps $\gamma_{\mathfrak{g}}$, following the methods in [AL, S2, S5]. As the arguments are entirely similar, we omit the details and only explain one most complicated case.

For F_4, we make use of Table 14 on induced nilpotent orbits. Here $\tilde{\mathcal{O}} = \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}$, 0 denotes the zero orbit in the third row, and the nilpotent orbits in $\mathfrak{l}$ correspond to a partition.

The most complicated case is the orbit $F_4(a_3)$. Let $x = e_{pqr} + e_{qrs} + e_{pq2r} + $

Orbit of x	ϕ	$\rho_{x,\phi}$	Orbit of x	ϕ	$\rho_{x,\phi}$	Orbit of x	ϕ	$\rho_{x,\phi}$
F_4	(1)	$\chi_{1,1}$	$F_4(a_3)$	(2^2)	$\chi_{6,2}$	$\widetilde{A_2}$	(1)	$\chi_{8,2}$
$F_4(a_1)$	(2)	$\chi_{4,2}$	$F_4(a_3)$	$(2,1,1)$	$\chi_{1,3}$	A_2	(2)	$\chi_{8,4}$
$F_4(a_1)$	(1^2)	$\chi_{2,3}$	$F_4(a_3)$	(1^4)	$-$	A_2	(1^2)	$\chi_{1,2}$
$F_4(a_2)$	(2)	$\chi_{9,1}$	$C_3(a_1)$	(2)	$\chi_{16,1}$	$A_1 + \widetilde{A_1}$	(1)	$\chi_{9,4}$
$F_4(a_2)$	(1^2)	$\chi_{2,1}$	$C_3(a_1)$	(1^2)	$\chi_{4,4}$	$\widetilde{A_1}$	(2)	$\chi_{4,5}$
B_3	(1)	$\chi_{8,1}$	B_2	(2)	$\chi_{9,2}$	$\widetilde{A_1}$	(1^2)	$\chi_{2,2}$
C_3	(1)	$\chi_{8,3}$	B_2	(1^2)	$\chi_{4,1}$	A_1	(1)	$\chi_{2,4}$
$F_4(a_3)$	(4)	$\chi_{12,1}$	$\widetilde{A_2} + A_1$	(1)	$\chi_{6,1}$	$\emptyset$	(1)	$\chi_{1,4}$
$F_4(a_3)$	$(3,1)$	$\chi_{9,3}$	$A_2 + \widetilde{A_1}$	(1)	$\chi_{4,3}$			

Table 10: Springer correspondence for $\mathfrak{g}$, type F_4 and char $\mathbf{k} = 3$

Orbit	representative x	dim $Z_G(x)$	$A_G(x)$	dim $\mathcal{B}_x$
G_2	$e_\alpha + e_\beta$	2	1	0
$G_2(a_1)$	$e_\beta + e_{2\alpha+\beta}$	4	S_3	1
$\widetilde{A_1}$	e_α	6	1	2
A_1	e_β	8	1	3
$\emptyset$	0	14	1	6

Table 11: Type G_2, nilpotent orbits, char $\mathbf{k} = 2$

e_{q2r2s} be an element in the orbit $F_4(a_3)$. By direct computations using (2) and (4), one verifies that

$$A_G(x) \cong \langle \gamma_1, \gamma_2, \gamma_3 \rangle \cong S_4,$$

where

$$\gamma_1 = h_p(-1)h_q(-1)h_r(-1)h_s(-1)x_p(-1)x_s(-1)x_{rs}(1),$$
$$\gamma_2 = h_p(-1)h_q(-1)h_r(-1)h_s(-1)\, n_p\, n_s,$$
$$\gamma_3 = h_p(-1)h_q(-1)x_p(-1)x_s(1)x_{rs}(1)x_r(-1)\, n_r\, x_r(-1)$$

with

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = 1, \ (\gamma_1\gamma_2)^3 = (\gamma_2\gamma_3)^3 = 1, \ (\gamma_1\gamma_3)^2 = 1.$$

In particular we have used the following formula

$$\mathrm{Ad}(u(\mathbf{t}))x = e_{pqr} + e_{qrs} + (1 - 2t_2)e_{pq2r} + t_2\, e_{q2rs} + (t_1 - t_3)\, e_{pqrs} + (1 - 2t_4)e_{q2r2s}$$
$$+ (t_1 t_2 - t_2 t_3 + t_3 + t_4)e_{pq2rs} + (t_1 + t_3^2 - 2t_4(t_1 - t_3))e_{pq2r2s}$$

where $u(\mathbf{t}) = x_p(t_1)\, x_r(t_2)\, x_s(t_3)\, x_{rs}(t_4)$, $t_i \in \mathbb{G}_a$. One can then proceed the same way as in the case of orbit $F_4(a_3)$ in $\mathfrak{g}^*$ in characteristic 2.

Orbit	representative x	$\dim Z_G(x)$	$A_G(x)$	$\dim \mathcal{B}_x$
G_2	$e_\alpha + e_\beta$	2	1	0
$G_2(a_1)$	$e_\beta + e_{2\alpha+\beta}$	4	S_2	1
$\widetilde{A_1}$	$e_{\alpha+\beta} + e_\beta$	6	1	2
$(\widetilde{A_1})_2$	e_α	8	1	3
A_1	e_β	8	1	3
$\emptyset$	0	14	1	6

Table 12: Type G_2, nilpotent orbits, char $\mathbf{k} = 3$

Orbit	representative x	$\dim Z_G(x)$	$A_G(x)$	$\dim \mathcal{B}_x$
F_4	$e_p + e_q + e_r + e_s$	4	1	0
$F_4(a_1)$	$e_p + e_{qr} + e_{q2r} + e_s$	6	S_2	1
$F_4(a_2)$	$e_p + e_{qr} + e_{rs} + e_{q2r2s}$	8	S_2	2
B_3	$e_p + e_{qr} + e_{q2r2s}$	10	1	3
C_3	$e_s + e_{q2r} + e_{pqr}$	10	1	3
$F_4(a_3)$	$e_{pqr} + e_{qrs} + e_{pq2r} + e_{q2r2s}$	12	S_4	4
$C_3(a_1)$	$e_{pqr} + e_{q2rs} + e_{q2r2s}$	14	S_2	5
B_2	$e_{pqr} + e_{q2r2s}$	16	S_2	6
$\widetilde{A_2} + A_1$	$e_{pqr} + e_{q2rs} + e_{p2q2r2s}$	16	1	6
$A_2 + \widetilde{A_1}$	$e_{p2q2r} + e_{q2r2s} + e_{pq2rs}$	18	1	7
$\widetilde{A_2}$	$e_{pqrs} + e_{q2rs}$	22	1	9
A_2	$e_{p2q2r} + e_{pq2r2s}$	22	S_2	9
$A_1 + \widetilde{A_1}$	$e_{p2q2r2s} + e_{p2q3rs}$	24	1	10
$\widetilde{A_1}$	$e_{p2q3r2s}$	30	S_2	13
A_1	$e_{2p3q4r2s}$	36	1	16
$\emptyset$	0	52	1	24

Table 13: Type F_4, nilpotent orbits, char $\mathbf{k} = 3$

$\widetilde{\mathcal{O}}$	$\widetilde{A_2}$	A_2	B_2	$F_4(a_3)$	$F_4(a_3)$	$F_4(a_1)$	$F_4(a_2)$
type of L	B_3	C_3	C_3	$\widetilde{A_2}A_1$	B_3	B_3	$A_1\widetilde{A_1}$
$\mathcal{O}$	0	0	$2^1 1^4$	0	$3^1 1^4$	$5^1 1^2$	0
$\rho^L_{x',1}$	$[- : 1^3]$	$[- : 1^3]$	$[1^3 : -]$	$[1^3] \times [1^2]$	$[1 : 1^2]$	$[2 : 1]$	$[2] \times [2]$

Table 14: Induced nilpotent orbits, type F_4, char $\mathbf{k} = 3$

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