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A Distributional Treatment of Relative Mirabolic Multiplicity One

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Abstract. We study the role of the mirabolic subgroup P of $G = \mathbf{GL}_n(F)$ (F a p-adic field) for smooth irreducible representations of G that are distinguished relative to a subgroup of the form $H_k = \mathbf{GL}_k(F) \times \mathbf{GL}_{n-k}(F)$. We show that if a non-zero H_1-invariant linear form exists on a representation, then the a priori larger space of $P \cap H_1$-invariant forms is one-dimensional. When $k > 1$, we give a reduction of the same problem to a question about invariant distributions on the nilpotent cone tangent to the symmetric space G/H_k. Some new distributional methods for non-reductive groups are developed.
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1. Introduction

We study the nature of smooth irreducible complex representations of $G_n = \mathbf{GL}_n(F)$ over a non-archimedean field F of characteristic 0, that are distinguished by subgroups of the form $H_{k,n} = \mathbf{GL}_k(F) \times \mathbf{GL}_{n-k}(F)$ (the maximal standard Levi subgroups). Given a smooth irreducible representation π of G_n on a space V, it is said that π is distinguished by a subgroup $H < G_n$ if there is a non-zero linear H-invariant functional on V. Equivalently, π appears as a sub-representation of $C^\infty(G_n/H)$. Apart from standalone interest in the representation theory of p-adic groups, distinction comes into play as a local accessory when dealing with periods of an automorphic representation.

For a H-distinguished representation (π, V), the dual V^* can be embedded as a G_n-space into the space of distributions $\mathcal{D}$ on G_n/H. This observation (see Section 2.3) allows one to transform geometric results from the space $\mathcal{D}$ to representation theory. Among the advantages of such an approach, which has long been known for its potency, is that it avoids a classification of the representations of G_n.

For one instance, the mentioned motivation from automorphic forms raises the question of multiplicity one, that is, must the H-invariants of V^* be one-

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dimensional? In the case of $H_{k,n}$ it was answered positively by Jacquet and Rallis [7], using novel distributional methods. Namely, they applied a transfer of distributions from the symmetric space $G_n/H_{k,n}$ to its linear tangent space. The elaborate inductive argument was rephrased by Aizenbud and Gourevitch in [1] as a part of a more geometric mechanism called the Harish-Chandra descent. In this account, we would like to explore further these ideas to address another distributional problem with an implication for representation theory.

We focus on the mirabolic subgroup $\mathbf{P}_n(F) < G_n$, that is, the stabilizer subgroup in G_n of the vector $(0,\ldots,0,1)$ in its natural action on the row space F^n. The role of the mirabolic subgroup is known to be ubiquitous in the representation theory of G_n. For example, the subgroup is seen in the Gelfand-Kazhdan theory of derivatives, which served as a foundation for the Zelevinsky classification. In [4], Bernstein showed a distributional result regarding this subgroup: Every $\mathbf{P}_n(F)$-conjugation-invariant distribution on the matrix space $\mathbf{M}_{n,n}(F)$ is also G_n-conjugation-invariant. Using this result, he showed that integration over the mirabolic group in the Whittaker model of generic representations defines a canonical inner product between a representation and its contragredient.

Our current study can be seen as a follow up to these findings. We would like to check to what extent the role of the mirabolic group is preserved in the relative setting. Our main result regards the collection of $H_{1,n}$-distinguished representations of G_n. In the following theorem we write $\mathbf{H} = \mathbf{GL}_1 \times \mathbf{GL}_{n-1}$.

Theorem 1.1. *Every $\mathbf{P}_n(F) \cap \mathbf{H}(F)$-invariant linear functional on a $\mathbf{H}(F)$-distinguished irreducible smooth representation of $\mathbf{GL}_n(F)$, is also $\mathbf{H}(F)$-invariant.*

In particular, this implies multiplicity one for $\mathbf{P}_n(F) \cap H_{1,n}$-invariant functionals on $H_{1,n}$-distinguished representations.

Note that $H_{1,n}$-distinction is equivalent to $1 \times \mathbf{GL}_{n-1}(F)$-distinction combined with a trivial central character. This allows for a classification of $H_{1,n}$-distinguished representations in terms of parabolic induction to be read from the results of [14]. Namely, by [14, Corollary 6.15] a smooth irreducible representation of G_n is $H_{1,n}$-distinguished, if and only if, it can be written in the form $\sigma \times 1_{n-2}$ (multiplication in the sense of parabolic induction), where 1_{n-2} is the trivial representation of G_{n-2} and σ is an infinite-dimensional irreducible representation of G_2 with a trivial central character. Thus, Theorem 1.1 concerns what can be roughly described as an embedding of the smooth spectrum of $\mathbf{PGL}_2(F)$ into that of G_n.

Let us also mention, that it follows, either from the mentioned classification or from the results of [8], that for $n > 3$, any irreducible $H_{1,n}$-distinguished representation must be non-generic.

Theorem 1.1 comes as a corollary of our study into the geometric question about the difference between $P_n(F) \cap H_{k,n}$-invariance and $H_{k,n}$-invariance of distributions on $G_n/H_{k,n}$. We show (Theorem 3.9) that for all $1 \leq k < n-2$ the question can be reduced to a problem concerning invariant distributions on a certain cone in a linear F-space. Thus, we complete the following reduction.

Lemma 1.2. *Let $1 \leq k \leq n-2$ be integers.*

For $m \leq l$, let $\mathcal{N}_{m,l}$ be the space of nilpotent matrices of the form

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \qquad B \in \mathbf{M}_{m,l-m}(F) \quad C \in \mathbf{M}_{l-m,m}(F).$$

Suppose that for all $1 \leq m \leq k$ and $m \leq l \leq n$, every $\mathbf{P}_l(F) \cap (\mathbf{GL}_m(F) \times \mathbf{GL}_{l-m}(F))$-conjugation invariant distribution on $\mathcal{N}_{m,l}$ is also $\mathbf{GL}_m(F) \times \mathbf{GL}_{l-m}(F)$-invariant. Then, every $\mathbf{P}_n(F) \cap (\mathbf{GL}_k(F) \times \mathbf{GL}_{n-k}(F))$-invariant linear functional on a $\mathbf{GL}_k(F) \times \mathbf{GL}_{n-k}(F)$-distinguished irreducible smooth representation of $\mathbf{GL}_n(F)$, is also $\mathbf{GL}_k(F) \times \mathbf{GL}_{n-k}(F)$-invariant.

In the context of distinction, Bernstein's distributional result was applied in [10] to prove a similar statement to Theorem 1.1 for $\mathbf{GL}_n(L)$-distinguished representations, where F/L is a quadratic field extension. In our case, i.e. $H_{k,n}$-distinction, we show that the geometry of the space requires a similar approach to be supplemented with the assumption of Lemma 1.2, that is, $\mathcal{D}(\mathcal{N}_{k,n})^{\mathbf{P}_n(F) \cap H_{k,n}} = \mathcal{D}(\mathcal{N}_{k,n})^{H_{k,n}}$. The relative analogy to Bernstein's result facilitates the conjecture that the assumption indeed holds (for all $1 \leq k \leq n-2$). We were successful in proving it for the case $k = 1$ by techniques of prolongation of invariant distributions.

Let us expound further on this. In our proof we are able to prolong $H_{1,n}$-invariant distributions on $\mathcal{N}_{1,n}$ from one open orbit to its closure. Yet, in [11, Section 4], an example of a similar case was shown in which a so-called Ranga-Rao type theorem does *not* hold. Namely, if G/H is a p-adic symmetric space, the H-invariant distributions on the nilpotent orbits of the tangent space of G/H may not possess such prolongation properties. Indeed, the validity of such a property in our case for $k > 1$ remains one of the obstacles when dealing with the general linear nilpotent problem (the assumption of Lemma 1.2).

We now describe the structure and content of the present paper. Section 2 sets the main tools needed for our analysis. We recall the Frobenius descent (Proposition 2.2) which allows for the most direct transfer of a distribution into a smaller space. Since the mirabolic group is not reductive, we are unable to apply "traditional" distributional techniques directly. For that reason we introduce a new refinement (Proposition 2.4) of the descent techniques, which eases the treatment of invariance under general p-adic groups. We also sketch the treatment of Luna's Slice Theorem in the setting of symmetric spaces which was developed in [1].

In addition, we give a formulation (Proposition 2.5) of what we call the dense Frobenius descent. Building upon ideas from [4], it will serve us as a tool for reducing problems of invariant prolongation of distributions into similar problems on smaller spaces.

We describe in Section 2.3 the Bessel distributions mechanism. That gives an embedding of the linear forms on a H-distinguished irreducible representation of G into the space of distributions on G/H.

Section 3 starts with rather classical arguments which complete our transfer of distributional results into representation theory. We then study the geometry of closed $H_{k,n}$-orbits on the space $G_n/H_{k,n}$ and their decomposition to $\mathbf{P}_n(F) \cap H_{k,n}$-

orbits. Using the mentioned Harish-Chandra descent techniques we are able to reduce a question on the distribution spaces of $G_n/H_{k,n}$ to that of distributions on its tangent space (Theorem 3.7). Furthermore, we show that the "core" of the problem lies in distributions on the nilpotent cones $\mathcal{N}_{k,n}$.

In Section 4, using the dense Frobenius descent, we construct the full space of $H_{1,n}$-invariant distributions on $\mathcal{N}_{1,n}$. It is shown that these are also invariant under the smaller $\mathbf{P}_n(F)\cap H_{1,n}$ (Theorem 4.2). The problem is eventually reduced to that of prolongation of $F^\times$-invariant distributions from $F^\times$ to F. This is a classical problem that was treated in [6] and [15]). That will allow the final deduction of Theorem 1.1 from Lemma 1.2.

The results reported in this account constitute a part of my Ph.D. thesis submitted at the Technion. I would like to thank Omer Offen for suggesting me this problem and providing guidance. Special thanks to Dima Gourevitch and Rami Aizenbud for useful discussions and suggestions.

2. Tools and preliminaries

2.1. The symmetric space. Given $0 \leq k < n$, there is a maximal standard Levi subgroup $M_{k,n-k} \simeq G_k \times G_{n-k}$ of G_n. In this chapter we will write $H_{k,n} := M_{k,n-k}$.

Let $\theta_{k,n} : \mathbf{GL}_n \to \mathbf{GL}_n$ be the involutive automorphism defined by

$$\theta_{k,n}(g) = \epsilon_{k,n} g \epsilon_{k,n} \qquad \epsilon_{k,n} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}.$$

Then, the fixed point subgroup of $\theta_{k,n}$ is exactly $\mathbf{GL}_k \times \mathbf{GL}_{n-k}$, which makes the quotient $\mathbf{GL}_n/(\mathbf{GL}_k \times \mathbf{GL}_{n-k})$ a *symmetric space*. Consider the left action of G_n on itself by $\theta_{k,n}$-twisted conjugation: $g \cdot x := g x \theta_{k,n}(g)^{-1}$. The stabilizer of the identity element will be the fixed point subgroup of $\theta_{k,n}$ inside G_n, i.e. $H_{k,n}$. Thus, the orbit of the identity relative to this action

$$X_{k,n} = \left\{ g\theta_{k,n}(g)^{-1} \ : \ g \in G_n \right\} \subset G_n$$

is identified with the quotient $G_n/H_{k,n}$.

Since the Galois cohomology space $H^1(F, \mathbf{GL}_k \times \mathbf{GL}_{n-k})$ is trivial, we may also identify $X_{k,n}$ with F-points of the variety $\mathbf{GL}_n/(\mathbf{GL}_k \times \mathbf{GL}_{n-k})$ (see e.g. [13, Proposition 12.3.4] for more details on such a consideration).

We write the symmetrization map $\rho_{k,n}(g) = g\theta_{k,n}(g)^{-1} = g \cdot I_n$ from G_n to $X_{k,n}$. Notice that the action of $H_{k,n}$ on $X_{k,n}$ is given by usual conjugation, and therefore the stabilizer group inside $H_{k,n}$ of a point $x \in X_{k,n}$ is the centralizer of x which we will denote by $H^x_{k,n}$.

For a subgroup $P < H_{k,n}$, we will write $P^x = H^x_{k,n} \cap P$.

Observe that on the Lie algebra of $\mathbf{GL}_n$ there is a linear involution $d\theta_{k,n} = Ad(\epsilon_{k,n})$. After identifying the algebra with the matrix space $\mathbf{Mat}_{n,n}$ it becomes useful to define the linear version of the symmetric space

$$\mathbf{L}_{k,n} = \{A \in \mathbf{Mat}_{n,n} \ : \ Ad(\epsilon_{k,n})(A) = -A\} =$$

$$= \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \ : \ B \in \mathbf{Mat}_{k,n-k}, C \in \mathbf{Mat}_{n-k,k} \right\} .$$

We will want to apply Luna's slice theorem on the action of $H_{k,n}$ on $X_{k,n}$. Therefore, we need to be able to describe the closed orbits of this action and the their corresponding normal spaces in concrete terms.

Proposition 2.1. *If $x \in X_{k,n}$ has a closed $H_{k,n}$-orbit $\mathcal{O}$, then x is a semisimple matrix, and the action of $H^x_{k,n}$ on the normal space $N_x\mathcal{O}$ is isomorphic to its action on $\mathbf{L}^{Ad(x)}_{k,n}(F)$ by conjugation.*

Proof. This is [1, Proposition 7.2.1], after noting that the normal space in $\mathbf{GL}_n$ relative to the action of $\mathbf{GL}_k \times \mathbf{GL}_{n-k}$ on both sides is isomorphic to the normal space of $\mathbf{GL}_n/\mathbf{GL}_k \times \mathbf{GL}_{n-k}$ relative to the one sided action. Moreover, the closed orbits of both actions are in correspondence, by definition of the quotient topology. ■

2.2. Distributions on p-adic spaces.

For a general locally compact totally disconnected (lctd) space X (such as the F-points of an algebraic group), we write $\mathcal{S}(X)$ for the space of locally constant compactly supported complex-valued functions on X. The space $\mathcal{D}(X)$ of distributions on X is defined to be the dual space of $\mathcal{S}(X)$.

Given an open subset Ω of such a space X, Ω and $X \setminus \Omega$ become lctd spaces themselves. We can then restrict functions in $\mathcal{S}(X)$ to $X \setminus \Omega$ to produce an element of $\mathcal{S}(X \setminus \Omega)$. We also can prolong functions from $\mathcal{S}(\Omega)$ to X by setting a zero value outside of Ω. In this manner, we have a short exact sequence (see e.g. [3])

$$0 \longrightarrow \mathcal{S}(\Omega) \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{S}(X \setminus \Omega) \longrightarrow 0 .$$

Dualizing, we see another short exact sequence of distribution spaces

$$0 \longrightarrow \mathcal{D}(X \setminus \Omega) \longrightarrow \mathcal{D}(X) \longrightarrow \mathcal{D}(\Omega) \longrightarrow 0 .$$

Thus, for $T \in \mathcal{D}(X)$ we can talk about its restriction $T|_\Omega$, or say it is *supported* in $X \setminus \Omega$. The support Supp T of a distribution T is the complement of the union of all open sets Ω for which $T|_\Omega = 0$.

A continuous left action of a lctd group G on a lctd space X induces a left linear action of G on $\mathcal{S}(X)$ by $(g \cdot f)(x) = f(g^{-1} \cdot x)$ $(g \in G, f \in \mathcal{S}(X), x \in X)$. Hence, G acts on $\mathcal{D}(X)$ by the dual action.

In particular, the left action of $G \times G$ on G by $(g_1, g_2) \cdot h = g_1 h g_2^{-1}$ gives rise to a left action of $G \times G$ on $\mathcal{D}(G)$.

A lcdt group G has a left (resp. right) Haar measure $\mu_{G,l}$ (resp. $\mu_{G,r}$) on it, which is unique up to a constant. Those give rise to the *modular character* $\delta_G : G \to \mathbb{R}^\times$, which satisfies the relation $\mu_{G,l} = \delta_H \cdot \mu_{G,r}$.

For a closed subgroup $H < G$, we will denote by $\delta_{G/H}$ the *relative modular character* $\delta_{G/H} = \delta_H \delta_G^{-1}|_H$ of H.

Frobenius descent and a localization principle

The following is a well-known rephrasing of the Frobenius reciprocity principle for group representations.

Proposition 2.2. ***Frobenius descent*** *[4, Lemma 1.5] Suppose that G is a lctd group acting continuously on a lctd space X and $\mathcal{O}$, with a G-equivariant continuous map $p : X \to \mathcal{O}$. Suppose furthermore, that the G-action on $\mathcal{O}$ is transitive. For a chosen point $x \in \mathcal{O}$ let $H < G$ be the stabilizer group of x. Then, for any character $\chi : G \to \mathbb{C}$ there is an isomorphism of distribution spaces*

$$\mathcal{D}(p^{-1}(x))^{H,\chi\delta_{G/H}^{-1}} \cong \mathcal{D}(X)^{G,\chi} ,$$

where $\delta_{G/H}$ is the relative modular character.

If a distribution $T' \in \mathcal{D}(p^{-1}(x))^{H,\chi\delta_{G/H}^{-1}}$ corresponds to $T \in \mathcal{D}(X)^{G,\chi}$, we have the equality $\operatorname{Supp} T = G \cdot \operatorname{Supp} T'$.

We further mention that if $\delta_{G/H}$ is trivial (with the assumptions of the above proposition), then $\mathcal{O}$ has a non-zero G-invariant distribution μ on it. In that case, the Frobenius descent map $\Phi_\mu : \mathcal{D}(p^{-1}(x))^H \to \mathcal{D}(X)^G$ is given by the formula

$$(\Phi_\mu(T))(f) = \int_{\mathcal{O}} T((g_z \cdot f)|_{p^{-1}(x)})\, d\mu(z) \qquad f \in \mathcal{S}(X)$$

where $g_z \in G$ are fixed elements such that $g_z \cdot z = x$.

The second known tool we will require is the localization principle proved by Bernstein in [4, 1.4].

Proposition 2.3. *Suppose G is a lctd group acting continuously on lctd spaces X. Let T be a lctd space, and $p : X \to T$ a continuous map, such that $p(g \cdot x) = p(x)$, for all $x \in X$ and $g \in G$.*

Suppose that $P < G$ is a subgroup, for which $\mathcal{D}(p^{-1}(t))^P = \mathcal{D}(p^{-1}(t))^G$ holds, for all $t \in T$. Then,

$$\mathcal{D}(X)^P = \mathcal{D}(X)^G .$$

We further develop some adaptations of the Frobenius descent technique. Given a lctd group G acting on a lctd space X and a subgroup $P < G$, we want to formulate a criterion for the equality of spaces $\mathcal{D}(X)^G$ and $\mathcal{D}(X)^P$ based on the Frobenius descent.

Proposition 2.4. *Suppose G is a lctd group, and $P < G$ is a closed subgroup. Suppose X and $\mathcal{O}$ are lctd spaces on which G acts, and $p : X \to \mathcal{O}$ is a G-equivariant surjection. Assume that:*
(i) The G-action on $\mathcal{O}$ is transitive.
(ii) There is a non-zero G-invariant distribution on $\mathcal{O}$.
(iii) There are finitely many P-orbits in $\mathcal{O}$.
Suppose further that there exists an open P-orbit $\mathcal{U} \subset \mathcal{O}$ and an element $x \in \mathcal{U}$

such that:
(iv) $\mathcal{D}(p^{-1}(x))^{Stab(P,x)} = \mathcal{D}(p^{-1}(x))^{Stab(G,x)}$.
(v) For every $y \in \mathcal{O} \setminus \mathcal{U}$, $\mathcal{D}(p^{-1}(y))^{Stab(P,y),\, \delta^{-1}_{P/Stab(P,y)}} = \{0\}$.
Then,

$$\mathcal{D}(X)^P = \mathcal{D}(X)^G.$$

Proof. Let $T \in \mathcal{D}(X)^P$ be a distribution. If $0 \neq \mu$ is an G-invariant distribution on $\mathcal{O}$, then $0 \neq \mu|_{\mathcal{U}}$ is P-invariant, and $\delta_{P/Stab(P,x)} = 1$ follows. Thus, we can use Proposition 2.2 on the mapping $p : p^{-1}(\mathcal{U}) \to \mathcal{U}$ to produce $\xi \in \mathcal{D}(p^{-1}(x))^{Stab(P,x)}$ corresponding to $T|_{p^{-1}(\mathcal{U})}$. From condition (iv) we know that ξ is also $Stab(G,x)$-invariant, hence, corresponds by another Frobenius descent to an G-invariant distribution S on X. Yet, looking on both descents through the formula given above we see that $S|_{p^{-1}(\mathcal{U})} = T|_{p^{-1}(\mathcal{U})}$, which means $T - S$ has its support in $p^{-1}(\mathcal{O} \setminus \mathcal{U})$. Therefore, it is enough to prove that $\mathcal{D}(p^{-1}(\mathcal{O} \setminus \mathcal{U}))^P$ is trivial to show that $T = S$ and to finish.
This is achieved by induction on the number of P-orbits in $\mathcal{O} \setminus \mathcal{U}$. Since this is a finite number, there is an open P-orbit $\mathcal{U}_1 \subset \mathcal{O} \setminus \mathcal{U}$. Choosing $y \in \mathcal{U}_1$ and combining condition (v) with a Frobenius descent, we see that there are no non-zero P-invariant distributions on $p^{-1}(\mathcal{U}_1)$. Thus $\mathcal{D}(p^{-1}(\mathcal{O} \setminus \mathcal{U}))^P = \mathcal{D}(p^{-1}(\mathcal{O} \setminus (\mathcal{U} \cup \mathcal{U}_1)))^P$, and so on. ∎

Another problem we can tackle with descent techniques is the prolongation of invariant distributions. Given a distribution on a locally closed subset Y of a larger space we would like to claim that it is the restriction of a distribution on the closure $\overline{Y}$ with the same invariance properties. The following proposition allows us under suitable conditions to reduce such a question to a smaller space. This argument has appeared implicitly in [4, 4.3].

Proposition 2.5. *"Dense Frobenius descent"*
Let G be a lctd group that acts on a lctd space X. Let $\mathcal{K}$ be a compact totally disconnected space on which G acts transitively. Suppose $Y \subset X$ is an G-invariant locally closed subset equipped with an G-equivariant surjective continuous map $i : Y \to \mathcal{K}$.
For a point $s \in \mathcal{K}$, denote $Y_s = i^{-1}(s)$ and $B = Stab(G,s) < G$. If there exists a distribution $T \in \mathcal{D}(X)^{B,\delta^{-1}_{G/B}}$ with Supp $T = \overline{Y_s}$, *then there also exists a distribution $T' \in \mathcal{D}(X)^G$ with* Supp $T' = \overline{Y}$.

Proof. Consider the product space $X \times \mathcal{K}$ with its G-action. Let $Q \subset X \times \mathcal{K}$ be the graph of the map i. We apply Proposition 2.2 relative to the projection $p_2 : X \times \mathcal{K} \to \mathcal{K}$ on the right component. This gives the correspondence $\mathcal{D}(p_2^{-1}(s))^{B,\Delta^{-1}_{G/B}} \cong \mathcal{D}(X \times \mathcal{K})^G$. Identifying $p_2^{-1}(s)$ with X, we see that the existence of T as described in the assumption supplies the existence of a G-invariant distribution T'' on $X \times \mathcal{K}$ whose support is $\overline{G \cdot (\overline{Y}_s \times \{s\})} = \overline{Q}$.
Consider the projection $p_1 : X \times \mathcal{K} \to X$ on the left component. Since $\mathcal{K}$ is compact, if f is in $\mathcal{S}(X)$, then $f \circ p_1 \in \mathcal{S}(X \times \mathcal{K})$. Hence, we can push distributions with $p_{1*} : \mathcal{D}(X \times \mathcal{K}) \to \mathcal{D}(X)$ defined by $p_{1*}(T)(f) = T(f \circ p_1)$.

We claim that the support of $T' := p_{1*}(T'')$ is exactly $\overline{Y}$. Indeed, suppose that $x \in Y$, and $x \in U' \subset X$ is an open neighborhood, which we are allowed to assume to satisfy $U' \cap Y = U' \cap \overline{Y}$ (Y locally closed). Since $(x, i(x)) \in \operatorname{Supp} T''$, there exists an open compact $U \times V \subset U' \times \mathcal{K}$ with $T''(\chi_{U \times V}) \neq 0$ (with χ denoting the characteristic function of a set). We choose open sets $W_1, W_2 \subset X$ such that $W_1 \cap Y = i^{-1}(V)$ and $W_2 \cap Y = i^{-1}(\mathcal{K} \setminus V)$. By compactness, we can also choose finitely many disjoint open compact sets $U_1, \ldots, U_k \subset U \cap W_1$ which cover $(U \cap \overline{Y}) \setminus W_2$. Denote $U'' = U \setminus \cup_i U_i$. Then, $U'' \times V$ and Q are disjoint, which means that $T''(\chi_{U'' \times V}) = 0$ and $T''(\chi_{U_j \times V}) \neq 0$ for some j. Similarly, $U_j \times (\mathcal{K} \setminus V)$ lies outside the support of T''. So, $T'(\chi_{U_j}) = T''(\chi_{U_j \times V}) \neq 0$ shows that $T'|_{U'} \neq 0$. ∎

Luna's Slice Theorem and applications

In [1], the authors developed a mechanism for the transfer of information on spaces of invariant distributions on F-varieties to invariant distributions on F-linear spaces. We will now review the required tools from the mentioned reference.

Suppose a linear reductive F-group $\mathbf{G}$ acts on an affine F-variety $\mathbf{Y}$. In such setting, there exists an affine F-variety $\mathbf{Q} = \mathbf{Y}/\mathbf{G}$ which is the *algebraic quotient* variety (see [9, Chapter 1.2]). It is equipped with a $\mathbf{G}$-invariant F-algebraic map $\pi : \mathbf{Y} \to \mathbf{Q}$ and satisfies the expected universal property of a quotient. Namely, for every F-variety $\mathbf{X}$ with a $\mathbf{G}$-invariant F-algebraic map $\pi' : \mathbf{Y} \to \mathbf{X}$, there is a map $\phi : \mathbf{Q} \to \mathbf{X}$, such that $\pi' = \phi \circ \pi$.

Let us denote by π^F the resulting continuous map from $\mathbf{Y}(F)$ to $\mathbf{Q}(F)$.

Definition 2.6. We will say that a subset $U \subset \mathbf{Y}(F)$ is *saturated* if there is an open set $V \subset \mathbf{Q}(F)$, such that $U = (\pi^F)^{-1}(V)$.

Proposition 2.7. *For a saturated subset $U \subset \mathbf{Y}(F)$, there is a continuous $\mathbf{G}(F)$-invariant map $\phi : \mathbf{Y}(F) \to F^m$ and an open set $V \subset F^m$ such that $U = \phi^{-1}(V)$.*

Proof. Since $\mathbf{Q}$ is affine, there is an embedding $\iota : \mathbf{Q}(F) \to F^m$. Then $\phi = \iota \circ \pi^F$. The statement now follows from the definition of a saturated subset. ∎

In the same setting, suppose $x \in \mathbf{Y}(F)$ is such that its $\mathbf{G}(F)$-orbit $\mathcal{O}$ is closed (equivalently, its $\mathbf{G}$-orbit is Zariski closed [1, 2.3.8]). The tangent spaces $T_x(\mathbf{G} \cdot x) \subset T_x\mathbf{Y}$ are equipped with the action of the stabilizer group $\mathbf{G}^x$ of x (by differentiation). So, $\mathbf{G}^x$ acts on their quotient which is the *normal space* $\mathbf{N}_x(\mathbf{G} \cdot x)$ whose F-points may be identified with the normal space $N_x\mathcal{O}$ of F-analytic manifolds.

Theorem 2.8. (Luna's Slice Theorem–see e.g. [1, 2.3.17])
Let $\mathbf{G}$ be a linear reductive F-group acting on an affine F-variety $\mathbf{Y}$, and $x \in \mathbf{Y}(F)$ such that its $\mathbf{G}(F)$-orbit $\mathcal{O}$ is closed. Then there exists an open $\mathbf{G}(F)$-invariant neighborhood $\mathcal{O} \subset U \subset \mathbf{Y}(F)$, a $\mathbf{G}(F)$-equivariant continuous retract $p : U \to \mathcal{O}$, and a $\mathbf{G}^x(F)$-equivariant F-analytic embedding $\psi : p^{-1}(x) \to N_x\mathcal{O}$

whose image is saturated and $\psi(x) = 0$.

Let us remark that the slices of the variety obtained in the above procedure exhaust the whole variety because of the following fact.

Proposition 2.9. *[1, 2.3.7] Suppose a linear F-group $\mathbf{G}$ acts on an affine F-variety $\mathbf{Y}$. If U is an open subset of $\mathbf{Y}(F)$ which contains all closed $\mathbf{G}(F)$-orbits on $\mathbf{Y}(F)$, then $U = \mathbf{Y}(F)$.*

2.3. Bessel distributions. The analysis of distribution spaces on lctd groups and their homogeneous spaces has long been known to be useful for the study of the representation theory of the group. Many of these representation-theoretic applications go through the formalism of Bessel distributions.

Let (π, V) be a smooth irreducible representation of a lctd group G. The dual representation V^* is not smooth, yet it is possible to embed it into the space of distributions on G.

For simplicity let us assume that G is a unimodular (i.e. δ_G is trivial) group such as G_n. We fix a Haar measure $d\mu$ on G. For every function $\phi \in \mathcal{S}(G)$ and every $\ell \in V^*$, we define a new linear functional by the formula

$$(\pi(\phi)\ell)(v) = \int_G \phi(g)\ell(\pi(g^{-1})v)\, d\mu(g) \quad \forall v \in V.$$

It is easily verified that $\pi(\phi)\ell$ is smooth, that is, $\pi(\phi)\ell \in V^\vee$. Hence, given any $\widehat{\ell} \in (V^\vee)^*$, we can define the distribution $\mathcal{B}_\pi(\ell \otimes \widehat{\ell})$ on G as

$$\mathcal{B}_\pi(\ell \otimes \widehat{\ell})(\phi) = \widehat{\ell}(\pi(\phi)\ell) \quad \forall \phi \in \mathcal{S}(G)\ .$$

As hinted by the notation, $\mathcal{B}_\pi$ naturally extends to a linear map $V^* \otimes (V^\vee)^* \to \mathcal{D}(G)$, that is $G \times G$-equivariant. This map is in fact injective (see [5, Lemma 2.2]). The distributions in the image of $\mathcal{B}_\pi$ are the *Bessel distributions.*

Lemma 2.10. *A choice of $0 \neq \widehat{\ell} \in (V^\vee)^*$ gives an embedding of the dual space V^* into $\mathcal{D}(G)$, which is G-equivariant with respect to the left action of G on the distribution space.*

In case that $\widehat{\ell}$ is invariant under a closed subgroup $H < G$, there is a G-equivariant embedding of V^* into $\mathcal{D}(G/H)$.

Proof. The embedding is given by $\mathcal{B}_\pi(\cdot \otimes \widehat{\ell})$. In case that $\widehat{\ell}$ is invariant under H, the image of the above embedding lies in $\mathcal{D}(G)^{1\times H}$. Yet, $\mathcal{D}(G)^{1\times H}$ and $\mathcal{D}(G/H)$ are isomorphic as G-spaces (see [10, 3.1]). ∎

3. Reduction to a linear problem

Consider the natural action of $\mathbf{GL}_n$ on the affine space $\mathbb{A}^n$. For a fixed vector $0 \neq v \in \mathbb{A}^n$, let $\mathbf{P}_n < \mathbf{GL}_n$ be the stabilizer group of v. When v is chosen to be in $\mathbb{A}^n(F) \simeq F^n$, $\mathbf{P}_n$ is defined over F, and we write $P_n = \mathbf{P}_n(F)$. It is sometimes referred to as the *mirabolic group* (as we will do in this work), and is described usually in more concrete terms.

Namely, let us consider F^n as row vectors. The linear group G_n, realized as matrices, will act on it by right matrix multiplication. We are free to choose v to be the row $e_n = (0, \dots, 0, 1)$. The group P_n then consists of the invertible matrices whose last row is $(0, \dots, 0, 1)$.

We will say that an irreducible smooth representation (π, V) of G_n is $H_{k,n}$-*distinguished*, if there is a non-zero linear functional on V, which is invariant under the dual $H_{k,n}$-action.

Let us write $P_{k,n} = H_{k,n} \cap P_n$. We are interested in the space of $P_{k,n}$-invariant functionals on smooth irreducible representations of G_n. More precisely, given an $H_{k,n}$-distinguished such (π, V), we would like to claim that $(V^*)^{P_{k,n}} = (V^*)^{H_{k,n}}$.

Note, that since the center Z of G_n is contained in $H_{k,n}$, it must act trivially on all $H_{k,n}$-distinguished representations. In case $k = n-1$, we have $H_{k,n} = P_{k,n}Z$, which trivially implies the equality that we seek. The cases $k = 0, n$ are also clearly of no interest. Thus, we deal with the cases $1 \le k \le n-2$.

Let us recount a rather standard argument for a deduction of representation-theoretic statements from the results of previous sections. Let (π, V) be a smooth irreducible $H_{k,n}$-distinguished representation of G_n. By the Gelfand-Kazhdan theorem, the contragredient representation $(\widetilde{\pi}, \widetilde{V})$ can be realized on the space V with the action $\widetilde{\pi}(g) = \pi({}^t g^{-1})$ (^{t} marks matrix transposition). Since $H_{k,n} = H_{k,n}^t$, the above realization shows that $(\widetilde{\pi}, \widetilde{V})$ is $H_{k,n}$-distinguished as well. Thus, going through the identification $G_n/H_{k,n} \cong X_{k,n}$ and through Lemma 2.10, we get the following proposition.

Proposition 3.1. *The equality $\mathcal{D}(X_{k,n})^{P_{k,n}} = \mathcal{D}(X_{k,n})^{H_{k,n}}$ would imply $(V^*)^{P_{k,n}} = (V^*)^{H_{k,n}}$ for every $H_{k,n}$-distinguished irreducible smooth representation of G_n.*

We would like to reduce the problem of proving the equality of the distribution spaces in the above proposition into an equality of distribution spaces on a given cone of a linear space.

Let us introduce some terminology. When a group G acts on a space X, we would like to call the pair (G, X) an *action space*. Given a set $\{(G^i, X^i)\}_{i=1}^t$ of action spaces, we will say that an action space (G, X) is their *product*, if there exists a group isomorphism $G \cong G^1 \times \cdots \times G^t$ and a bijection $X \to X^1 \times \cdots \times X^t$ which intertwines the G-action relative to the fixed isomorphism.

Suppose an action space (G, X) admits a decomposition into a product of $\{(G^i, X^i)\}_{i=1}^t$. We call a subgroup $H < G$ *admissible* to this decomposition if the fixed isomorphism $G \cong G^1 \times \cdots \times G^t$ sends H to a product of subgroups $H^i < G^i$. In this case (H, X) becomes the product of $\{(H^i, X^i)\}_{i=1}^t$.

Given an involution σ on any vector space X, we will denote by $X^{\widehat{\sigma}} = \{x \in X \ : \ \sigma(x) = -x\}$ the space of anti-invariants of X. E.g. $\mathbf{L}_{k,n}(K) = \mathbf{Mat}_{n,n}(K)^{\widehat{Ad(\epsilon_{k,n})}}$ for any field K.

3.1. Geometry of closed orbits. It is an easy fact that the modular character of P_n is given by the character $\nu = |\det|_F$. Note also, that the same formula remains true for the group $\mathbf{P}_n(E)$, where $F < E$ is a finite field extension, if the determinant of a matrix is taken as an F-linear operator.

Suppose $x \in X_{k,n}$ is a semisimple matrix (for some $k < n-1$). Our first mission is to decompose the conjugation actions of the centralizer $H^x_{k,n}$ and its subgroup $P^x_{k,n}$ on $\mathbf{L}^{Ad(x)}_{k,n}(F)$ into a product of simpler action spaces. Regarding the group $H^x_{k,n}$, such a decomposition can surely be deduced from similar descriptions in [7] and [1], but we prefer to reproduce it here directly for completeness.

Proposition 3.2. *1. The action space $\left(H^x_{k,n}, \mathbf{L}^{Ad(x)}_{k,n}(F)\right)$ decomposes as a product of the action spaces*

$$\{(\mathbf{GL}_{l_i}(E_i), \mathbf{Mat}_{l_i,l_i}(E_i))\}_{i=1}^t \quad \textit{and of} \quad \{(H_{m_i,l_i}, \mathbf{L}_{m_i,l_i}(F))\}_{t<i\le s},$$

where $0 \le t \le s \le t+2$, $F < E_i$ is a finite field extension and l_i is a positive integer for every $1 \le i \le t$; and $m_i \le k, l_i \le n$ are positive integers for every $t < i \le s$.

All actions in this decomposition are by conjugation.

2. *The subgroup $P^x_{k,n}$ of $H^x_{k,n}$ is admissible to the decomposition above. Therefore, the action space $\left(P^x_{k,n}, \mathbf{L}^{Ad(x)}_{k,n}(F)\right)$ is a product of*

$$\left\{(P^i, \mathbf{Mat}_{l_i,l_i}(E_i)\right\}_{i=1}^t \quad \textit{and of} \quad \{(P^i, \mathbf{L}_{m_i,l_i}(F)\}_{t<i\le s},$$

for certain subgroups P_i. Those are given as follows:

$$P^i = \begin{cases} \mathbf{GL}_{l_i}(E_i) & \alpha_i = 0 \quad 1 \le i \le t \\ \mathbf{P}_{l_i}(E_i) & \alpha_i = 1 \\ H_{m_i,l_i} & \alpha_i = 0 \quad t < i \le s \\ P_{m_i,l_i} & \alpha_i = 1 \end{cases},$$

where $\alpha(x) = (\alpha_1, \ldots, \alpha_s) \in \{0,1\}^s$ is a fixed tuple.

3. *After identifying $P^x_{k,n}$ with $\prod_{i=1}^s P^i$, the relative modular character of $P^x_{k,n}$ inside $P_{k.n}$ is given by*

$$\delta_{P_{k,n}/P^x_{k,n}}(g_1, \ldots, g_s) = \prod_{\{i\,:\,\alpha_i=0\}} |\det \widetilde{g_i}|^{-1},$$

where

$$\widetilde{g_i} = \begin{cases} g_i & 1 \le i \le t \\ h_i & t < i \le s \quad H_{m_i,l_i} = G_{m_i} \times G_{l_i-m_i},\ g_i = (f_i, h_i) \end{cases}.$$

Proof. **1.** Consider the action of G_n by right-multiplication on the row vector space F^n. The semisimple operator x gives rise to a decomposition of $F^n = \oplus_j V_j$ to a direct sum of vector spaces, such that V_j has an $E_j = F(\zeta_j)$-vector space structure for a finite field extension $F < E_j$, and $x|_{V_j}$ is acting as multiplication by ζ_j. Grouping isomorphic extensions, we can assume the actions of the ζ_j's are not isomorphic for distinct j's. Then it is easy to see that such a decomposition of F^n gives rise to a decomposition of the conjugation action space $\left(G_n^x, \mathbf{Mat}_{n,n}(F)^{Ad(x)}\right)$ into action spaces of the form $(\mathbf{GL}_{l_j}(E_j), \mathbf{Mat}_{l_j,l_j}(E_j))$, with $l_j \leq n$. It remains to see what happens when we add the requirement for commutation relations with $\epsilon_{k,n}$.

Since $x \in X_{k,n}$, we have $\epsilon_{k,n} x \epsilon_{k,n} = \theta_{k,n}(x) = x^{-1}$. Thus, the action of $\epsilon_{k,n}$ on F^n must permute the V_j's. Since $\epsilon_{k,n}$ is an involution, we conclude that $\left(H_{k,n}^x, \mathbf{L}_{k,n}^{Ad(x)}(F)\right)$ decomposes as a product of spaces either of the form

$$\left((\mathbf{GL}_{l_{j_1}}(E_{j_1}) \times \mathbf{GL}_{l_{j_2}}(E_{j_2}))^{\epsilon_{k,n}}, (\mathbf{Mat}_{l_{j_1},l_{j_1}}(E_{j_1}) \times \mathbf{Mat}_{l_{j_2},l_{j_2}}(E_{j_2}))^{\widehat{Ad(\epsilon_{k,n})}}\right)$$

where $j_1 \neq j_2$ are such that $\epsilon_{k,n}(V_{j_1}) = V_{j_2}$, or of the form

$$\left(\mathbf{GL}_{l_j}(E_j)^{\epsilon_{k,n}}, \mathbf{Mat}_{l_j,l_j}(E_j)^{\widehat{Ad(\epsilon_{k,n})}}\right)$$

where V_j is $\epsilon_{k,n}$-invariant. Let us show that all of these are isomorphic to the action spaces described in the statement.

In the former case (*case (1)*), the acting group is clearly given by

$$\{(g, \epsilon_{k,n} g \epsilon_{k,n}|_{V_{j_2}}) \,:\, g \in \mathbf{GL}_{l_{j_1}}(E_{j_1})\},$$

while the space is given by

$$\{(A, -\epsilon_{k,n} A \epsilon_{k,n}|_{V_{j_2}}) \,:\, A \in \mathbf{Mat}_{l_{j_1},l_{j_1}}(E_{j_1})\}.$$

Hence, the situation is isomorphic to $\mathbf{GL}_{l_{j_1}}(E_{j_1})$ acting on $\mathbf{Mat}_{l_{j_1},l_{j_1}}(E_{j_1})$.

The latter case should itself be separated into two cases. First, assume that $x^{-1}|_{V_j} = \theta_{k,n}(x)|_{V_j} \neq x|_{V_j}$ (*case (2)*). Then, on V_j, $\theta_{k,n}$ serves as a non-trivial F-linear involution of E_j. Let $F < L_j < E_j$ be its fixed sub-field ($[E_j : L_j] = 2$). This means we can write $V_j \cong E_j \otimes_{L_j} W_j$ for an L_j-subspace $W_j \subset V_j$, with $\epsilon_{k,n}$ acting as the non-trivial Galois automorphism in $Gal(E_j/L_j)$ on the L_j component. In these terms, we have

$$\mathbf{Mat}_{l_j,l_j}(E_j)^{\widehat{Ad(\epsilon_{k,n})}} \cong (E_j \otimes_{L_j} \mathbf{Mat}_{l_j,l_j}(L_j))^{\widehat{Ad(\epsilon_{k,n})}} = \tau \otimes_{L_j} \mathbf{Mat}_{l_j,l_j}(L_j),$$

where $\tau \in L_i$ is an element for which $\epsilon_{k,n}(\tau \otimes w) = -\tau \otimes w$. Similarly, we have $\mathbf{GL}_{l_j}(E_j)^{\epsilon_{k,n}} \cong \mathbf{GL}_{l_j}(L_j)$, and the action space is isomorphic to

$$(\mathbf{GL}_{l_j}(L_j), \mathbf{Mat}_{l_j,l_j}(L_j)).$$

Finally, still in the case of $\left(\mathbf{GL}_{l_j}(E_j)^{\epsilon_{k,n}}, \mathbf{Mat}_{l_j,l_j}(E_j)^{\widehat{Ad(\epsilon_{k,n})}}\right)$, if $x|_{V_j} = x^{-1}|_{V_j}$ (*case (3)*), then $x = \pm I_{V_j}$. That means at once that $E_j = F$ and that there are at most two of such factors in the product. Furthermore, we see that under a suitable change of basis, $\epsilon_{k,n}|_{V_j}$ equals ϵ_{m_j,l_j}. Since m_j is the dimension of the eigenspace of $\epsilon_{k,n}|_{V_j}$ corresponding to the eigenvalue 1, it obviously cannot exceed k.

2. Let $\{v_j \in V_j\}$ be the decomposition of the vector $e = (0, \ldots, 0, 1) \in F^n = \oplus_j V_j$. Since the subspaces V_j are invariant under the $H^x_{k,n}$-action on F^n and since $P^x_{k,n}$ is the stabilizer of e, the subgroup can also be described as the intersection of the stabilizers of the vectors v_j inside $H^x_{k,n}$. This clearly means that $P^x_{k,n}$ is an admissible subgroup for the above product decomposition. It is left to identify the stabilizer subgroups of v_j in each of the three cases appearing in the product.

In case (1), since $\epsilon_{k,n}(e) = -e$, we see that $v_{j_1} = -v_{j_2}$. If $v_{j_1} = 0$, the stabilizer is clearly the whole acting group, and we set $\alpha_i = 0$ for the i with which we are dealing. Otherwise, we set $\alpha_i = 1$ and the subgroup of $\mathbf{GL}_{l_{j_1}}(E_{j_1})$ which stabilizes $(v_{j_1}, -v_{j_2})$ becomes $\mathbf{P}_{l_{j_1}}(E_{j_1})$, up to a change of basis.

In case V_j is $\epsilon_{k,n}$-invariant, again we set $\alpha_i = 0$ for the corresponding i when $v_j = 0$, and the stabilizer is then the whole acting group. Otherwise we set $\alpha_i = 1$. In case (2), the condition $\epsilon_{k,n}(e) = -e$ imposes $v_j = \tau \otimes w$, whose stabilizer is again just a stabilizer of one non-zero vector in $\mathbf{GL}_{l_j}(L_j)$, i.e. isomorphic to $\mathbf{P}_{l_j}(L_j)$.

As for the last case, it is easily seen that given a vector v_j for which $\epsilon_{k,n}|_{V_j}(v_j) = -v_j$, a suitable change of basis exists such that H_{m_j,l_j} remains at place while the stabilizer of v_j becomes P_{m_j,l_j}.

3. Since $\mathbf{GL}_l(E)$ is a unimodular group and the modular character of $\mathbf{P}_l(E)$ is ν, it is easy to see that $\delta_{P^x_{k,n}}(g) = \prod_{i\,:\,\alpha_i=1} |\det(\widetilde{g}_i)|$, for all $g = (g_1, \ldots, g_s) \in \prod_{i=1}^s P^i \cong P^x_{k,n}$. Now, $\delta_{P_{k,n}}$ is given by the norm of the determinant of the lower block, that is, the determinant of the restriction of the operator to the -1 eigenspace of $\epsilon_{k,n}$. We need to compute it for each P^i.

In case (1), we are interested in the restriction of an operator in P^i to the space $\{(v, -\epsilon_{k,n}(v))\}_{v \in V_{j_1}}$, which is isomorphic to $\mathbf{GL}_{l_{j_1}}(E_{j_1})$. In case (2), we are looking on its action on the space $\tau \otimes W_j$ which gives the same conclusion. Finally, for case (3), the restriction is the projection on the right component of $H_{m_j,l_j} = G_{m_j} \times G_{l_j - m_j}$.

Thus, in all cases we have $\delta_{P_{k,n}}(g_i) = |\det(\widetilde{g}_i)|$, and the statement follows.

■

The above decomposition can also be applied for the study of the geometry of the $P_{k,n}$-action on $X_{k,n}$ in the following way.

Proposition 3.3. *Let $\mathcal{O}$ be a closed $H_{k,n}$-orbit in $X_{k,n}$. Then, $\mathcal{O}$ splits into a finite number of $P_{k,n}$-orbits, with a unique open (in $\mathcal{O}$) orbit $A \subset \mathcal{O}$. Furthermore, A consists exactly of those $x \in \mathcal{O}$ for which $\alpha(x) = (1, \ldots, 1)$.*

Proof. Since $H^x_{k,n}$ is the stabilizer of $x \in \mathcal{O}$ in $H_{k,n}$, we can fix one $x \in \mathcal{O}$ and identify $\mathcal{O} \cong H_{k,n}/H^x_{k,n}$. Note, that instead of counting orbits of $P_{n,k}$ on $H_{n,k}/H^x_{k,n}$, we can count the orbits of the right action of $H^x_{k,n}$ on the space $P_{n,k}\backslash H_{k,n}$. Consider $F^{n-k} \setminus \{0\}$ as a row space on which $H_{k,n}$ acts transitively by right matrix multiplication with its lower block. Then, $P_{k,n}$ will be the stabilizer of $e = (0, \ldots, 0, 1)$, and $F^{n-k} \setminus \{0\} \cong P_{n,k}\backslash H_{n,k}$.

By Proposition 2.1, x is a semisimple matrix, hence, we can apply the previous proposition. Since F^{n-k} lies inside F^n as the eigenspace of $\epsilon_{k,n}$ related to the value -1, we can use the same reasoning as in the previous proof to identify it as the set of vectors of the form $((w_i, -\epsilon_{k,n}(w_i)), (\tau \otimes w_i), w_i)$, with the notation that corresponds to the 3 cases classification of the previous proof. We are left to count the non-zero orbits of $H^x_{k,n}$ on this subspace.

Clearly, these are determined by the nullity of each of $\{w_i\}_{i=1}^s$, and we see that there are exactly $2^s - 1$ such orbits. In particular, there is only one open orbit among them, namely, the set of vectors for which all $w_i \neq 0$. We denote by A the corresponding unique open orbit in $\mathcal{O}$.

The $P_{k,n}$-orbit of x inside $\mathcal{O}$ corresponds to the $H^x_{k,n}$-orbit of the identity in $P_{n,k}\backslash H_{n,k}$, or in other words, of the vector e in F^{n-k}. Thus, the question of whether x belongs to A is equivalent to asking whether the components of e in the $\{w_i\}$ (or $\oplus_j V_j$) decomposition are all non-zero. This is equivalent to the condition $\alpha(x) = (1, \ldots, 1)$ from the way we defined $\alpha(x)$. ∎

3.2. Distributions on $X_{k,n}$.

Corollary 3.4. *If A is the open $P_{k,n}$-orbit inside a closed $H_{k,n}$-orbit $\mathcal{O} \subset X_{k,n}$ and $x \notin A$, then $\mathcal{D}(N_x\mathcal{O})^{P^x_{k,n}, \delta^{-1}_{P_{k,n}/P^x_{k,n}}}$ is trivial.*

Proof. First by Proposition 2.1, we may prove that $\mathcal{D}(\mathbf{L}^{Ad(x)}_{k,n}(F))^{P^x_{k,n}, \delta^{-1}_{P_{k,n}/P^x_{k,n}}}$ is trivial. Since $x \notin A$, it must have $\alpha_i = 0$, for some i, where $\alpha(x) = (\alpha_i)$ is as in the notation of Proposition 3.2. Also, as observed in Proposition 3.2, there must be a subgroup M inside $P^x_{k,n}$ isomorphic to either $\mathbf{GL}_{l_i}(E)$ or H_{m_i,l_i} whose action on $\mathbf{L}^{Ad(x)}_{k,n}(F)$ is isomorphic to a conjugation action on a specified matrix space. Note also, that by Proposition 3.2.3, $\delta_{P_{k,n}/P_x}(g) = |\det(\tilde{g})|$ for $g \in M$, with $\tilde{g}$ defined as in the mention proposition. Take $g = \omega I \in M$ for some $\omega \in F$ with $|\omega| > 1$. Then it is clear that $\delta_{P_{k,n}/P_x}(g) \neq 1$, while g, being a scalar operator, must act trivially on $\mathbf{L}^{Ad(x)}_{k,n}(F)$. This shows there cannot be any non-trivial $P^x_{k,n}, \delta^{-1}_{P_{k,n}/P_x}$-invariant distributions on that space. ∎

We are now ready to claim the first distributional reduction of the main problem.

Lemma 3.5. *Let k, n be positive integers such that $k < n - 1$. Let $\mathbf{Y}$ be a smooth affine F-variety equipped with a $\mathbf{GL}_k \times \mathbf{GL}_{n-k}$-action.*

Suppose that $U_X \subset X_{k,n}$ and $U_Y \subset \mathbf{Y}(F)$ are $H_{k,n}$-invariant open subsets with a $H_{k,n}$-equivariant diffeomorphism $s : U_X \to U_Y$.

Suppose further that $\mathcal{O} \subset \mathbf{Y}(F)$ is a closed $H_{k,n}$-orbit which is contained in U_Y and for which $s^{-1}(\mathcal{O})$ is closed in $X_{k,n}$,

If for every semisimple matrix $x \in s^{-1}(\mathcal{O})$ the equality

$$\mathcal{D}\left(\mathbf{L}^{Ad(x)}_{k,n}(F)\right)^{P^x_{k,n}} = \mathcal{D}\left(\mathbf{L}^{Ad(x)}_{k,n}(F)\right)^{H^x_{k,n}}$$

holds, where the group action is by conjugation, then there is an open $H_{k,n}$-invariant neighborhood $\mathcal{O} \subset U_{\mathcal{O}} \subset \mathbf{Y}(F)$ with

$$\mathcal{D}(U_{\mathcal{O}})^{P_{k,n}} = \mathcal{D}(U_{\mathcal{O}})^{H_{k,n}}.$$

Proof. Let $\mathcal{O}$ be a closed $H_{k,n}$-orbit as in the statement. By Luna's slice theorem (Theorem 2.8), there exists an open $H_{k,n}$-invariant subset $\mathcal{O} \subset U_{\mathcal{O}} \subset \mathbf{Y}(F)$, together with an $H_{k,n}$-equivariant retract $p : U_{\mathcal{O}} \to \mathcal{O}$ possessing certain properties.

Let $A \subset s^{-1}(\mathcal{O})$ be the open $P_{k,n}$-orbit which was shown to exist in Proposition 3.3. Choose $x \in A$, which is semisimple by Proposition 2.1. Naturally $\mathcal{O} \cong s^{-1}(\mathcal{O}) \cong H_{k,n}/H^x_{k,n}$. The result will be achieved by applying Proposition 2.4 on the retract p and the groups $P_{k,n} < H_{k,n}$. Indeed, condition (ii) holds because $H_{k,n}$ and $H^x_{k,n}$ are unimodular (this is a consequence of Proposition 3.2.1), and (iii) holds because of Proposition 3.3. We are left to show that conditions (iv) and (v) hold for $y = s(x)$.

Recall that Luna's slice comes with an $\mathrm{Stab}(H_{k,n}, y)$-equivariant embedding $\iota : p^{-1}(y) \to N_y\mathcal{O}$ whose image is open and saturated. Hence, there is an $\mathrm{Stab}(H_{k,n}, y)$-invariant map $\pi : N_y\mathcal{O} \to F^m$, and an open $V \subset F^m$ such that $i(p^{-1}(y)) = \pi^{-1}(V)$. Consider $\xi \in \mathcal{D}(p^{-1}(y))^{\mathrm{Stab}(P_{k,n},y)}$, and suppose there is an $h \in \mathrm{Stab}(H_{k,n}, y)$ with $\xi \neq \xi^h$. Pick $t \in \mathrm{Supp}(\xi - \xi^h)$ and an open compact neighborhood B of $\pi(\iota(t))$ in V. Then, $\pi^{-1}(B)$ is an open and closed subset of $N_y\mathcal{O}$ that contains $\iota(t)$ and contained in $\iota(p^{-1}(y))$. Now, we have a well-defined operator $\alpha_B : \mathcal{D}(p^{-1}(y)) \to \mathcal{D}(N_y\mathcal{O})$ given by $\alpha_B(\eta)(f) = \eta((f \circ \iota)|_{\pi^{-1}(B)})$ for $f \in \mathcal{S}(N_y\mathcal{O})$. Hence, $\alpha_B(\xi) \in \mathcal{D}(N_y\mathcal{O})^{\mathrm{Stab}(P_{k,n},y)}$.

Now, since s is a diffeomorphism, $N_y\mathcal{O}$ and $N_x(s^{-1}(\mathcal{O}))$ are isomorphic as spaces with a $\mathrm{Stab}(H_{k,n}, y) \cong H^x_{k,n}$ action. By Proposition 2.1 these are also isomorphic to the $H^x_{k,n}$-space $\mathbf{L}^{Ad(x)}_{k,n}(F)$. Thus by our assumption $\alpha_B(\xi) \in \mathcal{D}(N_y\mathcal{O})^{\mathrm{Stab}(H_{k,n},y)}$.

Therefore, $\alpha_B(\xi^h) = \alpha_B(\xi)^h = \alpha_B(\xi)$, which gives $\alpha_B(\xi - \xi^h) = 0$. This contradicts the clear fact that $\iota(t) \in \mathrm{Supp}(\alpha_B(\xi - \xi^h))$. Finally,

$$\mathcal{D}(p^{-1}(y))^{\mathrm{Stab}(P_{k,n},y)} = \mathcal{D}(p^{-1}(y))^{\mathrm{Stab}(H_{k,n},y)}$$

holds and condition (iv) of Proposition 2.4 is satisfied.

With similar arguments, we can deduce condition (v) from Corollary 3.4 by noting that $p^{-1}(z)$ can be embedded with an open saturated image inside $N_z\mathcal{O}$ for all $z \in \mathcal{O}$, and directly using [1, Lemma 3.1.3]. ∎

Proposition 3.6. *Let k, n be positive integers such that $k < n - 1$. Suppose that for every semisimple matrix $x \in X_{k,n}$ the equality*

$$\mathcal{D}\left(\mathbf{L}^{Ad(x)}_{k,n}(F)\right)^{P^x_{k,n}} = \mathcal{D}\left(\mathbf{L}^{Ad(x)}_{k,n}(F)\right)^{H^x_{k,n}}$$

holds. Then, we also have

$$\mathcal{D}(X_{k,n})^{P_{k,n}} = \mathcal{D}(X_{k,n})^{H_{k,n}}.$$

Proof. Using Lemma 3.5 with $\mathbf{Y} = \mathbf{GL}_n/(\mathbf{GL}_k \times \mathbf{GL}_{n-k})$, $U_X = U_Y = X_{k,n}$ and $s = id$ produces for each closed $H_{k,n}$-orbit $\mathcal{O} \subset X_{k,n}$ an open $H_{k,n}$-invariant neighborhood $\mathcal{O} \subset U_{\mathcal{O}} \subset X_{k,n}$ with $\mathcal{D}(U_{\mathcal{O}})^{P_{k,n}} = \mathcal{D}(U_{\mathcal{O}})^{H_{k,n}}$.

By [10, Lemma 3.2] we now that $\mathcal{D}(\bigcup_{\mathcal{O}} U_{\mathcal{O}})^{P_{k,n}} = \mathcal{D}(\bigcup_{\mathcal{O}} U_{\mathcal{O}})^{H_{k,n}}$, where the union is over all closed orbits $\mathcal{O}$. Now, by Proposition 2.9, $\bigcup_{\mathcal{O}} U_{\mathcal{O}} = X_{k,n}$. ■

Theorem 3.7. *Let k, n be positive integers such that $k < n-1$. Suppose that the equality*

$$\mathcal{D}\left(\mathbf{L}_{m,l}(F)\right)^{P_{m,l}} = \mathcal{D}\left(\mathbf{L}_{m,l}(F)\right)^{H_{m,l}}$$

holds, for all positive integers $m \leq k$ and $l \leq n$. Then, we also have

$$\mathcal{D}(X_{k,n})^{P_{k,n}} = \mathcal{D}(X_{k,n})^{H_{k,n}}$$

Proof. Suppose that $\{G^i\}$ are finitely many lctd groups acting on lctd spaces $\{X^i\}$ respectively, and $H^i < G^i$ are fixed subgroups. Let (H, X) be the product of all $\{(H^i, X^i)\}$'s, and (G, X) the product of $\{(G^i, X^i)\}$'s. If $\mathcal{D}(X^i)^{P^i} = \mathcal{D}(X^i)^{H^i}$ for all i, then it is easy to check that $\mathcal{D}(X)^H = \mathcal{D}(X)^G$. See, for example [2, 3.1.5].

So, combining that reasoning with Proposition 3.2 and Proposition 3.6, it is enough to show that $\mathcal{D}(\mathbf{Mat}_{l,l}(E))^{\mathbf{P}_l(E)} = \mathcal{D}(\mathbf{Mat}_{l,l}(E))^{\mathbf{GL}_l(E)}$ for certain finite field extensions $F < E$ and integers $l \leq n$, and that $\mathcal{D}\left(\mathbf{L}_{m,l}(F)\right)^{P_{m,l}} = \mathcal{D}\left(\mathbf{L}_{m,l}(F)\right)^{H_{m,l}}$ for certain integers $m \leq k$, $l \leq n$.

The former equality was proved by Bernstein in [4], while the latter is the assumption, when $m > 0$. When $m = 0$ there is, of course, nothing to prove. ■

3.3. Reduction to nilpotents. Let $\mathcal{N}_{k,n} \subset \mathbf{L}_{k,n}(F)$ be the cone of nilpotent matrices.

Before moving to tackle the equality of distribution spaces on the linear space, we would like to reduce our problem further to that of distribution spaces on $\mathcal{N}_{k,n}$.

Let us consider the matrix sets

$$V = \{g \in G_n \,:\, \det(g + I) \neq 0\},$$

$$W = \{x \in \mathbf{M}_{n,n}(F) \,:\, \det(x + I)\det(x - I) \neq 0\}$$

and the Cayley transform diffeomorphism $C : V \to W$ between them which is defined by $C(g) = (I - g)(I + g)^{-1}$. Since C commutes with matrix conjugation and $C(g^{-1}) = -C(g)$, we see that it restricts to a diffeomorphism from $V \cap Q_{k,n}$ to $W \cap \mathbf{L}_{k,n}(F)$, where $Q_{k,n} = \{g \in G_n \,:\, \theta_{k,n}(g) = g^{-1}\}$.

Furthermore, since by [12, Proposition 9.1] $X_{k,n}$ is an open subset of $Q_{k,n}$, we see that $\lambda := C|_{V \cap X_{k,n}}$ is a $H_{k,n}$-equivariant diffeomorphism from an open $H_{k,n}$-invariant subset of $X_{k,n}$ onto an open subset of $\mathbf{L}_{k,n}(F)$.

Proposition 3.8. *Suppose that the equality*

$$\mathcal{D}\left(\mathbf{L}_{m,l}(F)\right)^{P_{m,l}} = \mathcal{D}\left(\mathbf{L}_{m,l}(F)\right)^{H_{m,l}}$$

holds, for all positive integers $m \leq k$ and $l < n$, and suppose further that $\mathcal{D}(\mathcal{N}_{k,n})^{P_{k,n}} = \mathcal{D}(\mathcal{N}_{k,n})^{H_{k,n}}$. Then, $\mathcal{D}\left(\mathbf{L}_{k,n}(F)\right)^{P_{k,n}} = \mathcal{D}\left(\mathbf{L}_{k,n}(F)\right)^{H_{k,n}}$ holds.

Proof. Let $t : \mathbf{L}_{k,n}(F) \to F^n$ be the map given by the coefficients of the characteristic polynomials of elements of $\mathbf{L}_{k,n}(F)$ (as matrices). Clearly, t is $H_{k,n}$-invariant. We use the localization principle from Proposition 2.3 with the map t. By that principle, it is enough to prove the equality of invariant distribution spaces only on the fibers of t. For $t^{-1}(1,0,\ldots,0) = \mathcal{N}_{k,n}$ this is our assumption. Now let $\mathcal{N}_{k,n} \neq Z \subset \mathbf{L}_{k,n}(F)$ be a fixed fiber of t.

Let $\mathcal{O} \subset Z$ be a closed $H_{k,n}$-orbit. As in the proof of Proposition 3.6, if we are able to find for each such $\mathcal{O}$ an open $H_{k,n}$-invariant neighborhood $\mathcal{O} \subset U_{\mathcal{O}} \subset \mathbf{L}_{k,n}(F)$ for which $\mathcal{D}(U_{\mathcal{O}})^{P_{k,n}} = \mathcal{D}(U_{\mathcal{O}})^{H_{k,n}}$ holds, then $\mathcal{D}(Z)^{P_{k,n}} = \mathcal{D}(Z)^{H_{k,n}}$ and we are done.

Let $\mathcal{U} \subset \mathbf{L}_{k,n}(F)$ be the open image of the Cayley transform map λ. Since $0 = \lambda(I) \in \mathcal{U}$, $\alpha\mathcal{O} \subset \mathcal{U}$ for some $\alpha \in F^{\times}$. The $H_{k,n}$-orbit $\alpha\mathcal{O}$ is also closed. Since the action on $\mathbf{L}_{k,n}(F)$ is linear, finding a suitable neighborhood $U_{\alpha\mathcal{O}}$, would give $\alpha^{-1}U_{\alpha\mathcal{O}}$ as our desired neighborhood for $\mathcal{O}$. Thus, we can safely assume that $\mathcal{O} \subset \mathcal{U}$.

Note that since λ is defined on $V \cap X_{k,n}$ and the matrices in $\lambda^{-1}(\mathcal{O})$ all have the same characteristic polynomial, the orbit $\lambda^{-1}(\mathcal{O})$ remains closed in $X_{k,n}$. Therefore, we can construct $U_{\mathcal{O}}$ by applying Lemma 3.5 with $\mathbf{Y} = \mathbf{L}_{k,n}$ and $s = \lambda$ as the diffeomorphism between open subsets.

The lemma then states that the equality

$$\mathcal{D}\left(\mathbf{L}_{k,n}^{Ad(x)}(F)\right)^{P_{k,n}^{x}} = \mathcal{D}\left(\mathbf{L}_{k,n}^{Ad(x)}(F)\right)^{H_{k,n}^{x}} \tag{1}$$

for all semisimple $x \in \lambda^{-1}(\mathcal{O})$, would imply the existence of $U_{\mathcal{O}}$ as desired.

Indeed, since $\lambda(x) \neq 0$, we clearly have $H_{k,n}^{x} = H_{k,n}^{\lambda(x)} \subsetneq H_{k,n}$. Hence,

$$\dim \mathbf{L}_{k,n}^{Ad(x)}(F) = \dim N_x \iota^{-1}(\mathcal{O}) = \dim X_{k,n} - \dim H_{k,n}/H_{k,n}^{x} < \dim \mathbf{L}_{k,n}(F).$$

We see that the factor $\mathbf{L}_{k,n}(F)$ cannot appear in the decomposition of Proposition 3.2 for $\mathbf{L}_{k,n}^{Ad(x)}(F)$. Therefore, using the same technique as in the proof of Theorem 3.7, our assumptions are enough to prove the equality (1). ∎

Theorem 3.9. *Let k, n be positive integers such that $k < n-1$. Suppose that the equality*

$$\mathcal{D}\left(\mathcal{N}_{m,l}\right)^{P_{m,l}} = \mathcal{D}\left(\mathcal{N}_{m,l}\right)^{H_{m,l}}$$

holds, for all positive integers $m \leq k$ and $l \leq n$. Then, we also have

$$\mathcal{D}(X_{k,n})^{P_{k,n}} = \mathcal{D}(X_{k,n})^{H_{k,n}}$$

Proof. By induction on m and l, it is easy to use Proposition 3.8 at each step in such manner that the assumptions of Theorem 3.7 are satisfied. ∎

By Proposition 3.1 we see now that Lemma 1.2 follows from Theorem 3.9.

4. The case $k = 1$.

In this section we fix $n \geq 3$. We would like to characterize the space of $P_{1,n}$-conjugation-invariant distributions on the nilpotent cone $\mathcal{N}_{1,n}$. The eventual result will be that it is a 2-dimensional space spanned by the obvious point distribution centered on the zero vector, and by a second distribution ξ which we will need to construct, both of which are also $H_{1,n}$-invariant. To show the existence of such ξ we need to be able to prolong an $H_{1,n}$-invariant distribution on an open orbit onto its closure in such way that it remains invariant. This task of singularity resolution is carried out using the "dense Frobenius descent" (Proposition 2.5) technique, which reduces the problem to that of distribution prolongation from $F^\times$ to F.

The space $\mathbf{L}_{1,n}(F)$ is the set of matrices given by

$$\left\{ \begin{pmatrix} 0 & \overline{v} \\ \overline{w} & 0 \end{pmatrix} : \overline{v} \in \mathbf{Mat}_{1,n-1}(F),\ \overline{w} \in \mathbf{Mat}_{n-1,1}(F) \right\}.$$

Thus we can parametrize this space by pairs $(\overline{v}, \overline{w})$, where the first is a row a vector while the second is a column vector. It will sometimes be convenient to write the row vector as $\overline{v} = (\overline{v}'\ a)$ and the column vector as $\overline{w} = \begin{pmatrix} \overline{w}' \\ b \end{pmatrix}$, where $a, b \in F$ and $\overline{v}', \overline{w}'$ are of length $n-2$. In this notation we get a parametrization of $\mathbf{L}_{1,n}(F)$ by quadruples $(\overline{v}', a, \overline{w}', b)$.

The conjugation action of the group $H_{1,n}$, naturally viewed as $F^\times \times G_{n-1}$, is given in pairs notation by the formula $(\alpha, A) \cdot (\overline{v}, \overline{w}) = (\alpha \overline{v} A^{-1}, \alpha^{-1} A \overline{w})$. Also, it can be seen that in these terms the nilpotent cone $\mathcal{N}_{1,n}$ consists of matrices for which $\overline{v} \cdot \overline{w} = 0$. We denote by $U \subset \mathcal{N}_{1,n}$ the open $H_{1,n}$-orbit defined by the condition $\overline{v}, \overline{w} \neq 0$.

Lemma 4.1. *On $\mathcal{N}_{1,n}$ there is an $H_{1,n}$-invariant distribution, whose support is the whole space.*

Proof. Consider the compact projective space $\mathbb{P}^{n-2}(F)$ together with the $H_{1,n}$-action given by $(\alpha, A) \cdot [p] = [Ap]$ for any line $[p]$ represented by $p \in F^{n-1}$ and $(\alpha, A) \in F^\times \times G_{n-1}$. We define an $H_{1,n}$-equivariant map $i : U \to \mathbb{P}^1(F)$ by $i((\overline{v}, \overline{w})) = [\overline{w}]$.

Note, that the closure of $i^{-1}\left(\left[(0 \cdots 0\ 1)^t\right]\right)$ is

$$Y := \left\{ (\overline{v}', 0, \overline{0}, b) \,:\, \overline{v}' \in \mathbf{Mat}_{1,n-2}(F),\, b \in F \right\} \cong F^{n-2} \times F.$$

Note also, that

$$S := Stab\left(H_{1,n}, \left[(0 \cdots 0\ 1)^t\right] \in \mathbb{P}^{n-2}(F)\right) =$$

$$= F^\times \times \left\{ \begin{pmatrix} B & 0 \\ \gamma & d \end{pmatrix} \in G_{n-1}(F) \,:\, B \in G_{n-2}(F),\, d \in F^\times \right\}.$$

Since U is open and dense in $\mathcal{N}_{1,n}$, by Proposition 2.5, it is enough to exhibit a non-zero $S, \delta^{-1}_{H_{1,n}/S}$-invariant distribution supported on Y. The action of S on Y

is given by

$$\left(\alpha, \begin{pmatrix} B & 0 \\ \overline{\gamma} & d \end{pmatrix}\right) \cdot (\overline{v}', 0, \overline{0}, b) = (\alpha \overline{v}' B^{-1}, 0, \overline{0}, \alpha^{-1} d b).$$

Also, we have

$$\delta^{-1}_{H_{1,n}(F)/S}\left(\alpha, \begin{pmatrix} B & 0 \\ \overline{\gamma} & d \end{pmatrix}\right) = \delta_S^{-1}\left(\alpha, \begin{pmatrix} B & 0 \\ \overline{\gamma} & d \end{pmatrix}\right) = |\det(B) d^{2-n}|.$$

The last equality is easily derived after noting that S is a direct product of the unimodular $F^\times$, and of a parabolic subgroup of G_{n-1}.

Let $\chi(x) = |x|^{2-n}$ be a character on $F^\times$, and $0 \neq \mu \in \mathcal{D}(F^\times)^{F^\times,\chi}$ the corresponding distribution. It is known that such μ can be prolonged into a non-zero distribution $\widetilde{\mu} \in \mathcal{D}(F)^{F^\times,\chi}$ supported on all of F (see [6, Chapter 2, 2.3], or [15]). After identifying Y with $F^{n-2} \times F$, we can put the distribution $m \otimes \widetilde{\mu}$ on it, where m is the Haar measure on the vector space F^{n-2}. It is easy to see that the S-action transforms this distribution according to the formula for $\delta^{-1}_{H_{1,n}(F)/S}$. Thus, we have our desired distribution. ■

Let $\xi \in \mathcal{D}(\mathcal{N}_{1,n})^{H_{1,n}}$ be a distribution as provided by the lemma. Denote also by $\delta_0 \in \mathcal{D}(\mathcal{N}_{1,n})^{H_{1,n}}$ the distribution given by $\delta_0(f) = f(\overline{0}, \overline{0})$, for all $f \in \mathcal{S}(\mathcal{N}_{1,n})$.

Theorem 4.2. *The space of $P_{1,n}$-invariant distributions on $\mathcal{N}_{1,n}$ is spanned by δ_0 and ξ. In particular,*

$$\mathcal{D}(\mathcal{N}_{1,n})^{P_{1,n}} = \mathcal{D}(\mathcal{N}_{1,n})^{H_{1,n}}$$

Proof. Let $T \in \mathcal{D}(\mathcal{N}_{1,n})^{P_{1,n}}$ be given.

Note that $W = \{(\overline{v}', a, \overline{w}', b) \in U \ : \ b \neq 0\}$ is an open $P_{1,n}$-orbit in $\mathcal{N}_{1,n}$. Since an orbit can have at most one invariant distribution up to a constant and since $\xi|_W \neq 0$, there must be a constant c for which $T - c\xi|_W = 0$. Yet, $T - c\xi$ is still $P_{1,n}$-invariant, which means we can simply assume that T has its support inside $\mathcal{N}_{1,n} \setminus W$.

In this set, $O_1 = \{(\overline{0}, 0, \overline{w}', b) \ : \ b \neq 0\}$ is an open $P_{1,n}$-orbit lacking a non-zero invariant distribution. Indeed, for $s_1 = (\overline{0}, 0, \overline{0}, 1) \in O_1$, its stabilizer inside $P_{1,n}$ is the unimodular group

$$\{1\} \times \left\{ \begin{pmatrix} A & \\ & 1 \end{pmatrix} \ : \ A \in G_{n-2} \right\}.$$

It can be seen that $\delta_{P_{1,n}/Stab(P_{1,n},s_1)} \neq 1$. Thus, Supp $T \subset Y = \{(\overline{v}', a, \overline{w}', 0)\}$. Note, that we naturally have $Y \cong \mathcal{N}_{1,n-1} \times F$ as $P_{1,n}$-spaces, with the action on the right component given by $(\alpha, A) \cdot a = \alpha a$.

Under this decomposition we can write $T = \sum_i T_{i,1} \otimes T_{i,2}$, with $\{T_{i,1}\}$ being linearly independent distributions on $\mathcal{N}_{1,n-1}$. Since $\left(\alpha, \begin{pmatrix} \alpha I_{n-2} & \\ & 1 \end{pmatrix}\right) \in P_{1,n}$ acts trivially on the left component of the decomposition, $\{T_{i,2}\}$ are all actually

$F^{\times}$-invariant distributions on F, hence, by a well-known fact $\{T_{i,2}\}$ are supported on 0 (see, for example, [4, 0.7]). In other words, Supp $T \subset \{(\overline{v}', 0, \overline{w}', 0)\}$.

Now, there are only two $P_{1,n}$-orbits left that are fully contained inside the above set: The zero orbit, and $O_2 = \left\{(\overline{0}', 0, \overline{w}', 0) : \overline{w} \neq 0\right\}$. Thus, T can be viewed as a G_{n-2}-invariant distribution on the space $F^{n-2} \cong O_2 \cup \{0\}$, which means it must be supported on the zero vector, i.e. a multiple of δ_0. ■

A combination of the above with Theorem 3.9 immediately gives a result about distributions on a non-linear space.

Corollary 4.3. *The equality*

$$\mathcal{D}(X_{1,n})^{P_{1,n}} = \mathcal{D}(X_{1,n})^{H_{1,n}}$$

holds.

Again by Proposition 3.1 we see that Theorem 1.1 follows.

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