Rigidity of Bott–Samelson–Demazure–Hansen Variety for PSp(2n, ℂ)

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Abstract. Let $G = PSp(2n, \mathbb{C})$ ($n \geq 3$) and B be a Borel subgroup of G containing a maximal torus T of G. Let w be an element of the Weyl group W and $X(w)$ be the Schubert variety in the flag variety G/B corresponding to w. Let $Z(w, \underline{i})$ be the Bott–Samelson–Demazure–Hansen variety (the desingularization of $X(w)$) corresponding to a reduced expression $\underline{i}$ of w.

In this article, we study the cohomology groups of the tangent bundle on $Z(w_0, \underline{i})$, where w_0 is the longest element of the Weyl group W. We describe all the reduced expressions $\underline{i}$ of w_0 in terms of a Coxeter element such that all the higher cohomology groups of the tangent bundle on $Z(w_0, \underline{i})$ vanish (see Theorem 8.1).

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1. Introduction

Let G be a simple algebraic group of adjoint type over the field $\mathbb{C}$ of complex numbers. We fix a maximal torus T of G and let $W = N_G(T)/T$ denote the Weyl group of G with respect to T. We denote by R the set of roots of G with respect to T and by R^+ a set of positive roots. Let B^+ be the Borel subgroup of G containing T with respect to R^+. Let w_0 denotes the longest element of the Weyl group W. Let B be the Borel subgroup of G opposite to B^+ determined by T, i.e. $B = n_{w_0} B^+ n_{w_0}^{-1}$, where n_{w_0} is a representative of w_0 in $N_G(T)$. Note that the roots of B is the set $R^- := -R^+$ of negative roots. We use the notation $\beta < 0$ for $\beta \in R^-$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of all simple roots in R^+, where n is the rank of G. The simple reflection in the Weyl group corresponding to a simple root α is denoted by s_α. For simplicity of notation, the simple reflection corresponding to a simple root α_i is denoted by s_i.

For $w \in W$, let $X(w)(:= \overline{BwB/B})$ denote the Schubert variety in the flag

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variety G/B corresponding to w. Note that in general Schubert varieties are not smooth.

Given a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ of w, with the corresponding tuple $\underline{i} := (i_1, \ldots, i_r)$, we denote by $Z(w, \underline{i})$ the desingularization of the Schubert variety $X(w)$, which is now known as Bott–Samelson–Demazure–Hansen (for short BSDH) variety.

Demazure in [6] and Hansen in [8] independently constructed these desingularizations of Schubert varieties using the idea from [3]. In this paper we prove that two BSDH-varieties $Z(w, \underline{i})$ and $Z(w, \underline{j})$ are isomorphic if $\underline{i}$ and $\underline{j}$ differ only by commuting relations (see Theorem 3.1).

In [2], it is proved that all the higher cohomology groups $H^i(G/B, T_{G/B})$ for the tangent bundle $T_{G/B}$ on G/B vanish. In [16], it is proved that the higher cohomology groups of the restriction of $T_{G/B}$ to $X(w)$ vanish (see [16, Theorem 4.1 and Theorem 6.5]). In [5], we proved the following vanishing results of the tangent bundle $T_{Z(w,\underline{i})}$ on $Z(w, \underline{i})$ (see [5, Section 3]);

(1) $H^j(Z(w, \underline{i}), T_{Z(w,\underline{i})}) = 0$ for all $j \geq 2$.

(2) If G is simply laced, then $H^j(Z(w, \underline{i}), T_{Z(w,\underline{i})}) = 0$ for all $j \geq 1$.

As a consequence, it follows that the BSDH-varieties are rigid for simply laced groups and their deformations are unobstructed in general (see [5, Section 3]). The above vanishing result is independent of the choice of the reduced expression $\underline{i}$ of w. While computing the first cohomology group $H^1(Z(w, \underline{i}), T_{Z(w,\underline{i})})$ for non simply laced group, we observed that this cohomology group very much depend on the choice of a reduced expression $\underline{i}$ of w.

It is a natural question to ask that for which reduced expressions $\underline{i}$ of w, the cohomology group $H^1(Z(w, \underline{i}), T_{Z(w,\underline{i})})$ does vanish? In this article, we give a partial answer to this question for $w = w_0$ when $G = PSp(2n, \mathbb{C})$.

Recall that a Coxeter element is an element of the Weyl group having a reduced expression of the form $s_{i_1}s_{i_2}\cdots s_{i_n}$ such that $i_j \neq i_l$ whenever $j \neq l$ (see [10, p. 56, Section 4.4]). Note that for any Coxeter element c, there is a decreasing sequence of integers $n \geq a_1 > a_2 > \ldots > a_k = 1$ such that $c = \prod_{j=1}^{k}[a_j, a_{j-1} - 1]$, where $a_0 := n + 1$, $[i, j] := s_i s_{i+1}\cdots s_j$ for $i \leq j$.

Throughout this paper, for simplicity, we denote the product $w_1 \cdots w_k$ of elements in W by $\prod_{j=1}^{k} w_j$. For instance, we use the product $c = \prod_{j=1}^{k}[a_j, a_{j-1}-1]$ as above to denote $c = [a_1, a_0 - 1][a_2, a_1 - 1]\cdots[a_k, a_{k-1} - 1]$.

In this paper we prove the following theorem.

Theorem. *Let $G = PSp(2n, \mathbb{C})$ $(n \geq 3)$ and let $c \in W$ be a Coxeter element. Let $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^n)$ be a sequence corresponding to a reduced expression of w_0, where $\underline{i}^r$ $(1 \leq r \leq n)$ is a sequence of reduced expressions of c (see Lemma 4.4). Then, $H^j(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$ for all $j \geq 1$ if and only if $c = \prod_{j=1}^{k}[a_j, a_{j-1} - 1]$, where $a_0 := n + 1$ and $a_j \neq n - 1$ for every $j = 1, 2, \ldots, k$.*

By the above vanishing results, we conclude that if $G = PSp(2n, \mathbb{C})$ $(n \geq 3)$ and $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^n)$ is a reduced expression of w_0 as above, then the BSDH-variety $Z(w_0, \underline{i})$ is rigid.

The organization of the paper is as follows: In Section 2, we recall some

preliminaries on BSDH-varieties. In Section 3, we prove that two BSDH-varieties $Z(w, \underline{i})$ and $Z(w, \underline{j})$ are isomorphic if $\underline{i}$ and $\underline{j}$ differ only by commuting relations. We deal with the special case $G = PSp(2n, \mathbb{C})$ $(n \geq 3)$ in Sections 4, 5, 6, 7 and 8. In Section 4, we write down explicit reduced expressions of c^i $(1 \leq i \leq n)$ for each Coxeter element c. In Section 5 (respectively, Section 6) we compute the weight spaces of H^0 (respectively, H^1) of the relative tangent bundle of BSDH-varieties associated to some elements of the Weyl group. In Section 7, we prove some results on cohomology modules of tangent bundle of BSDH varieties. In Section 8, we prove the main result using the results from the previous sections.

2. Preliminaries

We refer to [4], [9], [11], [12], [15] and [18] for preliminaries in Algebraic geometry, Algebraic groups and Lie algebras. For a simple root $\alpha \in S$, we denote by P_α the minimal parabolic subgroup of G containing B and s_α. We recall that the BSDH-variety corresponds to a reduced expression $\underline{i}$ of $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is defined by

$$Z(w, \underline{i}) = \frac{P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}}{B \times \cdots \times B},$$

where the action of $B \times \cdots \times B$ on $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}$ is given by $(p_1, \ldots, p_r)(b_1, \ldots, b_r) = (p_1 \cdot b_1, b_1^{-1} \cdot p_2 \cdot b_2, \ldots, b_{r-1}^{-1} \cdot p_r \cdot b_r)$, $p_j \in P_{\alpha_{i_j}}$, $b_j \in B$ and $\underline{i} = (i_1, i_2, \ldots, i_r)$ (see [6, p.73, Definition 1], [4, p.64, Definition 2.2.1]).

We note that for each reduced expression $\underline{i}$ of w, $Z(w, \underline{i})$ is a smooth projective variety. We denote by ϕ_w, the natural birational surjective morphism from $Z(w, \underline{i})$ to $X(w)$.

Let $f_r : Z(w, \underline{i}) \longrightarrow Z(w s_{i_r}, \underline{i}')$ denote the map induced by the projection $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}} \longrightarrow P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_{r-1}}}$, where $\underline{i}' = (i_1, i_2, \ldots, i_{r-1})$. Then we observe that f_r is a $P_{\alpha_{i_r}}/B \simeq \mathbb{P}^1$-fibration.

For a B-module V, let $\mathcal{L}(w, V)$ denote the restriction of the associated homogeneous vector bundle on G/B to $X(w)$. By abuse of notation, we denote the pullback of $\mathcal{L}(w, V)$ via ϕ_w to $Z(w, \underline{i})$ also by $\mathcal{L}(w, V)$, when there is no cause for confusion. Since for any B-module V the vector bundle $\mathcal{L}(w, V)$ on $Z(w, \underline{i})$ is the pullback of the homogeneous vector bundle from $X(w)$, we conclude that the cohomology modules

$$H^j(Z(w, \underline{i}), \ \mathcal{L}(w, V)) \cong H^j(X(w), \ \mathcal{L}(w, V))$$

for all $j \geq 0$ (see [4, Theorem 3.3.4 (b)]) are independent of the choice of the reduced expression $\underline{i}$. Hence we denote $H^j(Z(w, \underline{i}), \ \mathcal{L}(w, V))$ by $H^j(w, V)$. In particular, if λ is a character of B, then we denote the cohomology modules $H^j(Z(w, \underline{i}), \ \mathcal{L}_\lambda)$ by $H^j(w, \lambda)$.

We recall the following short exact sequences of B-modules from [5], we call it *SES*.

1. $H^0(w, V) \simeq H^0(s_\gamma, H^0(s_\gamma w, V))$.

2. $0 \to H^1(s_\gamma, H^0(s_\gamma w, V)) \to H^1(w, V) \to H^0(s_\gamma, H^1(s_\gamma w, V)) \to 0$.

Let α be a simple root and $\lambda \in X(T)$ be such that $\langle \lambda, \alpha \rangle \geq 0$. Let $\mathbb{C}_\lambda$ denote the one-dimensional B-module associated to λ. Here, we recall the following result due to Demazure [7, Page 1] on a short exact sequence of B-modules:

Lemma 2.1. *Let α be a simple root and $\lambda \in X(T)$ be such that $\langle \lambda, \alpha \rangle \geq 0$. Let $ev : H^0(s_\alpha, \lambda) \longrightarrow \mathbb{C}_\lambda$ be the evaluation map. Then we have*

1. *If $\langle \lambda, \alpha \rangle = 0$, then $H^0(s_\alpha, \lambda) \simeq \mathbb{C}_\lambda$.*

2. *If $\langle \lambda, \alpha \rangle \geq 1$, then $\mathbb{C}_{s_\alpha(\lambda)} \hookrightarrow H^0(s_\alpha, \lambda)$ and there is a short exact sequence of B-modules:*

$$0 \longrightarrow H^0(s_\alpha, \lambda - \alpha) \longrightarrow H^0(s_\alpha, \lambda)/\mathbb{C}_{s_\alpha(\lambda)} \xrightarrow{ev} \mathbb{C}_\lambda \longrightarrow 0.$$

Furthermore, $H^0(s_\alpha, \lambda - \alpha) = 0$ when $\langle \lambda, \alpha \rangle = 1$.

3. *Let $n = \langle \lambda, \alpha \rangle$. As a B-module, $H^0(s_\alpha, \lambda)$ has a composition series*

$$0 \subsetneq V_n \subsetneq V_{n-1} \subsetneq \ldots \subsetneq V_0 = H^0(s_\alpha, \lambda)$$

such that $V_i/V_{i+1} \simeq \mathbb{C}_{\lambda - i\alpha}$ for $i = 0, 1, \ldots, n-1$ and $V_n = \mathbb{C}_{s_\alpha(\lambda)}$.

We define the dot action by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is the half sum of positive roots. As a consequence of the exact sequences of Lemma 2.1, we can prove the following.

Lemma 2.2. *Let $w = \tau s_\alpha$, $l(w) = l(\tau) + 1$. Then we have*

1. *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^j(\tau, H^0(s_\alpha, \lambda))$ for all $j \geq 0$.*

2. *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^{j+1}(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*

3. *If $\langle \lambda, \alpha \rangle \leq -2$, then $H^{j+1}(w, \lambda) = H^j(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*

4. *If $\langle \lambda, \alpha \rangle = -1$, then $H^j(w, \lambda)$ vanishes for every $j \geq 0$.*

The following consequence of Lemma 2.2 will be used to compute cohomology modules in this paper. Now onwards we will denote the Levi subgroup of P_α ($\alpha \in S$) containing T by L_α and the subgroup $L_\alpha \cap B$ by B_α. Let $\pi : \widetilde{G} \longrightarrow G$ be the universal cover. Let $\widetilde{L}_\alpha$ (respectively, $\widetilde{B}_\alpha$) be the inverse image of L_α (respectively, of B_α).

Lemma 2.3. *Let V be an irreducible L_α-module. Let λ be a character of B_α. Then, we have*

1. *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda)$ is isomorphic as an L_α-module to the tensor product of V and $H^0(L_\alpha/B_\alpha, \mathbb{C}_\lambda)$, and $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ for every $j \geq 1$.*

2. *If $\langle \lambda, \alpha \rangle \leq -2$, $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$, and $H^1(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda)$ is isomorphic to the tensor product of V and $H^0(L_\alpha/B_\alpha, \mathbb{C}_{s_\alpha \cdot \lambda})$.*

3. *If* $\langle \lambda, \alpha \rangle = -1$, *then* $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ *for every* $j \geq 0$.

Recall the structure of indecomposable B_α-modules (respectively, $\widetilde{B}_\alpha$-modules) (see [1, p.130, Corollary 9.1]).

Lemma 2.4.

1. *Any finite-dimensional indecomposable* $\widetilde{B}_\alpha$-*module* V *is isomorphic to* $V' \otimes \mathbb{C}_\lambda$ *for some irreducible representation* V' *of* $\widetilde{L}_\alpha$ *and for some character* λ *of* $\widetilde{B}_\alpha$.

2. *Any finite-dimensional indecomposable* B_α-*module* V *is isomorphic to* $V' \otimes \mathbb{C}_\lambda$ *for some irreducible representation* V' *of* $\widetilde{L}_\alpha$ *and for some character* λ *of* $\widetilde{B}_\alpha$.

Recall the following result from [16] (see [16, Corollary 5.6]).

Corollary 2.5.   *Let* α *be a short root such that* $-\alpha \notin S$, *let* $w \in W$. *Then we have,* $H^i(w, \alpha) = 0$ *for* $i \geq 1$.

3.   Reduced expressions differing only by commuting relations

Let $w \in W$ and let $\underline{i} := (i_1, \ldots, i_r)$ be the tuple corresponding to a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ of w. Note that if $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle = 0$ for some $1 \leq k \leq r-1$, then $(i_1, \ldots, i_{k-1}, i_{k+1}, i_k, i_{k+2}, \ldots, i_r)$ is also a tuple corresponding to a reduced expression of w.

Two reduced expressions $\underline{i}$ and $\underline{j}$ of w are said to differ only by commuting relations if $\underline{j}$ is obtained from $\underline{i}$ by a sequence of process as above.

In this section we prove the following:

Theorem 3.1.   *Let* $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ *and* $w = s_{j_1} s_{j_2} \cdots s_{j_r}$ *be two reduced expressions* $\underline{i}$ *and* $\underline{j}$ *for* w *which differ only by commuting relations. Then, there is a* B-*equivariant isomorphism from* $Z(w, \underline{i})$ *onto* $Z(w, \underline{j})$.

Proof.   By the recursion, without loss of generality, we may assume that there is a $1 \leq k \leq r-1$ such that $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle = 0$ and satisfying

$$i_l = j_l \text{ for } l \neq k, \ k+1; \ i_k = j_{k+1}, \ i_{k+1} = j_k.$$

Let P be the parabolic subgroup of G corresponding to the subset $\{\alpha_{i_k}, \alpha_{i_{k+1}}\}$ of S. Since the simple roots α_{i_k} and $\alpha_{i_{k+1}}$ are orthogonal, the product map $P_{\alpha_{i_k}} \times P_{\alpha_{i_{k+1}}} \to P$ is surjective.

Let

$$X := (P_{\alpha_{i_1}} \times \cdots \times P_{\alpha_{i_{k-1}}} \times P \times P_{\alpha_{i_{k+2}}} \times \cdots \times P_{\alpha_{i_r}})/B^{r-1}$$

be the quotient variety where the action of B^{r-1} on

$$P_{\alpha_{i_1}} \times \cdots \times P_{\alpha_{i_{k-1}}} \times P \times P_{\alpha_{i_{k+2}}} \times \cdots \times P_{\alpha_{i_r}}$$

is given by

$$(p_1, \ldots, p_{k-1}, p, p_{k+2}, \ldots, p_r) \cdot (b_1, b_2, \ldots, b_{k-1}, b_{k+1}, \ldots, b_r) =$$
$$= (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{k-1}^{-1} p b_{k+1}, b_{k+1}^{-1} p_{k+2} b_{k+2}, \ldots, b_{r-1}^{-1} p_r b_r),$$

$p_j \in P_{\alpha_{i_j}}$, $p \in P$ and $b_j \in B$.

Note that X is a smooth projective variety. Now consider the map

$$P_{\alpha_{i_1}} \times \cdots \times P_{\alpha_{i_{k-1}}} \times P_{\alpha_{i_k}} \times P_{\alpha_{i_{k+1}}} \times \cdots \times P_{\alpha_{i_r}} \to P_{\alpha_{i_1}} \times \cdots \times P_{\alpha_{i_{k-1}}} \times P \times P_{\alpha_{i_{k+2}}} \times \cdots \times P_{\alpha_{i_r}}$$

given by

$$(p_1, \ldots, p_{k-1}, p_k, p_{k+1}, p_{k+2}, \ldots, p_r) \mapsto (p_1, \ldots, p_{k-1}, p_k p_{k+1}, p_{k+2}, \ldots, p_r).$$

This induces a birational surjective B-equivariant morphism $f : Z(w, \underline{i}) \to X$.

Claim: f is injective.

We denote by $[(p_1, p_2, \ldots, p_k p_{k+1}, p_{k+2}, \ldots, p_r)]$ be the point in X corresponding to

$$(p_1, p_2, \ldots, p_{k-1}, p_k p_{k+1}, p_{k+2}, \ldots, p_r) \in P_{\alpha_{i_1}} \times \cdots \times P_{\alpha_{i_{k-1}}} \times P \times P_{\alpha_{i_{k+2}}} \times \cdots \times P_{\alpha_{i_r}}.$$

If $[(p_1, p_2, \ldots, p_{k-1}, p_k p_{k+1}, \ldots, p_r)] = [(p_1', p_2', \ldots, p_{k-1}', p_k' p_{k+1}', \ldots, p_r')]$, then there exists $(b_1, b_2, \ldots, b_{r-2}, b_r)$ such that $p_1 = p_1' b_1$, $p_l = b_{l-1}^{-1} p_l' b_l$ for all $l \neq k, k+1$ and $p_k p_{k+1} = b_{k-1}^{-1} p_k' p_{k+1}' b_{k+1}$. Hence we have

$$(b_{k-1}^{-1} p_k')^{-1} p_k = p_{k+1}' b_{k+1} p_{k+1}^{-1} \ (\text{say} = b_k) \in P_{\alpha_{i_k}} \cap P_{\alpha_{i_{k+1}}} = B.$$

Therefore $p_k = b_{k-1}^{-1} p_k' b_k$ and $p_{k+1} = b_k^{-1} p_{k+1}' b_{k+1}$. Thus, we have that $(p_1, \ldots, p_{k-1}, p_k, p_{k+1}, p_{k+2}, \ldots, p_r)$ and $(p_1', \ldots, p_{k-1}', p_k', p_{k+1}', p_{k+2}', \ldots, p_r')$ represents the same element in $Z(w, \underline{i})$. Hence f is injective.

Since X is normal and f is bijective birational, by Zariski main theorem [18, p. 85, Theorem 5.2.8], we conclude that f is an isomorphism. Similarly, we see that there is a B-equivariant isomorphism from $Z(w, \underline{j})$ onto X. Thus, there is a B-equivariant isomorphism from $Z(w, \underline{i})$ onto $Z(w, \underline{j})$. ∎

Corollary 3.2.   *Let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ and $w = s_{j_1} s_{j_2} \cdots s_{j_r}$ be two reduced expressions $\underline{i}$ and $\underline{j}$ for w which differ only by commuting relations. Then, we have*

1. *The automorphism groups $Aut^0(Z(w, \underline{i}))$ and $Aut^0(Z(w, \underline{j}))$ are isomorphic.*

2. *The first cohomology groups $H^1(Z(w, \underline{i}), T_{(w, \underline{i})})$ and $H^1(Z(w, \underline{j}), T_{(w, \underline{j})})$ are isomorphic.*

Proof.   The proof follows from Proposition 3.1. ∎

Recall from [19] and [20] the definition of a fully commutative element of a Coxeter group. An element $w \in W$ is said to be fully commutative if it has the property that any reduced expression for w can be obtained from any other by using only the Coxeter relations that involve commuting generators. These elements are characterized in [19] and [20].

Then we have

Corollary 3.3. *Let w be a fully commutative element in W. Then, $Z(w, \underline{i})$ is independent of the choice of the reduced expression $\underline{i}$ of w.*

4. Reduced expressions of some elements of W in type C_n

Now onwards we will assume that $G = PSp(2n, \mathbb{C})$ $(n \geq 3)$. First note that if $n \geq 3$, then the highest short root β_0 is ω_2 and $w_0 = -id$. We recall the following proposition from [21] (see [21, Proposition 1.3]). We use the notation as in [21].

Proposition 4.1. *Let $c \in W$ be a Coxeter element, let ω_i be a fundamental weight corresponding to the simple root α_i. Then, there exists a least positive integer $h(i, c)$ such that $c^{h(i,c)}(\omega_i) = w_0(\omega_i)$.*

Now we can deduce the following:

Lemma 4.2. *Let $c \in W$ be a Coxeter element. Then, we have*

1. *$w_0 = c^n$.*

2. *For any sequence $\underline{i}^r$ $(1 \leq r \leq n)$ of reduced expressions of c; the sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^n)$ is a reduced expression of w_0.*

Proof. Proof of (1): Let $\eta : S \longrightarrow S$ be the involution of S defined by $i \longmapsto i^*$, where i^* is given by $\omega_{i^*} = -w_0(\omega_i)$. Since G is of type C_n, $w_0 = -id$ and hence $\omega_{i^*} = \omega_i$ for every i. Therefore, we have $i = i^*$ for every i. Let h be the Coxeter number. By [21, Proposition 1.7], we have $h(i, c) + h(i^*, c) = h$. Since $h = \frac{2|R^+|}{n}$ (see [13, Proposition 3.18]) and $i = i^*$, we have $h(i, c) = n$.

By Proposition 4.1, we have $c^n(\omega_i) = -\omega_i$ for all $1 \leq i \leq n$. Since $\{\omega_i : 1 \leq i \leq n\}$ forms a $\mathbb{R}$-basis of $X(T) \otimes \mathbb{R}$, it follows that $c^n = -id$. Hence, we have $w_0 = c^n$.

The assertion (2) follows from the fact that $l(c) = n$ and $l(w_0) = |R^+| = n^2$ (see [11, p.66, Table 1]). ∎

We note that there are reduced expressions of w_0 in C_n which are neither of the form as in Lemma 4.2(2) nor differing from such a form by commuting relations. For example, we can take $n = 3$. Then, the reduced expression $w_0 = s_2 s_3 s_2 s_3 s_1 s_2 s_3 s_2 s_1$ is one such.

Lemma 4.3. *Let $n \geq a_1 > a_2 > a_3 > \ldots > a_r \geq 1$ be a decreasing sequence of integers. Then,*

$$w = \left(\prod_{j=a_1}^{n} s_j \right) \left(\prod_{j=a_2}^{n} s_j \right) \left(\prod_{j=a_3}^{n} s_j \right) \cdots \left(\prod_{j=a_{r-1}}^{n} s_j \right) \left(\prod_{j=a_r}^{n-1} s_j \right)$$

is a reduced expression of w.

Proof.   If $r = 1$, then $w = \prod_{j=a_1}^{n-1} s_j$ and clearly it is a reduced expression of w. Let

$$w_1 = \left(\prod_{j=a_1}^{n} s_j\right)\left(\prod_{j=a_2}^{n} s_j\right) \cdots \left(\prod_{j=a_{r-2}}^{n} s_j\right)\left(\prod_{j=a_{r-1}}^{n-1} s_j\right) \quad \text{and} \quad w_2 = s_n\left(\prod_{j=a_r}^{n-1} s_j\right).$$

Then, we have $w = w_1 w_2$. Now, we will prove that the expression

$$\left(\prod_{j=a_1}^{n} s_j\right)\left(\prod_{j=a_2}^{n} s_j\right)\left(\prod_{j=a_3}^{n} s_j\right) \cdots \left(\prod_{j=a_{r-1}}^{n} s_j\right)\left(\prod_{j=a_r}^{n-1} s_j\right)$$

of $w_1 w_2$ is reduced.

By induction on r,

$$w_1 = \left(\prod_{j=a_1}^{n} s_j\right)\left(\prod_{j=a_2}^{n} s_j\right) \cdots \left(\prod_{j=a_{r-1}}^{n-1} s_j\right)$$

is a reduced expression. Note that, since G is of type C_n,

$$R^+(w_2^{-1}) = \left\{\alpha_n, \sum_{j=a_r}^{n} \alpha_j\right\} \cup \left\{\sum_{j=a_r}^{m} \alpha_j : a_r \leq m \leq n-2\right\}.$$

Since $n \geq a_1 > a_2 > a_3 > \ldots > a_r \geq 1$, it follows that

$$R^+(w_1) \cap R^+(w_2^{-1}) = \emptyset.$$

Thus we have $l(w) = l(w_1) + l(w_2)$. Hence

$$w = \left(\prod_{j=a_1}^{n} s_j\right)\left(\prod_{j=a_2}^{n} s_j\right)\left(\prod_{j=a_3}^{n} s_j\right) \cdots \left(\prod_{j=a_{r-1}}^{n} s_j\right)\left(\prod_{j=a_r}^{n-1} s_j\right)$$

is a reduced expression of w. This completes the proof of the lemma. ∎

Let c be a Coxeter element in W. We take a reduced expression $c = [a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$, where $[i, j] = s_i s_{i+1} \cdots s_j$ for $i \leq j$ and $n \geq a_1 > a_2 > \ldots > a_k = 1$.

Then we have the following.

Lemma 4.4.

1. *For all $1 \leq i \leq k - 1$,*

$$c^i = \left(\prod_{l_1=1}^{i} [a_{l_1}, n]\right)\left(\prod_{l_2=i+1}^{k} [a_{l_2}, a_{l_2-i} - 1]\right)\left(\prod_{l_3=1}^{i-1} [a_k, a_{k-i+l_3} - 1]\right).$$

2. *For all $k \leq j \leq n$,*

$$c^j = \left(\prod_{l_1=1}^{k-1} [a_{l_1}, n] \right) \left([a_k, n]^{j+1-k} \right) \left(\prod_{l_2=1}^{k-1} [a_k, a_{l_2} - 1] \right).$$

3. *The expressions of c^i for $1 \leq i \leq n$ as in (1) and (2) are reduced.*

Proof. Proof of (1) is by induction on i and by commuting relation in the Weyl group. Assume that for a fixed $1 \leq i \leq k - 2$,

$$c^i = \left(\prod_{l_1=1}^{i} [a_{l_1}, n] \right) \left(\prod_{l_2=i+1}^{k} [a_{l_2}, a_{l_2-i} - 1] \right) \left(\prod_{l_3=1}^{i-1} [a_k, a_{k-i+l_3} - 1] \right).$$

Now we will prove,

$$c^{i+1} = \left(\prod_{l_1=1}^{i+1} [a_{l_1}, n] \right) \left(\prod_{l_2=i+2}^{k} [a_{l_2}, a_{l_2-(i+1)} - 1] \right) \left(\prod_{l_3=1}^{i} [a_k, a_{k-(i+1)+l_3} - 1] \right).$$

Let $w_1 := \prod_{l_1=1}^{i} [a_{l_1}, n]$, $w_2 := \prod_{l_2=i+1}^{k} [a_{l_2}, a_{l_2-i} - 1]$ and $w_3 := \prod_{l_3=1}^{i-1} [a_k, a_{k-i+l_3} - 1]$. Let $w_1' := \prod_{l_1=1}^{i+1} [a_{l_1}, n]$, $w_2' := \prod_{l_2=i+2}^{k} [a_{l_2}, a_{l_2-(i+1)} - 1]$ and $w_3' := \prod_{l_3=1}^{i} [a_k, a_{k-(i+1)+l_3} - 1]$.

Let $w_{2,1} := [a_{i+1}, a_1 - 1]$, and let $v_1 := [a_1, n]$. Therefore, we have $w_1' = w_1 w_{2,1} v_1$. Further, let $w_{2,j} := [a_{i+j}, a_j - 1]$ $(2 \leq j \leq k - i)$. Also, let $w_{3,j} := [a_k, a_j - 1]$ $(2 \leq j \leq k - i)$, and $v_j := [a_j, a_{j-1} - 1]$ $(2 \leq j \leq k)$.

Now look at $c^i \cdot c = w_1 w_2 w_3 v_1 \cdots v_k$. It is easy to derive the following commuting relations:

- v_1 commutes with each $w_{2,j}$ and $w_{3,j}$ $(2 \leq j \leq k - i)$.

- For each $2 \leq j \leq k - i$, v_j commutes with $w_{2,l}$ for each $j + 1 \leq l \leq k - i$ and $w_{3,l}$ for each $(k + 1 - i \leq l \leq k - 1)$.

Using these relations, we see that

$$c^i c = (w_1 w_{2,1} v_1) \left(\prod_{j=2}^{k-i} w_{2,j} v_j \right) \left(\prod_{j=k+1-i}^{k-1} w_{3,j} v_j \right) v_k.$$

Now, the statement (1) can be obtained from the following facts:

- $w_1' = w_1 w_{2,1} v_1$,

- $w_{2,j} v_j = [a_{i+j}, a_{j-1} - 1]$ for each $2 \leq j \leq k - i$ and

- $w_{3,j} v_j = [a_k, a_{j-1} - 1]$ for each $k + 1 - i \leq j \leq k - 1$.

Proof of (2) is similar to the proof of (1). Proof of (3) follows from the fact that $l(c^i) = in$ for all $1 \leq i \leq n$ and Lemma 4.3. ∎

Example 4.5.   Let G is of type C_3. In this case we have the following four Coxeter elements;

- $c_1 = s_1 s_2 s_3$; with $k = 1; a_1 = 1$.

- $c_2 = s_3 s_1 s_2$; with $k = 2; a_1 = 3, a_2 = 1$.

- $c_3 = s_2 s_3 s_1$; with $k = 2; a_1 = 2, a_2 = 1$.

- $c_4 = s_3 s_2 s_1$; with $k = 3; a_1 = 3, a_2 = 2, a_3 = 1$.

Note that $h(i, c_j) = 3$ for all $1 \leq i \leq 3$ and $1 \leq j \leq 4$.

The reduced expressions of c_1^j $(1 \leq j \leq 3)$ appearing in Lemma 4.4 are:

- $c_1 = s_1 s_2 s_3$.

- $c_1^2 = (s_1 s_2 s_3)(s_1 s_2 s_3)$.

- $c_1^3 = (s_1 s_2 s_3)(s_1 s_2 s_3)(s_1 s_2 s_3)$.

The reduced expressions of c_2^j $(1 \leq j \leq 3)$ appearing in Lemma 4.4 are:

- $c_2 = s_3 s_1 s_2$

- $c_2^2 = s_3(s_1 s_2)(s_3 s_1 s_2)$

- $c_2^3 = s_3(s_1 s_2 s_3)(s_1 s_2 s_3)(s_1 s_2)$

5.   Cohomology module H^0 of the relative tangent bundle

In this section we describe the weights of H^0 of the relative tangent bundle.

For a B-module V and a character $\mu \in X(T)$, we denote by V_μ the space of all vectors v in V such that $t \cdot v = \mu(t)v$ for all $t \in T$. Let R_s (respectively, R_s^-) be the set of all short roots (respectively, negative short roots).

Lemma 5.1.   Let $w \in W$ and let V be a B-module such that $V_\mu = 0$ unless $\mu \in R_s^- \setminus (-S)$. Then $H^0(w, V)_\mu = 0$ unless $\mu \in R_s^- \setminus (-S)$.

Proof.   Fix $\alpha \in S$ and let $\lambda_0 = -\alpha$. Then by [5, Lemma 4.1 (1)], $H^0(w, V)_\mu = 0$ unless $\mu \in R_s^-$ and $\mu < -\alpha$. Since $\alpha \in S$ is arbitrary, we have $H^0(w, V)_\mu = 0$ unless $\mu \in R_s^- \setminus (-S)$. ∎

Lemma 5.2.   Fix $1 \leq j \leq n - 2$.

1. $H^0(s_j s_{j+1} \cdots s_{n-1}, \alpha_{n-1})_\mu \neq 0$ if and only if either $\mu = 0$, or $\mu = s_t s_{t+1} \cdots s_{n-1}(\alpha_{n-1})$ for some $j \leq t \leq n - 1$.

2. $H^0(s_n s_j s_{j+1} \cdots s_{n-1}, \alpha_{n-1})_\mu \neq 0$ if and only if either $\mu = 0$, or $\mu = v s_t s_{t+1} \cdots s_{n-1}(\alpha_{n-1})$ for some $j \leq t \leq n - 1$ and $v \in \{id, s_n\}$.

3. *Let a_1, a_2 be two integers such that $n \geq a_1 > a_2 \geq 1$. Then, $H^0(s_{a_1} \cdots s_n s_{a_2} \cdots s_{n-1}, \alpha_{n-1})_\mu \neq 0$ if and only if $\mu = s_t \cdots s_{n-1} s_n s_j \cdots s_{n-1}(\alpha_{n-1})$ for some $a_2 \leq j \leq n-1$ and $a_1 \leq t \leq n$ such that $t > j$.*

Proof.   Proof of (1): If $j = n - 2$, then by SES we have

$$H^0(s_{n-2} s_{n-1}, \alpha_{n-1}) = H^0(s_{n-2}, H^0(s_{n-1}, \alpha_{n-1})).$$

Note that

$$H^0(s_{n-1}, \alpha_{n-1}) = sl_{2, \alpha_{n-1}} = \mathbb{C}_{\alpha_{n-1}} \oplus \mathbb{C}h_{\alpha_{n-1}} \oplus \mathbb{C}_{-\alpha_{n-1}},$$

where $h_{\alpha_{n-1}}$ is a zero weight vector in $sl_{2, \alpha_{n-1}}$. Since $\langle \alpha_{n-1}, \alpha_{n-2} \rangle = -1$, by Lemma 2.2, we see that

$$H^0(s_{n-2} s_{n-1}, \alpha_{n-1}) = \mathbb{C}h_{\alpha_{n-1}} \oplus \mathbb{C}_{-\alpha_{n-1}} \oplus \mathbb{C}_{-(\alpha_{n-1} + \alpha_{n-2})}.$$

We prove the result by recursion. Now assume that the statement holds for $H^0(s_{l+1} \cdots s_{n-1}, \alpha_{n-1})$ and $j \leq l \leq n-3$. Then, for any $\mu \in X(T)$ such that $H^0(s_{l+1} \cdots s_{n-1}, \alpha_{n-1})_\mu \neq 0$, we have $\langle \mu, \alpha_j \rangle \in \{0, 1\}$. Thus, by Lemma 2.1 and by using SES, we see that $H^0(s_l s_{l+1} \cdots s_{n-1}, \alpha_{n-1})_\mu \neq 0$ if and only if either $\mu = 0$, or $\mu = s_t s_{t+1} \cdots s_{n-1}(\alpha_{n-1})$ for some $l \leq t \leq n-1$. Therefore, proof of (1) follows by recursion.

Proof of (2): By SES, we have

$$H^0(s_n s_j s_{j+1} \cdots s_{n-1}, \alpha_{n-1}) = H^0(s_n, H^0(s_j s_{j+1} \cdots s_{n-1}, \alpha_{n-1})).$$

Note that $\langle s_t \cdots s_{n-1}(\alpha_{n-1}), \alpha_n \rangle = 1$ for all $j \leq t \leq n-1$. Now the proof of (2) follows from Lemma 2.1 and (1).

Proof of (3) is similar to the proof of (1) and (2).   ∎

Now onwards we fix the following notation:

Let c be a Coxeter element in W. We take a reduced expression $c = [a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$, where $[i, j] = s_i s_{i+1} \cdots s_j$ for $i \leq j$ and $n \geq a_1 > a_2 > \ldots > a_k = 1$.

Fix $1 \leq r \leq k$. Let $n \geq a_1 > a_2 > a_3 > \ldots > a_r \geq 1$ be a decreasing sequence of integers.

Let

$$w_r = \left(\prod_{j=a_1}^{n} s_j \right) \left(\prod_{j=a_2}^{n} s_j \right) \left(\prod_{j=a_3}^{n} s_j \right) \cdots \left(\prod_{j=a_{r-1}}^{n} s_j \right) \left(\prod_{j=a_r}^{n} s_j \right)$$

and let

$$\tau_r = \left(\prod_{j=a_1}^{n} s_j \right) \left(\prod_{j=a_2}^{n} s_j \right) \left(\prod_{j=a_3}^{n} s_j \right) \cdots \left(\prod_{j=a_{r-1}}^{n} s_j \right) \left(\prod_{j=a_r}^{n-1} s_j \right).$$

Note that $l(w_r) = l(\tau_r) + 1$ and $c\tau_k = w_k[a_k, n-1]$.

The following lemma describes the weights of H^0 of the relative tangent bundles.

Lemma 5.3.　　*Let $3 \leq r \leq k$. Then, we have*

1. $H^0(\tau_r, \alpha_{n-1})_\mu \neq 0$ *if and only if* $\mu = (\prod_{j=t_1}^{n} s_j)(\prod_{j=t_2}^{n-1} s_j)(\alpha_{n-1})$ *for some* $a_r \leq t_2 < t_1$ *and* $a_{r-1} \leq t_1 \leq a_{r-2} - 1$. *In such a case,* $\dim(H^0(\tau_r, \alpha_{n-1})_\mu) = 1$.

2. $H^0(\tau_r, \alpha_{n-1})$ *is a cyclic B-module generated by a weight vector of weight* $\mu = (\prod_{j=a_{r-2}-1}^{n} s_j)(\prod_{j=a_{r-2}-2}^{n} s_j)(\alpha_{n-1})$.

Proof.　　Note that there exists a $w_1 \in W$ such that $w_1(\alpha_{n-1}) = \beta_0$ and $l(\tau_r w_1^{-1}) = l(\tau_r) + l(w_1)$, where β_0 is the highest short root. As a consequence, we have $H^0(\tau_r, \alpha_{n-1}) \subset H^0(\tau_r w_1^{-1}, \beta_0)$. Hence it follows that for every non zero weight μ of $H^0(\tau_r, \alpha_{n-1})$, we have $\dim(H^0(\tau_r, \alpha_{n-1})_\mu) = 1$ (see the proof of [5, Lemma 4.4]).

　　　Proof of (1): Let $u_1 = s_{a_r} s_{a_r+1} \cdots s_{n-1}$, $u_2 = s_{a_{r-1}} \cdots s_n$ and $u_3 = s_{a_{r-2}} \cdots s_n$. Let $w' = u_2 u_1 w_1^{-1}$.

　　　<u>Step 1</u>: We prove $H^0(u_2 u_1, \alpha_{n-1})_\mu \neq 0$ if and only if $\mu \in R_s^-$ and $\mu = v_1(\alpha_{n-1})$, where $s_n s_{n-1} \leq v_1 \leq u_2 u_1$.

　　　<u>Claim:</u> $H^0(u_2 u_1, \alpha_{n-1})$ is the B-submodule of $H^0(w', \beta_0)$ generated by $\mathbb{C}_{s_n s_{n-1}(\alpha_{n-1})}$.

　　　Consider the following commutative diagram of B-modules:

$$H^0(s_{n-1}s_n u_1, \alpha_{n-1}) \xrightarrow{\quad ev_1 \quad} H^0(s_n u_1, \alpha_{n-1})$$

with ev_3 and ev_2 mapping to

$$H^0(u_1, \alpha_{n-1})$$

By Lemma 5.2(1), if $H^0(u_1, \alpha_{n-1})_\mu \neq 0$, then $\mu = 0$ or $\mu = s_j s_{j+1} \cdots s_{n-1}(\alpha_{n-1})$ for some $a_r \leq j \leq n-1$.

　　　First, observe that $\mathbb{C}h_{\alpha_{n-1}} \oplus \mathbb{C}_{-\alpha_{n-1}}$ is an indecomposable $B_{\alpha_{n-1}}$-module (see [5, p. 11] and [16, p. 8]). By Lemma 2.4, we have

$$\mathbb{C}h_{\alpha_{n-1}} \oplus \mathbb{C}_{-\alpha_{n-1}} = V \otimes \mathbb{C}_{-\omega_{n-1}},$$

where V is the 2-dimensional irreducible representation of $\widetilde{L}_{\alpha_{n-1}}$.

　　　Note that $\mathbb{C}h_{\alpha_{n-1}} \oplus \mathbb{C}_{-\alpha_{n-1}}$ is an indecomposable $B_{\alpha_{n-1}}$-summand of $H^0(s_n u_1, \alpha_{n-1})$. By Lemma 2.3, we have

$$H^0(s_{n-1}, \mathbb{C}h_{\alpha_{n-1}} \oplus \mathbb{C}_{-\alpha_{n-1}}) = H^0(s_{n-1}, V \otimes \mathbb{C}_{-\omega_{n-1}}) = V \otimes H^0(s_{n-1}, \mathbb{C}_{-\omega_{n-1}}) = 0.$$

Also, note that for each μ such that $H^0(u_1, \alpha_{n-1})_\mu \neq 0$ with $\mu \notin \{0, -\alpha_{n-1}\}$, $\mathbb{C}_\mu$ is an indecomposable $B_{\alpha_{n-1}}$-summand of $H^0(s_n u_1, \alpha_{n-1})$. Since $\langle s_t \ldots s_{n-1}(\alpha_{n-1}), \alpha_{n-1} \rangle = -1$ for all $j \leq t \leq n-2$, using Lemma 5.2(1), we see that the evaluation map ev_3 is zero.

　　　Now consider the following commutative diagram of B-modules.

$$H^0(u_2 u_1, \alpha_{n-1}) \xrightarrow{\quad ev' \quad} H^0(s_{n-1} s_n u_1, \alpha_{n-1})$$

with ev and ev_3 mapping to

$$H^0(u_1, \alpha_{n-1})$$

By the above arguments and commutativity of the above diagram, we see that the evaluation map ev is zero. Note that $\langle s_n s_{n-1}(\alpha_{n-1}), \alpha_{n-1} \rangle = 0$ and $\langle s_n s_{n-1}(\alpha_{n-1}), \alpha_j \rangle \geq 0$ for $1 \leq j \leq n-2$. Hence $\mathbb{C}_{s_n s_{n-1}(\alpha_{n-1})}$ is an indecomposable $B_{\alpha_{n-1}}$-summand of $H^0(s_n u_1, \alpha_{n-1})$. Therefore, $\mathbb{C}_{s_n s_{n-1}(\alpha_{n-1})}$ is in the image of the evaluation map

$$ev' : H^0(u_2 u_1, \alpha_{n-1}) \longrightarrow H^0(s_{n-1} s_n u_1, \alpha_{n-1}).$$

Hence, $s_n s_{n-1}(\alpha_{n-1})$ is the highest weight of $H^0(u_2 u_1, \alpha_{n-1})$. Since β_0 is dominant weight, the restriction map of B-modules

$$H^0(G/B, \mathcal{L}(\beta_0)) = H^0(w_0, \beta_0) \longrightarrow H^0(w', \beta_0)$$

is surjecive.

Since $H^0(u_2 u_1, \alpha_{n-1})$ is B-submodule of $H^0(w', \beta_0)$, the multiplicity of $s_n s_{n-1}(\alpha_{n-1})$ in $H^0(u_2 u_1, \alpha_{n-1})$ is one. Also, note that the lowest weight of $H^0(u_2 u_1, \alpha_{n-1})$ is $u_2 u_1(\alpha_{n-1})$. Let $\{e_\beta : \beta \in R\} \cup \{h_\alpha : \alpha \in S\}$ be the Chevalley basis for $\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G (refer to [11, Chapter VII])). Hence we conclude that

$$H^0(u_2 u_1, \alpha_{n-1}) = \bigoplus_{a_r \leq t_2 < t_1 \leq n-1} \mathbb{C} x_{t_1, t_2},$$

where $x_{t_1, t_2} = e_{-\alpha_{t_1}} \cdots e_{-\alpha_{n-1}} e_{-\alpha_{t_2}} \cdots e_{-\alpha_{n-2}}(y)$ for some non zero vector $y \in \mathbb{C}_{s_n s_{n-1}(\alpha_{n-1})}$.

Therefore, $H^0(u_2 u_1, \alpha_{n-1})$ is the B-submodule of $H^0(w', \beta_0)$ generated by $\mathbb{C}_{s_n s_{n-1}(\alpha_{n-1})}$. This proves the claim.

Now Step 1 follows from the claim.

Step 2: Fix $a_{r-2} \leq j \leq n$. $H^0(s_j \cdots s_n u_2 u_1, \alpha_{n-1})_\mu \neq 0$ if and only if $\mu \in R_s^-$ and $\mu = v_1(\alpha_{n-1})$, where $s_{n-1} s_n s_{n-2} s_{n-1} \leq v_1 \leq u_2 u_1$ and $\langle \mu, \alpha_t \rangle = 0$ for all $j \leq t \leq n$.

Let $\mu_j = s_j \cdots s_{n-1}(\alpha_{n-1}), a_r \leq j \leq n-1$. Since ev_3 is zero, $H^0(u_2 u_1, \alpha_{n-1})_{\mu_j} = 0$. Further, we have $H^0(u_2 u_1, \alpha_{n-1})_{s_n(\mu_j)} = \mathbb{C}_{s_n(\mu_j)}$ (by Step 1). Since $\langle s_n(\mu_j), \alpha_n \rangle = -1$, by Lemma 2.2, $\mathbb{C}_{s_n(\mu_j)}$ is not in the image of the evaluation map

$$ev : H^0(s_n u_2 u_1, \alpha_{n-1}) \to H^0(u_2 u_1, \alpha_{n-1}).$$

In particular, we have

$$H^0(s_n u_2 u_1, \alpha_{n-1})_{\mu_j} = H^0(s_n u_2 u_1, \alpha_{n-1})_{s_n(\mu_j)} = 0.$$

By recursion assume that $H^0(s_{j+1} \cdots s_n u_2 u_1, \alpha_{n-1})_\mu \neq 0$ if and only if $\mu = v_1(\alpha_{n-1})$ for some $s_{n-1} s_n s_{n-2} s_{n-1} \leq v_1 \leq u_2 u_1$ and $\langle \mu, \alpha_t \rangle = 0$ for all $j+1 \leq t \leq n$. Let μ be such that $H^0(s_{j+1} \cdots s_n u_2 u_1, \alpha_{n-1})_\mu \neq 0$. If $\langle \mu, \alpha_j \rangle = 0$, then by Lemma 2.2, $\mathbb{C}_\mu$ is in the image of the evaluation map

$$H^0(s_j s_{j+1} \cdots s_n u_2 u_1, \alpha_{n-1}) \to H^0(s_{j+1} \cdots s_n u_2 u_1, \alpha_{n-1}).$$

Otherwise, we have $\langle \mu, \alpha_j \rangle = -1$ (as $\mu \in R_s^- \setminus \{-\alpha_j\}$). Note that by recursion $\langle \mu + \alpha_j, \alpha_{j+1} \rangle = -1$. Therefore again using recursion, we see that

$H^0(s_{j+1} \cdots s_n u_2 u_1, \alpha_{n-1})_{\mu+\alpha_j} = 0$. Hence $\mathbb{C}_\mu$ is an indecomposable B_{α_j}-summand of $H^0(s_{j+1} \cdots s_n u_2 u_1, \alpha_{n-1})$. Thus, $\mathbb{C}_\mu$ is not in the image of the evaluation map

$$ev : H^0(s_j s_{j+1} \cdots s_n u_2 u_1, \alpha_{n-1}) \longrightarrow H^0(s_{j+1} \cdots s_n u_2 u_1, \alpha_{n-1}).$$

In particular, we have

$$H^0(s_j s_{j+1} \cdots s_n u_2 u_1, \alpha_{n-1})_\mu = 0.$$

Hence, the assertion of Step 2 follows by recursion. In particular, for $j = a_{r-2}$ we have $H^0(u_3, H^0(u_2 u_1, \alpha_{n-1}))_\mu \neq 0$ if and only if $\mu \in R_s^-$ and $\mu = v_1(\alpha_{n-1})$, where $s_{n-1} s_n s_{n-2} s_{n-1} \leq v_1 \leq u_2 u_1$ and $\langle \mu, \alpha_t \rangle = 0$ for all $a_{r-2} \leq t \leq n$.

Hence we see that $H^0(u_3, H^0(u_2 u_1, \alpha_{n-1}))_\mu \neq 0$ if and only if $\mu \in R_s^-$ and $\mu = v_1(\alpha_{n-1})$, where $s_{a_{r-2}-1} \cdots s_{n-1} s_n s_{a_{r-2}-2} \cdots s_{n-2} s_{n-1} \leq v_1 \leq u_2 u_1$.

Step 3: Let $J = S \setminus \{\alpha_1, \ldots, \alpha_{a_{r-2}}\}$. Let W_J be the subgroup of W generated by $\{s_{\alpha_j} : j \in J\}$. Let $v' \in W_J$. We prove $H^0(v' u_3 u_2 u_1, \alpha_{n-1})_\mu \neq 0$ if and only if $\mu \in R_s^-$ and $\mu = v_1(\alpha_{n-1})$, where $s_{a_{r-2}-1} \cdots s_{n-1} s_n s_{a_{r-2}-2} \cdots s_{n-2} s_{n-1} \leq v_1 \leq u_2 u_1$.

By Step 2, we see that if $H^0(u_3 u_2 u_1, \alpha_{n-1})_\mu \neq 0$, then $\mu \in R_s^-$ and $\mu = v_1(\alpha_{n-1})$, where $s_{n-1} s_n s_{n-2} s_{n-1} \leq v_1 \leq u_2 u_1$ and $\langle \mu, \alpha_j \rangle = 0$ for all $a_{r-2} \leq j \leq n$. Hence, by Lemma 2.1(1) and Lemma 2.2(1) we conclude the proof of Step 3.

From Step 3, we see that $H^0(\tau_r, \alpha_{n-1})_\mu \neq 0$ if and only if $\mu = (\prod_{j=t_1}^n s_j)(\prod_{j=t_2}^{n-1} s_j)(\alpha_{n-1})$, where $t_2 < t_1$, $a_r \leq t_2 \leq a_{r-2} - 2$ and $a_{r-1} \leq t_1 \leq a_{r-2} - 1$. This completes the proof of (1).

Proof of (2) follows from (1). ∎

Lemma 5.4. $H^0(w_k[a_k, n-1], \alpha_{n-1})_\mu \neq 0$ *if and only if*

$$\mu = \left(\prod_{j=t_1}^n s_j \right) \left(\prod_{j=t_2}^{n-1} s_j \right) (\alpha_{n-1})$$

for some $a_k \leq t_2 < t_1 \leq a_{k-1} - 1$. *In such a case,*

$$\dim(H^0(w_k[a_k, n-1], \alpha_{n-1})_\mu) = 1.$$

Proof.   If $H^0(\tau_k, \alpha_{n-1})_\mu \neq 0$, then by Lemma 5.3, we have $H^0(\tau_k, \alpha_{n-1})_\mu = \mathbb{C}_\mu$. Let $\mu = -\left(\alpha_n + 2 \sum_{j=a_{k-1}}^{n-1} \alpha_j + \sum_{j=t_2}^{a_{k-1}-1} \alpha_j \right)$. Note that $\langle \mu, \alpha_{a_{k-1}-1} \rangle = 1$. Then $H^0(s_{a_{k-1}-1} \tau_k, \alpha_{n-1})_{\mu_1} \neq 0$, where $\mu_1 = \mu - \alpha_{a_{k-1}-1}$. By recursion, we see that $H^0(s_i \cdots s_{a_{k-1}-1} \tau_k, \alpha_{n-1})_{\mu_i} \neq 0$ for some $\mu_i = \mu - \sum_{j=a_{k-1}-i}^{a_{k-1}-1} \alpha_j$ with $1 \leq i \leq a_{k-1} - (t_2 + 1)$. Hence, we conclude that $H^0([a_k, a_{k-1} - 1]\tau_k, \alpha_{n-1})_\mu \neq 0$ if and only if $\mu = (\prod_{j=t_1}^n s_j)(\prod_{j=t_2}^{n-1} s_j)(\alpha_{n-1})$ for some $a_k \leq t_2 < t_1 \leq a_{k-2} - 1$.

Claim: $H^0([a_{k-1}, a_{k-2} - 1][a_k, a_{k-1} - 1]\tau_k, \alpha_{n-1})_\mu \neq 0$ if and only if $\mu = (\prod_{j=t_1}^n s_j)(\prod_{j=t_2}^{n-1} s_j)(\alpha_{n-1})$ for some $a_k \leq t_2 < t_1 \leq a_{k-1} - 1$.

Fix m such that $a_{k-2} - 1 \leq m \leq a_{k-1}$. Let $\mu_m = (\alpha_n + 2 \sum_{j=m}^{n-1} \alpha_j + \sum_{j=t_2}^{m-1} \alpha_j)$. Note that $\langle \mu_m, \alpha_m \rangle = -1$. Hence the claim follows from SES and Lemma 2.2.

Let $\mu = (\prod_{j=t_1}^{n} s_j)(\prod_{j=t_2}^{n-1} s_j)(\alpha_{n-1})$ for some $a_k \le t_2 < t_1 \le a_{k-1} - 1$. Since $\langle \mu, \alpha_l \rangle = 0$ for all $a_{k-2} \le l \le n$, we conclude that $H^0(w_k[a_k, n-1], \alpha_{n-1})_\mu \ne 0$ if and only if $\mu = (\prod_{j=t_1}^{n} s_j)(\prod_{j=t_2}^{n-1} s_j)(\alpha_{n-1})$ for some $a_k \le t_2 < t_1 \le a_{k-1} - 1$. ∎

Let $h(\alpha_n) \in \mathfrak{h}$ be the fundamental dominant coweight corresponding to α_n. That is, $\alpha_n(h(\alpha_n)) = 1$ and $\alpha_i(h(\alpha_n)) = 0$ for $i \ne n$. These coweights are used in studing the indecomposable B_α-modules in [16, p. 8, Lemma 3.3]).

Let $v'_r = s_{a_r} \cdots s_{n-1}$ and $v_r = s_n s_{a_r} \cdots s_{n-1}$. Let

$$V' = H^0(v'_r, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}) \quad \text{and} \quad V = H^0(v_r, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}).$$

Lemma 5.5.

1. $V'_\mu \ne 0$ if and only if μ is of the form
   (i) $\mu \in \{0, -\alpha_n\}$ or
   (ii) $\mu = -\left(\sum_{j=t}^{n} \alpha_j\right)$ for some $a_r \le t < n$ or
   (iii) $\mu = -\left(\alpha_n + 2\sum_{j=t}^{n-1} \alpha_j\right)$ for some $a_r \le t \le n-1$ or
   (iv) $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$ for some $a_r \le t_2 < t_1 \le n-1$.

2. $V_\mu \ne 0$ if and only if μ is of the form:
   (i) $\mu = -\left(\alpha_n + 2\sum_{j=t}^{n-1} \alpha_j\right)$ for some $a_r \le t \le n-1$ or
   (ii) $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$ for some $a_r \le t_2 < t_1 \le n-1$.

Proof. Proof of (1): Note that $H^0(s_{a_r} \cdots s_{n-1}, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n})$ is a cyclic B-module generated by $h(\alpha_n)$. Therefore the weights are of the form (each with multiplicity one):

$$\{0, -\alpha_n\} \cup \left\{ -\sum_{j=t}^{n} \alpha_j : a_r \le t < n \right\} \cup \left\{ -\left(\alpha_n + 2\sum_{j=t}^{n-1} \alpha_j\right) : a_r \le t \le n-1 \right\} \cup$$

$$\cup \left\{ -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right) : a_r \le t_2 < t_1 \le n-1 \right\}.$$

Proof of (2): By Lemma 2.4, $\mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n} = V_1 \otimes \mathbb{C}_{-\omega_n}$, where V_1 is the 2-dimensional irreducible $\tilde{L}_{\alpha_n}$-module. Since $\langle -\sum_{j=t}^{n} \alpha_j, \alpha_n \rangle = -1$ for all $a_r \le t \le n-1$, by SES we see that, the weights of $H^0(s_n s_{a_r} \cdots s_n, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n})$ are of the form (each with multiplicity one):

$$\left\{ -\left(\alpha_n + 2\sum_{j=t}^{n-1} \alpha_j\right) : a_r \le t \le n-1 \right\} \cup$$

$$\cup \left\{ -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right) : a_r \le t_2 < t_1 \le n-1 \right\}. \quad ∎$$

Lemma 5.6. $H^0(s_{a_{r-1}} \cdots s_{n-1}, V)_\mu \neq 0$ *if and only if μ is of the form:*

(i) $\mu = -\left(\alpha_n + 2\sum_{j=t}^{n-1} \alpha_j\right)$ *for some $a_r \leq t \leq a_{r-1} - 1$ or*

(ii) $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$ *for some $a_r \leq t_2 < t_1 \leq a_{r-1} - 1$.*

Proof.   We have $H^0(s_{n-1}, V) = H^0(s_{n-1}, H^0(s_n s_{a_r} \cdots s_{n-1}, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n})$. Since $\langle -\alpha_n + 2\alpha_{n-1}, \alpha_{n-1}\rangle = -2$ and $\langle -\alpha_n + 2\alpha_{n-1} + \sum_{j=t_2}^{n-2} \alpha_j, \alpha_{n-1}\rangle = -1$ for $a_r \leq t_2 \leq n-2$, by Lemma 2.3 and Lemma 5.5, we see that the weights of $H^0(s_{n-1}, V)$ are of the form

$$\mu = -\left(\alpha_n + 2\sum_{j=t}^{n-1} \alpha_j\right), \quad \text{where } a_r \leq t \leq n-2, \quad \text{or}$$

$$\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right), \quad \text{where } a_r \leq t_2 < t_1 \leq n-2.$$

Now the proof of the lemma follows by similar arguments as above. ∎

Proposition 5.7.   *Let $2 \leq r \leq k$. Then, $H^0(w_r, \alpha_n)_\mu \neq 0$ if and only if μ is of the form:*

(i) $\mu = -\left(\alpha_n + 2\sum_{j=t}^{n-1} \alpha_j\right)$ *for some $a_r \leq t \leq a_{r-1} - 1$ or*

(ii) $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$ *for some $a_r \leq t_2 < t_1 \leq a_{r-1} - 1$. In such a case, $\dim(H^0(w_r, \alpha_n)_\mu) = 1$.*

Proof.   Let $u = (\prod_{j=a_1}^n s_j)(\prod_{j=a_2}^n s_j)(\prod_{j=a_3}^n s_j) \cdots (\prod_{j=a_{r-3}}^n s_j)(\prod_{j=a_{r-2}}^n s_j)$ and let $v = (\prod_{j=a_{r-1}}^n s_j)(\prod_{j=a_r}^n s_j)$. Observe that $w_r = uv$, $l(w_r) = l(uv) = l(u) + l(v)$ and by SES, we have $H^0(w_r, \alpha_n) = H^0(u, H^0(v, \alpha_n))$.

Since $\langle \alpha_n, \alpha_{n-1}\rangle = -2$ and by SES, $H^0(v, \alpha_n) = H^0(s_{a_{r-1}} \cdots s_{n-1}, V)$, where $V = H^0(v_r, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n})$. By Lemma 5.6, $H^0(v, \alpha_n)_\mu \neq 0$ if and only if μ is of the form:

(i) $\mu = -\left(\alpha_n + 2\sum_{j=t}^{n-1} \alpha_j\right)$, where $a_r \leq t \leq a_{r-1} - 1$ or

(ii) $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$, where $a_r \leq t_2 < t_1 \leq a_{r-1} - 1$.

Note that for any such μ, we have $\langle \mu, \alpha_j\rangle = 0$ for all $a_{r-1} \leq j \leq n$. Hence by Lemma 2.1 and by SES, we conclude that $H^0(w_r, \alpha_n)_\mu \neq 0$ if and only if μ is as in the statement. ∎

6.   Cohomology module H^1 of the relative tangent bundle

In this section we describe the weights of H^1 of a relative tangent bundle.

Let $\mathfrak{g}'_{<\alpha_n}$ be the B-submodule of $\mathfrak{g}$ generated by $h(\alpha_n)$. Note that

$$\mathfrak{g}'_{<\alpha_n} = \mathbb{C}h(\alpha_n) \oplus \bigoplus_{\beta \leq -\alpha_n} \mathfrak{g}_\beta.$$

Lemma 6.1.   *Let $w \in W$. Let V be a B-submodule of $\mathfrak{g}'_{<\alpha_n}$ such that either $(\mathbb{C}h(\alpha_n) \oplus \mathfrak{g}_{-\alpha_n}) \cap V = 0$ or $V = \mathfrak{g}'_{<\alpha_n}$. Then, $H^1(w, V)_\mu = 0$ unless $\mu \in R_s^- \setminus (-S)$.*

Proof.   The proof is by induction on $l(w)$. Assume that $l(w) = 1$. Then $w = s_i$ for some $1 \le i \le n$. If $H^1(s_i, V)_\mu \ne 0$, then there exists an indecomposable B_{α_i}-direct summand V_1 of V such that $H^1(s_i, V_1) \ne 0$. By Lemma 2.4, we have $V_1 = V' \otimes \mathbb{C}_{a\omega_i}$ for some irreducible $\tilde{L}_{\alpha_i}$-module V' and an integer a. Since $H^1(s_i, V_1) \ne 0$ and G is of type C_n, by Lemma 2.3, we have $\dim(V_1) = 1$ and $a = -2$. Further, we have

$$H^1(s_i, V_1) = V' \otimes H^1(s_i, -2\omega_i) = V' \otimes H^0(s_i, -2\omega_i + \alpha_i).$$

Therefore, $H^1(s_i, V_1) = \mathbb{C}_{\mu_1 + \alpha_i}$, where μ_1 is the lowest weight of V_1. By the hypothesis on V, we have $\mu_1 = -\beta$ for some $\beta \in R^+ \setminus S$ with $\langle \beta, \alpha_i \rangle = 2$. Therefore β is a long root and $-\beta + \alpha_i \in R_s^- \setminus (-S)$.

Assume that $l(w) > 1$. Choose $1 \le i \le n$ such that $l(ws_i) = l(w) - 1$. By [16, Lemma 6.1], we have the following exact sequence of B-modules:

$$H^1(ws_i, H^0(s_i, V)) \longrightarrow H^1(w, V) \longrightarrow H^0(ws_i, H^1(s_i, V)) \qquad (6.1.1)$$

<u>Claim:</u> $H^0(s_i, V) \cap (\mathbb{C}h(\alpha_n) \oplus \mathfrak{g}_{-\alpha_n}) = 0$.
    Assume that $i = n$. By Lemma 2.4, we have

$$\mathbb{C}h(\alpha_n) \oplus \mathfrak{g}_{-\alpha_n} = V_2 \otimes \mathbb{C}_{-\omega_n},$$

V_2 is the 2-dimensional irreducible $\tilde{L}_{\alpha_n}$-module. Hence we have

$$H^0(s_n, \mathbb{C}h(\alpha_n) \oplus \mathfrak{g}_{-\alpha_n}) = V_2 \otimes H^0(s_n, \mathbb{C}_{-\omega_n}) = 0.$$

Assume that $i \ne n$. If $V = \mathfrak{g}'_{<\alpha_n}$, then $H^0(s_i, V) = V$. Otherwise, since $H^0(s_i, V) \subset V$, we have

$$H^0(s_i, V) \cap (\mathbb{C}h(\alpha_n) \oplus \mathfrak{g}_{-\alpha_n}) = 0.$$

By induction on $l(w)$, if $H^1(ws_i, H^0(s_i, V))_\mu \ne 0$, then $\mu \in R_s^- \setminus (-S)$.
    By above (as in the case of $l(w) = 1$), there is a descending sequence of B-modules:

$$H^1(s_i, V) \supsetneq V^1 \supsetneq V^2 \supsetneq \ldots \supsetneq V^r = 0$$

such that $V^i/V^{i+1} \simeq \mathbb{C}_{\beta_i}$ for some $\beta_i \in R_s^- \setminus (-S)$. By Lemma 5.1, if $H^0(ws_i, H^1(s_i, V))_\mu \ne 0$, then $\mu \in R_s^- \setminus (-S)$. Hence by the above exact sequence (6.1.1), we conclude that if $H^1(w, V)_\mu \ne 0$ then $\mu \in R_s^- \setminus (-S)$. This completes the proof. ∎

Proposition 6.2.   *Let $u \in W$ and an integer $1 \le a \le n - 2$ be such that $l(us_a s_{a+1} \cdots s_n) = l(u) + (n + 1 - a)$. Let $w = us_a s_{a+1} \cdots s_n$. Then we have:*

1. If $u = id$, then $H^1(w, \alpha_n) = 0$.

2. $H^j(ws_n, \alpha_n) = 0$ *for all $j \geq 0$.*

3. $H^1(w, \alpha_n) = H^1(ws_n, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n})$.

4. $H^1(w, \alpha_n)_\mu = 0$ *unless $\mu \in R_s^- \setminus (-S)$.*

Proof.   Proof of (2): Since $l(ws_n s_{n-1}) = l(ws_n) - 1$, by Lemma 2.2, we have

$$H^0(ws_n, \alpha_n) = 0 \quad \text{and}$$

$$H^j(ws_n, \alpha_n) = H^{j-1}(ws_n s_{n-1}, H^0(s_{n-1}, s_{n-1} \cdot \alpha_n)) \quad \text{for all} \quad j \geq 1.$$

Since $s_{n-1} \cdot \alpha_n = \alpha_{n-1} + \alpha_n$ and $H^0(s_{n-1}, \alpha_{n-1} + \alpha_n) = \mathbb{C}_{\alpha_{n-1}+\alpha_n}$, we have

$$H^j(ws_n, \alpha_n) = H^{j-1}(ws_n s_{n-1}, \alpha_{n-1} + \alpha_n) \quad \text{for all} \quad j \geq 1.$$

Since $\langle \alpha_{n-1} + \alpha_n, \alpha_{n-2} \rangle = -1$ and $l(ws_n s_{n-1} s_{n-2}) = l(ws_n s_{n-1}) - 1$, by Lemma 2.2, we see that $H^{j-1}(ws_n s_{n-1}, \alpha_{n-1} + \alpha_n) = 0$ for all $j \geq 1$.

Therefore, $H^j(ws_n, \alpha_n) = 0$ for every $j \geq 0$. This completes the proof of (2).

Proof of (3): Consider the following short exact sequences of B-modules:

$$0 \longrightarrow K_1 \longrightarrow H^0(s_n, \alpha_n) \longrightarrow \mathbb{C}_{\alpha_n} \longrightarrow 0 \qquad (6.2.1)$$

where K_1 is the kernel of the evaluation map $ev : H^0(s_n, \alpha_n) \longrightarrow \mathbb{C}_{\alpha_n}$. Note that $K_1 = \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}$.

By applying $H^0(ws_n, -)$ to (6.2.1), we have the following long exact sequence of B-modules:

$$\cdots \longrightarrow H^0(ws_n, \alpha_n) \longrightarrow H^1(ws_n, K_1) \longrightarrow H^1(ws_n, H^0(s_n, \alpha_n)) \longrightarrow$$
$$\longrightarrow H^1(ws_n, \alpha_n) \longrightarrow \cdots$$

By (2), we conclude that

$$H^1(ws_n, K_1) = H^1(ws_n, H^0(s_n, \alpha_n)) = H^1(w, \alpha_n).$$

Proof of (4): Let $K_2 = \sum_{\mu < -\alpha_n} \mathfrak{g}_\mu$. Clearly, K_2 is a B-submodule of $\mathfrak{g}'_{<\alpha_n}$ and

$$\frac{\mathfrak{g}'_{<\alpha_n}}{K_2} \simeq K_1.$$

Then we have a following short exact sequence of B-modules:

$$0 \longrightarrow K_2 \longrightarrow \mathfrak{g}'_{<\alpha_n} \longrightarrow K_1 \longrightarrow 0 \qquad (6.2.2)$$

By applying $H^0(ws_n, -)$ to (6.2.2), we have the following long exact sequence of B-modules:

$$\cdots \longrightarrow H^1(ws_n, K_2) \longrightarrow H^1(ws_n, \mathfrak{g}'_{<\alpha_n}) \longrightarrow H^1(ws_n, K_1) \longrightarrow H^2(ws_n, K_2) \longrightarrow \cdots$$

By [16, Lemma 6.2], we have $H^2(ws_n, K_2) = 0$. Hence by Lemma 6.1 we see that if $H^1(ws_n, \mathfrak{g}'_{<\alpha_n})_\mu \neq 0$, then $\mu \in R_s^- \setminus (-S)$. Therefore, $H^1(ws_n, K_1)_\mu = 0$ unless $\mu \in R_s^- \setminus (-S)$.

By (3), we have $H^1(w, \alpha_n) = H^1(ws_n, K_1)$. Hence we conclude that $H^1(w, \alpha_n)_\mu = 0$ unless $\mu \in R_s^- \setminus (-S)$.

Proof of (1) is similar to the proof of (2).   ■

Lemma 6.3.   *Let $V = H^0(s_n s_{a_r} \cdots s_{n-1}, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n})$.*
Then $H^1(s_{a_{r-1}} \cdots s_{n-1}, V)_\mu \neq 0$ if and only if μ is of the form:
(i) $\mu = -\sum_{j=t}^{n} \alpha_j$ for some $a_{r-1} \leq t \leq n-1$ or
(ii) $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$ for some $a_{r-1} \leq t_2 < t_1 \leq n-1$.

Proof.   Case 1: $a_{r-1} = n-1$. By Lemma 5.5, it follows that $V_\mu \neq 0$ if and only if μ is of the form:

(i) $\mu = -\left(\alpha_n + 2\sum_{j=t}^{n-1} \alpha_j\right)$, where $a_r \leq t \leq n-1$ or

(ii) $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$, where $a_r \leq t_2 < t_1 \leq n-1$.

If $t < n-1$ (or $t_1 < n-1$), then we have $\langle \mu, \alpha_{n-1} \rangle = 0$. Hence by Lemma 2.2, we see that $H^1(s_{n-1}, \mu) = 0$.

If $t = n-1$ in type (i), then $\mu = -(\alpha_n + 2\alpha_{n-1})$ and $\langle \mu, \alpha_{n-1} \rangle = -2$. Then by Lemma 2.2,

$$H^1(s_{n-1}, -(\alpha_n + 2\alpha_{n-1})) = H^0(s_{n-1}, s_{n-1} \cdot (-(\alpha_n + 2\alpha_{n-1}))).$$

Since $s_{n-1} \cdot (-(\alpha_n + 2\alpha_{n-1})) = -(\alpha_n + \alpha_{n-1})$, we see that

$$H^1(s_{n-1}, -(\alpha + 2\alpha_{n-1})) = \mathbb{C}_{-(\alpha_n + \alpha_{n-1})}.$$

If $t_1 = n-1$ in type (ii), then $\mu = -\left(\alpha_n + 2\alpha_{n-1} + \sum_{j=t_2}^{n-2} \alpha_j\right)$ and $\langle \mu, \alpha_{n-1} \rangle = -1$. Then by Lemma 2.2, we conclude that $H^1(s_{n-1}, \mu) = 0$.

Case 2: $a_{r-1} \neq n-1$. Fix $a_{r-1} \leq i \leq n-2$. By recursion we assume that $H^1(s_{i+1} \cdots s_{n-1}, V)_\mu \neq 0$ if and only if $\mu = -\sum_{j=t}^{n} \alpha_j$ for some $i+1 \leq t \leq n-1$ or $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$ for some $i+1 \leq t_2 < t_1 \leq n-1$.

By SES, we have the following short exact sequence of B-modules:

$$0 \longrightarrow H^1(s_i, H^0(s_{i+1} \cdots s_{n-1}, V)) \longrightarrow H^1(s_i s_{i+1} \cdots s_{n-1}, V) \longrightarrow$$
$$\longrightarrow H^0(s_i, H^1(s_{i+1} \cdots s_{n-1}, V)) \longrightarrow 0.$$

By the above discussion, we see that $H^0(s_i, H^1(s_{i+1} \cdots s_{n-1}, V))_\mu \neq 0$ if and only if $\mu = -\sum_{j=t}^{n} \alpha_j$, where $i \leq t \leq n-1$ or $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$, where $i \leq t_2 < t_1 \leq n-1$ and $t_1 \geq i+2$.

Further, by Lemma 5.6, we see that $H^1(s_i, H^0(s_{i+1} \cdots s_{n-1}, V))_\mu \neq 0$ if and only if $\mu = -\left(\alpha_n + 2\sum_{j=i+1}^{n-1} \alpha_j + \alpha_i\right)$. By the recursion, we conclude the proof of the lemma. ∎

Let $v_r = s_n s_{a_r} \cdots s_{n-1}$, $v_{r-1} = s_{a_{r-1}} \cdots s_{n-1}$ and $v_{r-2} = s_{a_{r-2}} \cdots s_{n-1} s_n$. Then we have

Lemma 6.4.   *1. $H^1(v_{r-1} v_r s_n, \alpha_n)_\mu \neq 0$ if and only if μ is of the form:*
(i) $\mu = -\sum_{j=t}^{n} \alpha_j$ for some $a_{r-1} \leq t \leq n-1$ or
(ii) $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$ for some $a_{r-1} \leq t_2 < t_1 \leq n-1$.

2. *Let* $w = v_{r-2}v_{r-1}v_r s_n$. *Then,* $H^1(w, \alpha_n)_\mu \neq 0$ *if and only if* $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$ *for some* $a_{r-1} \leq t_2 < t_1 \leq a_{r-2} - 1$.

Proof.   <u>Step 1</u>: We prove

$$H^1(v_{r-2}v_{r-1}, V) = H^0(v_{r-2}, H^1(v_{r-1}, V)),$$

where V is as in Lemma 6.3.

By SES we have the following short exact sequence of B-modules:

$$0 \longrightarrow H^1(s_n, H^0(v_{r-1}, V)) \longrightarrow H^1(s_n v_{r-1}, V) \longrightarrow H^0(s_n, H^1(v_{r-1}, V)) \longrightarrow 0.$$

By Lemma 5.6, if $H^0(v_{r-1}, V)_\mu \neq 0$, then we see that $\langle \mu, \alpha_i \rangle = 0$ for all $a_{r-2} \leq i \leq n$. By Lemma 2.2, we have

$$H^1(s_n, H^0(v_{r-1}, V)) = 0 \quad \text{and}$$

$$H^0(s_n, H^0(v_{r-1}, V)) = H^0(v_{r-1}, V).$$

Hence by the above short exact sequence, we see that

$$H^1(s_n v_{r-1}, V) = H^0(s_n, H^1(v_{r-1}, V)).$$

By recursion, we have

$$H^1(s_{i+1}\cdots s_n v_{r-1}, V) = H^0(s_{i+1}\cdots s_n, H^1(v_{r-1}, V)).$$

By SES, we have the following short exact sequence of B-modules:

$$0 \longrightarrow H^1(s_i, H^0(s_{i+1}\cdots s_n v_{r-1}, V)) \longrightarrow H^1(s_i \cdots s_n v_{r-1}, V) \longrightarrow$$
$$\longrightarrow H^0(s_i, H^1(s_{i+1}\cdots s_n v_{r-1}, V)) \longrightarrow 0.$$

Since

$$H^0(s_{i+1}\cdots s_n v_{r-1}, V) = H^0(v_{r-1}, V) \quad \text{and} \quad H^1(s_i, H^0(s_{i+1}\cdots s_n v_{r-1}, V)) = 0,$$

we have

$$H^1(s_i s_{i+1}\cdots s_n v_{r-1}, V) = H^0(s_i s_{i+1}\cdots s_n, H^1(v_{r-1}, V)).$$

Therefore, we conclude that

$$H^1(v_{r-2}v_{r-1}, V) = H^0(v_{r-2}, H^1(v_{r-1}, V)).$$

This completes the proof of Step 1.

<u>Step 2</u>: We prove

$$H^1(v_{r-1}v_r, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}) = H^1(v_{r-1}, V)$$

and

$$H^1(v_{r-2}v_{r-1}v_r, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}) = H^1(v_{r-2}v_{r-1}, V).$$

First note that $H^i(s_j \cdots s_{n-1}, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}) = 0$ for each $a_r \leq j \leq n - 1$ for all $i \geq 1$ and $H^i(s_n s_{a_r} \cdots s_{n-1}, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}) = 0$ for all $i \geq 1$. By using SES repeatedly, we see that

$$H^1(v_{r-1}v_r, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}) = H^1(v_{r-1}, H^0(v_r, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}))$$

and

$$H^1(v_{r-2}v_{r-1}v_r, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}) = H^1(v_{r-2}v_{r-1}, H^0(v_r, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n})).$$

Hence we have

$$H^1(v_{r-1}v_r, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}) = H^1(v_{r-1}, V)$$

and

$$H^1(v_{r-2}v_{r-1}v_r, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}) = H^1(v_{r-2}v_{r-1}, V).$$

From Step 1, Step 2 and Proposition 6.2(3), we see that

$$H^1(v_{r-1}v_r s_n, \alpha_n) = H^1(v_{r-1}, V) \tag{6.4.1}$$

and

$$H^1(w, \alpha_n) = H^0(v_{r-2}, H^1(v_{r-1}, V)) \tag{6.4.2}$$

By Lemma 6.3, $H^1(v_{r-1}, V)_\mu \neq 0$ if and only if μ is of the form:
(i) $\mu = -\sum_{j=t}^n \alpha_j$, where $a_{r-1} \leq t \leq n - 1$ or
(ii) $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$, where $a_{r-1} \leq t_2 < t_1 \leq n - 1$.

Proof of (1) is immediate from (6.4.1).

Proof of (2): If μ is of type (i), then $\langle \mu, \alpha_n \rangle = -1$. By Lemma 2.2, (6.4.2) and SES, we see that $H^1(w, \alpha_n)_\mu = 0$.

Fix $a_{r-2} \leq l \leq n - 1$. By recursion, we assume that

$$H^0(s_{l+1} \ldots s_n, H^1(v_{r-1}, V))_\mu \neq 0$$

if and only if $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$ for some $a_{r-1} \leq t_2 < t_1 \leq l$.

On the other hand, we have $\langle -(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j), \alpha_l \rangle = -1$. Therefore, $H^0(s_l, H^0(s_{l+1} \cdots s_n, H^1(v_{r-1}, V)))_\mu \neq 0$ if and only if $\mu = -\left(\alpha_n + 2\sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j\right)$ for some $a_{r-1} \leq t_2 < t_1 \leq l - 1$. Now the proof of (2) follows by recursion. ∎

Proposition 6.5. *Let $3 \leq r \leq k$. Then, $H^1(w_r, \alpha_n)_\mu \neq 0$ if and only if $\mu = \left(\prod_{j=t_1}^n s_j\right)\left(\prod_{j=t_2}^{n-1} s_j\right)(\alpha_{n-1})$ for some $a_{r-1} \leq t_2 < t_1 \leq a_{r-2} - 1$. In such a case, $\dim(H^1(w_r, \alpha_n)_\mu) = 1$.*

Proof. Let $u = \left(\prod_{j=a_1}^n s_j\right)\left(\prod_{j=a_2}^n s_j\right)\left(\prod_{j=a_3}^n s_j\right) \cdots \left(\prod_{j=a_{r-3}}^n s_j\right)$ and let $v = \left(\prod_{j=a_{r-2}}^n s_j\right)\left(\prod_{j=a_{r-1}}^n s_j\right)\left(\prod_{j=a_r}^n s_j\right)$. Observe that $w_r = uv$, $l(w_r) = l(uv) = l(u) + l(v)$.

<u>Claim:</u> $H^1(w_r, \alpha_n) = H^0(u, H^1(v, \alpha_n))$.

By SES we have the following short exact sequence of B-modules:

$$0 \longrightarrow H^1(s_n, H^0(v, \alpha_n)) \longrightarrow H^1(s_nv, \alpha_n)) \longrightarrow H^0(s_n, H^1(v, \alpha_n)) \longrightarrow 0.$$

By Lemma 5.7, if $H^0(v, \alpha_n)_\mu \neq 0$, then we see that $\langle \mu, \alpha_j \rangle = 0$ for all $a_{r-3} \leq j \leq n - 1$. Therefore, the vector bundle $\mathcal{L}(H^0(v, \alpha_n))$ on $X(u')$ is trivial for each $u' \leq u$. Thus, $H^i(u', H^0(v, \alpha_n)) = 0$ for each $u' \leq u$ and for all $i \geq 1$.

Hence by using SES recursively, we see that

$$H^1(uv, \alpha_n) = H^0(u, H^1(v, \alpha_n)).$$

Therefore, we have $H^1(w_r, \alpha_n) = H^0(u, H^1(v, \alpha_n))$.

By Lemma 6.4, $H^1(v, \alpha_n)_\mu \neq 0$ if and only if μ is of the form:

$$\mu = -\left(\alpha_n + 2 \sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j \right), \quad \text{where } a_{r-1} \leq t_2 < t_1 \leq a_{r-2} - 1.$$

Note that $\langle \mu, \alpha_j \rangle = 0$ for all $a_{r-2} \leq j \leq n$. Therefore, $\mathcal{L}(H^1(v, \alpha_n))$ is the trivial vector bundle on $X(u)$. Hence $H^0(u, H^1(v, \alpha_n)) = H^1(v, \alpha_n)$. Thus, we conclude that $H^1(w_r, \alpha_n)_\mu \neq 0$ if and only if μ is of the form:

$$\mu = -\left(\alpha_n + 2 \sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j \right),$$

where $a_{r-1} \leq t_2 < t_1 \leq a_{r-2} - 1$. Note that

$$\mu = -\left(\alpha_n + 2 \sum_{j=t_1}^{n-1} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j \right) = \left(\prod_{j=t_1}^{n} s_j \right) \left(\prod_{j=t_2}^{n-1} s_j \right)(\alpha_{n-1}).$$

This completes the proof of the proposition. ■

Corollary 6.6. *Let $2 \leq r \leq k$. If $H^1(w_r, \alpha_n)_\mu \neq 0$, then $H^0(\tau_r, \alpha_{n-1})_\mu \neq 0$.*

Proof. <u>Case 1:</u> $3 \leq r \leq k$. If $H^1(w_r, \alpha_n)_\mu \neq 0$ then by Proposition 6.5, we have $\mu = \left(\prod_{j=t_1}^{n} s_j \right) \left(\prod_{j=t_2}^{n-1} s_j \right)(\alpha_{n-1})$ for some integers $t_2 < t_1$ such that $a_{r-1} \leq t_2 < t_1 \leq a_{r-2} - 1$. Then by Lemma 5.3, we have $H^0(\tau_r, \alpha_{n-1})_\mu \neq 0$.

<u>Case 2:</u> $r = 2$. The proof follows from Lemma 5.2(3) and Lemma 6.4(1). ■

Corollary 6.7. *Let $2 \leq r \leq k$. If $H^1(w_r, \alpha_n)_\mu \neq 0$, then $H^0(w_{r-1}, \alpha_n)_\mu \neq 0$.*

Proof. <u>Case 1:</u> $3 \leq r \leq k$. If $H^1(w_r, \alpha_n)_\mu \neq 0$, by Proposition 6.5, $\mu = -\left(\alpha_n + 2 \sum_{j=t_1}^{n} \alpha_j + \sum_{j=t_2}^{t_1-1} \alpha_j \right)$ for some integers $t_2 < t_1$ such that $a_{r-1} \leq t_2 < t_1 \leq a_{r-2} - 1$. For each such μ, by Proposition 5.7(ii), we see that $H^0(w_{r-1}, \alpha_n)_\mu \neq 0$.

<u>Case 2:</u> $r = 2$. First note that

$$H^0(w_1, \alpha_n) = H^0(s_{a_1} \cdots s_{n-1}, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}) \quad \text{and}$$

$$H^1(w_2, \alpha_n) = H^1(s_{a_1} \cdots s_{n-1}, H^0(s_n s_{a_2} \cdots s_{n-1}, \mathbb{C}h(\alpha_n) \oplus \mathbb{C}_{-\alpha_n}).$$

Now the proof follows from Lemma 5.5(1) and Lemma 6.3. ■

Recall $[a_k, n] = s_{a_k} \cdots s_n$ and $[a_k, n-1] = s_{a_k} \cdots s_{n-1}$. Then, we have

Lemma 6.8. $\quad H^1(w_k[a_k, n], \alpha_n) = H^0(w_k[a_k, n-1], \alpha_{n-1})$.

Proof. Note that $l(w_k[a_k, n]s_{n-1}) = l(w_k[a_k, n]) - 1$. Since $\langle \alpha_n, \alpha_{n-1} \rangle = -2$, by Lemma 2.2, we have

$$H^1(w_k[a_k, n], \alpha_n) = H^0(w_k[a_k, n], s_{\alpha_{n-1}} \cdot \alpha_n) \qquad (6.8.1)$$

By SES and $s_{\alpha_{n-1}} \cdot \alpha_n = \alpha_n + \alpha_{n-1}$, we have

$$H^0(w_k[a_k, n], s_{\alpha_{n-1}} \cdot \alpha_n) = H^0(w_k[a_k, n-1], H^0(s_n, \alpha_{n-1} + \alpha_n)) \qquad (6.8.2)$$

By applying $H^0(w_k[a_k, n-1], -)$ to the following short exact sequence,

$$0 \longrightarrow \mathbb{C}_{\alpha_{n-1}} \longrightarrow H^0(s_n, \alpha_n + \alpha_{n-1}) \longrightarrow \mathbb{C}_{\alpha_{n-1}+\alpha_n} \longrightarrow 0.$$

We obtain the following long exact sequence:

$$0 \longrightarrow H^0(w_k[a_k, n-1], \alpha_{n-1}) \longrightarrow H^0(w_k[a_k, n-1], H^0(s_n, \alpha_n + \alpha_{n-1})) \longrightarrow$$
$$\longrightarrow H^0(w_k[a_k, n-1], \alpha_{n-1} + \alpha_n) \longrightarrow H^1(w_k[a_k, n-1], \alpha_{n-1}) \longrightarrow \cdots$$

Note that $\langle \alpha_n + \alpha_{n-1}, \alpha_{n-1} \rangle = 0$ and $\langle \alpha_n + \alpha_{n-1}, \alpha_{n-2} \rangle = -1$. Hence by Lemma 2.2 and SES, we see that

$$H^0(w_k[a_k, n-1], \alpha_{n-1} + \alpha_n) = 0.$$

Therefore, we have

$$H^0(w_k[a_k, n-1], \alpha_{n-1}) = H^0(w_k[a_k, n-1], H^0(s_n, \alpha_n + \alpha_{n-1})) \qquad (6.8.3)$$

Hence by using (6.8.1), (6.8.2) and (6.8.3), we conclude that

$$H^1(w_k[a_k, n], \alpha_n) = H^0(w_k[a_k, n-1], \alpha_{n-1}). \qquad \blacksquare$$

Let $1 \le r \le k$. Let $M_r := \{\mu \in X(T) : H^1(w_r, \alpha_n)_\mu \ne 0\}$ and $M_0 := \{\mu \in X(T) : H^1(w_k[a_k, n], \alpha_n)_\mu \ne 0\}$. Then, we have

Corollary 6.9.

1. $M_r \cap M_{r'} = \emptyset$ whenever $r \ne r'$.

2. $M_0 \cap M_r = \emptyset$ for every $1 \le r \le k$.

Proof. Note that by Proposition 6.2(1), $H^1(w_1, \alpha_n) = 0$. By Lemma 6.8, we have

$$H^1(w_k[a_k, n], \alpha_n) = H^0(w_k[a_k, n-1], \alpha_{n-1}).$$

Now the proof follows from Lemma 5.3, Lemma 5.4 and Corollary 6.6. $\qquad \blacksquare$

Lemma 6.10. *Let* $u, w \in W$, *let* $v := \left(\prod_{j=1}^{n} s_j\right)^l$ *for some positive integer* $l \leq n$ *such that* $w = uv$ *and* $l(w) = l(u) + l(v)$. *If* $l \geq 3$, *then* $H^i(w, \alpha_n) = 0$ *for all* $i \geq 0$.

Proof. First note that by SES,

$$H^0(w, \alpha_n) = H^0(u, H^0(v, \alpha_n)).$$

By similar arguments as in the proof of Lemma 6.4, we see that

$$H^1(w, \alpha_n) = H^0(u, H^1(v, \alpha_n)).$$

We now prove that $H^0(v, \alpha_n) = 0$ and $H^1(v, \alpha_n) = 0$. Let $c = \prod_{j=1}^{n} s_j$. Note that for each $1 \leq r \leq n$, $c^r(\alpha_j) < 0$ for $n + 1 - r \leq j \leq n$. In particular, we have $l(vs_{n-1}) = l(v) - 1$ and $l(vs_{n-1}s_{n-2}) = l(v) - 2$.

Therefore, by Lemma 2.2 and using SES, we have

$$H^0(v, \alpha_n) = H^0(vs_{n-1}, H^0(s_{n-1}, \alpha_n)) = 0.$$

Further, by Lemma 2.2, we have

$$H^1(v, \alpha_n) = H^0(vs_{n-1}, H^1(s_{n-1}, \alpha_n)) = H^0(vs_{n-1}, \mathbb{C}_{\alpha_{n-1}+\alpha_n}).$$

Since $\langle \alpha_{n-1} + \alpha_n, \alpha_{n-2} \rangle = -1$ and $l(vs_{n-1}s_{n-2}) = l(vs_{n-1}) - 1$; by Lemma 2.2, we have $H^0(vs_{n-1}, \alpha_{n-1} + \alpha_n) = 0$. Hence $H^1(v, \alpha_n) = 0$.

Therefore, by [16, Corollary 6.4], we see that $H^i(v, \alpha_n) = 0$ for all $i \geq 0$. Hence, we conclude that $H^i(w, \alpha_n) = 0$ for all $i \geq 0$. ∎

7. Cohomology modules of the tangent bundle of $Z(w, \underline{i})$

Let $w \in W$ and let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression for w and let $\underline{i} = (i_1, i_2, \ldots, i_r)$. Let $\tau = s_{i_1} s_{i_2} \cdots s_{i_{r-1}}$ and $\underline{i}' = (i_1, i_2, \ldots, i_{r-1})$.

Recall the following long exact sequence of B-modules from [5] (see [5, Proposition 3.1]):

$$0 \longrightarrow H^0(w, \alpha_{i_r}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(\tau, \underline{i}'), T_{(\tau,\underline{i}')}) \longrightarrow$$
$$\longrightarrow H^1(w, \alpha_{i_r}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(Z(\tau, \underline{i}'), T_{(\tau,\underline{i}')}) \longrightarrow$$
$$\longrightarrow H^2(w, \alpha_{i_r}) \longrightarrow H^2(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^2(Z(\tau, \underline{i}'), T_{(\tau,\underline{i}')}) \longrightarrow$$
$$\longrightarrow H^3(w, \alpha_{i_r}) \longrightarrow \cdots$$

By [16, Corollary 6.4], we have $H^j(w, \alpha_{i_r}) = 0$ for every $j \geq 2$. Thus we have the following exact sequence of B-modules:

$$0 \longrightarrow H^0(w, \alpha_{i_r}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(\tau, \underline{i}'), T_{(\tau,\underline{i}')}) \longrightarrow$$
$$\longrightarrow H^1(w, \alpha_{i_r}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(Z(\tau, \underline{i}'), T_{(\tau,\underline{i}')}) \longrightarrow 0.$$

Now onwards we call this exact sequence by LES.

Let $w_0 = s_{j_1} s_{j_2} \cdots s_{j_N}$ be a reduced expression of w_0 and $\underline{j} = (j_1, j_2, \ldots, j_N)$ such that $(j_1, j_2, \ldots, j_r) = \underline{i}$.

Lemma 7.1.   *The natural homomorphism*

$$f : H^1(Z(w_0, \underline{j}), T_{(w_0, \underline{j})}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w, \underline{i})})$$

of B-modules is surjecive.

Proof.   If $w = w_0$, we are done. Otherwise, let $\underline{i}_1 = (j_1, \ldots, j_r, j_{r+1})$. Note that by [16, Corollary 6.4], we have $H^2(ws_{i_{r+1}}, \alpha_{r+1}) = 0$. Then by LES, the natural homomorphism

$$H^1(Z(ws_{i_{r+1}}, \underline{i}_1), T_{(ws_{i_{r+1}}, \underline{i}_1)}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w, \underline{i})})$$

is surjective. By descending induction on $l(w)$, the natural homomorphism $H^1(Z(w_0, \underline{j}), T_{(w_0, \underline{j})}) \longrightarrow H^1(Z(ws_{i_{r+1}}, \underline{i}_1), T_{(ws_{i_{r+1}}, \underline{i}_1)})$ of B-modules is surjecive. Hence the natural homomorphism

$$f : H^1(Z(w_0, \underline{j}), T_{(w_0, \underline{j})}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w, \underline{i})})$$

of B-modules is surjective. ∎

Lemma 7.2.   Let $J = S \setminus \{\alpha_n\}$. Let $v \in W_J$ and $u \in W$ be such that $l(uv) = l(u) + l(v)$. Let $u = s_{i_1} \cdots s_{i_r}$ and $v = s_{i_{r+1}} \cdots s_{i_t}$ be reduced expressions of u and v respectively. Let $\underline{i} = (i_1, i_2, \ldots, i_r)$ and $\underline{j} = (i_1, i_2, \ldots, i_r, i_{r+1}, \ldots, i_t)$. Then, we have

1. The natural homomorphism

   $$H^0(Z(uv, \underline{j}), T_{(uv, \underline{j})}) \to H^0(Z(u, \underline{i}), T_{(u, \underline{i})})$$

   of B-modules is surjecive.

2. The natural homomorphism

   $$H^1(Z(uv, \underline{j}), T_{(uv, \underline{j})}) \longrightarrow H^1(Z(u, \underline{i}), T_{(u, \underline{i})})$$

   of B-modules is an isomorphism.

Proof.   Let $r + 1 \leq l \leq t$. Let $v_l = us_{i_{r+1}} \cdots s_{i_l}$ and $\underline{i}_l = (\underline{i}, i_{r+1}, \ldots, i_l)$.

Proof of (1): By Corollary 2.5, we have $H^i(v_t, \alpha_{i_t}) = 0$. Therefore, using LES, we see that the natural homomorphism

$$H^0(Z(v_t, \underline{i}_t), T_{(v_t, \underline{i}_t)}) \to H^0(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1}, \underline{i}_{t-1})})$$

is surjective. By the recursion, the natural homomorphism

$$H^0(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1}, \underline{i}_{t-1})}) \to H^0(Z(u, \underline{i}), T_{(u, \underline{i})})$$

is surjective. Hence we conclude that the natural homomorphism

$$H^0(Z(uv, \underline{j}), T_{(uv, \underline{j})}) \to H^0(Z(u, \underline{i}), T_{(u, \underline{i})})$$

is surjective.

Proof of (2): Proof is by induction on $l(v)$. By LES, we have the following exact sequence of B-modules:

$$0 \longrightarrow H^0(uv, \alpha_{i_t}) \longrightarrow H^0(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^0(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})}) \longrightarrow$$
$$\longrightarrow H^1(uv, \alpha_{i_t}) \longrightarrow H^1(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^1(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})}) \longrightarrow 0.$$

By induction on $l(v)$, the natural homomorphism $H^1(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})}) \longrightarrow H^1(Z(u, \underline{i}), T_{(u,\underline{i})})$ is an isomorphism.

By Corollary 2.5, $H^1(uv, \alpha_{i_t}) = 0$. Therefore, by the above exact sequence, we see that $H^1(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^1(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})})$ is an isomorphism. Hence, we conclude that the homomorphism

$$H^1(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^1(Z(u, \underline{i}), T_{(u,\underline{i})})$$

of B-modules is an isomorphism. ∎

Recall that by Lemma 4.2 and Lemma 4.4 we have

$$w_0 = \left(\prod_{l_1=1}^{k-1} [a_{l_1}, n] \right) \left([a_k, n]^{n+1-k} \right) \left(\prod_{l_2=1}^{k-1} [a_k, a_{l_2} - 1] \right)$$

is a reduced expression for w_0. Let $\underline{i}$ be the tuple corresponding to this reduced expression of w_0. Let $u_1 = w_k[a_k, n]$ and $\underline{i}_1$ be the tuple corresponding to the reduced expression $(\prod_{l_1=1}^{k} [a_{l_1}, n])([a_k, n])$. Note that $a_k = 1$. With this notation, we have

Lemma 7.3.

1. *The natural homomorphism*

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})$$

   of B-modules is an isomorphism.

2. *The natural homomorphism*

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})$$

   of B-modules is an isomorphism.

Proof. Let $u_j = w_k[a_k, n]^j$ and $\underline{i}_j$ be the tuple corresponding to the reduced expression $(\prod_{l_1=1}^{k} [a_{l_1}, n])([a_k, n]^j)$ (see Lemma 4.4). By Lemma 7.2, the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_{n-k}, \underline{i}_{n-k}), T_{(u_{n-k},\underline{i}_{n-k})})$$

is surjective and the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_{n-k}, \underline{i}_{n-k}), T_{(u_{n-k},\underline{i}_{n-k})})$$

is an isomorphism.

If $j \geq 2$, then by Lemma 6.10, we have $H^1(u_j, \alpha_n) = 0$. Hence by LES and Lemma 7.2, for each $2 \leq j \leq n - k$ we observe that the natural homomorphism

$$H^0(Z(u_j, \underline{i}_j), T_{(u_j, \underline{i}_j)}) \longrightarrow H^0(Z(u_{j-1}, \underline{i}_{j-1}), T_{(u_{j-1}, \underline{i}_{j-1})})$$

is surjective and

$$H^1(Z(u_j, \underline{i}_j), T_{(u_j, \underline{i}_j)}) \longrightarrow H^1(Z(u_{j-1}, \underline{i}_{j-1}), T_{(u_{j-1}, \underline{i}_{j-1})})$$

is an isomorphism. Therefore, the homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})$$

of B-modules is surjecive and the homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})$$

of B-modules is an isomorphism.

Further since $u_1^{-1}(\alpha_0) < 0$, by [5, Lemma 6.2], we have

$$H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})_{-\alpha_0} \neq 0.$$

By [5, Theorem 7.1], $H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})})$ is a parabolic Lie subalgebra of $\mathfrak{g}$ and hence there is a unique B-stable line in $H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})})$, namely $\mathfrak{g}_{-\alpha_0}$. Therefore we conclude that the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})$$

of B-modules is an isomorphism. ∎

The following is a useful corollary.

Corollary 7.4. *If $\mu \in X(T) \setminus \{0\}$, then $\dim(H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})_\mu) \leq 1$.*

Proof. By [5, Theorem 7.1], $H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})})$ is a parabolic Lie subalgebra of $\mathfrak{g}$. By Lemma 7.3(1), we have

$$H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \simeq H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)}) \quad \text{(as } B\text{-modules)}.$$

Hence for any $\mu \in X(T) \setminus \{0\}$, we have

$$\dim(H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})_\mu) \leq 1. \qquad \blacksquare$$

Let $u_1' = w_k[a_k, n-1]$ and let $\underline{i}_1'$ be the tuple corresponding to the reduced expression $(\prod_{l_1=1}^k [a_{l_1}, n])[a_k, n-1]$. Then, we have

Lemma 7.5. *Let $\mu \in X(T) \setminus \{0\}$.*

1. If $H^1(u_1, \alpha_n)_\mu = 0$, then $\dim(H^0(Z(u_1', \underline{i}_1'), T_{(u_1', \underline{i}_1')})_\mu) \leq 1$.

2. If $H^1(u_1, \alpha_n)_\mu \neq 0$, then $\dim(H^0(Z(u'_1, \underline{i}'_1), T_{(u'_1, \underline{i}'_1)})_\mu) = 2$.

Proof. By LES, we have the following long exact sequence of B-modules:

$$0 \longrightarrow H^0(u_1, \alpha_n) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)}) \longrightarrow H^0(Z(u'_1, \underline{i}'_1), T_{(u'_1, \underline{i}'_1)}) \longrightarrow$$
$$\longrightarrow H^1(u_1, \alpha_n) \longrightarrow \cdots \quad (7.5.1)$$

Proof of (1): If $H^1(u_1, \alpha_n)_\mu = 0$ and $\mu \in X(T) \setminus \{0\}$, then by the above exact sequence the natural homomorphism (of T-modules)

$$H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})_\mu \longrightarrow H^0(Z(u'_1, \underline{i}'_1), T_{(u'_1, \underline{i}'_1)})_\mu$$

is surjective. By Corollary 7.4, we have $\dim(H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})_\mu) \leq 1$. Hence we conclude that $\dim(H^0(Z(u'_1, \underline{i}'_1), T_{(u'_1, \underline{i}'_1)})_\mu) \leq 1$.

Proof of (2): If $H^1(u_1, \alpha_n)_\mu \neq 0$, then by Proposition 6.2, we have $\mu \in R_s^- \setminus (-S)$. Also note that $\dim(H^1(u_1, \alpha_n)_\mu) = 1$. Hence by the above arguments, we see that if $H^1(u_1, \alpha_n)_\mu \neq 0$, then

$$\dim(H^0(Z(u'_1, \underline{i}'_1), T_{(u'_1, \underline{i}'_1)})_\mu) \leq 2 \quad (7.5.2)$$

Therefore, we have the following observations;

- Note that $H^0(u_1, \alpha_n) = 0$. Hence by (7.5.1), we see that $H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})$ is a B-submodule of $H^0(Z(u'_1, \underline{i}'_1), T_{(u'_1, \underline{i}'_1)})$.

- $H^0(u'_1, \alpha_{n-1})$ is a B-submodule of $H^0(Z(u'_1, \underline{i}'_1), T_{(u'_1, \underline{i}'_1)})$.

- Since $-\alpha_0$ is a long root, we note that $H^0(u'_1, \alpha_{n-1})_{-\alpha_0} = 0$.

By Lemma 7.3, we have $H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})$. By [5, Theorem 7.1], $H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})})$ is a parabolic Lie subalgebra of $\mathfrak{g}$ and hence it has a unique B-stable one-dimensional subspace, namely $\mathfrak{g}_{-\alpha_0}$. By above discussion, the B-submodule $H^0(u'_1, \alpha_{n-1}) \cap H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})$ of $H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})})$ does not contain $\mathfrak{g}_{-\alpha_0}$ and so it is zero.

By Lemma 6.8, we have $H^1(u_1, \alpha_n) = H^0(u'_1, \alpha_{n-1})$. Therefore, if $H^1(u_1, \alpha_n)_\mu \neq 0$, then we have

$$\dim(H^0(Z(u'_1, \underline{i}'_1), T_{(u'_1, \underline{i}'_1)})_\mu) \geq 2 \quad (7.5.3)$$

Hence from (7.5.2) and (7.5.3), we conclude that

$$\dim(H^0(Z(u'_1, \underline{i}'_1), T_{(u'_1, \underline{i}'_1)})_\mu) = 2.$$

This completes the proof of (2). ∎

Corollary 7.6. *The natural homomorphism*

$$H^0(Z(u'_1, \underline{i}'_1), T_{(u'_1, \underline{i}'_1)}) \longrightarrow H^1(u_1, \alpha_n)$$

is surjective.

Proof.   Note that by Proposition 6.2, if $H^1(u_1, \alpha_n)_\mu \neq 0$ then $\mu \in R_s^- \setminus (-S)$. By Corollary 7.4, we have $\dim(H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})_\mu) \leq 1$. By Lemma 7.5, if $H^1(u_1, \alpha_n)_\mu \neq 0$, then

$$\dim(H^0(Z(u_1', \underline{i}_1'), T_{(u_1', \underline{i}_1')})_\mu) = 2.$$

Hence by the exact sequence (7.5.1) we conclude that the natural homomorphism

$$H^0(Z(u_1', \underline{i}_1'), T_{(u_1', \underline{i}_1')})_\mu \longrightarrow H^1(u_1, \alpha_n)_\mu$$

is surjective. ∎

Let $1 \leq r \leq k$. Let $\underline{j}_r = (i_{a_1}, \ldots, n, i_{a_2}, \ldots, n, \ldots, i_{a_r}, \ldots, n)$ and $\underline{j}_r' = (i_{a_1}, \ldots, n, i_{a_2}, \ldots, n, \ldots, i_{a_{r-1}}, \ldots, n, i_{a_r}, \ldots, n-1)$.

We now prove

Lemma 7.7.

1. If $H^1(w_m, \alpha_n)_\mu = 0$ for all $r \leq m \leq k$ and $H^1(u_1, \alpha_n)_\mu = 0$, then $\dim(H^0(Z(\tau_r, \underline{j}_r'), T_{(\tau_r, \underline{j}_r')})_\mu) \leq 1$.

2. If $H^1(w_r, \alpha_n)_\mu \neq 0$, then $\dim(H^0(Z(\tau_r, \underline{j}_r'), T_{(\tau_r, \underline{j}_r')})_\mu) = 2$ and hence the natural homomorphism

$$H^0(Z(\tau_r, \underline{j}_r'), T_{(\tau_r, \underline{j}_r')})_\mu \longrightarrow H^1(w_r, \alpha_n)_\mu$$

is surjective.

Proof.   Proof of (1): If $H^1(u_1, \alpha_n)_\mu = 0$. Then by Lemma 7.5, we have $\dim(H^0(Z(u_1', \underline{i}_1'), T_{(u_1', \underline{i}_1')})_\mu) \leq 1$. By Lemma 7.2, the natural homomorphism

$$H^0(Z(u_1', \underline{i}_1'), T_{(u_1', \underline{i}_1')})_\mu \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)})_\mu$$

is surjective. If $H^1(w_m, \alpha_n)_\mu = 0$ for all $r \leq m \leq k$, by using LES, we see that the natural homomorphism

$$H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)})_\mu \longrightarrow H^0(Z(\tau_r, \underline{j}_r'), T_{(\tau_r, \underline{j}_r')})_\mu$$

is surjective. Therefore, we have $\dim(H^0(Z(\tau_r, \underline{j}_r'), T_{(\tau_r, \underline{j}_r')})_\mu) \leq 1$.

Proof of (2): If $H^1(w_r, \alpha_n)_\mu \neq 0$, then by Corollary 6.9, we have $H^1(w_m, \alpha_n)_\mu = 0$ for all $r + 1 \leq m \leq k$ and $H^1(u_1, \alpha_n)_\mu = 0$. Then by (1) we have

$$\dim(H^0(Z(\tau_{r+1}, \underline{j}_{r+1}'), T_{(\tau_{r+1}, \underline{j}_{r+1}')})_\mu) \leq 1.$$

By Lemma 7.2, the natural homomorphism

$$H^0(Z(\tau_{r+1}, \underline{j}_{r+1}'), T_{(\tau_{r+1}, \underline{j}_{r+1}')})_\mu \longrightarrow H^0(Z(w_r, \underline{j}_r), T_{(w_r, \underline{j}_r)})_\mu$$

is surjecive. Hence $\dim(H^0(Z(w_r, \underline{j}_r), T_{(w_r, \underline{j}_r)})_\mu) \leq 1$.

By LES we have the following long exact sequence of B-modules:

$$0 \longrightarrow H^0(w_r, \alpha_n) \longrightarrow H^0(Z(w_r, \underline{j}_r), T_{(w_r, \underline{j}_r)}) \longrightarrow H^0(Z(\tau_r, \underline{j}'_r), T_{(\tau_r, \underline{j}'_r)}) \longrightarrow$$
$$\longrightarrow H^1(w_r, \alpha_n) \longrightarrow \cdots$$

Since $\dim(H^0(Z(w_r, \underline{j}_r), T_{(w_r, \underline{j}_r)})_\mu) \leq 1$ and $\dim(H^1(w_r, \alpha_n)_\mu) = 1$, we see that

$$\dim(H^0(Z(\tau_r, \underline{j}'_r), T_{(\tau_r, \underline{j}'_r)})_\mu) \leq 2 \qquad (7.7.1)$$

Let $\underline{j}''_r$ be the tuple corresponding to the reduced expression $(\prod_{l_1=1}^{k-1}[a_{l_1}, n])[a_k, n-2]$ of $\tau_r s_{n-1}$. By LES we have the following exact sequence of B-modules:

$$0 \longrightarrow H^0(\tau_r, \alpha_{n-1}) \longrightarrow H^0(Z(\tau_r, \underline{j}'_r), T_{(\tau_r, \underline{j}'_r)}) \longrightarrow$$
$$\longrightarrow H^0(Z(\tau_r s_{n-1}, \underline{j}''_r), T_{(\tau_r s_{n-1}, \underline{j}''_r)}) \longrightarrow 0 \quad (7.7.2)$$

By Lemma 7.2, the natural homomorphism

$$H^0(Z(\tau_r s_{n-1}, \underline{j}''_r), T_{(\tau_r s_{n-1}, \underline{j}''_r)}) \longrightarrow H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})})$$

is surjective.

Observe that, by LES we have $H^0(w_{r-1}, \alpha_n)$ is a B-submodule of $H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})})$. If $H^1(w_r, \alpha_n)_\mu \neq 0$ then by Corollary 6.6, we have $H^0(\tau_r, \alpha_{n-1})_\mu \neq 0$ and by Corollary 6.7, we have $H^0(w_{r-1}, \alpha_n)_\mu \neq 0$. Therefore, we have

$$\dim(H^0(Z(\tau_r s_{n-1}, \underline{j}''_r), T_{(\tau_r s_{n-1}, \underline{j}''_r)})_\mu) \geq \dim(H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})})_\mu) \geq 1.$$

Hence by above exact sequence (7.7.1), we observe that

$$\dim(H^0(Z(\tau_r, \underline{j}'_r), T_{(\tau_r, \underline{j}'_r)})_\mu) \geq 2 \qquad (7.7.3)$$

Therefore, from (7.7.1) and (7.7.3) we conclude that $\dim(H^0(Z(\tau_r, \underline{j}'_r), T_{(\tau_r, \underline{j}'_r)})_\mu) = 2$. This completes the proof of (2). ∎

8. Main Theorem

In this section we prove the main theorem.

Recall that $G = PSp(2n, \mathbb{C})$ $(n \geq 3)$, and let c be a Coxeter element in W. Also, recall that c has a reduced expression $c = [a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$, where $[i, j]$ for $i \leq j$ denotes $s_i s_{i+1} \cdots s_j$ and $n \geq a_1 > a_2 > \ldots > a_k = 1$.

Let $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^n)$ be a sequence corresponding to a reduced expression of w_0, where $\underline{i}^r$ $(1 \leq r \leq n)$ is a sequence of reduced expressions of c (see Lemma 4.4). Then, we have

Theorem 8.1. $H^j(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ for all $j \geq 1$ if and only if $a_1 \neq n - 1$ and $a_2 \leq n - 2$.

Proof.   By [5, Proposition 3.1], we have $H^j(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ for all $j \geq 2$. So, by Corollary 3.2(2), it is enough to prove the following:

$H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ for all $j \geq 1$ if and only if c is of the form $[a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$ with $a_1 \neq n - 1$ and $a_2 \leq n - 2$.

Proof of ($\Longrightarrow$): If $a_1 = n - 1$, then $c = s_{n-1}s_n v$ for some $v \in W_J$, where $J = S \setminus \{\alpha_n, \alpha_{n-1}\}$. Take $w = s_{n-1}s_n$ and $\underline{i}' = (n - 1, n)$ in LES, then we have

$$0 \longrightarrow H^0(w, \alpha_n) \longrightarrow H^0(Z(w, \underline{i}'), T_{(w, \underline{i}')}) \longrightarrow H^0(s_{n-1}, \alpha_{n-1}) \longrightarrow$$
$$\longrightarrow H^1(w, \alpha_n) \xrightarrow{f} H^1(Z(w, \underline{i}'), T_{(w, \underline{i}')}) \longrightarrow 0.$$

By a simple calculation, we see that $H^1(s_{n-1}s_n, \alpha_n) = \mathbb{C}_{\alpha_n + \alpha_{n-1}}$ and $H^0(s_{n-1}, \alpha_{n-1})_{\alpha_n + \alpha_{n-1}} = 0$. Hence f is a non zero homomorphism. Hence $H^1(Z(w, \underline{i}'), T_{(w, \underline{i}')}) \neq 0$. By Lemma 7.1, the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \longrightarrow H^1(Z(w, \underline{i}'), T_{(w, \underline{i}')})$$

is surjective. Hence we have

$$H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \neq 0.$$

If $a_2 = n - 1$, then $c = s_n s_{n-1} v$ for some $v \in W_J$, where $J = S \setminus \{\alpha_n, \alpha_{n-1}\}$. Let $\underline{i}''$ be a reduced expression of $c = s_n s_{n-1} v$ and $\underline{i}'''$ be a be the reduced expression of cs_n such that $\underline{i}''' = (i'_1, \ldots, i'_n, n)$ with $(i'_1, \ldots, i'_n) = \underline{i}''$. Since $\langle \alpha_n, \alpha_j \rangle = 0$ for $j \in \{1, 2, \ldots, n - 2\}$, we have

$$H^i(cs_n, \alpha_n) = H^i(s_n s_{n-1} s_n, \alpha_n) \text{ for } \geq 0.$$

By *SES*, we see that $H^1(s_n s_{n-1} s_n, \alpha_n) = \mathbb{C}_{\alpha_{n-1}}$. By [5, Proposition 6.3], we have

$$H^0(Z(c, \underline{i}''), T_{(c, \underline{i}'')})_{\alpha_{n-1}} = 0.$$

Hence, by LES, we conclude that $H^1(Z(cs_n, \underline{i}'''), T_{(cs_n, \underline{i}''')})_{\alpha_{n-1}} \neq 0$. By Lemma 7.1, the natural homomorphism $H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \longrightarrow H^1(Z(cs_n, \underline{i}'''), T_{(cs_n, \underline{i}''')})$ is surjective. Hence, we have

$$H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \neq 0.$$

Proof of ($\Longleftarrow$): Assume that $a_1 \neq n - 1$ and $a_2 \leq n - 2$. By Lemma 7.3(2), the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})$$

of B-modules is an isomorphism. By LES, we have the following exact sequence of B-modules:

$$\cdots \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)}) \longrightarrow H^0(Z(u'_1, \underline{i}'_1), T_{(u'_1, \underline{i}'_1)}) \xrightarrow{h_1}$$
$$\longrightarrow H^1(u_1, \alpha_n) \longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)}) \longrightarrow H^1(Z(u'_1, \underline{i}'_1), T_{(u'_1, \underline{i}'_1)}) \longrightarrow 0.$$

By Corollary 7.6, we see that the natural homomorphism

$$h_1 : H^0(Z(u_1', \underline{i}_1'), T_{(u_1',\underline{i}_1')}) \longrightarrow H^1(u_1, \alpha_n)$$

is surjecive. Therefore, the natural homomorphism

$$H^1(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)}) \longrightarrow H^1(Z(u_1', \underline{i}_1'), T_{(u_1',\underline{i}_1')})$$

is an isomorphism. Hence the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_1', \underline{i}_1'), T_{(u_1',\underline{i}_1')})$$

is an isomorphism. By Lemma 7.2(2), the natural homomorphism

$$H^1(Z(u_1', \underline{i}_1'), T_{(u_1',\underline{i}_1')}) \longrightarrow H^1(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)})$$

is an isomorphism. Therefore, the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)})$$

is an isomorphism. By LES, we have the following exact sequence of B-modules:

$$\cdots \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \longrightarrow H^0(Z(\tau_k, \underline{j}_k'), T_{(\tau_k,\underline{j}_k')}) \xrightarrow{h_2}$$
$$\longrightarrow H^1(w_k, \alpha_n) \longrightarrow H^1(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \xrightarrow{h_3} H^1(Z(\tau_k, \underline{j}_k'), T_{(\tau_k,\underline{j}_k')}) \longrightarrow 0.$$

By Lemma 7.7(2), we see that the map

$$h_2 : H^0(Z(\tau_k, \underline{j}_k'), T_{(\tau_k,\underline{j}_k')}) \longrightarrow H^1(w_k, \alpha_n)$$

is surjecive. Therefore, the map

$$h_3 : H^1(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \longrightarrow H^1(Z(\tau_k, \underline{j}_k'), T_{(\tau_k,\underline{j}_k')})$$

is an isomorphism.

Thus, the natural map

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(\tau_k, \underline{j}_k'), T_{(\tau_k,\underline{j}_k')})$$

is an isomorphism. By using Lemma 7.2(2) and Lemma 7.7(2) repeatedly, we see that the natural map

$$H^1(Z(\tau_k, \underline{j}_k'), T_{(\tau_k,\underline{j}_k')}) \longrightarrow H^1(Z(\tau_r, \underline{j}_r'), T_{(\tau_r,\underline{j}_r')})$$

is an isomorphism for all $1 \leq r \leq k-1$.

Note that $\tau_1 \in W_{S\setminus\{\alpha_n\}}$. By taking $u = id$ and $v = \tau_1$ in Lemma 7.2(2), we see that $H^1(Z(\tau_1, \underline{j}_1'), T_{(\tau_1,\underline{j}_1')}) = 0$. Therefore, we conclude that $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$. This completes the proof of the theorem. ∎

Corollary 8.2. *Let* c *be a Coxeter element such that* c *is of the form* $[a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$ *with* $a_1 \neq n - 1$, $a_2 \leq n - 2$ *and* $a_k = 1$. *Let* $(w_0, \underline{i})$ *be a reduced expression of* w_0 *in terms of* c *as in Theorem 8.1. Then,* $Z(w_0, \underline{i})$ *has no deformations.*

Proof. By Theorem 8.1 and by [5, Proposition 3.1], we have $H^i(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ for all $i > 0$. Hence, by [14, p. 272, Proposition 6.2.10], we see that $Z(w_0, \underline{i})$ has no deformations. ∎

References

[1] Balaji, V., S. S. Kannan, and K. V. Subrahmanyam, *Cohomology of line bundles on Schubert varieties-I*, Transformation Groups **9** (2004), 105–131.

[2] Bott, R., *Homogeneous vector bundles*, Annals of Math., Ser. 2, **66** (1957), 203–248.

[3] Bott, R., and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964–1029.

[4] Brion, M., and S. Kumar, "Frobenius splitting methods in geometry and representation theory," Progress in Mathematics **231**, Birkhäuser, Boston, MA, 2005.

[5] Chary, B. N., S. S. Kannan, and A. J. Parameswaran, *Automorphism group of a Bott–Samelson–Demazure–Hansen Variety*, Transformation Groups **20** (2015), 665–698.

[6] Demazure, M., *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. **7** (1974), 53–88.

[7] —, *A very simple proof of Bott's theorem*, Invent. Math. **33** (1976), 271–272.

[8] Hansen, H. C., *On cycles on flag manifolds*, Math. Scand. **33** (1973), 269–274.

[9] Hartshorne, R., "Algebraic Geometry," Graduate Texts in Mathematics, Springer-Verlag, Berlin–Heidelberg, 1977.

[10] Humphreys, J. E., "Conjugacy classes in semisimple algebraic groups," Math. Surveys Monographs **43**, Amer. Math. Soc., Providence, RI, 1995.

[11] —, "Introduction to Lie algebras and Representation theory,", Springer-Verlag, Berlin–Heidelberg–New York, 1972.

[12] —, "Linear Algebraic Groups,", Springer-Verlag, Berlin–Heidelberg–New York, 1975.

[13] —, "Reflection groups and Coxeter groups,", Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, 1992.

[14] Huybrechts, D., "Complex Geometry: An Introduction," Springer-Verlag, Berlin–Heidelberg–New York, 2005.

[15] Jantzen, J. C., "Representations of Algebraic Groups," Pure and Appl. Math., Academic Press, 1987.

[16] Kannan, S. S., *On the Automorphism Group of a Smooth Schubert Variety*, Algebras and Representation Theory **19** (2016), 761–782.

[17] Seshadri, C. S., "Introduction to Standard Monomial theory," Lecture Notes **4**, Brandeis University, Waltham, MA, 1985.

[18] Springer, T. A., "Linear Algebraic Groups," Progress in Mathematics **9**, Birkhäuser, Boston, MA, 2009.

[19] Stembridge, J. R., *On the fully commutative elements of Coxeter groups*, Journal of Algebraic Combinatorics **5** (1996), 353–385.

[20] ―, *Some combinatorial aspects of reduced words in finite Coxeter groups*, Trans. Amer. Math. Soc. **349** (1997), 1285–1332.

[21] Yang, S. W., and A. Zelevinsky, *Cluster algebras of finite type via Coxeter elements and principal minors*, Transformation Groups **13** (2008), 855–895.

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