Toda Field Theories and Integral Curves
of Standard Differential Systems

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Abstract. This paper establishes three relations between the Toda field theory associated to a simple Lie algebra and the integral curves of the standard differential system on the corresponding complete flag variety. The motivation comes from the viewpoint on the Toda field theories as Darboux integrable differential systems as developed in [1]. First, we establish an isomorphism concerning regular functions on the jet space and on the unipotent subgroup in the setting of a simple Lie group. Using this result, we then show that in the sense of differential systems, after restricting one independent variable to a constant the Toda field theory becomes the system for integral curves of the standard differential system on a complete flag variety. Finally, we establish that, in terms of differential invariants, the Toda field theory is the quotient of the product of two such systems by a natural group action.
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1. Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank ℓ, and let (a_{ij}) be its Cartan matrix. The (conformal) Toda field theory associated to $\mathfrak{g}$ is the following system of semilinear PDEs

$$u_{i,xy} = -\exp\Big(\sum_{j=1}^{\ell} a_{ij}u_j\Big), \quad 1 \le i \le \ell, \tag{1}$$

where x and y are the independent variables and the u_i are the unknown functions.

When $\mathfrak{g} = \mathfrak{sl}_2$, the corresponding Toda field theory is the ubiquitous Liouville equation. Toda field theories are fundamental examples of integrable systems and have been studied thoroughly in the literature (see the books [20, 2]). They have zero curvature representations, and there have been detailed studies of their (local) solutions [18, 19]. The solution structure has close relations to the representation theory of $\mathfrak{g}$ and its Lie group G [19, 17]. In this regard, the author

would also like to mention the recent classification results for the elliptic versions of the Toda systems (1), where one replaces $u_{i,xy}$ by $u_{i,z\bar{z}} = \frac{1}{4}\Delta u_i$ (see [22, 25, 21]). In fact, the Liouville equation was first studied in this elliptic form in relation to conformal metrics of constant Gaussian curvature [23].

In this paper, we are concerned with the aspects of Toda field theories from the viewpoint of differential systems. By a differential system, we mean a Pfaffian system of 1-forms with constant rank on a manifold M [4]. In the dual viewpoint, the differential system is given by a subbundle of the tangent bundle TM, called a distribution, defined as the kernel of the Pfaffian system. We treat Toda field theories as Darboux integrable differential systems and will establish the quotient structure from [1, Theorem 1.4] for such systems. We take a direct approach to establishing our results, and the proofs are crucially based on the works of Kostant [16], of Feigin and Frenkel [9, 11], of Leznov and Saveliev [19, 20], and of the author himself [24].

A system of hyperbolic equations on the plane is called Darboux integrable if there is a sufficient number of characteristic integrals, which are also called intermediate integrals in [1] and integrals of motion in [10]. A *characteristic integral* for the Toda field theory (1) is a polynomial of the derivatives of the u_i with respect to one independent variable whose derivative with respect to the other independent variable is zero if the u_i are solutions. For example, for the Liouville equation $u_{xy} = -e^{2u}$, $I = u_{xx} - u_x^2$ is a characteristic integral on the x-side since $I_y = 0$ for a solution u. Of course so is $u_{yy} - u_y^2$ on the y-side. Such integrals have been intensively studied in the literature. We refer the reader to [10] for their cohomological interpretation and theoretical structure. These characteristic integrals form the $\mathcal{W}$-algebra and as such have been thoroughly studied in [3] and for more general gradings in [8]. In [24], the author has directly established that for the Toda field theory (1) associated to a Lie algebra of rank ℓ, there are ℓ basic characteristic integrals I_j on one side. Furthermore, there is a completely algebraic and explicit algorithm for constructing them (see (9)). This very algorithm will actually be important when we prove Theorem 1.1 on the relation of jet spaces and unipotent subgroups. In this connection, the author would also like to mention the interesting application of characteristic integrals to the classification of solutions to elliptic Toda systems with finite energy and with singular sources [22, 25, 21].

Let us now briefly discuss the key results of this paper. One underlying theme of this paper is the relation of jet spaces and unipotent subgroups. Such relations, especially for affine Lie algebras, lie at the heart of the soliton equations and can be said to be the reason for their existence. This approach was thoroughly developed in a series of papers by Feigin, Frenkel and Enriquez [10, 9, 7] and was beautifully surveyed in [11]. The following Theorem 1.1 is an analogue of [9, Prop. 4] (see also [11, Theorem 1.1]) in the setting of finite-dimensional simple Lie groups. This theorem is proved in Section 2.

We first introduce some notation. Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$. Let $\Delta = \Delta^+ \cup \Delta^-$ be a decomposition of the set of roots of $\mathfrak{g}$ into the sets of positive and negative ones, and let $\{\alpha_1, \ldots, \alpha_\ell\}$ be the set of positive simple roots. Let e_α be a root vector in the root space $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$. We will specify

our normalization of the e_α when needed. Let G be any connected Lie group integrating $\mathfrak{g}$, and let B_+ and N_- be the Borel and lower unipotent subgroups corresponding to the Lie subalgebras $\mathfrak{b}_+ = \mathfrak{h} \oplus \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \oplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$. Then N_- is diffeomorphic to a Euclidean space [13].

The homogeneous space G/B_+ is called a complete flag variety, and it has a natural transitive left G-action by $g(kB_+) = gkB_+$ for $g \in G$ and $kB_+ \in G/B_+$. It is well-known that the composition $N_- \hookrightarrow G \to G/B_+$ of the inclusion and the projection is injective and its image is a big cell (see e.g. [17, 20, 9]). We will mostly work with this big cell in the local pictures and continue to denote it by N_-. In particular, N_- has a $\mathfrak{g}$-action induced from the above left G-action.

Theorem 1.1. *Let $\mathbb{C}[\mathbf{U}]$ be the ring of polynomials on the variables $u_{i,x}^{(n)}$ for $1 \leq i \leq \ell$, $n \geq 1$ equipped with an action of the derivative ∂_x which sends $u_{i,x}^{(n)}$ to $u_{i,x}^{(n+1)}$. Let $\mathcal{I}$ be the ideal of $\mathbb{C}[\mathbf{U}]$ generated by the characteristic integrals of the Toda field theory (1) on the x-side and their derivatives $\partial_x^m I_j$ for $1 \leq j \leq \ell$, $m \geq 0$. Moreover, let $\mathbb{C}[N_-]$ be the ring of regular functions on N_-. Then we have a canonical isomorphism*

$$\mathbb{C}[\mathbf{U}]/\mathcal{I} \cong \mathbb{C}[N_-],$$

under which the derivative ∂_x on the left is identified with the derivation $\mathcal{L}_e$ on the right for the infinitesimal action of the principal nilpotent element $e = \sum_{i=1}^\ell e_{\alpha_i} \in \mathfrak{g}$.

By [1, Theorem 1.4], a Darboux integrable differential system is the quotient, in the sense of differential systems, of the product of two Pfaffian systems by the action of a common Lie group G. One of the main goals of this paper is to explicitly demonstrate this for the Toda field theory (1) in a direct way, and this is achieved in our Theorem 1.3. First we need to introduce the two Pfaffian systems in our result.

There exists a so-called grading element $H_0 \in \mathfrak{h}$ such that $\alpha_i(H_0) = 1$, $\forall 1 \leq i \leq \ell$. The principal grading of $\mathfrak{g}$ is

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_i, \quad \text{where } \mathfrak{g}_i = \{X \in \mathfrak{g} \,|\, [H_0, X] = iX\}. \tag{2}$$

Identify the tangent space of G/B_+ at the point $o = 1 \cdot B_+$ with $\mathfrak{g}/\mathfrak{b}_+$, where $1 \in G$ is the identity element. Define the distribution $D \subset T(G/B_+)$ as the G-invariant distribution equal to $\mathfrak{g}_{-1}$ mod $\mathfrak{b}_+$ at o. Clearly $\mathfrak{g}_{-1} = \oplus_{i=1}^\ell \mathfrak{g}_{-\alpha_i}$ is the direct sum of negative simple root spaces. The Pfaffian system dual to the distribution D is called the *standard differential system* for the principal grading [27], and we call D the *standard distribution*. For an interval $I \subset \mathbb{R}^1$, a curve $\Phi : I \to G/B_+$ is called an *integral curve of the standard differential system* if

$$\Phi'(y) = d\Phi\left(\frac{\partial}{\partial y}\right) \in D_{\Phi(y)}, \quad \forall y \in I \tag{3}$$

(see [5] for geometric studies of such curves). Again, we mostly work with $N_- \subset G/B_+$ which has a standard differential system naturally induced by restriction.

Our second main result concerns the differential system representations of Toda field theories. The proof is contained in Section 3 and uses Theorem 1.1 in an essential way.

Theorem 1.2.   *In the differential system for the Toda field theory* (1), *if we restrict x to a constant and discard jet prolongations (see* (31) *and the discussion after it), then we obtain the differential system on the first jet space $J_y^1(N_-, D)$ of the integral curves to the standard differential system on the unipotent Lie group N_- in the following sense: there is an isomorphism between the global vector field generators for the corresponding distributions which respects all Lie brackets.*

The above theorem can be used to determine the two Pfaffian systems in [1, Theorem 1.4] with the other one obtained by restricting y to a constant. Therefore, it is natural to consider the quotient of the product of two such systems by the diagonal G-action. For differential systems with symmetry, quotients are defined in terms of differential invariants. Our third main result studies such differential invariants, and it is proved in Section 4.

Theorem 1.3.   *Let $J_y^1(N_-, D)$ denote the 1st jet space of the integral curves to the standard distribution D on N_- for the independent variable y. For the other independent variable x, consider the similarly defined $J_x^1(N_+, \tilde{D})$ (see the paragraph containing* (48)*). Then there exist ℓ differential invariants of the prolonged natural G-action on the product $J_x^1(N_+, \tilde{D}) \times J_y^1(N_-, D)$, which satisfy the Toda field theory* (1)*.*

2.   Jet spaces and unipotent subgroups

In this section, we prove Theorem 1.1, which will be used in a crucial way in proving Theorem 1.2. Theorem 1.1 has its own interest and is the analogue of [9, Prop. 4], also stated as [11, Theorem 1.1], in the setting of simple Lie groups. Interestingly in this finite-dimensional case, the characteristic integrals naturally come up. Now we recall the construction of such integrals from [24].

Let us introduce the zero curvature representation of (1). We normalize the root vectors e_{α_i} and $e_{-\alpha_i}$ for $1 \le i \le \ell$ such that $\alpha_i(H_{\alpha_i}) = 2$, where $H_{\alpha_i} = [e_{\alpha_i}, e_{-\alpha_i}]$. Then by [12, p. 208],

$$\alpha_i(H_{\alpha_j}) = a_{ij}, \quad 1 \le i, j \le \ell. \tag{4}$$

Let

$$\mathbf{u} = \sum_{i=1}^{\ell} u_{i,x} H_{\alpha_i}, \quad e = \sum_{i=1}^{\ell} e_{\alpha_i}, \quad Y = \sum_{i=1}^{\ell} e^{\rho_i} e_{-\alpha_i}, \tag{5}$$

where throughout the paper we use the shorthand notation

$$\rho_i = \sum_{j=1}^{\ell} a_{ij} u_j. \tag{6}$$

Then the Toda field theory (1) is equivalent to the following zero curvature equation

$$[\partial_x + e + \mathbf{u}, \; \partial_y - Y] = 0. \tag{7}$$

With respect to the principal grading (2) of $\mathfrak{g}$, let $\mathfrak{s}$ be a homogeneous complement of $[e, \mathfrak{g}]$ in $\mathfrak{g}$, that is,

$$\mathfrak{g} \cong \mathfrak{s} \oplus [e, \mathfrak{g}]. \tag{8}$$

Then by [15], $\mathfrak{s} \subset \mathfrak{n}_-$ and $\dim \mathfrak{s} = \ell$. We call $\mathfrak{s}$ a Kostant slice, and let $\{s_j\}_{j=1}^{\ell}$ be a homogeneous basis of $\mathfrak{s}$ with nonincreasing principal gradings $-m_j$. The m_j are called the *exponents* of the Lie algebra $\mathfrak{g}$.

By [6] and [24, Remark 2.1] (see also the proof of Proposition 2.1), we can bring the first element in (7) into its Drinfeld-Sokolov gauge in a unique way. That is, there exists a unique element $M \in N_-$ (whose entries are differential polynomials of the u_i) such that

$$M(\partial_x + e + \mathbf{u})M^{-1} = \partial_x + e + \mathbf{I}, \quad \mathbf{I} = \sum_{j=1}^{\ell} I_j s_j \in \mathfrak{s}, \tag{9}$$

where the I_j are differential polynomials of the u_i. The uniqueness of M is easily proved by induction on the principal grading (2) using the fact that

$$\ker \mathrm{ad}_e \cap (\mathfrak{h} \oplus \mathfrak{n}_-) = 0.$$

Then Theorem 2.1 in [24] proves directly that the I_j are the basic characteristic integrals of the Toda field theory (1) on the x-side. See also [10, Prop. 2.4.7 and §2.4.1].

We first show the following.

Proposition 2.1. *There exists a set of generators for the ring $\mathbb{C}[\mathbf{U}]/\mathcal{I}$ whose cardinality is $\dim N_-$.*

Proof. For a differential monomial in the u_i, we call by its *degree* the sum of the orders of differentiation multiplied by the algebraic degrees of the corresponding factors. For example $I = u_{xx} - u_x^2$ for the Liouville equation has a homogeneous degree 2. According to [10, Prop. 2.4.7], the degrees d_j of the homogeneous characteristic integrals I_j are $d_j = m_j + 1$, and this can also be seen from the algorithm (9).

Define

$$s_j^k = (-\mathrm{ad}_e)^k s_j = [\cdots[s_j, \underbrace{e], \cdots, e]}_{k}, \quad 0 \leq k \leq 2m_j, \; 1 \leq j \leq \ell.$$

It is known from [15] that the above s_j^k form a basis of $\mathfrak{g}$. Hence $\{H_{\alpha_i}\}_{i=1}^{\ell}$ and $\{s_j^{m_j}\}_{j=1}^{\ell}$ are two bases of $\mathfrak{h}$. We define the matrix (c_{ji}) by

$$H_{\alpha_i} = \sum_{j=1}^{\ell} c_{ji} s_j^{m_j}, \quad 1 \leq i \leq \ell.$$

Clearly (c_{ji}) is nondegenerate. Now we show that

$$I_j = \sum_{i=1}^{\ell} c_{ji} u_{i,x}^{(d_j)} + l.o.t., \tag{10}$$

where l.o.t. stands for terms which are products of lower order derivatives.

We will use this opportunity first to give more details on the existence and the uniqueness of M in (9), which satisfies

$$-\partial_x M \cdot M^{-1} + M(e + \mathbf{u})M^{-1} = e + \mathbf{I}. \tag{11}$$

It is well-known that the exponential map $\exp : \mathfrak{n}_- \to N_-$ is a diffeomorphism [13, Cor. 1.126, Thm. 6.46]. Using the so-called coordinates of the second kind [13, p. 76], write

$$M = e^{a_1} \cdots e^{a_{m_\ell}}, \quad a_i \in \mathfrak{g}_{-i}, \ i = 1, \ldots, m_\ell, \tag{12}$$

where m_ℓ is the largest exponent of $\mathfrak{g}$. We will uniquely determine the a_i inductively.

For $i \geq 1$, consider $M_{i-1} = e^{a_1} \cdots e^{a_{i-1}}$ with the convention $M_0 = 1$. Define

$$L_{i-1} := -\partial_x M_{i-1} \cdot M_{i-1}^{-1} + M_{i-1}(e + \mathbf{u})M_{i-1}^{-1}.$$

In general, for $X \in \mathfrak{n}_-$, let X_j denote its component in $\mathfrak{g}_{-j}$ in the principal grading (2). Inductively, assume that

$$(L_{i-1})_j \in \mathfrak{s} \cap \mathfrak{g}_{-j} \quad \text{for } 0 \leq j \leq i - 2.$$

Note that this inductive hypothesis is vacuous when $i = 1$. By (8), the component of L_{i-1} in $\mathfrak{g}_{-(i-1)}$ can be uniquely written as

$$(L_{i-1})_{i-1} = [e, a_i] + \mathbf{J}_{i-1} \tag{13}$$

with $a_i \in \mathfrak{g}_{-i}$ and $\mathbf{J}_{i-1} \in \mathfrak{s} \cap \mathfrak{g}_{-(i-1)}$.

For a general $\tilde{a}_i \in g_{-i}$ and with $\tilde{M}_i = M_{i-1}e^{\tilde{a}_i}$, we have

$$\begin{aligned} \tilde{L}_i &= -\partial_x \tilde{M}_i \cdot \tilde{M}_i^{-1} + \tilde{M}_i(e + \mathbf{u})\tilde{M}_i^{-1} \\ &= -\partial_x M_{i-1} \cdot M_{i-1}^{-1} - M_{i-1}(\partial_x e^{\tilde{a}_i} \cdot e^{-\tilde{a}_i})M_{i-1}^{-1} + M_{i-1}e^{\tilde{a}_i}(e + \mathbf{u})e^{-\tilde{a}_i}M_{i-1}^{-1}. \end{aligned} \tag{14}$$

Since $\tilde{a}_i \in \mathfrak{g}_{-i}$, we see that

$$\begin{aligned} (\tilde{L}_i)_j &= (L_{i-1})_j \in \mathfrak{s} \cap \mathfrak{g}_{-j}, \quad 0 \leq j \leq i - 2, \\ (\tilde{L}_i)_{i-1} &= (L_{i-1})_{i-1} + [\tilde{a}_i, e]. \end{aligned} \tag{15}$$

If and only if we choose $\tilde{a}_i = a_i$ from (13), we see that

$$(L_i)_{i-1} = (L_{i-1})_{i-1} + [a_i, e] = \mathbf{J}_{i-1} \in \mathfrak{s} \cap \mathfrak{g}_{-(i-1)}, \tag{16}$$

where L_i denotes $\tilde{L}_i$ with $\tilde{a}_i = a_i$. Therefore,

$$(L_i)_j \in \mathfrak{s} \cap \mathfrak{g}_{-j} \quad \text{for } 0 \leq j \leq i - 1.$$

The inductive proof for the existence and uniqueness of M is completed.

Furthermore, in view of (11), (12), (15), (16) and (13), we have that

$$\mathbf{I}_{i-1} = (L_{m_\ell})_{i-1} = (L_i)_{i-1} = \text{component of } (L_{i-1})_{i-1} \text{ in } \mathfrak{s} \cap \mathfrak{g}_{i-1}. \qquad (17)$$

That is, the component of $(L_{i-1})_{i-1}$ in $\mathfrak{s} \cap \mathfrak{g}_{i-1}$ is equal to the component of $\mathbf{I}$ from (11) in $\mathfrak{g}_{-(i-1)}$.

We now show that the component of L_i in $\mathfrak{g}_{-i}$ is

$$(L_i)_i = -\partial_x a_i + l.o.t. \qquad (18)$$

by showing that the other terms in $(L_i)_i$ from (14) contain only products of lower order derivatives. We have that

$$-\partial_x M_{i-1} \cdot M_{i-1}^{-1} = -\sum_{j=1}^{i-1} M_{j-1}(\partial_x e^{a_j} \cdot e^{-a_j})M_{j-1}^{-1}.$$

Since $a_j \in g_{-j}$ and $1 \leq j \leq i-1$, we see that all the terms in $(-\partial_x M_{i-1} \cdot M_{i-1}^{-1})_i$ contain only products of lower order derivatives. The same argument also applies to the last element $M_{i-1}e^{a_i}(e + \mathbf{u})e^{-a_i}M_{i-1}^{-1}$. Since $a_i \in \mathfrak{g}_{-i}$, we see that

$$(-M_{i-1}(\partial_x e^{a_i} \cdot e^{-a_i})M_{i-1}^{-1})_i = -\partial_x a_i.$$

Actually, $-\partial_x a_i$ may contain products too, but it is the only term that contains non-products.

Now we consider more specifically the highest order derivative terms. Continuing with the above notation, we have

$$(L_0)_0 = (e + \mathbf{u})_0 = \sum_{i=1}^{\ell} u_{i,x} H_{\alpha_i} = \sum_{j,i=1}^{\ell} c_{ji} u_{i,x} s_j^{m_j}.$$

From (13), we have

$$a_1 = -\sum_{j,i=1}^{\ell} c_{ji} u_{i,x} s_j^{m_j-1}.$$

By (18), we have

$$(L_1)_1 = -\partial_x a_1 + l.o.t. = \sum_{j,i=1}^{\ell} c_{ji} u_{i,x}^{(2)} s_j^{m_j-1} + l.o.t..$$

For all simple Lie algebras, $m_1 = 1$ and $m_2 > 1$ by [14]. Therefore, by (17) we see that $I_1 = \sum_{i=1}^{\ell} c_{1i} u_{i,x}^{(2)} + l.o.t..$

Continuing this way, we see that (10) holds for all $j = 1, \ldots, \ell$.

Then through the Gaussian elimination, there exists a permutation σ in the symmetric group S_ℓ such that the following set

$$\{u_{\sigma(i),x}^{(k)} \mid 1 \leq k \leq m_i, \ 1 \leq i \leq \ell\} \qquad (19)$$

is a set of generators of $\mathbb{C}[\mathbf{U}]/\mathcal{I}$ in the sense that the other

$$u_{\sigma(i),x}^{(k)}, \quad k \geq d_i, \ 1 \leq i \leq \ell$$

are solved as polynomials in them modulo $\mathcal{I}$. The cardinality of the set (19) is

$$m_1 + \cdots + m_\ell = \dim N_-$$

by [14]. ∎

We now study the restriction of the left G-action on G/B_+ to N_-. The natural multiplication map $N_- \times B_+ \to G$ is injective and its image G_r is an open subset, called the regular part of G (see [17, Eq. (2.4.4)], [20, Eq. (1.5.6)]). For $n_1 \in N_-$, there exists an open set $1 \in U \subset G$ such that $gn_1 \in G_r$ for $g \in U$. The action $g \in U$ on n_1 is obtained by the following normalization procedure:

$$\text{if } gn_1 = \tilde{n}_1 p \in N_- B_+, \text{ then } g \cdot n_1 := \tilde{n}_1. \tag{20}$$

For $a \in \mathfrak{g}$, we denote by $\mathcal{L}_a$ the infinitesimal action of a on $\mathbb{C}[N_-]$. Explicitly,

$$(\mathcal{L}_a f)(n) = \frac{d}{dt}\Big|_{t=0} f(\exp(-ta) \cdot n), \quad n \in N_-, \ f \in \mathbb{C}[N_-]. \tag{21}$$

In particular, we have the vector field $\mathcal{L}_e$ for $e = \sum_{i=1}^{\ell} e_{\alpha_i}$ defined on N_-.

Following [11, Lemma 1.1], we have the following lemma about the action of $\mathcal{L}_a$. Here we take a faithful representation of N_- and represent an element $K \in N_-$ by a matrix whose (i,j)th entry f_{ij} is considered as a regular function in $\mathbb{C}[N_-]$. Then $\mathcal{L}_a K$ is the matrix whose (i,j)th entry is $\mathcal{L}_a f_{ij}$.

Lemma 2.2.   *For $a \in \mathfrak{g}$ and $K \in N_-$, we have*

$$K^{-1}\mathcal{L}_a K = -(K^{-1}aK)_-, \tag{22}$$

where $(\cdot)_- : \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b}_+ \to \mathfrak{n}_-$ is the projection.

Proof. Choose a one-parameter subgroup $a(t)$ of G such that $a(t)=1-ta+o(t)$. We have $a(t)K = K - taK + o(t)$. For small t, we can factor $a(t)K$ into a product $L_- L_+$, where $L_- = K + tL_-^{(1)} + o(t) \in N_-$ and $L_+ = 1 + tL_+^{(1)} + o(t) \in B_+$. Therefore we have

$$KL_+^{(1)} + L_-^{(1)} = -aK.$$

From this we see that $L_-^{(1)} = -K(K^{-1}aK)_-$. This proves formula (22). ∎

For $1 \leq i \leq \ell$, let $\omega_i \in \mathfrak{h}^*$ be the ith fundamental weight defined by the conditions that $\omega_i(H_{\alpha_j}) = \delta_{ij}$ for $1 \leq j \leq \ell$.

Proof of Theorem 1.1.   Let $K \in N_-$. Following Kostant [16] and [9, 11], we define the following functions on N_-

$$\begin{aligned} v_i &:= (\omega_i, K^{-1}eK), \quad 1 \leq i \leq \ell, \\ v_i^{(n)} &:= \mathcal{L}_e^n v_i, \qquad\qquad n \geq 0. \end{aligned} \tag{23}$$

Here $(\cdot, \cdot)$ is the Killing form on $\mathfrak{g}$, and ω_i is regarded as an element of $\mathfrak{h}$ via the Killing form. For simplicity, we write $u_i^{(k)}$ for $u_{i,x}^{(k)}$. Define a ring homomorphism

$$\varphi : \mathbb{C}[\mathbf{U}] \to \mathbb{C}[N_-], \quad u_i^{(k)} \mapsto v_i^{(k-1)}, \ k \geq 1. \tag{24}$$

We note that under this homomorphism, ∂_x corresponds to $\mathcal{L}_e$ by the above definitions. Now we show that the images $\varphi(I_j)$ of the characteristic integrals I_j are zero. That is, if we replace the $u_i^{(k)}$ by the $v_i^{(k-1)}$ in I_j, we get zero.

We find the images $\varphi(I_j)$ by adapting our algorithm in (9) for computing I_j. So we replace ∂_x by $\mathcal{L}_e$ and $\mathbf{u}$ by $\mathbf{v} = \sum_{i=1}^{\ell} v_i H_{\alpha_i}$. Then by the same reason as for (9), there exists a unique element $M_1 \in N_-$, whose entries are functions on N_-, such that

$$M_1(\mathcal{L}_e + e + \mathbf{v})M_1^{-1} = \mathcal{L}_e + e + \sum_{j=1}^{\ell} J_j s_j. \tag{25}$$

Then $J_j = \varphi(I_j)$ for $1 \leq j \leq \ell$.

We will show below that the unique M_1 in (25) is the K in (23), and (25) becomes

$$K(\mathcal{L}_e + e + \mathbf{v})K^{-1} = \mathcal{L}_e + e. \tag{26}$$

Therefore, $\varphi(I_j) = J_j = 0$ for $1 \leq j \leq \ell$.

It is clear that $K(\mathcal{L}_e K^{-1}) = -(\mathcal{L}_e K)K^{-1}$. Therefore by (22), we have

$$\begin{aligned}
K(\mathcal{L}_e + e + \mathbf{v})K^{-1} &= \mathcal{L}_e - (\mathcal{L}_e K)K^{-1} + K(e + \mathbf{v})K^{-1} \\
&= \mathcal{L}_e + K(K^{-1}eK)_- K^{-1} + K(e + \mathbf{v})K^{-1} \\
&= \mathcal{L}_e + K\big((K^{-1}eK)_- + e + \mathbf{v}\big)K^{-1} \\
&= \mathcal{L}_e + K(K^{-1}eK)K^{-1} \\
&= \mathcal{L}_e + e.
\end{aligned}$$

Here we have used that $K^{-1}eK = e + \mathbf{v} + (K^{-1}eK)_-$ by (23) since the Cartan component of $K^{-1}eK$ is equal to $\sum_{i=1}^{\ell}(\omega_i, K^{-1}eK)H_{\alpha_i} = \mathbf{v}$.

Furthermore, since ∂_x corresponds to $\mathcal{L}_e$, we have $\varphi(\partial_x^m I_j) = \mathcal{L}_e^m J_j = 0$ for $m \geq 0$. Therefore, the morphism φ in (24) descends to a morphism, which we continue to denote by the same notation,

$$\varphi : \mathbb{C}[\mathbf{U}]/\mathcal{I} \to \mathbb{C}[N_-]. \tag{27}$$

Now we show that this morphism is an isomorphism, and this part is analogous to the proof of Theorem 1.1 in [11].

To prove that φ is injective, we have to show that the images $\varphi(u_{\sigma(i)}^{(k)}) = v_{\sigma(i)}^{(k-1)}$ for $1 \leq k \leq m_i$, $1 \leq i \leq \ell$ of the set (19) are algebraically independent. We will do that by showing that at the identity $1 \in N_-$, the $dv_{\sigma(i)}^{(k-1)}\big|_1$ for $1 \leq k \leq m_i$, $1 \leq i \leq \ell$ are linearly independent in the cotangent space $T_1^* N_-$, which is identified with $\mathfrak{n}_+ = \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ through the Killing form. It can be checked that $dv_i\big|_1 = e_{-\alpha_i}^*$, which is identified with $\frac{(\alpha_i, \alpha_i)}{2}e_{\alpha_i}$ since $(e_{-\alpha_i}, e_{\alpha_i}) = \frac{2}{(\alpha_i, \alpha_i)}$ by

our normalization (4). Furthermore by (23), we have $dv_i^{(n)}\big|_1 = \mathrm{ad}_e^n(dv_i\big|_1)$. The set

$$\{\mathrm{ad}_e^{k-1}(dv_{\sigma(i)}\big|_1)\,|\,1 \leq k \leq m_i,\ 1 \leq i \leq \ell\}$$

is a basis of $\mathfrak{n}_+$ by [14] and hence is linearly independent.

To prove that φ is surjective, we introduce gradings and compare them. On $\mathbb{C}[\mathbf{U}]/\mathcal{I}$, we set $\deg u_{\sigma(i)}^{(k)} = k$ for $1 \leq k \leq m_i$, $1 \leq i \leq \ell$. The above algebraic independence also shows that $\mathbb{C}[\mathbf{U}]/\mathcal{I}$ is a polynomial algebra with b_i generators of degree i by [14], where $b_i = \dim \mathfrak{g}_i$ from (2) for $1 \leq i \leq m_\ell$.

On $\mathbb{C}[N_-]$, we take the derivation $\mathcal{L}_{H_0}$ as the grading operator, where H_0 is the grading element for the principal grading (2). Then it can be checked that $\deg v_i = 1$. Since $[H_0, e] = e$, the degree of $\mathcal{L}_e$ also equals 1 and so $\deg v_i^{(k-1)} = k$. Therefore, the morphism φ in (27) preserves degrees. Since $\exp : \mathfrak{n}_- \to N_-$ is an isomorphism, we see that $\mathbb{C}[N_-]$ is also a polynomial algebra on b_i generators of degree i for $1 \leq i \leq m_\ell$. Hence the character of $\mathbb{C}[N_-]$ coincides with the character of $\mathbb{C}[\mathbf{U}]/\mathcal{I}$, and φ is an isomorphism. Here by a character of a $\mathbb{Z}$-graded vector space V we understand the formal power series

$$\mathrm{ch}\,V = \sum_{n\in\mathbb{Z}} \dim V_n q^n,$$

where V_n is the homogeneous subspace of V of degree n. Concretely,

$$\mathrm{ch}\,\mathbb{C}[\mathbf{U}]/\mathcal{I} = \mathrm{ch}\,\mathbb{C}[N_-] = \prod_{i=1}^{m_\ell}(1 - q^i)^{-b_i}.\qquad\blacksquare$$

Example 2.3.   To illustrate our results, let us consider the example of A_2 Toda field theory (1) as

$$\begin{cases} u_{1,xy} = -e^{2u_1 - u_2} \\ u_{2,xy} = -e^{-u_1 + 2u_2}. \end{cases}$$

Let E_{ij} denote the matrix of dimension 3 whose only nonzero element is 1 at position (i, j). We use standard choices of $e = E_{12} + E_{23}$, $H_{\alpha_1} = E_{11} - E_{22}$, $H_{\alpha_2} = E_{22} - E_{33}$, $s_1 = E_{21}$, and $s_2 = E_{31}$. Then the transformation matrix from $(H_{\alpha_1}, H_{\alpha_2})$ to (s_1^1, s_2^2) is $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$.

Our formula from (9) computes that

$$\begin{aligned} I_1 &= -u_{1,xx} - u_{2,xx} + u_{1,x}^2 - u_{1,x}u_{2,x} + u_{2,x}^2, \\ I_2 &= -u_{2,xxx} + 2u_{2,xx}u_{2,x} - u_{1,xx}u_{2,x} + u_{1,x}^2 u_{2,x} - u_{1,x}u_{2,x}^2. \end{aligned} \qquad (28)$$

Note that (10) is thus checked.

From these we see that a set of ring generators for $\mathbb{C}[\mathbf{U}]/\mathcal{I}$ in Proposition 2.1 can be chosen as $(u_{1,x}, u_{2,x}, u_{2,xx})$ or $(u_{1,x}, u_{2,x}, u_{1,xx})$.

Let us also show the content of Theorem 1.1 using this example. An element $K \in N_-$ has the following form

$$K = \begin{pmatrix} 1 & & \\ v_1 & 1 & \\ v_3 & v_2 & 1 \end{pmatrix}, \qquad (29)$$

where v_1, v_2, v_3 are the coordinates on N_-. We see that the definitions of v_1 and v_2 in (23) are compatible with the above. By (21) and the action (20), we compute that

$$\mathcal{L}_e = (v_1^2 - v_3)\frac{\partial}{\partial v_1} + (v_2^2 + v_3 - v_1 v_2)\frac{\partial}{\partial v_2} + v_1 v_3 \frac{\partial}{\partial v_3}.$$

It is easy to check directly that for the characteristic integrals in (28) and the φ in (24), we have $\varphi(I_1) = 0$ and $\varphi(I_2) = 0$. Then the map in (27) is

$$\varphi : \mathbb{C}[\mathbf{U}]/\mathcal{I} \to \mathbb{C}[N_-]; \ u_{1,x} \mapsto v_1, \ u_{2,x} \mapsto v_2, \tag{30}$$
$$u_{2,xx} \mapsto v_2^{(1)} = \mathcal{L}_e v_2 = v_2^2 + v_3 - v_1 v_2.$$

3. From Toda field theories to standard differential systems

In this section, we prove Theorem 1.2 by utilizing Theorem 1.1.

Proof of Theorem 1.2. We represent the Toda field theory (1) as the Pfaffian system on the following infinite jet space with coordinates

$$(x, y, u_i, u_{i,x}^{(k)}, u_{i,y}^{(k)}), \quad 1 \le i \le \ell, \ k \ge 1.$$

By convention, $u_i = u_{i,x}^{(0)} = u_{i,y}^{(0)}$, and $u_{i,x} = u_{i,x}^{(1)}$, $u_{i,y} = u_{i,y}^{(1)}$. Let ∂_x and ∂_y be the derivative with respect to x and y respectively. We have $\partial_x u_{i,x}^{(k)} = u_{i,x}^{(k+1)}$ and $\partial_y u_{i,y}^{(k)} = u_{i,y}^{(k+1)}$. Using the shorthand (6), the differential 1-forms defining the Toda field theory (1) are

$$\begin{cases} du_i - u_{i,x}dx - u_{i,y}dy, \\ du_{i,x}^{(k)} - u_{i,x}^{(k+1)}dx + \partial_x^{k-1}e^{\rho_i}dy, & k \ge 1, \\ du_{i,y}^{(k)} + \partial_y^{k-1}e^{\rho_i}dx - u_{i,y}^{(k+1)}dy, & k \ge 1. \end{cases}$$

Now we restrict x to a constant in the jet space while the $u_{i,x}^{(k)}$ are still coordinates. Then $dx = 0$ and the differential system becomes

$$\begin{cases} du_{i,y}^{(k)} - u_{i,y}^{(k+1)}dy, & k \ge 0 \\ du_{i,x}^{(k)} + \partial_x^{k-1}e^{\rho_i}dy, & k \ge 1. \end{cases} \tag{31}$$

Define B_i^k via

$$\partial_x^k e^{\rho_i} = e^{\rho_i}B_i^k, \quad \text{for } 1 \le i \le \ell, \ k \ge 0.$$

The B_i^k are differential polynomials of the u_j with respect to x and they clearly satisfy that for $1 \le i \le \ell$,

$$B_i^0 = 1, \qquad B_i^k = \partial_x B_i^{k-1} + \rho_{i,x}B_i^{k-1}, \ k \ge 1. \tag{32}$$

It is in this way that we make sense of the second set of 1-forms in (31). The first set of 1-forms in (31) are just some jet relations. Therefore disregarding them,

we represent our Toda field theory (1) restricted to $x = $ constant on the manifold with coordinates

$$(y, u_i, u_{i,x}^{(k)}), \quad 1 \le i \le \ell, \ k \ge 1$$

by the system of 1-forms

$$du_{i,x}^{(k)} + e^{\rho_i} B_i^{k-1} dy, \quad 1 \le i \le \ell, \ k \ge 1.$$

We use the dual viewpoint and choose the following basis of vector fields that generate at each point the corresponding distribution

$$
\begin{aligned}
U_j &:= \sum_{i=1}^{\ell} a^{ij} \frac{\partial}{\partial u_i}, \quad 1 \le j \le \ell, \\
Y &:= \frac{\partial}{\partial y} - \sum_{i=1}^{\ell} e^{\rho_i} \sum_{k \ge 1} B_i^{k-1} \frac{\partial}{\partial u_{i,x}^{(k)}},
\end{aligned}
\tag{33}
$$

where $(a^{ij}) = (a_{ij})^{-1}$.

Taking Lie brackets and recalling (6), for $1 \le j \le \ell$ we have

$$V_j := [Y, U_j] = \sum_{m,i=1}^{\ell} a^{mj} a_{im} e^{\rho_i} \sum_{k \ge 1} B_i^{k-1} \frac{\partial}{\partial u_{i,x}^{(k)}} = e^{\rho_j} \sum_{k \ge 1} B_j^{k-1} \frac{\partial}{\partial u_{j,x}^{(k)}}. \tag{34}$$

Then $Y = \frac{\partial}{\partial y} - \sum_{j=1}^{\ell} V_j$, and the bracket relations so far are

$$[\frac{\partial}{\partial y}, U_j] = 0, \quad [\frac{\partial}{\partial y}, V_j] = 0, \quad [U_i, V_j] = \delta_{ij} V_j. \tag{35}$$

Now we study the bracket relations among the V_j, and we will show that they generate $\mathfrak{n}_-$.

Define

$$\widetilde{V}_j = e^{-\rho_j} V_j = \sum_{k \ge 1} B_j^{k-1} \frac{\partial}{\partial u_{j,x}^{(k)}}. \tag{36}$$

Then for $1 \le j_1, \cdots, j_m \le \ell$, we have

$$[V_{j_1}, \cdots, [V_{j_{m-1}}, V_{j_m}] \cdots] = e^{\rho_{j_1} + \cdots + \rho_{j_m}} [\widetilde{V_{j_1}}, \cdots, [\widetilde{V_{j_{m-1}}}, \widetilde{V_{j_m}}] \cdots].$$

Now we show that the $\widetilde{V}_j$ can be viewed as derivations on the ring $\mathbb{C}[\mathbf{U}]/\mathcal{I}$ from Theorem 1.1.

First we note that the vector fields Y and the U_j in (33) annihilate all the characteristic integrals I_i and their derivatives on the x-side. This is the case for Y by the definition of characteristic integrals and that Y is the total derivative vector ∂_y. This is the case for the U_j since the characteristic integrals I_i contain at least the first-order derivatives in view of (9) and the definition of $\mathbf{u}$ in (5). Therefore, by (34) and (36), we have $\mathcal{L}_{\widetilde{V}_j} \mathcal{I} \subset \mathcal{I}$ and the $\widetilde{V}_j$ descend as derivations on the ring $\mathbb{C}[\mathbf{U}]/\mathcal{I}$.

By the isomorphism in Theorem 1.1, derivations on $\mathbb{C}[N_-]$ or equivalently (algebraic) vector fields on N_- canonically correspond to derivations on $\mathbb{C}[\mathbf{U}]/\mathcal{I}$.

For $1 \leq j \leq \ell$, let $e^R_{-\alpha_j}$ be the vector field for the infinitesimal action of $e_{-\alpha_j}$ under the right multiplication of N_- on $N_- \subset G/B_+$. Explicitly, we have

$$(e^R_{-\alpha_j} f)(n) = \frac{d}{dt}\Big|_{t=0} f(n \exp(te_{-\alpha_j})), \quad n \in N_-, \ f \in \mathbb{C}[N_-]. \tag{37}$$

By [16, Proposition 3.5 and Theorem 2.2], we know that as vector fields on N_-,

$$[\mathcal{L}_e, e^R_{-\alpha_j}] = -(\alpha_j, K^{-1}eK)e^R_{-\alpha_j}, \tag{38}$$

where $K \in N_-$ and $(\alpha_j, K^{-1}eK)$ is a function on N_-.

We denote by $\widetilde{e^R_{-\alpha_j}}$ the derivation on $\mathbb{C}[\mathbf{U}]/\mathcal{I}$ corresponding to $e^R_{-\alpha_j}$ on $\mathbb{C}[N_-]$. Note that $\mathcal{L}_e$ corresponds to ∂_x under the isomorphism in Theorem 1.1. By $\alpha_j = \sum_{m=1}^{\ell} a_{jm}\omega_m$ and in view of (23) and (24), (38) gives

$$[\widetilde{e^R_{-\alpha_j}}, \partial_x] = \Big(\sum_{m=1}^{\ell} a_{jm} u_{m,x}\Big)\widetilde{e^R_{-\alpha_j}} = \rho_{j,x}\widetilde{e^R_{-\alpha_j}}. \tag{39}$$

As a derivation on $\mathbb{C}[\mathbf{U}]/\mathcal{I}$, we have

$$\partial_x = \sum_{i=1}^{\ell} \sum_{k \geq 1} u_{i,x}^{(k+1)} \frac{\partial}{\partial u_{i,x}^{(k)}}.$$

Write

$$\widetilde{e^R_{-\alpha_j}} = \sum_{i=1}^{\ell} \sum_{k \geq 1} C_{j,i}^{k-1} \frac{\partial}{\partial u_{i,x}^{(k)}},$$

where the $C_{j,i}^{k-1}$ are functions of $\{u_{i,x}^{(m)}\}_{1 \leq i \leq \ell}^{m \geq 1}$. Then from (39), we see that

$$C_{j,i}^k = \partial_x C_{j,i}^{k-1} + \rho_{j,x} C_{j,i}^{k-1}, \quad k \geq 1.$$

Using (37), we see that $C_{j,i}^0 = \delta_{ij}$. Hence $C_{j,i}^k = 0$ if $j \neq i$, and $C_{i,i}^k = B_i^k$ in view of (32). Therefore,

$$\widetilde{V}_j = \widetilde{e^R_{-\alpha_j}}, \tag{40}$$

as derivations on $\mathbb{C}[\mathbf{U}]/\mathcal{I}$. This shows, in particular, that the $\widetilde{V}_j$ and hence the V_j generate $\mathfrak{n}_-$.

Now we show that the Lie algebra generated by the vector fields in (33) is isomorphic to a Lie algebra of vector fields on the first jet space of the integral curves of the standard differential system on the big cell $N_- \subset G/B_+$.

Two integral curves $\Phi_1, \Phi_2 : I \to N_-$ of the standard differential system (see (3)) are called 1-equivalent at a point $y_0 \in I$ if their graphs have a contact of order 1 at the point $\Phi_1(y_0) = \Phi_2(y_0)$. The equivalence class of Φ with respect to this relation is denoted by $[\Phi]^1_{y_0}$ and is called the 1-jet of Φ at y_0. For an interval $I \subset \mathbb{R}^1$, the first jet space is

$$J^1_y(N_-, D) = \{[\Phi]^1_{y_0} \mid \Phi : I \to N_-, y_0 \in I, \Phi \text{ satisfies (3)}\}$$

with a natural structure of manifold. Let $\omega \in \Omega^1(N_-, \mathfrak{n}_-)$ be the Maurer-Cartan form. Then Φ satisfies (3) iff $\omega(\Phi'(y)) \in \mathfrak{g}_{-1}$. Hence we can write

$$\omega(\Phi'(y)) = \sum_{i=1}^{\ell} \phi_i(y) e_{-\alpha_i} \tag{41}$$

using some functions $\phi_i(y)$.

Therefore, $J_y^1(N_-, D)$ is a manifold with coordinates

$$(y, \Phi^0, \phi_i^0), \quad 1 \le i \le \ell,$$

where Φ^0 is the set of global coordinates on N_-. Here the superscripts 0 stand for 0-jets (the map values), and are used to distinguish the coordinates on the jet space from the actual representative maps. By (41), the Pfaffian system on $J_y^1(N_-, D)$ is defined by the components of the $\mathfrak{g}$-valued 1-form

$$\omega - \sum_{i=1}^{\ell} \phi_i^0 \, dy \otimes e_{-\alpha_i}. \tag{42}$$

where ω is expressed in terms of the coordinates Φ^0.

Let us now consider the dual viewpoint in terms of distributions. We identify the tangent spaces of N_-, through the left translation to the identity, with the Lie algebra $\mathfrak{n}_-$. Then $\{\frac{\partial}{\partial y}, \frac{\partial}{\partial \phi_i^0}, e_{-\alpha}\}$ is a basis for the tangent spaces to $J_y^1(N_-, D)$, where $1 \le i \le \ell$ and the $-\alpha \in \Delta^-$ are the negative roots. The distribution dual to (42) is globally generated, where $\phi_i^0 \ne 0$, by

$$\phi_i^0 \frac{\partial}{\partial \phi_i^0}, \quad 1 \le i \le \ell, \quad \text{and}$$

$$\overline{Y} = \frac{\partial}{\partial y} + \sum_{i=1}^{\ell} \phi_i^0 e_{-\alpha_i}.$$

since the Maurer-Cartan form satisfies $\omega(a) = a$ for $a \in \mathfrak{n}_-$.

Analogously to (34), define

$$\overline{V}_j := [\overline{Y}, \phi_j^0 \frac{\partial}{\partial \phi_j^0}] = -\phi_j^0 e_{-\alpha_j}, \quad 1 \le j \le \ell. \tag{43}$$

The $\overline{V}_j$ clearly generate $\mathfrak{n}_-$. We have $\overline{Y} = \frac{\partial}{\partial y} - \sum_{j=1}^{\ell} \overline{V}_j$ and also the bracket relations

$$[\frac{\partial}{\partial y}, \phi_i^0 \frac{\partial}{\partial \phi_i^0}] = 0, \quad [\frac{\partial}{\partial y}, \overline{V}_j] = 0, \quad [\phi_i^0 \frac{\partial}{\partial \phi_i^0}, \overline{V}_j] = \delta_{ij} \overline{V}_j. \tag{44}$$

By the comparison of (35) with (44) and (36) with (43), we see using (40) that the map

$$Y \mapsto \overline{Y}, \quad U_i \mapsto \phi_i^0 \frac{\partial}{\partial \phi_i^0}, \ 1 \le i \le \ell, \tag{45}$$

establishes an isomorphism between the two sets of vector field generators of the two distributions which respects their successive Lie brackets. This completes the proof of Theorem 1.2. ∎

Example 3.1.   We continue to use Example 2.3 to illustrate our results. Using $e_{-\alpha_1} = E_{21}$, $e_{-\alpha_2} = E_{32}$ and the coordinates in (29), we see that

$$e^R_{-\alpha_1} = \frac{\partial}{\partial v_1} + v_2 \frac{\partial}{\partial v_3}, \quad e^R_{-\alpha_2} = \frac{\partial}{\partial v_2}.$$

By the transformation (30) and the chain rule, we see that

$$\widetilde{e^R_{-\alpha_1}} = \frac{\partial}{\partial u_{1,x}}, \quad \widetilde{e^R_{-\alpha_2}} = \frac{\partial}{\partial u_{2,x}} + (2u_{2,x} - u_{1,x})\frac{\partial}{\partial u_{2,xx}}.$$

These are clearly the $\widetilde{V}_1$ and $\widetilde{V}_2$ from (36) expressed in the generators $u_{1,x}$, $u_{2,x}$ and $u_{2,xx}$ of $\mathbb{C}[\mathbf{U}]/\mathcal{I}$.

4.   From standard differential systems to Toda field theories

In this section, we prove Theorem 1.3, which realizes the quotient structure of [1, Theorem 1.4] for Toda field theories as Darboux integrable differential systems.

Proof of Theorem 1.3.   By (3), an integral curve in N_- of the standard differential system is locally a map $\Phi : I \to N_-$ such that

$$a := \Phi^{-1}\Phi_y \in \mathfrak{n}_- \text{ lies in the subspace } \mathfrak{g}_{-1}, \tag{46}$$

where $\Phi_y = \Phi'(y)$.

Similarly there is the following "transposed" version on the x-side. We have the similarly defined subgroups N_+ and B_- of G integrating the Lie subalgebras $\mathfrak{n}_+ = \oplus_{i>0}\mathfrak{g}_i$ and $\mathfrak{b}_- = \oplus_{i\leq 0}\mathfrak{g}_i$. Furthermore the space $B_-\backslash G$ of right cosets contains N_+ as a big cell. The *left* action of G on $B_-\backslash G$ is defined by $g(B_-k) = B_-kg^{-1}$ for $g \in G$ and $B_-k \in B_-\backslash G$. The action of g in an open set of G (which contains the identity) on $n_2 \in N_+$ is defined by

$$\text{if } n_2g^{-1} = q\tilde{n}_2 \in B_-N_+, \text{ then } g \cdot n_2 := \tilde{n}_2. \tag{47}$$

The standard distribution $\tilde{D}$ on $B_-\backslash G$ is defined as the G-invariant distribution equal to $\mathfrak{g}_1$ mod $\mathfrak{b}_-$ at $\tilde{o} = B_- \cdot 1$. An integral curve in N_+ of the standard differential system is thus locally a map $\Psi : J \to N_+$, where J is an interval, such that

$$b := \Psi_x\Psi^{-1} \in \mathfrak{n}_+ \text{ lies in the subspace } \mathfrak{g}_1, \tag{48}$$

where $\Psi_x = \Psi'(x)$, $x \in J$.

We write the a and b in (46) and (48) as

$$a = \sum_{i=1}^{\ell} \phi_i(y)e_{-\alpha_i} \quad \text{and} \quad b = \sum_{i=1}^{\ell} \psi_i(x)e_{\alpha_i}. \tag{49}$$

Therefore, we have the following coordinates

$$(x, y, \Psi^0, \psi_i^0, \Phi^0, \phi_i^0), \quad 1 \leq i \leq \ell.$$

on the product $J_x^1(N_+, \tilde{D}) \times J_y^1(N_-, D)$ of jet spaces. An element $g \in G$ acts trivially on x and y, acts on Φ^0 and Ψ^0 by (20) and (47) when defined, and acts on ϕ_i^0 and ψ_i^0 through the prolonged action. That is, given a point $(y_0, \Phi_0^0, \phi_{i,0}^0) \in J_y^1(N_-, D)$, we choose a representative map $\Phi : I \to N_-$ such that $\Phi(y_0) = \Phi_0^0$ and $\Phi(y_0)^{-1}\Phi_y(y_0) = \sum_{i=1}^\ell \phi_{i,0}^0 e_{-\alpha_i}$, and then we define

$$\sum_{i=1}^\ell (g \cdot \phi_{i,0}^0) e_{-\alpha_i} := (g \cdot \Phi(y_0))^{-1} \frac{d}{dy}\Big|_{y=y_0} (g \cdot \Phi(y))$$

where g is close to the identity so that $g \cdot \Phi(y_0)$ is defined as in (20). The definition of $g \cdot \psi_i^0$ is similar. We refer the reader to [26, Chapter 5] for background on prolonged group action and differential invariants.

The number of coordinates that G acts on is $\dim N_- + \dim N_+ + 2\ell = \dim G + \ell$. Now we construct ℓ differential invariants and show that they satisfy the Toda field theory (1). In this proof, we will work with representatives of the jet spaces. That is, we will work with actual integral curves $\Phi : I \to N_-$ and $\Psi : J \to N_+$ satisfying (46) and (48).

We use the bracket notation following the physicists [20]. For $1 \le i \le \ell$, we denote by $|i\rangle$ the highest weight vector for the ith fundamental representation V_i of G [12], and by $\langle i|$ the lowest weight vector for the dual representation V_i^*, where G acts from the right. There are the induced representations of $\mathfrak{g}$ and its universal enveloping algebra $U(\mathfrak{g})$ (see [13]) on V_i and V_i^*. For $\mu, \nu \in U(\mathfrak{g})$ and $g \in G$, $\langle i|\nu g\mu|i\rangle$ denotes the pairing of $\langle i|\nu$ in V_i^* with $g(\mu|i\rangle)$ in V_i. We require that for the identity element $1 \in G$, we have $\langle i|1|i\rangle = 1$. Then consider the following function

$$\xi_i := \langle i|\Psi(x)\Phi(y)|i\rangle.$$

This function is not invariant under the G-action, but its transformation under $g \in G$ is simple. Using (20) and (47), we have

$$g^{-1}\xi_i = \langle i|(g \cdot \Psi(x))(g \cdot \Phi(y))|i\rangle = \langle i|Q(x)^{-1}\Psi(x)g^{-1}g\Phi(y)P(y)^{-1}|i\rangle = \xi_i p(y)q(x). \tag{50}$$

Here $Q(x) \in B_-$ and $P(y) \in B_+$ are group elements for the normalizations from (20) and (47). Since $\langle i|$ and $|i\rangle$ are the lowest and highest weight vectors, we have

$$\langle i|Q(x)^{-1} = q(x)\langle i|, \quad P(y)^{-1}|i\rangle = p(y)|i\rangle,$$

for some functions $p(y)$ and $q(x)$. Therefore the locally defined

$$-(\log \xi_i)_{xy}$$

is G-invariant.

Now we define

$$u_i = -\log \xi_i + \sum_{j=1}^\ell a^{ij} \log(\phi_j(y)\psi_j(x)), \quad 1 \le i \le \ell. \tag{51}$$

The $u_{i,xy} = -(\log \xi_i)_{xy}$ are also differential invariants.

Now we compute that

$$u_{i,xy} = -\frac{\xi_i \xi_{i,xy} - \xi_{i,x} \xi_{i,y}}{\xi_i^2}.$$

By (46), (48) and (49), we have

$$\xi_{i,y} = \langle i|\Psi(x)\Phi_y(y)|i\rangle = \langle i|\Psi(x)\Phi(y)\sum_{j=1}^{\ell}\phi_j(y)e_{-\alpha_j}|i\rangle = \phi_i(y)\langle i|\Psi(x)\Phi(y)e_{-\alpha_i}|i\rangle,$$

$$\xi_{i,x} = \langle i|\Psi_x(x)\Phi(y)|i\rangle = \langle i|\sum_{j=1}^{\ell}\psi_j(x)e_{\alpha_j}\Psi(x)\Phi(y)|i\rangle = \psi_i(x)\langle i|e_{\alpha_i}\Psi(x)\Phi(y)|i\rangle,$$

$$\xi_{i,xy} = \phi_i(y)\psi_i(x)\langle i|e_{\alpha_i}\Psi(x)\Phi(y)e_{-\alpha_i}|i\rangle.$$

Here we have used the well-known fact that for $j \neq i$ we have $e_{-\alpha_j}|i\rangle = 0$ and similarly $\langle i|e_{\alpha_j} = 0$ (see [20, Eq. (1.4.19)]). Therefore

$$u_{i,xy} = -\phi_i(y)\psi_i(x)\frac{\Delta_i}{\xi_i^2},$$

where

$$\Delta_i = \begin{vmatrix} \langle i|\Psi(x)\Phi(y)|i\rangle & \langle i|\Psi(x)\Phi(y)e_{-\alpha_i}|i\rangle \\ \langle i|e_{\alpha_i}\Psi(x)\Phi(y)|i\rangle & \langle i|e_{\alpha_i}\Psi(x)\Phi(y)e_{-\alpha_i}|i\rangle \end{vmatrix}.$$

Now the so-called Jacobi identity [20, Eq. (1.6.16)] asserts that

$$\Delta_i = \prod_{j\neq i}\xi_j^{-a_{ij}}. \tag{52}$$

Therefore since $a_{ii} = 2$, we have

$$u_{i,xy} = -\phi_i(y)\psi_i(x)\prod_{j=1}^{\ell}\xi_j^{-a_{ij}} = -\exp\left(\sum_{j=1}^{\ell}a_{ij}u_j\right)$$

by the definition of u_i in (51). This shows that the u_j for $1 \leq j \leq \ell$ are functions in the $u_{i,xy}$ and hence are differential invariants themselves.

Therefore, the following functions

$$u_i^0 = -\log\langle i|\Psi^0\Phi^0|i\rangle + \sum_{j=1}^{\ell}a^{ij}\log(\phi_j^0\psi_j^0), \quad 1 \leq i \leq \ell, \tag{53}$$

on $J_x^1(N_+, \tilde{D}) \times J_y^1(N_-, D)$ are invariant under the G action, and they satisfy the Toda field theory (1). ∎

It is interesting to note that the above differential invariants in (51) are exactly the general local solutions of the Toda field theory (1) constructed in [19, 20].

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