Reduced Hyperbolic Root Systems of Rank 2

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Abstract. We define hyperbolic root systems and classify the special case, namely, reduced hyperbolic root systems of rank 2. We also determine the Weyl groups of the root systems.
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1. Introduction

K. Saito generalized the classical concept of root systems in 1985 as follows [S]:

Definition 1.1. Let V be a vector space over $\mathbb{R}$ with a symmetric bilinear form $(\cdot, \cdot)$. A subset Δ of V is called a *generalized root system* or simply a *root system* if Δ satisfies:

(A1) $(\alpha, \alpha) \neq 0$ for all $\alpha \in \Delta$;

(A2) Δ spans V;

(A3) $\langle \alpha, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$, where $\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}$
(crystallographic condition);

(A4) $\sigma_\alpha(\beta) \in \Delta$ for all $\alpha, \beta \in \Delta$, where $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$;

(A5) $\Delta = \Delta' \cup \Delta''$ and $(\Delta', \Delta'') = 0$ imply $\Delta' = \emptyset$ or $\Delta'' = \emptyset$ (irreducibility).

The dimension of V is called the *rank* of Δ, and Δ is called *reduced* if $2\alpha \notin \Delta$ for all $\alpha \in \Delta$.

This concept contains many examples which are crucial in Lie theory.

Example 1.2. (1) Δ is a *finite irreducible root system* if $\dim V < \infty$ and $(\cdot, \cdot)$ is positive definite. It has type A_ℓ, B_ℓ, C_ℓ, D_ℓ, BC_ℓ, E_6, E_7, E_8, F_4 or G_2.

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(2) Δ is a *locally finite irreducible root system* if $(\cdot, \cdot)$ is positive definite. It has type A_∞, B_∞, C_∞, D_∞ or BC_∞ (see [LN1], [NS], [St]).

(3) Δ is an *affine root system* if $\dim V < \infty$ and $(\cdot, \cdot)$ is positive semidefinite of radical dimension 1 (see [M]). Reduced affine root systems are exactly the set of *real roots* of affine Kac-Moody Lie algebras.

(4) Δ is an *elliptic root system* if $\dim V < \infty$ and $(\cdot, \cdot)$ is positive semidefinite of radical dimension 2 (see [S]).

(5) Δ is an *extended affine root system* if $\dim V < \infty$ and $(\cdot, \cdot)$ is positive semidefinite (see [S]). There is a different definition for an extended affine root system in [A-P] which contains null roots. But both concepts are essentially same (see [A]).

(6) Δ is a *locally affine root system* if $(\cdot, \cdot)$ is positive semidefinite of radical dimension 1 (see [N], [Y1], [MY2]).

(7) Δ is a *locally extended affine root system* if $(\cdot, \cdot)$ is positive semidefinite (see [MY1]). There is a more general concept of root systems in [LN2] which contain null roots.

We note that when the vector space V has non-trivial radical, it is more natural to define root systems over $\mathbb{Q}$ (see [Y1]).

We define the isomorphisms between root systems.

Definition 1.3. A generalized root system (V, Δ) and (V', Δ') are called *isomorphic*, denoted by $\Delta \cong \Delta'$, if there is a linear isomorphism $\varphi : V \longrightarrow V'$ such that $\varphi(\Delta) = \Delta'$ and $\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Delta$.

Now we define a special type of root systems, which are not yet studied much.

Definition 1.4. A generalized root system Δ of V with $\dim V = n \geq 2$ is called a *hyperbolic root system* if $(\cdot, \cdot)$ has the signature $(n-1, 1)$. (One can say that V is a Minkowski space.)

We have classified the reduced hyperbolic root systems of rank 2. There are 10 types, up to isomorphisms. The main purpose of this paper is to explain the classification. Let us briefly describe it.

We fix a basis $\{v_1, v_2\}$ of V such that $(v_1, v_1) = 1 = -(v_2, v_2)$ and $(v_1, v_2) = 0$. Using hyperbolic trigonometric functions, an example of such a root system is

$$\Delta^\theta = \{\pm \alpha_{k\theta} \mid k \in \mathbb{Z}\} \quad \text{whrere} \quad \alpha_\phi = \begin{pmatrix} \cosh \phi \\ \sinh \phi \end{pmatrix}$$

for some fixed angle θ satifying $2\cosh\theta \in \mathbb{Z}_{\geq 3}$. Note that $(\alpha, \alpha) = 1$ for all $\alpha \in \Delta^\theta$. In general, one can consider the subsets of roots of a fixed hyperbolic length by putting for $r \in \mathbb{R}_{>0}$,

$$\Delta_r = \{\alpha \in \Delta \mid (\alpha, \alpha) = r^2\} \quad \text{and} \quad \Delta_{-r} = \{\alpha \in \Delta \mid (\alpha, \alpha) = -r^2\}.$$

Then any root system Δ as defined above decomposes

$$\Delta = \bigsqcup_{0 \neq s \in \mathbb{R}} \Delta_s.$$

The main results of the paper can be summarized in terms of this partition of Δ as follows.

- Each non-empty Δ_s is an infinite set, and in fact, a reduced hyperbolic root system of rank 2 (Lemma 5.1).

- Up to canonical isomorphisms (scaling and rotating), one can assume that

$$\emptyset \neq \Delta_1 \subset \Delta \quad \text{and} \quad \Delta_1 = \Delta^\theta,$$

  as defined above for an appropriate θ (Proposition 5.4).

- Exactly one of the following 5 cases occurs where all r, r_i are positive real numbers (Section 6):

  (1) $\Delta = \Delta^\theta$

  (2) $\Delta = \Delta^\theta \sqcup \Delta_r$

  (3) $\Delta = \Delta^\theta \sqcup \Delta_{-r}$ (3 types)

  (4) $\Delta = \Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2}$ (3 types)

  (5) $\Delta = \Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2} \sqcup \Delta_{-r_3}$ (2 types)

  Thus, there are altogether 10 types.

- Each of the 10 types is described in more detail in the sections 7-15.

The cases (1) and (2) are precisely the real roots of a hyperbolic Kac-Moody Lie algebra of rank 2 (see [Mo], [LM]). No realization of the other cases as roots of a Lie algebra or Lie group seems to be known.

A hyperbolic root system usually means the roots of a hyperbolic Kac-Moody Lie algebra or the roots of a hyperbolic Coxeter group, but our hyperbolic root systems are more general (cf. [B], [KM], [Mor]). Note that hyperbolic Coxeter groups can only exist in rank ≥ 3 (see [H, p.140 Exercise]). There is another definition of a hyperbolic root system in [Ni], but this is also imposing some strong conditions. Thus, the classification of hyperbolic root systems of rank 3 in our sense is far beyond the goal.

In the last section we determined the Weyl groups of the root systems in Theorem 17.1. Finally, we thank Professor Jun Morita and the referee for helpful suggestions.

2. Some properties of hyperbolic functions

For the classification of hyperbolic root systems, we show some properties of $\cosh$ and $\sinh$. We often use the addition formulas:

$$\cosh(\phi + \psi) = \cosh\phi\cosh\psi + \sinh\phi\sinh\psi$$
$$\sinh(\phi + \psi) = \sinh\phi\cosh\psi + \cosh\phi\sinh\psi.$$

First, adding the following two equations

$$\cosh(k+1)\theta = \cosh k\theta\cosh\theta + \sinh k\theta\sinh\theta$$
$$\cosh(k-1)\theta = \cosh k\theta\cosh\theta - \sinh k\theta\sinh\theta$$

for $k \in \mathbb{Z}$, we get

$$\cosh(k+1)\theta = -\cosh(k-1)\theta + 2\cosh k\theta\cosh\theta.$$

If we put $T_k := \cosh k\theta$, then we have

$$T_{k+1} = 2T_k\cosh\theta - T_{k-1}, \tag{1}$$

which can be considered as an integral polynomial of $\cosh\theta$. Note that T_k is the Chebyshev (Tschebyscheff) polynomial of the first kind. So, if $\cosh\theta \in \mathbb{Z}$, then $T_k \in \mathbb{Z}$ since $T_0 = 1$ and $T_1 = \cosh\theta$.

Multiplying by 2 on both sides of (1), we get

$$2T_{k+1} = (2\cosh\theta)2T_k - 2T_{k-1}, \tag{2}$$

which implies that $2T_k$ is an integral polynomial of $2T_1 = 2\cosh\theta$ since $2T_0 = 2$ and $2T_1 = 2\cosh\theta$. Thus we obtain:

Lemma 2.1.   (i) If $\cosh\theta \in \mathbb{Z}$, then $T_k = \cosh k\theta \in \mathbb{Z}$ for all $k \in \mathbb{Z}$.
      (ii) If $2\cosh\theta \in \mathbb{Z}$, then $2T_k = 2\cosh k\theta \in \mathbb{Z}$ for all $k \in \mathbb{Z}$.   ∎

Let

$$C_n(t) := t\cosh\tfrac{n\theta}{2}$$

for $t \in \mathbb{R}$. Adding two equations

$$\cosh(k+\tfrac{1}{2})\theta = \cosh k\theta\cosh\tfrac{\theta}{2} + \sinh k\theta\sinh\tfrac{\theta}{2}$$
$$\cosh(k-\tfrac{1}{2})\theta = \cosh k\theta\cosh\tfrac{\theta}{2} - \sinh k\theta\sinh\tfrac{\theta}{2}$$

and multiplying with t on both sides, we get

$$C_{2k+1}(t) + C_{2k-1}(t) = 2tT_k\cosh\tfrac{\theta}{2}. \tag{3}$$

Thus:

Lemma 2.2.   If $2\cosh\theta \in \mathbb{Z}$ and $t\cosh\tfrac{\theta}{2} \in \mathbb{Z}$, then $C_{2k+1}(t) = t\cosh(k+\tfrac{1}{2})\theta \in \mathbb{Z}$ for all $k \in \mathbb{Z}$.

Proof.   We have $C_1(t) = t \cosh \frac{\theta}{2} \in \mathbb{Z}$, and $2T_k \in \mathbb{Z}$ by Lemma 2.1. Hence, by the recursive formula (3) and induction on k, we obtain $C_{2k+1}(t) \in \mathbb{Z}$.   ∎

Next, adding the following two equations

$$\sinh(k+1)\theta = \sinh k\theta \cosh \theta + \cosh k\theta \sinh \theta \tag{4}$$
$$\sinh(k-1)\theta = \sinh k\theta \cosh \theta - \cosh k\theta \sinh \theta$$

for $k \in \mathbb{Z}$, we get

$$\sinh(k+1)\theta = -\sinh(k-1)\theta + 2\sinh k\theta \cosh \theta. \tag{5}$$

If we put $U_k := \frac{\sinh k\theta}{\sinh \theta}$, then we have

$$U_{k+1} = 2U_k \cosh \theta - U_{k-1},$$

which can be considered as an integral polynomial of $2\cosh\theta$ since $U_0 = 0$ and $U_1 = 1$. Note that U_k is the Chebyshev polynomial of the second kind. (Note that T_k is not an integral polynomial of $2\cosh\theta$ since $T_1 = \cosh\theta$.) Thus we obtain:

Lemma 2.3.   *If $2\cosh\theta \in \mathbb{Z}$, then $U_k \in \mathbb{Z}$ for all $k \in \mathbb{Z}$.*

We will use the following relation

$$U_{k+1} = U_k \cosh \theta + T_k \tag{6}$$

which follows from (4).

Remark 2.4.   If $2\cosh\theta = 3$, then the sequence $\{U_k\}_{k\geq 0}$ is a subsequence of Fibonacci sequence $\{f_k\}_{k\geq 0}$, i.e., $f_{k+1} = f_k + f_{k-1}$ with $f_0 = 0$ and $f_1 = 1$. More precisely, we have $U_k = f_{2k}$ for $k \geq 0$. Also, by (6), we have $T_k = U_{k+1} - \frac{3}{2}U_k = f_{2k+2} - \frac{3}{2}f_{2k}$ (see [K, p.77]).

Let
$$S_k(t) := t \sinh k\theta$$

for $t \in \mathbb{R}$. Multiplying with t on both sides of (5), we get the recursive formula $S_{k+1}(t) = 2S_k(t)\cosh\theta - S_{k-1}(t)$. Thus, as in Lemma 2.2, we obtain:

Lemma 2.5.   *If $2\cosh\theta \in \mathbb{Z}$ and $S_1(t) = t\sinh\theta \in \mathbb{Z}$, then $S_k(t) \in \mathbb{Z}$ for all $k \in \mathbb{Z}$.*

Finally, let
$$V_n(t) := t \sinh \frac{2\theta}{n}$$

for $t \in \mathbb{R}$. We claim that

$$\sinh(k+\frac{1}{2})\theta = (U_{k+1} + U_k)\sinh\frac{\theta}{2}. \tag{7}$$

In fact, we have

$$
\begin{aligned}
\sinh(k + \tfrac{1}{2})\theta &= \sinh k\theta \cosh \tfrac{\theta}{2} + \cosh k\theta \sinh \tfrac{\theta}{2} \\
&= U_k \sinh \theta \cosh \tfrac{\theta}{2} + T_k \sinh \tfrac{\theta}{2} \\
&= (2U_k \cosh^2 \tfrac{\theta}{2} + T_k) \sinh \tfrac{\theta}{2} \\
&= (U_k \cosh \theta + U_k + T_k) \sinh \tfrac{\theta}{2} \quad \text{since } 2\cosh^2 \tfrac{\theta}{2} = \cosh \theta + 1. \\
&= (U_{k+1} + U_k) \sinh \tfrac{\theta}{2} \quad \text{by (6).}
\end{aligned}
$$

Thus:

Lemma 2.6.   *If* $2\cosh \theta \in \mathbb{Z}$ *and* $t \sinh \tfrac{\theta}{2} \in \mathbb{Z}$, *then* $V_{2k+1}(t) = t \sinh(k + \tfrac{1}{2})\theta \in \mathbb{Z}$ *for all* $k \in \mathbb{Z}$.

Proof.   This follows from (7) and Lemma 2.3.   ■

3.   Basic properties

In a generalized root system Δ, we have $\sigma_\alpha(\alpha) = -\alpha$. Thus, letting

$$
-\Delta := \{-\alpha \mid \alpha \in \Delta\},
$$

Lemma 3.1. $-\Delta = \Delta$.

We call (α, α) the *length* of α for convenience. So there may be negative lengths. A linear isomorphism $\varphi : V \longrightarrow V'$ is called *orthogonal* if $(\varphi(v), \varphi(w)) = (v, w)$ for all $v, w \in V$.

Lemma 3.2.   *Suppose that* $\varphi : V \longrightarrow V'$ *is an orthogonal linear map. Then we have* $\Delta \cong \varphi(\Delta)$. *In particular,* $\Delta \cong \sigma_\beta(\Delta)$ *for any* $\beta \in V$ *with* $(\beta, \beta) \neq 0$.
     Moroever, for any $r \in \mathbb{R}^\times$, *we have* $\Delta \cong r \cdot \varphi(\Delta)$. *In particular, for the identity map* $id : V \longrightarrow V$, *we have* $\Delta \cong r \cdot id(\Delta) = r\Delta = \{r\alpha \mid \alpha \in \Delta\}$.
     Conversely, if φ *is an isomorphism, then* φ *preserves the ratio of lengths. More precisely, we have* $\frac{(\alpha,\alpha)}{(\beta,\beta)} = \frac{(\varphi(\alpha),\varphi(\alpha))}{(\varphi(\beta),\varphi(\beta))}$ *for all* $\alpha, \beta \in \Delta$.

Proof.   This is easily checked.   ■

Proposition 3.3.   *Suppose that* V *is finite-dimensional and* $(\cdot, \cdot)$ *is nondegenerate. Then* Δ *is a subset of a lattice in* V.

Proof.   Let $\{v_1, \ldots, v_n\} \subset \Delta$ be a basis of V. Snce V is finite-dimensional, the dual space of V is isomorphic to V. Thus, there exists a basis $\{w_1, \ldots, w_n\}$ of V such that $(w_i, v_j) = \delta_{i,j}$. Let $\alpha = \sum_{i=1}^n a_i w_i$ for $a_i \in \mathbb{R}$. Then we have $\langle \alpha, v_i \rangle = \frac{2a_i}{(v_i,v_i)} \in \mathbb{Z}$ for all i, and hence $a_i \in \mathbb{Z}b_i$, where $b_i = \frac{(v_i,v_i)}{2}$. Therefore, Δ is a subset of the lattice $\bigoplus_{i=1}^n \mathbb{Z}b_i w_i$.   ■

Now we specialize to the case of hyperbolic root systems of rank 2. Without

loss of generality, we can assume that $(\cdot, \cdot)$ is defined by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to some fixed basis. Thus the hyperbola $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x^2 - y^2 = 1 \right\}$ is the set of length 1, and one can put $\begin{cases} x = \pm \cosh \phi \\ y = \sinh \phi \end{cases}$ for $\phi \in \mathbb{R}$. Let

$$\alpha_\phi = \begin{pmatrix} \cosh \phi \\ \sinh \phi \end{pmatrix} \quad \text{and} \quad \beta_\phi = \begin{pmatrix} \sinh \phi \\ \cosh \phi \end{pmatrix}.$$

Thus any root can be written as $r\alpha_\phi$ or $s\beta_\psi$ for some $r, s, \phi, \psi \in \mathbb{R}$ $(r, s \neq 0)$.

We see that Δ is orthogonally transformed by the hyperbolic rotation

$$R_\theta = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix},$$

which satifies $R_\theta(\alpha_\phi) = \alpha_{\phi+\theta}$ and $R_\theta(\beta_\phi) = \beta_{\phi+\theta}$. Thus, by Lemma 3.2, we have $\Delta \cong R_\theta(\Delta)$ for any $\theta \in \mathbb{R}$. Note that

$$\begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}^k = \begin{pmatrix} \cosh k\theta & \sinh k\theta \\ \sinh k\theta & \cosh k\theta \end{pmatrix}$$

for all $k \in \mathbb{Z}$.

Remark 3.4. We often use later the euclidean reflection $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$S : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}.$$

This is not orthogonal, but one can easily show that $\Delta \cong S(\Delta)$.

The matrices of σ_{α_θ} and σ_{β_θ} are

$$\begin{pmatrix} -\cosh 2\theta & \sinh 2\theta \\ -\sinh 2\theta & \cosh 2\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cosh 2\theta & -\sinh 2\theta \\ \sinh 2\theta & -\cosh 2\theta \end{pmatrix},$$

respectively, and using these matrices, one can easily check the following:

Lemma 3.5. *For any $\phi, \psi \in \mathbb{R}$, we have:*
  (i) $\sigma_{\beta_\phi} = -\sigma_{\alpha_\phi}$ (ii) $\sigma_{\alpha_\phi}(\alpha_\psi) = -\alpha_{2\phi-\psi}$ (iii) $\sigma_{\alpha_\phi}(\beta_\psi) = \beta_{2\phi-\psi}$

  Hence, $\sigma_{\beta_\phi}(\alpha_\psi) = \alpha_{2\phi-\psi}$ and $\sigma_{\beta_\phi}(\beta_\psi) = -\beta_{2\phi-\psi}$.

4. Notations

Let Δ be a hyperbolic root system of rank 2 in $\mathbb{R}^2$ with the bilinear form $(\cdot, \cdot)$ defined by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to some fixed basis. We define some notations for the classification. Let

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid |x| > |y| \right\} \subset \mathbb{R}^2$$

be the positive cone. Let $\Delta^P := \Delta \cap P$ and $\Delta^N := \Delta \setminus \Delta^P$ so that

$$\Delta = \Delta^P \sqcup \Delta^N.$$

For $r \in \mathbb{R}_{>0}$, we define

$$\Delta_r := \{\alpha \in \Delta \mid (\alpha, \alpha) = r^2\} \quad \text{and} \quad \Delta_{-r} := \{\alpha \in \Delta \mid (\alpha, \alpha) = -r^2\}.$$

So we have

$$\Delta_r \subset \{\begin{pmatrix} x \\ y \end{pmatrix} \mid x^2 - y^2 = r^2\} \quad \text{and} \quad \Delta_{-r} \subset \{\begin{pmatrix} x \\ y \end{pmatrix} \mid y^2 - x^2 = r^2\}.$$

Let

$$\ell(\Delta^P) := \{r \in \mathbb{R}_{>0} \mid (\alpha, \alpha) = r^2, \ \alpha \in \Delta^P\}$$
$$\text{and} \quad \ell(\Delta^N) := \{-r \in \mathbb{R}_{<0} \mid (\alpha, \alpha) = -r^2, \ \alpha \in \Delta^N\}.$$

Thus we have

$$\Delta^P = \bigsqcup_{r \in \ell(\Delta^P)} \Delta_r \quad \text{and} \quad \Delta^N = \bigsqcup_{-s \in \ell(\Delta^N)} \Delta_{-s},$$

and define

$$\ell(\Delta) := \ell(\Delta^P) \cup \ell(\Delta^N),$$

which is, of course, disjoint.

5.   Classification of Δ_r

As already mentioned in Section 3, we have

$$\Delta \subset \{\pm r\alpha_\phi, \pm s\beta_\psi \mid r, s \in \mathbb{R}_{>0}, \phi, \psi \in \mathbb{R}\}.$$

Thus, for $r \in \mathbb{R}_{>0}$, we see that

$$\begin{cases} \Delta_r \subset \{\pm r\alpha_\phi \mid \phi \in \mathbb{R}\} & \text{if } r \in \ell(\Delta) \\ \Delta_{-r} \subset \{\pm r\beta_\phi \mid \phi \in \mathbb{R}\} & \text{if } -r \in \ell(\Delta). \end{cases}$$

Lemma 5.1.   *Let $s \in \ell(\Delta)$. Then Δ_s is an infinite set.*
     Moreover, Δ_s is a reduced hyperbolic root system of rank 2.

Proof.   By Remark 3.4, it is enough to show the case $s > 0$. We know that $\pm s\alpha_\phi \in \Delta_s$ for some $\phi \in \mathbb{R}$ (see Lemma 3.1). If $\Delta \subset \{\pm s\alpha_\phi\} \cup \bigcup_r \{\pm r\alpha_\phi, \pm r\beta_\phi\}$, then Δ cannot span $\mathbb{R}^2$, or Δ is reducible, i.e., Δ does not satisfy (A5). Thus there exists $r\alpha_\psi \in \Delta$ or $r\beta_\psi \in \Delta$ for some $\psi \neq \phi$. Hence, by Lemma 3.5, $\sigma_{\gamma_\psi}(s\alpha_\phi) = \pm s\alpha_{2\psi - \phi} \in \Delta_s$, where γ is either α or β. Since $2\psi - \phi \neq \phi$, we have $|\Delta_s| \geq 4$. Suppose that Δ_s is finite. Then we can assume that

$$\Delta_s = \{\pm s\alpha_{\phi_1}, \dots, \pm s\alpha_{\phi_k} \mid 0 \leq \phi_1 \leq \cdots \leq \phi_k\}$$

for some $k \geq 2$. So we have $\sigma_{\alpha_{\phi_k}}(s\alpha_{\phi_1}) = -\alpha_{2\phi_k - \phi_1} \in \Delta_s$. But we have $2\phi_k - \phi_1 > \phi_k$, which is a contradiction. Therefore, Δ_s is infinite.
     For the second statement, clearly (A1), (A2) and (A3) hold. (A4) follows from Lemma 3.2. For (A5), the subspace orthogonal to $\pm s\alpha_\phi$ is the line $\mathbb{R}\beta_\phi$ in N, and the subspace orthogonal to $\pm s\beta_\phi$ is the line $\mathbb{R}\alpha_\phi$ in P. Thus Δ_s is irreducible since $\Delta_s \subset P$ or $\Delta_s \subset N$.   ■

Lemma 5.2. (1) *If $r\alpha_\phi \in \Delta$, then $r\alpha_{-\phi} \in \Delta$, and so $\pm r\alpha_{\pm\phi} \in \Delta$.*
(2) *If $r\beta_\phi \in \Delta$, then $r\beta_{-\phi} \in \Delta$, and so $\pm r\beta_{\pm\phi} \in \Delta$.*

Proof. For (1), since $\alpha_0 \in \Delta$, we have $\sigma_{\alpha_0}(r\alpha_\phi) = -r\alpha_{-\phi} \in \Delta$ by Lemma 3.5. Hence, $r\alpha_{-\phi} \in \Delta$. Similarly, (2) follows from Lemma 3.5. ∎

By Remark 3.4 and Lemma 3.2, we can assume that

$$\alpha_0 \in \Delta_1 \subset \Delta. \tag{8}$$

In fact, if $r\alpha_\phi \in \Delta$, then use the isomorphism $f := R_{-\phi} \circ r^{-1} \cdot \mathrm{id}$, and we get $f(\Delta_r) = \Delta_1$. If $r\beta_\phi \in \Delta$, then use the isomorphism $g := S \circ R_{-\phi} \circ r^{-1} \cdot \mathrm{id}$, where S is defined in Remark 3.4, and we get $g(\Delta_r) = \Delta_1$. Thus, from now on, we assume (8).

Let

$$A_\Delta := \{\phi \in \mathbb{R} \mid \alpha_\phi \in \Delta\}.$$

Then we know that $0 \in A_\Delta$, and that

$$2\phi - \psi \in A_\Delta \tag{9}$$

for all $\phi, \psi \in A_\Delta$ by Lemma 3.5 and Lemma 5.2. (Hence A_Δ is a pointed reflection space of $\mathbb{R}$ in the sense of [Y2].) Moreover, if $A_\Delta = \{0\}$, then $\Delta = \{\pm\alpha_0\}$ cannot span $\mathbb{R}^2$. Hence there is a positive element in A_Δ by Lemma 5.2.

Note that $\{\alpha_\phi \mid \phi \in A_\Delta\}$ is a subset of a lattice by Proposition 3.3. Thus there exists a closest α_θ to α_0 in $\{\alpha_\phi \mid \phi \in A_\Delta\}$ in the euclidean distance since, in general, any bounded neighborhood of a point in a lattice contains only finitely many points of the lattice. Also, one can assume that $\theta > 0$ by Lemma 5.2. Thus

$$\theta \text{ is a minimal positive element in } A_\Delta$$

since the euclidean distance from α_0 to α_ϕ for $\phi > 0$ is a monotonic increasing function.

Lemma 5.3. *We have $A_\Delta = \mathbb{Z}\theta$.*

Proof. For the inclusion $A_\Delta \supset \mathbb{Z}\theta$, since $0, \theta \in A_\Delta$, it is enough to show that $k\theta \in A_\Delta$ for $k \in \mathbb{Z}_{>1}$ by Lemma 5.2. Suppose that it is true for k and $k-1$. Then $(k+1)\theta = 2k\theta - (k-1)\theta \in A_\Delta$ by (9), and hence the inclusion holds by induction on k.

For the inclusion $A_\Delta \subset \mathbb{Z}\theta$, suppose that $\phi \in A_\Delta \setminus \mathbb{Z}\theta$. Then there exists $k \in \mathbb{Z}$ such that $k\theta < \phi < (k+1)\theta$. If k is even, then $2\frac{k}{2}\theta - \phi = k\theta - \phi \in A_\Delta$, and hence $\phi - k\theta \in A_\Delta$. This is a contradiction since $0 < \phi - k\theta < \theta$. If k is odd, then $2\frac{k+1}{2}\theta - \phi = (k+1)\theta - \phi \in A_\Delta$. This is also a contradiction since $0 < (k+1)\theta - \phi < \theta$. ∎

Thus, for the minimal element θ defined above, i.e., $\alpha_\theta \in \Delta$ such that $\alpha_\phi \notin \Delta$ if $0 < \phi < \theta$, we have shown that

$$\Delta^\theta := \{\pm\alpha_{k\theta} \mid k \in \mathbb{Z}\}$$

is always a subset of Δ and $\Delta^\theta = \Delta_1$, assuming (8). We state this result:

Proposition 5.4.   *Let Δ be a hyperbolic root system of rank 2. Then we can assume that $\Delta^\theta \subset \Delta$ for some $\theta \in \mathbb{R}_{>0}$ such that $2\cosh\theta \in \mathbb{Z}_{\geq 3}$.*

Proof.   The condition for θ follows from (A3). In fact, we have $\langle \alpha_\theta, \alpha_0 \rangle = 2\cosh\theta \in \mathbb{Z}$, which is bigger than 2 since $\theta \neq 0$.   ∎

By Remark 3.4 and Lemma 3.2, we can also assume that

$$|\ell(\Delta^N)| \leq |\ell(\Delta^P)| \text{ and } 1 \text{ is the minimum in } \ell(\Delta^P). \tag{10}$$

In fact, if $|\ell(\Delta^N)| > |\ell(\Delta^P)|$, then use the isomorphism S defined in Remark 3.4. After this, use the isomorphism $s^{-1} \cdot \mathrm{id}$ for the minimal element s in $\ell(\Delta^P)$. Thus, from now on, we assume (10).

Proposition 5.5.   *Let $r \in \ell(\Delta)$.*

(i) *If $r > 0$ and $r \neq 1$ (so $r > 1$ by (10)), then*

$$\Delta_r = \{\pm r\alpha_{k\eta+\omega} \mid k \in \mathbb{Z}\},$$

   where $(\eta, \omega) = (\frac{\theta}{2}, 0)$, $(\theta, 0)$, $(\theta, \frac{\theta}{2})$, $(2\theta, 0)$ or $(2\theta, \theta)$.

(ii) *If $r < 0$, then we can assume that*

$$\Delta_r = \{\pm r\beta_{k\eta+\omega} \mid k \in \mathbb{Z}\},$$

   where the possibility of (η, ω) is the same as in (i).

   Also, we have $-r \neq 1$ if $(\eta, \omega) = (\theta, 0)$.

(iii) *If $|\ell(\Delta^P)| = |\ell(\Delta^N)| = 1$ with $\ell(\Delta^N) = \{r\}$, then we can assume that $\Delta_r = \{\pm r\beta_{k\eta+\omega} \mid k \in \mathbb{Z}\}$, where $(\eta, \omega) = (\frac{\theta}{2}, 0)$, $(\theta, 0)$ or $(\theta, \frac{\theta}{2})$.*

   Moreover, if $\eta = \theta$, then we can assume that $-r \geq 1$.

   In particular, $-r > 1$ if $(\eta, \omega) = (\theta, 0)$.

(iv) *Suppose that Δ contains $\Delta_r = \{\pm r\alpha_{k\theta+\frac{\theta}{2}} \mid k \in \mathbb{Z}\}$, and suppose that $r' \in \ell(\Delta^N)$. Then we can assume that $\Delta_{r'} = \{\pm r'\beta_{k\eta+\omega} \mid k \in \mathbb{Z}\}$, where $(\eta, \omega) = (\frac{\theta}{2}, 0)$, $(\theta, 0)$ or $(\theta, \frac{\theta}{2})$.*

Proof.   For (i), let $r\alpha_\omega \in \Delta_r$. Then through the isomorphism $R_{-\omega} \circ r^{-1} \cdot \mathrm{id}$ of Δ, Δ_r is mapped to Δ_1 satisfying (8). Thus Δ_r is mapped to $\Delta^\eta = \{\pm\alpha_{k\eta} \mid k \in \mathbb{Z}\}$ for some $\eta \in \mathbb{R}_{>0}$ by Proposition 5.4. Therefore, through the inverse, we get $\Delta_r = r \cdot \mathrm{id} \circ R_\omega(\Delta^\eta) = \{\pm r\alpha_{k\eta+\omega} \mid k \in \mathbb{Z}\}$. Moroever, we can clearly assume that $0 \leq \omega < \eta$.

By Lemma 3.5, we have $\sigma_{\alpha_{k\eta+\omega}}(\alpha_\theta) = -\alpha_{2k\eta+2\omega-\theta}$. Thus we get $2\omega-\theta \in \mathbb{Z}\theta$ for $k = 0$ and $2\eta + 2\omega - \theta \in \mathbb{Z}\theta$ for $k = 1$. Therefore, we obtain $2\omega, 2\eta \in \mathbb{Z}\theta$. On the other hand, we have $\sigma_{\alpha_{-\theta}}(r\alpha_\omega) = -r\alpha_{-2\theta-\omega}$, and hence $2\theta + \omega = \ell\eta + \omega$ for some $\ell \in \mathbb{Z}$ and we get $2\theta = \ell\eta$. Hence $4\theta = 2\ell\eta = \ell n\theta$ for some $n \in \mathbb{Z}$. Thus $\ell n = 4$, and since $\ell > 0$, we get $\ell = 1, 2, 4$. Therefore, $\eta = 2\theta$, θ or $\frac{\theta}{2}$, and we obtain $(\eta, \omega) = (2\theta, 0)$, $(2\theta, \theta)$, $(\theta, 0)$, $(\theta, \frac{\theta}{2})$ or $(\frac{\theta}{2}, 0)$.

For (ii), let $r\beta_\omega \in \Delta_r$. Then through the isomorphism $S \circ R_{-\omega} \circ r^{-1} \cdot \mathrm{id}$ of Δ, where S is defined in Remark 3.4, Δ_r is mapped to Δ_1 satisfying (8). Thus Δ_r is mapped to $\Delta^\eta = \{\pm\alpha_{k\eta} \mid k \in \mathbb{Z}\}$ for some $\eta \in \mathbb{R}_{>0}$ by Proposition 5.4. Therefore, through the inverse, we get $\Delta_r = r \cdot \mathrm{id} \circ R_\omega \circ S(\Delta^\eta) = \{\pm r\beta_{k\eta+\omega} \mid k \in \mathbb{Z}\}$, and we can assume that $0 \le \omega < \eta$. By Lemma 3.5, we have $\sigma_{\beta_{k\eta+\omega}}(\alpha_\theta) = \alpha_{2k\eta+2\omega-\theta}$ and $\sigma_{\alpha_{-\theta}}(r\beta_\omega) = r\beta_{-2\theta-\omega}$. Thus, by the same way as in (i), we get the same (η, ω).

For the second assertion, let $m := 2\cosh\theta \ge 3$, and then $\sinh\theta = \pm\frac{\sqrt{m^2-4}}{2}$. We need $\langle \alpha_0, \beta_\theta \rangle \in \mathbb{Z}$ if $-r = 1$. But then $\sqrt{m^2-4} \in \mathbb{Z}$, which is impossible (because $m^2 - 4 = k^2 \Rightarrow (m-k)(m+k) = 4$ but $m \ge 3$).

For (iii), we already know by (ii) that $\Delta = \Delta^\theta \sqcup \Delta_r$, where $\Delta_r = \{\pm r\beta_{k\eta+\omega} \mid k \in \mathbb{Z}\}$ with $(\eta, \omega) = (\frac{\theta}{2}, 0)$, $(\theta, 0)$, $(\theta, \frac{\theta}{2})$, $(2\theta, 0)$ or $(2\theta, \theta)$. If $(\eta, \omega) = (2\theta, 0)$, then, using $S \circ r^{-1} \cdot \mathrm{id}$ (S is defined in Remark 3.4) and replacing 2θ by θ, Δ is isomorphic to $\Delta^\theta \sqcup \Delta_r$ with $(\eta, \omega) = (\frac{\theta}{2}, 0)$. If $(\eta, \omega) = (2\theta, \theta)$, then, using $S \circ R_{-\theta} \circ r^{-1} \cdot \mathrm{id}$ and replacing 2θ by θ, Δ is again isomorphic to $\Delta^\theta \sqcup \Delta_r$ with $(\eta, \omega) = (\frac{\theta}{2}, 0)$. Thus we have proved the first assertion.

Suppose that $\eta = \theta$ and $-r < 1$. If $\omega = 0$, then using $S \circ r^{-1} \cdot \mathrm{id}$, we can assume that $-r > 1$. If $\omega = \theta/2$, then using $S \circ R_{-\theta/2} \circ r^{-1} \cdot \mathrm{id}$, we can assume that $-r > 1$. Thus we can assume that $-r \ge 1$. The last statement follows from (ii).

(iv) Suppose that $(\eta, \omega) = (2\theta, 0)$. Using Lemma 3.5, we have $\sigma_{r\alpha_{\theta/2}}(r'\beta_{2\theta}) = -r'\beta_{-\theta} \notin \Delta_{r'}$. Hence this case cannot happen. Suppose that $(\eta, \omega) = (2\theta, \theta)$. Using Lemma 3.5, we have $\sigma_{r\alpha_{\theta/2}}(r'\beta_\theta) = -r'\beta_0 \notin \Delta_{r'}$. Hence this case cannot happen either. ∎

When Δ is reduced, we have the following simplification.

Corollary 5.6. *Assume that Δ is reduced. Let $r \in \ell(\Delta)$.*

(i) *If $r > 0$ and $r \ne 1$, then we can assume that $r > 1$ and*
$$\Delta_r = \{\pm r\alpha_{k\theta+\frac{\theta}{2}} \mid k \in \mathbb{Z}\}.$$
In particular, $|\ell(\Delta^P)| \le 2$ (so $|\ell(\Delta^N)| \le 2$ by (10)).

(ii) *If $|\ell(\Delta^N)| = 2$ with $\ell(\Delta^N) = \{s, s'\}$, then we can assume that $(\eta, \omega) = (\theta, 0)$ or $(\theta, \frac{\theta}{2})$, and $-s, -s' \ge 1$.*
In particular, if $s\beta_\theta \in \Delta$ and $-s' > -s$, then $-s, -s' > 1$.

Proof. (i) By Proposition 5.5(i), the only possibility for (η, ω) is $(\theta, \frac{\theta}{2})$ since Δ is reduced. Thus $\ell(\Delta^P) = \{1, r\}$ in this case.

(ii) We know that $|\ell(\Delta^P)| = 2$ by (10) and (i) above. Thus $\Delta = \Delta^\theta \sqcup \Delta_r \sqcup \Delta_s \sqcup \Delta_{s'}$, where $\Delta_r = \{\pm r\alpha_{k\theta+\frac{\theta}{2}} \mid k \in \mathbb{Z}\}$, $\Delta_s = \{\pm s\beta_{k\eta+\omega} \mid k \in \mathbb{Z}\}$ and $\Delta_{s'} = \{\pm s'\beta_{k\eta'+\omega'} \mid k \in \mathbb{Z}\}$, and the possibility of (η, ω) and (η', ω') is, by Proposition 5.5(iv), $(\frac{\theta}{2}, 0)$, $(\theta, 0)$ or $(\theta, \frac{\theta}{2})$. But since Δ is reduced, only the cases $(\theta, 0)$ and $(\theta, \frac{\theta}{2})$ remain. Next, let $s'' := \min\{-s, -s'\}$. If $s'' \ge 1$, there is nothing to prove. Otherwise, i.e., $s'' < 1$, using $U := S \circ s''^{-1} \cdot \mathrm{id}$, we get $\ell(U(\Delta)^P) = \{-s/s'', -s'/s''\}$ (one element is 1 and the other is > 1) and $\ell(U(\Delta)^N) = \{1/s'', r/s''\}$ (both elements are > 1). Thus we can assume that $-s, -s' \ge 1$. The last statement follows from Proposition 5.5(ii). ∎

6. Dividing into 10 types

By Corollary 5.6, we can divide the reduced hyperbolic root systems of rank 2 into 10 types. First, we call Δ Type I when $|\ell(\Delta)| = 1$. By Proposition 5.4, we can assume that $\Delta = \Delta^\theta$ for Type I, i.e.,

$$\Delta = \Delta^\theta = \{\pm\alpha_{k\theta} \mid k \in \mathbb{Z}\} \tag{11}$$

for some $\theta \in \mathbb{R}_{>0}$ such that $2\cosh\theta \in \mathbb{Z}_{\geq 3}$. Next, we call Δ Type II when $|\ell(\Delta)| > 1$ and $\ell(\Delta^N) = \emptyset$. By Corollary 5.6(i), we find that $|\ell(\Delta)| = 2$, and $\Delta = \Delta^\theta \sqcup \Delta_r$, where $r > 1$ and

$$\Delta_r = \{\pm r\alpha_{k\theta+\frac{\theta}{2}} \mid k \in \mathbb{Z}\}. \tag{12}$$

Note that Δ_r is uniquely determined by θ and r.

When $|\ell(\Delta^P)| = |\ell(\Delta^N)| = 1$, there are three possibilities by Proposition 5.5(iii). Namely, such a system Δ is either

$$\Delta^\theta \sqcup \{\pm r\beta_{\frac{2k+1}{2}\theta} \mid k \in \mathbb{Z}\}, \quad \Delta^\theta \sqcup \{\pm r\beta_{k\theta} \mid k \in \mathbb{Z}\} \quad \text{or} \quad \Delta^\theta \sqcup \{\pm r\beta_{\frac{k}{2}\theta} \mid k \in \mathbb{Z}\}$$

for some $r \in \mathbb{R}_{>0}$ and $r \geq 1$ for the first case and $r > 1$ for the second case (see Proposition 5.5(ii)). We call the first case Type III, the second case Type IV and the third case Type V. Let

$$\Delta_{-r} := \{\pm r\beta_{\frac{2k+1}{2}\theta} \mid k \in \mathbb{Z}\} \tag{13}$$

$$\Delta_{-r}^\theta := \{\pm r\beta_{k\theta} \mid k \in \mathbb{Z}\} \tag{14}$$

$$\Delta_{-r}^{\theta/2} := \{\pm r\beta_{\frac{k\theta}{2}} \mid k \in \mathbb{Z}\} \tag{15}$$

for convenience.

When $|\ell(\Delta^P)| = 2$ and $|\ell(\Delta^N)| = 1$, there are three possibilities by Corollary 5.6(i) and Proposition 5.5(iv). Using the notation above, such a system Δ is either

$$\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2}, \quad \Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2}^\theta \quad \text{or} \quad \Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2}^{\theta/2}$$

for some $r_1, r_2 \in \mathbb{R}_{>0}$ and $r_1 > 1$. We call the first case Type VI, the second case Type VII and the third case Type VIII.

Finally, we consider the case $|\ell(\Delta^P)| = |\ell(\Delta^N)| = 2$. This is the last case by Corollary 5.6(i), and we only have two subcases, namely, for some $r_1, r_3 \in \mathbb{R}_{>1}$ and $r_2 \in \mathbb{R}_{\geq 1}$,

$$\Delta(\theta, r_1, r_2, r_3) := \Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2} \sqcup \Delta_{-r_3}^\theta \quad \text{with } r_2 < r_3 \text{ or } r_3 < r_2 \tag{16}$$

by Corollary 5.6(ii). We call the former case, i.e., $r_2 < r_3$, Type IX, and the latter case, i.e., $r_3 < r_2$, Type X. Note that $r_3 > 1$ for both types, by Corollary 5.6(ii).
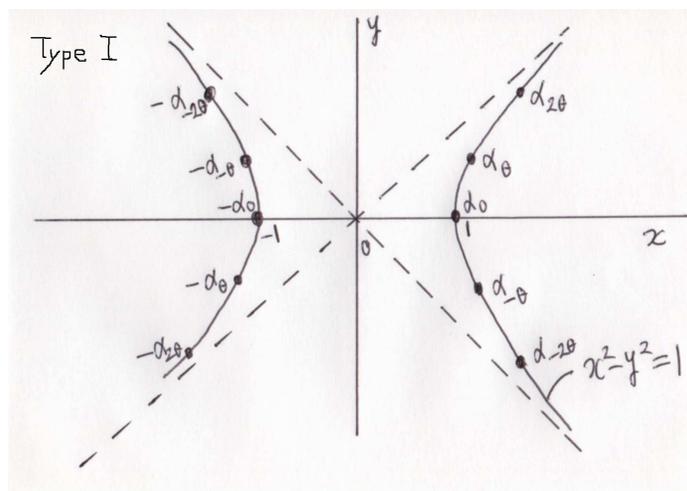
Finally, we note that through the isomorphisms
$$R_{-\theta/2} \circ r^{-1} \cdot \text{id}, \ S \circ R_{-\theta/2} \circ r^{-1} \cdot \text{id}, \text{ or } S \circ r^{-1} \cdot \text{id}$$
(S is defined in Remark 3.4) used in the proof of Proposition 5.5, we have

$$\Delta^\theta \cong \Delta_r \cong \Delta_{-r} \cong \Delta_{-r}^\theta \tag{17}$$

for the root systems defined in (11), (12), (13) and (14).

7. Type I: $\Delta = \Delta^\theta$

We have the following for Type I.

Theorem 7.1. *If $\theta \in \mathbb{R}_{>0}$ satisfies $2\cosh\theta \in \mathbb{Z}_{\geq 3}$, then Δ^θ defined as in (11) is always a hyperbolic root system of rank 2.*
Moreover, $\Delta^\theta \not\cong \Delta^\eta$ for $\eta > 0$ if $\eta \neq \theta$.

Proof. To show that Δ^θ is a hyperbolic root system of rank 2, we only need to check (A3), (A4) and (A5) since Δ^θ spans $\mathbb{R}^2$ and $\Delta^\theta \subset P$. First, (A3) follows from Lemma 2.1(ii) and the identity

$$\langle \alpha_{k\theta}, \alpha_{l\theta} \rangle = 2(\cosh k\theta \cosh l\theta - \sinh k\theta \sinh l\theta) = 2\cosh(k-l)\theta.$$

Next, (A4) follows from Lemma 3.5. For (A5), the subspace orthogonal to $\pm\alpha_\phi$ is the line $\mathbb{R}\beta_\phi$ in N. Thus Δ^θ is irreducible since $\Delta^\theta \subset P$.

Finally, suppose that $\Delta^\theta \cong \Delta^\eta = \{\pm\alpha_{k\eta} \mid k \in \mathbb{Z}\}$. Then we have

$$2\cosh\theta = \min\{|\langle\delta,\gamma\rangle| \mid \delta \neq \gamma \in \Delta^\theta\} = \min\{|\langle\delta,\gamma\rangle| \mid \delta \neq \gamma \in \Delta^\eta\} = 2\cosh\eta.$$

Hence $\eta = \theta$. ∎

Remark 7.2. Let $\alpha := -\alpha_{-\theta}$. Then we have

$$\begin{pmatrix} \langle\alpha_0,\alpha_0\rangle & \langle\alpha_0,\alpha\rangle \\ \langle\alpha,\alpha_0\rangle & \langle\alpha,\alpha\rangle \end{pmatrix} = \begin{pmatrix} 2 & -2\cosh\theta \\ -2\cosh\theta & 2 \end{pmatrix},$$

which is the Cartan matrix of a simply-laced hyperbolic Kac-Moody Lie algebra of rank 2. Since $2\cosh\theta \in \mathbb{Z}_{\geq 3}$, our simply-laced hyperbolic root systems of rank 2 are exactly the real roots of a simply-laced hyperbolic Kac-Moody Lie algebra of rank 2 (see [Mo], [LM]).

Remark 7.3. We explain the relation between the root system Δ^θ and Pell's equation. If $(a,b) \in \mathbb{Z}_{>0}^2$ is the fundamental solution (i.e., the minimal positive integral solution) of Pell's equation $x^2 - dy^2 = 1$ for some non-square $d \in \mathbb{Z}_{\geq 2}$, then

$(a, b\sqrt{d})$ is a solution of $x^2 - y^2 = 1$. Let $\alpha_\theta := (a, b\sqrt{d})$, and so $\cosh\theta = a \in \mathbb{Z}_{\geq 2}$ and $b = \frac{\sinh\theta}{\sqrt{d}} \in \mathbb{Z}_{\geq 1}$. Then, the solution set $\Gamma := \{(x, y) \in \mathbb{Z}^2 \mid x^2 - dy^2 = 1\}$ is in one-to-one correspondence with the root system $\Delta^\theta = \{\pm\alpha_{k\theta} \mid k \in \mathbb{Z}\}$. In fact, Γ is usually described as

$$\Gamma = \{\pm(s_k, t_k) \in \mathbb{Z}^2 \mid s_k + t_k\sqrt{d} = (a + b\sqrt{d})^k, \ k \in \mathbb{Z}\}.$$

We know that $(a + b\sqrt{d})^k = (\cosh\theta + \sinh\theta)^k = e^{k\theta} = \cosh k\theta + \sinh k\theta$, but what is (s_k, t_k)? Note that $\begin{pmatrix} s_k \\ t_k \end{pmatrix} = \begin{pmatrix} a & db \\ b & a \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and so Γ is also described as

$$\Gamma = \left\{ \pm \begin{pmatrix} \cosh\theta & \sqrt{d}\sinh\theta \\ \frac{\sinh\theta}{\sqrt{d}} & \cosh\theta \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid k \in \mathbb{Z} \right\}.$$

Note also that

$$\begin{pmatrix} \cosh\theta & \sqrt{d}\sinh\theta \\ \frac{\sinh\theta}{\sqrt{d}} & \cosh\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{d}} \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{d} \end{pmatrix}.$$

Hence, we get

$$\begin{pmatrix} s_k \\ t_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{d}} \end{pmatrix} \begin{pmatrix} \cosh k\theta & \sinh k\theta \\ \sinh k\theta & \cosh k\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{d} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh k\theta \\ \frac{\sinh k\theta}{\sqrt{d}} \end{pmatrix}.$$

Thus, using a linear isomorphism f of $\mathbb{R}^2$ such that $f(x, y) = (x, \sqrt{d}y)$, we have $f(s_k, t_k) = \alpha_{k\theta}$ and $f(\Gamma) = \Delta^\theta$. If we think that Γ sits in the hyperbolic plane defined by $\begin{pmatrix} 1 & 0 \\ 0 & -d \end{pmatrix}$, then Γ is a hyperbolic root system isomorphic to Δ^θ through the isometry f. Note that, using the Chebyshev polynomials T_k and U_k (see Section 2), we have

$$\begin{pmatrix} s_k \\ t_k \end{pmatrix} = \begin{pmatrix} \cosh k\theta \\ \frac{\sinh k\theta}{\sqrt{d}} \end{pmatrix} = \begin{pmatrix} T_k \\ bU_k \end{pmatrix}.$$

We say that s is half-integral if $2s \in \mathbb{Z}$. If $(a/2, b/2)$ is the fundamental half-integral solution (i.e., the minimal positive half-integral solution) of $x^2 - dy^2 = 1$ with an odd a (automatically b is odd, and $d \equiv 1 \mod 4$), then $(a/2, b\sqrt{d}/2)$ is a solution of $x^2 - y^2 = 1$. Let $\alpha_\theta := (a/2, b\sqrt{d}/2)$. Then, the half-integral solution set Γ of the Pell's equation is isomorphic to Δ^θ. In fact, Γ is described as

$$\Gamma = \{\pm(s_k, t_k) \in \mathbb{Z}^2 \mid s_k + t_k\sqrt{d} = (a/2 + b\sqrt{d}/2)^k, \ k \in \mathbb{Z}\},$$

and by the same argument above, we have

$$\begin{pmatrix} s_k \\ t_k \end{pmatrix} = \begin{pmatrix} \cosh k\theta \\ \frac{\sinh k\theta}{\sqrt{d}} \end{pmatrix} = \begin{pmatrix} T_k \\ bU_k/2 \end{pmatrix}.$$

Note that s_k and t_k are half-integral with $s_k + t_k \in \mathbb{Z}$.

Conversely, if $\cosh\theta \in \mathbb{Z}$ for some $\theta \in \mathbb{R}_{>0}$, then $d := \sinh^2\theta = \cosh^2\theta - 1$ is not a square. So $(\cosh\theta, 1)$ is the fundamental solution of the Pell's equation

$x^2 - dy^2 = 1$. Thus, by the argument above for $\sqrt{d} = \sinh \theta$, the solution set is equal to $\Gamma = \left\{ \pm \begin{pmatrix} \cosh k\theta \\ \frac{\sinh k\theta}{\sinh \theta} \end{pmatrix} \mid k \in \mathbb{Z} \right\}$, which is isomorphic to Δ^θ. Note that $\begin{pmatrix} \cosh k\theta \\ \frac{\sinh k\theta}{\sinh \theta} \end{pmatrix} = \begin{pmatrix} T_k \\ U_k \end{pmatrix}$, and so we find that $T_k, U_k \in \mathbb{Z}$ if $T_1 \in \mathbb{Z}$, which is already shown in Lemma 2.1(i) and Lemma 2.3.

If $m = 2 \cosh \theta \in \mathbb{Z}$ and m is odd, then $d := 4 \sinh^2 \theta = m^2 - 4$ is not a square (and $d \equiv 1 \mod 4$). So $(\frac{\cosh \theta}{2}, \frac{1}{2})$ is the fundamental half-integral solution of $x^2 - dy^2 = 1$. Thus, by the argument above for $\sqrt{d} = 2 \sinh \theta$, the half-integral solution set is equal to $\Gamma = \left\{ \pm \begin{pmatrix} \cosh k\theta \\ \frac{\sinh k\theta}{2 \sinh \theta} \end{pmatrix} \mid k \in \mathbb{Z} \right\}$, which is isomorphic to Δ^θ. Note that $\begin{pmatrix} \cosh k\theta \\ \frac{\sinh k\theta}{2 \sinh \theta} \end{pmatrix} = \begin{pmatrix} T_k \\ U_k/2 \end{pmatrix}$, and so we find that $2T_k, U_k \in \mathbb{Z}$ if $2T_1 \in \mathbb{Z}$, which is already shown in Lemma 2.1(ii) and Lemma 2.3.

Consequently, the solution set or the half-integral solution set of an arbitrary Pell's equation $x^2 - dy^2 = 1$ is isomorphic to some Δ^θ, and any Type I root system Δ^θ is isomorphic to the solution set or the half-integral solution set of some Pell's equation.

For example, if $2 \cosh \theta = 3$, then Δ^θ is isomorphic to the half-integral solution set of the Pell's equation $x^2 - 5y^2 = 1$. The fundamental half-integral solution is $(3/2, 1/2)$ and the next solution is $(7/2, 3/2)$. The third one is $(9, 4)$, which is the fundamental solution of $x^2 - 5y^2 = 1$. Note that $(3, 1)$ is the fundamental solution of $x^2 - 5y^2 = 4$.

We have to say that there is no contribution to find a fundamental solution. For example, the fundamental solution of $x^2 - 13y^2 = 1$ is $(649, 180)$ which is usually found from the continued fraction of $\sqrt{13}$. So the solution set is isomorphic to Δ^θ, where $\cosh \theta = 649$. Also, Δ^θ is isomorphic to the solution set of $x^2 - (649^2 - 1)y^2 = 1$ as above. Note that $649^2 - 1 = 42120 = 180^2 \cdot 13$. However, from the information $\cosh \theta = 649$, we do not know whether the solution sets of $x^2 - 180^2 \cdot 13y^2 = 1$ and $x^2 - 13y^2 = 1$ are isomorphic or not. We find it only when $(649, 180)$ is already known to be the fundamental solution of $x^2 - 13y^2 = 1$.

8. Type II: $\Delta = \Delta^\theta \sqcup \Delta_r$

Recall the root system $\Delta^\theta \sqcup \Delta_r$ of Type II, where Δ_r is defined in (12). We let

$$m := 2 \cosh \theta \in \mathbb{Z}_{\geq 3}$$

from now on.

Theorem 8.1. *For Type II, we have* $r = \frac{\sqrt{m+2}}{s} > 1$*, where s is a positive integer such that $s \mid m + 2$.*

Conversely, such a set $\Delta^\theta \sqcup \Delta_r$ determined by θ, r and s is always a reduced hyperbolic root system of rank 2.

Moreover, $(\Delta^\theta \sqcup \Delta_r) \not\cong (\Delta^{\theta'} \sqcup \Delta_{r'})$ if $\theta \neq \theta'$ or $r \neq r'$.
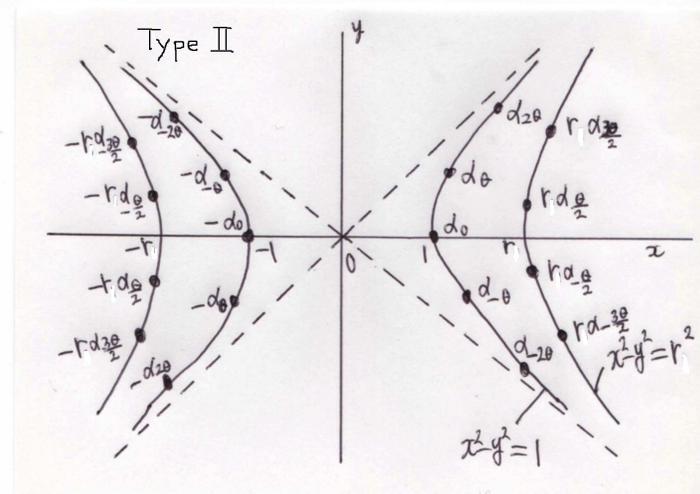
Proof.　Applying the crystallographic condition (A3) for α_0 and $r\alpha_{\theta/2}$, we get

$$\tfrac{2}{r}\cosh\tfrac{\theta}{2}\in\mathbb{Z}\quad\text{and}\quad 2r\cosh\tfrac{\theta}{2}\in\mathbb{Z}.\tag{18}$$

Since $\cosh\frac{\theta}{2}=\frac{\sqrt{m+2}}{2}$, we obtain that $\frac{\sqrt{m+2}}{r}\in\mathbb{Z}$ and $r\sqrt{m+2}\in\mathbb{Z}$. Let s be the former integer, i.e., $s=\frac{\sqrt{m+2}}{r}$. Then the latter condition implies that $s\mid m+2$. Thus we have proved the first statement.

　　Conversely, to show that $\Delta^\theta\sqcup\Delta_r$ is a hyperbolic root system of rank 2, we only need to check (A3), (A4) and (A5) since Δ^θ already spans $\mathbb{R}^2$ and $\Delta^\theta\sqcup\Delta_r\subset P$. We already checked (A3) for pairs from Δ^θ. Thus we need to check (A3) for $\langle\alpha_{k\theta},r\alpha_{l\theta+\frac{\theta}{2}}\rangle$, $\langle r\alpha_{l\theta+\frac{\theta}{2}},\alpha_{k\theta}\rangle$ and $\langle r\alpha_{k\theta+\frac{\theta}{2}},r\alpha_{l\theta+\frac{\theta}{2}}\rangle$ for $k,l\in\mathbb{Z}$, which are equal to $\frac{2}{r}\cosh\frac{2k-2l-1}{2}\theta$, $2r\cosh\frac{2l-2k+1}{2}\theta$ and $2\cosh(k-l)\theta$, respectively. Thus we only need to check the first two. Our assumptions imply that $\frac{2}{r}\cosh\frac{\theta}{2}\in\mathbb{Z}$ and $2r\cosh\frac{\theta}{2}\in\mathbb{Z}$. Then we have $\frac{2}{r}\cosh(k+\frac{1}{2})\theta\in\mathbb{Z}$ and $2r\cosh(k+\frac{1}{2})\theta\in\mathbb{Z}$ for all $k\in\mathbb{Z}$, by Lemma 2.2. Hence (A3) holds. (A4) follows from Lemma 3.5. Since Δ^θ is irreducible, so is Δ_r. Since $0\notin(\Delta^\theta,\Delta_r)$, (A5) holds. Thus $\Delta^\theta\sqcup\Delta_r$ is a reduced hyperbolic root system of rank 2.

　　Finally, suppose that $\Delta^\theta\sqcup\Delta_r\cong\Delta^{\theta'}\sqcup\Delta_{r'}$ via φ. Since an isomorphism of root systems keeps the ratio of lengths, we have $\varphi(\Delta^\theta)=\Delta^{\theta'}$ and $\varphi(\Delta_r)=\Delta_{r'}$. Then, by Theorem 7.1, we get $\theta=\theta'$, and by the map preserving ratio, we obtain $r=r'$.　■

　　Note that for a fixed m, there are some possibilities of s. For example, if $m=10$, then $s=1$, 2 or 3 (since $s<\sqrt{12}$).

Remark 8.2.　Let $\alpha:=-r\alpha_{-\frac{\theta}{2}}=\begin{pmatrix}-\frac{m+2}{2s}\\\frac{\sqrt{m^2-4}}{2s}\end{pmatrix}$. Then

$$\begin{pmatrix}\langle\alpha_0,\alpha_0\rangle&\langle\alpha_0,\alpha\rangle\\\langle\alpha,\alpha_0\rangle&\langle\alpha,\alpha\rangle\end{pmatrix}=\begin{pmatrix}2&-\frac{m+2}{s}\\-s&2\end{pmatrix},$$

which is the Cartan matrix of a hyperbolic Kac-Moody Lie algebra of rank 2. Since the determinant is equal to $2-m\le-1$, our hyperbolic root systems of Type I

and II are exactly the real roots of a hyperbolic Kac-Moody Lie algebra of rank 2 (see [Mo], [LM]).

9. Type III: $\Delta = \Delta^\theta \sqcup \{\pm r\beta_{\frac{2k+1}{2}\theta} \mid k \in \mathbb{Z}\}$
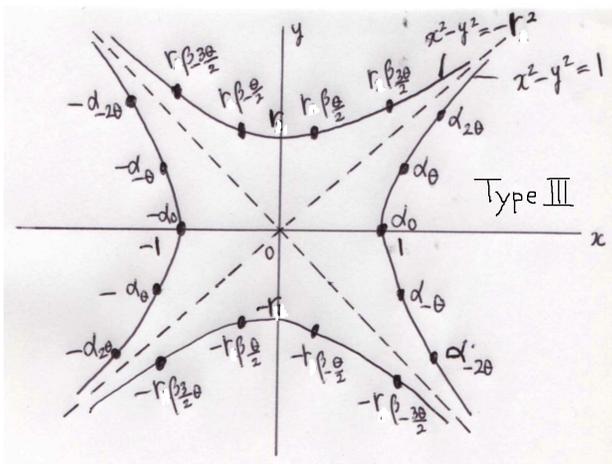
Recall the root system $\Delta^\theta \sqcup \Delta_{-r}$ of Type III, where Δ_{-r} is defined in (13).

Theorem 9.1. *For Type III, we have $r = \frac{\sqrt{m-2}}{s} \geq 1$, where s is a positive integer such that $s \mid m-2$.*

Conversely, such a set $\Delta^\theta \sqcup \Delta_{-r}$ determined by θ, r and s is always a reduced hyperbolic root system of rank 2.

Moreover, $(\Delta^\theta \sqcup \Delta_{-r}) \not\cong (\Delta^{\theta'} \sqcup \Delta_{-r'}^{\theta'/2})$ if $\theta \neq \theta'$ or $r \neq r'$.

Proof. Applying the crystallographic condition (A3) for α_0 and $r\beta_{\theta/2}$, we get

$$\tfrac{2}{r}\sinh\tfrac{\theta}{2} \in \mathbb{Z} \quad \text{and} \quad 2r\sinh\tfrac{\theta}{2} \in \mathbb{Z}. \tag{19}$$

Since $\sinh\frac{\theta}{2} = \pm\frac{\sqrt{m-2}}{2}$, we obtain that $\frac{\sqrt{m-2}}{r} \in \mathbb{Z}$ and $r\sqrt{m-2} \in \mathbb{Z}$. Let s be the former integer, i.e., $s = \frac{\sqrt{m-2}}{r}$. Then the latter condition implies that $s \mid m-2$. Thus we have proved the first statement.

Conversely, to show that $\Delta^\theta \sqcup \Delta_{-r}$ is a hyperbolic root system of rank 2, we only need to check (A3), (A4) and (A5) since Δ^θ already spans $\mathbb{R}^2$ and $\Delta_{-r} \subset N$. We already checked (A3) for pairs from Δ^θ. Thus we need to check (A3) for $\langle \alpha_{k\theta}, r\beta_{l\theta+\frac{\theta}{2}} \rangle$, $\langle r\beta_{l\theta+\frac{\theta}{2}}, \alpha_{k\theta} \rangle$ and $\langle r\beta_{k\theta+\frac{\theta}{2}}, r\beta_{l\theta+\frac{\theta}{2}} \rangle$ for $k,l \in \mathbb{Z}$, which are equal to $\frac{2}{r}\sinh\frac{2k-2l-1}{2}\theta$, $2r\sinh\frac{2l-2k+1}{2}\theta$ and $2\cosh(k-l)\theta$, respectively. Thus we only need to check the first two, but by the condition (19) and Lemma 2.6, we get (A3). (A4) follows from Lemma 3.5. Since Δ^θ is irreducible, so is Δ_{-r}. Since $0 \notin (\Delta^\theta, \Delta_{-r})$, (A5) holds. Thus $\Delta^\theta \sqcup \Delta_{-r}$ is a reduced hyperbolic root system of rank 2.

Finally, suppose that $\Delta^\theta \sqcup \Delta_{-r} \cong \Delta^{\theta'} \sqcup \Delta_{-r'}$ via φ. Suppose also that $r \neq 1$. Since an isomorphism of root systems keeps the ratio of lengths, we have $\varphi(\Delta^\theta) = \Delta^{\theta'}$ and $\varphi(\Delta_{-r}) = \Delta_{-r'}$. Then, by Theorem 7.1, we get $\theta = \theta'$, and by

the map preserving ratio, we obtain $r = r'$. If $r = 1$, then $r' = 1$ by the map preserving ratio. Thus $\varphi(\Delta^\theta) = \Delta^{\theta'}$ or Δ_{-1}. Since $\Delta^{\theta'} \cong \Delta_{-1}$ (see (17)), we get $\theta = \theta'$ by Theorem 7.1. ∎

Note that for a fixed m, there are some possibilities of s. For example, if $m = 14$, then $s = 1$, 2 or 3.

Remark 9.2. When $r = 1$ (e.g. if $m = 18$ and $s = 4$, then $r = \sqrt{m-2}/s = 1$), we hava a different description for $\Delta^\theta \sqcup \Delta_{-1}$, which is isomorphic to the solution set or the half-integral solution set of some Pell's equations $x^2 - dy^2 = \pm 1$. In fact, if $(a,b) \in \mathbb{Z}_{>0}^2$ is the fundamental solution of $x^2 - dy^2 = -1$, then $(a, b\sqrt{d})$ is a solution of $x^2 - y^2 = -1$. Let $\beta_{\theta/2} := (a, b\sqrt{d})$, and so $\cosh\theta = \cosh^2\frac{\theta}{2} + \sinh^2\frac{\theta}{2} = b^2 d + a^2 = 2a^2 + 1 \in \mathbb{Z}_{\geq 3}$. As in Remark 7.3, the solution set is described as $\Gamma = \{\pm(s_k, t_k) \in \mathbb{Z}^2 \mid s_k + t_k\sqrt{d} = (a + b\sqrt{d})^k,\ k \in \mathbb{Z}\}$, and from
$$\begin{pmatrix} s_k \\ t_k \end{pmatrix} = \begin{pmatrix} a & db \\ b & a \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ we get}$$

$$\Gamma = \left\{ \pm \begin{pmatrix} \sinh\frac{\theta}{2} & \sqrt{d}\cosh\frac{\theta}{2} \\ \frac{\cosh\frac{\theta}{2}}{\sqrt{d}} & \sinh\frac{\theta}{2} \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid k \in \mathbb{Z} \right\}.$$

Note that

$$\begin{pmatrix} \sinh\frac{\theta}{2} & \sqrt{d}\cosh\frac{\theta}{2} \\ \frac{\cosh\frac{\theta}{2}}{\sqrt{d}} & \sinh\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{d}} \end{pmatrix} \begin{pmatrix} \sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \\ \cosh\frac{\theta}{2} & \sinh\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{d} \end{pmatrix}$$

and

$$\begin{pmatrix} \sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \\ \cosh\frac{\theta}{2} & \sinh\frac{\theta}{2} \end{pmatrix}^k = \begin{cases} \begin{pmatrix} \cosh\ell\theta & \sinh\ell\theta \\ \sinh\ell\theta & \cosh\ell\theta \end{pmatrix} & \text{if } k = 2\ell \\ \begin{pmatrix} \sinh(\ell+\frac{1}{2})\theta & \cosh(\ell+\frac{1}{2})\theta \\ \cosh(\ell+\frac{1}{2})\theta & \sinh(\ell+\frac{1}{2})\theta \end{pmatrix} & \text{if } k = 2\ell+1. \end{cases}$$

So we obtain

$$\begin{pmatrix} s_k \\ t_k \end{pmatrix} = \begin{cases} \begin{pmatrix} \cosh\ell\theta \\ \frac{\sinh\ell\theta}{\sqrt{d}} \end{pmatrix} & \text{if } k = 2\ell \\ \begin{pmatrix} \sinh(\ell+\frac{1}{2})\theta \\ \frac{\cosh(\ell+\frac{1}{2})\theta}{\sqrt{d}} \end{pmatrix} & \text{if } k = 2\ell+1. \end{cases}$$

Thus, using the linear isomorphism f of $\mathbb{R}^2$ defined by $f(x,y) = (x, \sqrt{d}y)$, we have $f(s_{2\ell}, t_{2\ell}) = \alpha_{\ell\theta}$, $f(s_{2\ell+1}, t_{2\ell+1}) = \beta_{\frac{2\ell+1}{2}\theta}$ and $f(\Gamma) = \Delta^\theta \sqcup \Delta_{-1}$. If we think that Γ sits in the hyperbolic plane defined by $\begin{pmatrix} 1 & 0 \\ 0 & -d \end{pmatrix}$, then Γ is a hyperbolic root system isomorphic to $\Delta^\theta \sqcup \Delta_{-1}$ through the isometry f. Note that $(\cosh\ell\theta, \frac{\sinh\ell\theta}{\sqrt{d}}) = (T_\ell, 2abU_\ell)$ and

$$\left(\sinh(\ell+\tfrac{1}{2})\theta, \tfrac{\cosh(\ell+\frac{1}{2})\theta}{\sqrt{d}} \right) = \left(V_{2\ell+1}(1), C_{2\ell+1}(\tfrac{1}{\sqrt{d}}) \right), \tag{20}$$

where $V_n(t)$ and $C_n(t)$ are defined in Section 2.

Similarly, if $(a/2, b/2)$ is a fundamental half-integral solution for an odd a of a Pell's equation $x^2 - dy^2 = -1$, then $\beta_{\theta/2} := (a/2, b\sqrt{d}/2)$ is a solution of $x^2 - y^2 = -1$. Note that $2\cosh\theta = 2(1 + 2\sinh^2\frac{\theta}{2}) = 4a^2 + 2 \in \mathbb{Z}_{\geq 3}$, and so the half-integral solution set of $x^2 - dy^2 = \pm 1$ is isomorphic to $\Delta^\theta \sqcup \Delta_{-1}$. This time, we have $(\cosh\ell\theta, \frac{\sinh\ell\theta}{\sqrt{d}}) = (T_\ell, abU_\ell/2)$ for $k = 2\ell$, but the same expression as (20) for $k = 2\ell + 1$ with $V_{2\ell+1}(2) \in \mathbb{Z}$ and $C_{2\ell+1}(\frac{2}{\sqrt{d}}) \in \mathbb{Z}$.

Conversely, suppose that $n := \cosh\theta \in \mathbb{Z}$ for some $\theta \in \mathbb{R}_{>0}$ and $2\sinh\frac{\theta}{2} = \sqrt{2n-2} \in \mathbb{Z}$. So, $2n - 2$ has to be square, and hence, n should be odd. Let $d := \frac{n+1}{2}$. Then $(\sinh\frac{\theta}{2}, 1)$ is the fundamental solution of $x^2 - dy^2 = \pm 1$ (since $\sqrt{2n-2}$ is even). Thus the solution set of $x^2 - dy^2 = \pm 1$ is isomorphic to $\Delta^\theta \sqcup \Delta_{-1}$.

Similarly, if $m := 2\cosh\theta \in \mathbb{Z}$ for some $\theta \in \mathbb{R}_{>0}$ is odd and $2\sinh\frac{\theta}{2} = \sqrt{m-2} \in \mathbb{Z}$, then $m - 2$ has to be square. Let $d := m + 2$. Then $(\sinh\frac{\theta}{2}, \frac{1}{2})$ is the fundamental half-integral solution of $x^2 - dy^2 = \pm 1$. Thus the half-integral solution set of $x^2 - dy^2 = \pm 1$ is isomorphic to $\Delta^\theta \sqcup \Delta_{-1}$.

Consequently, the solution set or the half-integral solution set of an arbitrary Pell's equation $x^2 - dy^2 = \pm 1$ is isomorphic to some $\Delta^\theta \sqcup \Delta_{-1}$, and any $\Delta^\theta \sqcup \Delta_{-1}$ is isomorphic to the solution set or the half-integral solution set of some Pell's equations $x^2 - dy^2 = \pm 1$.

Example 9.3. (1) The fundamental solution of the Pell's equation $x^2 - 13y^2 = -1$ is $(18, 5)$. Let $\sinh\frac{\theta}{2} = 18$. Then the solution sets of $x^2 - 13y^2 = \pm 1$ is isomorphic to $\Delta^\theta \sqcup \Delta_{-1}$.

(2) If $2\cosh\theta = 3$, then $\Delta^\theta \sqcup \Delta_{-1}$ is isomorphic to the half-integral solution set of the Pell's equations $x^2 - 5y^2 = \pm 1$. The fundamental half-integral solution is $(1/2, 1/2)$ and the next solution $(3/2, 1/2)$ is the fundamental half-integral solution of $x^2 - 5y^2 = 1$. The third one $(2, 1)$ is the fundamental solution of $x^2 - 5y^2 = -1$, and the forth one is $(7/2, 3/2)$. The fifth one is $(11/2, 5/2)$, and the sixth one $(9, 4)$ is the fundamental solution of $x^2 - 5y^2 = 1$.
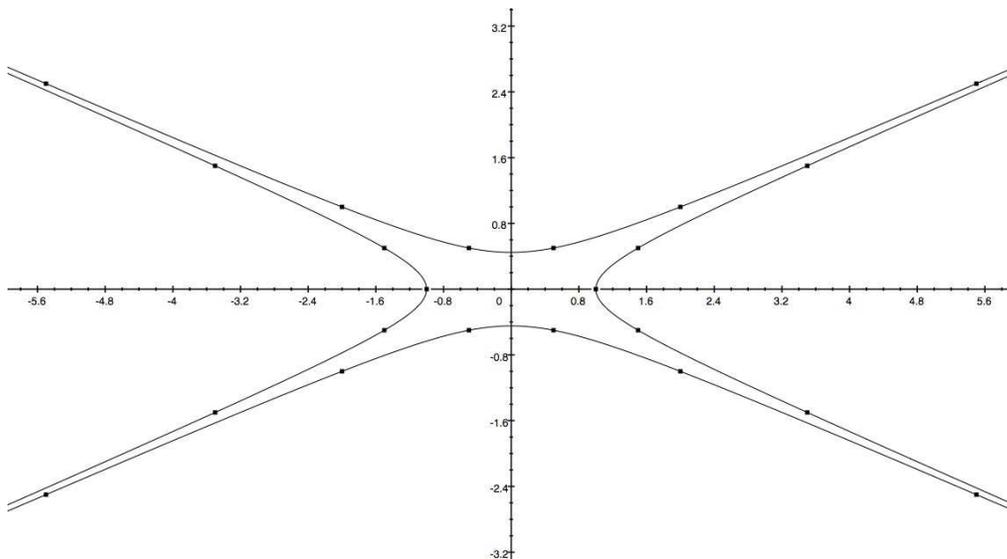
10. Type IV: $\Delta = \Delta^\theta \sqcup \{\pm r\beta_{k\theta} \mid k \in \mathbb{Z}\}$
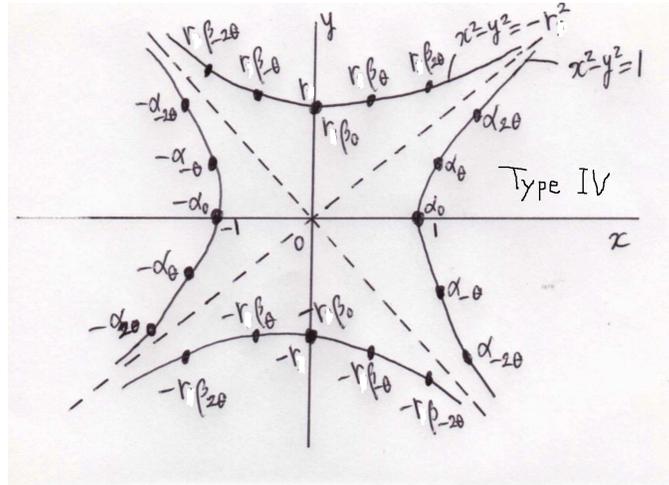
Recall the root system $\Delta^\theta \sqcup \Delta^\theta_{-r}$ of Type IV, where Δ^θ_{-r} is defined in (14).

Theorem 10.1. *For Type IV, we have* $r = \frac{\sqrt{m^2-4}}{s} > 1$, *where* s *is a positive integer such that* $s \mid m^2 - 4$.

Conversely, such a set $\Delta^\theta \sqcup \Delta^\theta_{-r}$ *determined by* θ, r *and* s *is always a reduced hyperbolic root system of rank* 2.

Moreover, $(\Delta^\theta \sqcup \Delta^\theta_{-r}) \not\cong (\Delta^{\theta'} \sqcup \Delta^{\theta'}_{-r'})$ *if* $\theta \neq \theta'$ *or* $r \neq r'$.

Proof. Applying the crystallographic condition (A3) for α_0 and $r\beta_\theta$, we get

$$\tfrac{2}{r}\sinh\theta \in \mathbb{Z} \quad \text{and} \quad 2r\sinh\theta \in \mathbb{Z}. \tag{21}$$

Since $\sinh\theta = \pm\frac{\sqrt{m^2-4}}{2}$, we obtain that $\frac{\sqrt{m^2-4}}{r} \in \mathbb{Z}$ and $r\sqrt{m^2-4} \in \mathbb{Z}$. Let s be the former integer, i.e., $s = \frac{\sqrt{m^2-4}}{r}$. Then the latter condition implies that $s \mid m^2 - 4$. We also have $r \neq 1$ by Proposition 5.5(ii). Thus we have proved the first statement.

Conversely, to show that $\Delta^\theta \sqcup \Delta^\theta_{-r}$ is a hyperbolic root system of rank 2, we only need to check (A3), (A4) and (A5) since Δ^θ already spans $\mathbb{R}^2$ and $\Delta^\theta_{-r} \subset N$. We already checked (A3) for pairs from Δ^θ. Thus we need to check (A3) for $\langle\alpha_{k\theta}, r\beta_{l\theta}\rangle$, $\langle r\beta_{l\theta}, \alpha_{k\theta}\rangle$ and $\langle r\beta_{k\theta}, r\beta_{l\theta}\rangle$ for $k,l \in \mathbb{Z}$, which are equal to $\frac{2}{r}\sinh(k-l)\theta$, $2r\sinh(l-k)\theta$ and $2\cosh(k-l)\theta$, respectively. Thus we only need to check the first two, but by the condition (21) and Lemma 2.5, we get (A3). (A4) follows from Lemma 3.5. Since Δ^θ is irreducible, so is Δ^θ_{-r}. Hence, if $\Delta = \Delta' \cup \Delta''$ is an orthogonal decomposition and $\alpha_0 \in \Delta'$, then $\Delta' = \Delta^\theta$ and $\Delta'' = \Delta^\theta_{-r}$. But the orthogonal elements to α_0 are only $\pm\beta_0$, and so $\Delta^\theta_{-r} \subset \{\pm\beta_0\}$, which is absurd. Therefore, (A5) holds. Thus $\Delta^\theta \sqcup \Delta^\theta_{-r}$ is a reduced hyperbolic root system of rank 2.

Finally, suppose that $\Delta^\theta \sqcup \Delta^\theta_{-r} \cong \Delta^{\theta'} \sqcup \Delta^{\theta'}_{-r'}$ via φ. Since $r \neq 1$ (and since an isomorphism of root systems keeps the ratio of lengths), we have $\varphi(\Delta^\theta) = \Delta^{\theta'}$ and $\varphi(\Delta^\theta_{-r}) = \Delta^{\theta'}_{-r'}$. Then, by Theorem 7.1, we get $\theta = \theta'$, and by the map preserving ratio, we obtain $r = r'$. ∎

Note that for a fixed m, there are some possibilities of s. For example, if $m = 4$, then $s = 1$, 2 or 3.

We also note that $A_1 \times A_1 := \{\pm\alpha_0, \pm\beta_0\}$ satisfies (A1-4), but not (A5). So $A_1 \times A_1$ is not a root system in our sense.

11. Type V: $\Delta = \Delta^\theta \sqcup \{\pm r\beta_{\frac{k\theta}{2}} \mid k \in \mathbb{Z}\}$

Recall the root system $\Delta^\theta \sqcup \Delta^{\theta/2}_{-r}$ of Type V, where $\Delta^{\theta/2}_{-r}$ is defined in (15).

Theorem 11.1. *For Type V, $m + 2$ is square, and $r = \frac{\sqrt{m-2}}{s} > 0$ with $r \neq 1$, where s is a positive integer such that $s \mid m - 2$.*

Conversely, such a set $\Delta^\theta \sqcup \Delta^{\theta/2}_{-r}$ determined by θ, r and s is always a reduced hyperbolic root system of rank 2.

Moreover, $(\Delta^\theta \sqcup \Delta^{\theta/2}_{-r}) \ncong (\Delta^{\theta'} \sqcup \Delta^{\theta'/2}_{-r'})$ if $\theta \neq \theta'$ or $r \neq r'$.

Proof. Since $\Delta^\theta \sqcup \Delta_{-r}$ (which is of Type III) is contained in $\Delta^\theta \sqcup \Delta^{\theta/2}_{-r}$, we have $r = \frac{\sqrt{m-2}}{s}$ with $s \mid m - 2$ for some $s \in \mathbb{Z}_{>0}$, as in Type III. Moreover, since $\Delta^\theta \sqcup \Delta^\theta_{-r}$ (which is of Type IV) is contained in $\Delta^\theta \sqcup \Delta^{\theta/2}_{-r}$, we have $r = \frac{\sqrt{m^2-4}}{s'}$ with $s' \mid m^2 - 4$ for some $s' \in \mathbb{Z}_{>0}$, as in Type IV. Hence, we have $r = \frac{\sqrt{m^2-4}}{s'} = \frac{\sqrt{m-2}}{s}$, and get $s\sqrt{m+2} = s'$. Thus, $m + 2$ has to be square. If $r = 1$, then $\sqrt{m^2 - 4}$ has to be square, which is impossible.

Conversely, to show that $\Delta^\theta \sqcup \Delta^{\theta/2}_{-r}$ is a hyperbolic root system of rank 2, we only need to check (A3), (A4) and (A5) since Δ^θ already spans $\mathbb{R}^2$ and $\Delta^{\theta/2}_{-r} \subset N$. We already checked (A3) for pairs from Δ^θ. Thus we need to check (A3) for $\langle \alpha_{k\theta}, r\beta_{\frac{l\theta}{2}} \rangle$, $\langle r\beta_{\frac{l\theta}{2}}, \alpha_{k\theta} \rangle$ and $\langle r\beta_{\frac{k\theta}{2}}, r\beta_{\frac{l\theta}{2}} \rangle$ for $k, l \in \mathbb{Z}$, which are equal to $\frac{2}{r} \sinh \frac{2k-l}{2}\theta$, $2r \sinh \frac{2l-k}{2}\theta$ and $-2\cosh \frac{k-l}{2}\theta$, respectively. Since $2\cosh \frac{\theta}{2} \in \mathbb{Z}$, $2r \sinh \frac{\theta}{2} \in \mathbb{Z}$ and $\frac{2}{r} \sinh \frac{\theta}{2} \in \mathbb{Z}$, we get (A3) by Lemma 2.1 and Lemma 2.6. (A4) follows from Lemma 3.5. Since Δ^θ is irreducible, so is $\Delta^{\theta/2}_{-r}$. Hence, by the same reason as in the proof of Theorem 10.1, (A5) holds. Thus $\Delta^\theta \sqcup \Delta^{\theta/2}_{-r}$ is a reduced hyperbolic root system of rank 2.
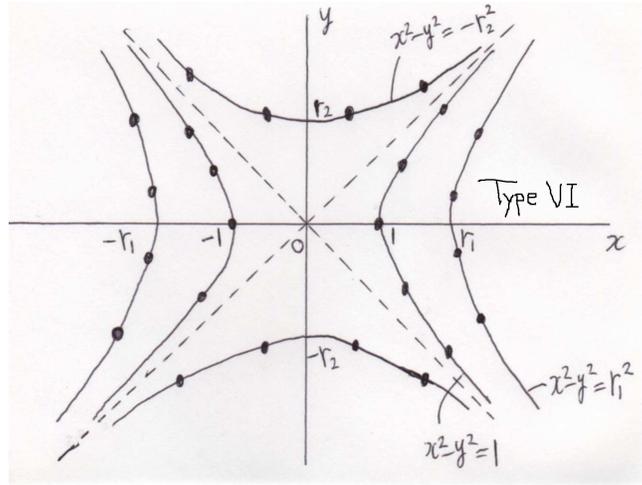
Finally, suppose that $\Delta^\theta \sqcup \Delta^{\theta/2}_{-r} \cong \Delta^{\theta'} \sqcup \Delta^{\theta'/2}_{-r'}$ via φ. Since $r \neq 1$ (and since an isomorphism of root systems keeps the ratio of lengths), we have $\varphi(\Delta^\theta) = \Delta^{\theta'}$ and $\varphi(\Delta^{\theta/2}_{-r}) = \Delta^{\theta'/2}_{-r'}$. Then, by Theorem 7.1, we get $\theta = \theta'$, and by the map preserving ratio, we obtain $r = r'$. ∎

Note that for a fixed m, there are some possibilities of s. For example, if $m = 14$, then s can be any divisor of 12.

12.　Type VI:　$\boldsymbol{\Delta = \Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2}}$

Recall the root system $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2}$ of Type VI (see (11), (12) and (13)).

Theorem 12.1.　*For Type VI, we have* $r_1 = \frac{\sqrt{m+2}}{s_1} > 1$ *and* $r_2 = \frac{\sqrt{m-2}}{s_2} > 0$, *where* s_1 *and* s_2 *are positive integers such that*

$$s_1 \mid m+2 \quad and \quad s_2 \mid m-2.$$

Conversely, such a set $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2}$ *determined by* θ, r_1, r_2, s_1 *and* s_2 *is always a reduced hyperbolic root system of rank* 2.

Moreover, $(\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2}) \ncong (\Delta^{\theta'} \sqcup \Delta_{r_1'} \sqcup \Delta_{-r_2'})$ *if* $\theta \neq \theta'$, $r_1 \neq r_1'$ *or* $r_2 \neq r_2'$.

Proof.　Type VI contains Type II and Type III, and so the first statement is clear. Conversely, to show that $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2}$ is a hyperbolic root system of rank 2, we only need to check (A3), (A4) and (A5) since Δ^θ already spans $\mathbb{R}^2$, and since $\Delta^\theta \sqcup \Delta_{r_1} \subset P$ and $\Delta_{-r_2} \subset N$. We already checked (A3) for pairs from $\Delta^\theta \sqcup \Delta_{r_1}$ in Type II and from $\Delta^\theta \sqcup \Delta_{-r_2}$ in Type III. Thus we need to check (A3) for $\langle r_1 \alpha_{k\theta + \frac{\theta}{2}}, r_2 \beta_{l\theta + \frac{\theta}{2}} \rangle$ and $\langle r_2 \beta_{k\theta + \frac{\theta}{2}}, r_1 \alpha_{l\theta + \frac{\theta}{2}} \rangle$ for $k, l \in \mathbb{Z}$, which are equal to $\frac{2r_1}{r_2} \sinh(l-k)\theta$ and $\frac{2r_2}{r_1} \sinh(k-l)\theta$, respectively. We note that

$$\frac{2r_1}{r_2} \sinh\theta = 2\frac{\sqrt{m+2}}{s_1} \frac{s_2}{\sqrt{m-2}} \frac{\sqrt{m^2-4}}{2} = \frac{s_2(m+2)}{s_1} \in \mathbb{Z}$$

$$\frac{2r_2}{r_1} \sinh\theta = 2\frac{\sqrt{m-2}}{s_2} \frac{s_1}{\sqrt{m+2}} \frac{\sqrt{m^2-4}}{2} = \frac{s_1(m-2)}{s_2} \in \mathbb{Z}.$$

Hence we get (A3) by Lemma 2.5. (A4) follows from Lemma 3.5. (A5) is clear since $\Delta^\theta \sqcup \Delta_{r_1}$ and $\Delta^\theta \sqcup \Delta_{-r_2}$ are irreducible. Thus $\Delta^\theta \sqcup \Delta_{-r}$ is a reduced hyperbolic root system of rank 2.

Finally, suppose that $(\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2}) \cong (\Delta^{\theta'} \sqcup \Delta_{r_1'} \sqcup \Delta_{-r_2'})$ via φ. Since an isomorphism of root systems keeps the ratio of lengths, we have $\varphi(\Delta^\theta) = \Delta^{\theta'}$, and $r_1 = r_1'$ and $r_2 = r_2'$ (even if $r_2 = 1$). Also, by Theorem 7.1, we get $\theta = \theta'$. $\blacksquare$

We note that there is no extra condition for s_1 and s_2 besides the conditions in Type II and Type III.

13. Type VII: $\Delta = \Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^\theta_{-r_2}$
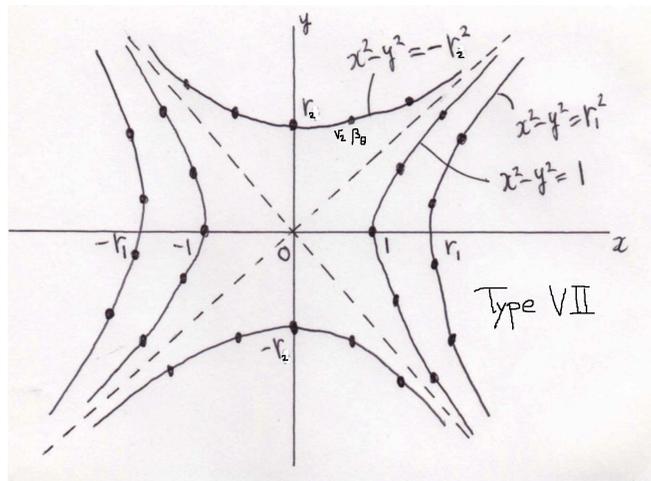
Recall the root system $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^\theta_{-r_2}$ of Type VII (see (11), (12) and (14)).

Theorem 13.1. *For Type VII, we have* $r_1 = \frac{\sqrt{m+2}}{s_1} > 1$ *and* $r_2 = \frac{\sqrt{m^2-4}}{s_2} > 0$ *with* $r_2 \neq 1$, *where* s_1 *and* s_2 *are positive integers such that*

$$s_1 \mid m+2, \quad s_1 \mid s_2 \quad and \quad s_2 \mid (m-2)s_1.$$

Conversely, such a set $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^\theta_{-r_2}$ *determined by* θ, r_1, r_2, s_1 *and* s_2 *is always a reduced hyperbolic root system of rank* 2.

Moreover, $(\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^\theta_{-r_2}) \ncong (\Delta^{\theta'} \sqcup \Delta_{r_1'} \sqcup \Delta^{\theta'}_{-r_2'})$ *if* $\theta \neq \theta'$, $r_1 \neq r_1'$ *or* $r_2 \neq r_2'$.

Proof. Type VII contains Type II and Type IV, and so we have $r_1 > 1$, $r_2 > 0$, $s_1 \mid m+2$ and $s_2 \mid m^2 - 4$, $r_1 = \frac{\sqrt{m+2}}{s_1} > 1$ and $r_2 = \frac{\sqrt{m^2-4}}{s_2} > 0$. We note that

$$\langle r_1\alpha_{\frac{\theta}{2}}, r_2\beta_0 \rangle = \frac{2r_1}{r_2}\sinh\frac{\theta}{2} = 2\frac{\sqrt{m+2}}{s_1}\frac{s_2}{\sqrt{m^2-4}}\frac{\sqrt{m-2}}{2} = \frac{s_2}{s_1} \in \mathbb{Z} \tag{22}$$

$$\langle r_2\beta_0, r_1\alpha_{\frac{\theta}{2}} \rangle = -\frac{2r_2}{r_1}\sinh\frac{\theta}{2} = -2\frac{\sqrt{m^2-4}}{s_2}\frac{s_1}{\sqrt{m+2}}\frac{\sqrt{m-2}}{2} = -\frac{s_1(m-2)}{s_2} \in \mathbb{Z}.$$

If $r_2 = 1$, then $\sqrt{m^2 - 4}$ has to be square, which is impossible. Hence we have shown the first statement. Note that the condition $s_2 \mid m^2 - 4$ is obtained by the other conditions.

Conversely, to show that $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^\theta_{-r_2}$ is a hyperbolic root system of rank 2, we only need to check (A3), (A4) and (A5) since Δ^θ already spans $\mathbb{R}^2$, and since $\Delta^\theta \sqcup \Delta_{r_1} \subset P$ and $\Delta^\theta_{-r_2} \subset N$. We already checked (A3) for pairs from $\Delta^\theta \sqcup \Delta_{r_1}$ in Type II and from $\Delta^\theta \sqcup \Delta^\theta_{-r_2}$ in Type IV. Thus we need to check (A3) for $\langle r_1 \alpha_{k\theta + \frac{\theta}{2}}, r_2 \beta_{l\theta} \rangle$ and $\langle r_2 \beta_{k\theta}, r_1 \alpha_{l\theta + \frac{\theta}{2}} \rangle$ for $k, l \in \mathbb{Z}$, which are equal to $\frac{2r_1}{r_2} \sinh(l - k - \frac{1}{2})\theta$ and $\frac{2r_2}{r_1} \sinh(k - l - \frac{1}{2})\theta$, respectively. Thus we get (A3) by (22) above and Lemma 2.5. (A4) follows from Lemma 3.5. (A5) is clear since $\Delta^\theta \sqcup \Delta_{r_1}$ and $\Delta^\theta \sqcup \Delta^\theta_{-r_2}$ are irreducible. Thus $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^\theta_{-r_2}$ is a reduced hyperbolic root system of rank 2.

Finally, suppose that $(\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^\theta_{-r_2}) \cong (\Delta^{\theta'} \sqcup \Delta_{r_1'} \sqcup \Delta^{\theta'}_{-r_2'})$ via φ. Since an isomorphism of root systems keeps the ratio of lengths, we have $\varphi(\Delta^\theta) = \Delta^{\theta'}$, and $r_1 = r_1'$ and $r_2 = r_2'$. Also, by Theorem 7.1, we get $\theta = \theta'$.   ■

14.   Type VIII: $\boldsymbol{\Delta = \Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^{\theta/2}_{-r_2}}$
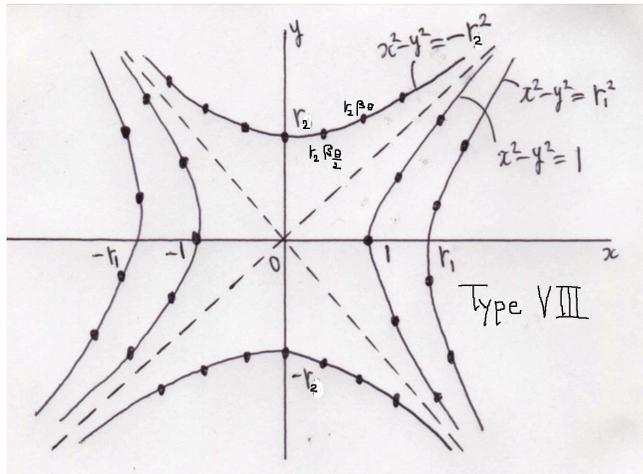
Recall the root system $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^{\theta/2}_{-r_2}$ of Type VIII (see (11), (12) and (15)).

Theorem 14.1.   *For Type VIII, $m + 2$ is square, $r_1 = \frac{\sqrt{m+2}}{s_1} > 1$ and $r_2 = \frac{\sqrt{m-2}}{s_2} > 0$ with $r_2 \neq 1$, where s_1 and s_2 are positive integers such that*

$$s_1 \mid m + 2, \quad s_2 \mid m - 2 \quad \text{and} \quad s_1 \mid s_2 \sqrt{m + 2} \mid s_1(m - 2).$$

Conversely, such a set $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^{\theta/2}_{-r_2}$ determined by θ, r_1, r_2, s_1 and s_2 is always a reduced hyperbolic root system of rank 2.

Moreover, $(\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^{\theta/2}_{-r_2}) \not\cong (\Delta^{\theta'} \sqcup \Delta_{r_1'} \sqcup \Delta^{\theta'/2}_{-r_2'})$ if $\theta \neq \theta'$, $r_1 \neq r_1'$ or $r_2 \neq r_2'$.

Proof. Type VIII contains Type II and Type V, and so we have $r_1 > 1$, $r_2 > 0$, $s_1 \mid m + 2$, $s_2 \mid m - 2$, and $m + 2$ is square. We note that

$$\langle r_1 \alpha_{\frac{\theta}{2}}, r_2 \beta_0 \rangle = \frac{2r_1}{r_2} \sinh \frac{\theta}{2} = 2 \frac{\sqrt{m+2}}{s_1} \frac{s_2}{\sqrt{m-2}} \frac{\sqrt{m-2}}{2} = \frac{s_2 \sqrt{m+2}}{s_1} \in \mathbb{Z} \tag{23}$$

$$\langle r_2 \beta_0, r_1 \alpha_{\frac{\theta}{2}} \rangle = -\frac{2r_2}{r_1} \sinh \frac{\theta}{2} = -2 \frac{\sqrt{m-2}}{s_2} \frac{s_1}{\sqrt{m+2}} \frac{\sqrt{m-2}}{2} = -\frac{s_1 (m-2)}{s_2 \sqrt{m+2}} \in \mathbb{Z}.$$

If $r_2 = 1$, then $\sqrt{m^2 - 4}$ has to be square, which is impossible. Hence we have shown the first statement.

Conversely, to show that $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^{\theta/2}_{-r_2}$ is a hyperbolic root system of rank 2, we only need to check (A3), (A4) and (A5) since Δ^θ already spans $\mathbb{R}^2$, and since $\Delta^\theta \sqcup \Delta_{r_1} \subset P$ and $\Delta^{\theta/2}_{-r_2} \subset N$. We already checked (A3) for pairs from $\Delta^\theta \sqcup \Delta_{r_1}$ in Type II and from $\Delta^\theta \sqcup \Delta^{\theta/2}_{-r_2}$ in Type V. Thus we need to check (A3) for $\langle r_1 \alpha_{k\theta + \frac{\theta}{2}}, r_2 \beta_{\frac{l\theta}{2}} \rangle$ and $\langle r_2 \beta_{\frac{k\theta}{2}}, r_1 \alpha_{l\theta + \frac{\theta}{2}} \rangle$ for $k, l \in \mathbb{Z}$, which are equal to $\frac{2r_1}{r_2} \sinh(\frac{l}{2} - k - \frac{1}{2})\theta$ and $\frac{2r_2}{r_1} \sinh(\frac{k}{2} - l - \frac{1}{2})\theta$, respectively. Thus we get (A3) by (23) above and Lemma 2.5. (A4) follows from Lemma 3.5. (A5) is clear since $\Delta^\theta \sqcup \Delta_{r_1}$ and $\Delta^\theta \sqcup \Delta^{\theta/2}_{-r_2}$ are irreducible. Thus $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^{\theta/2}_{-r_2}$ is a reduced hyperbolic root system of rank 2.
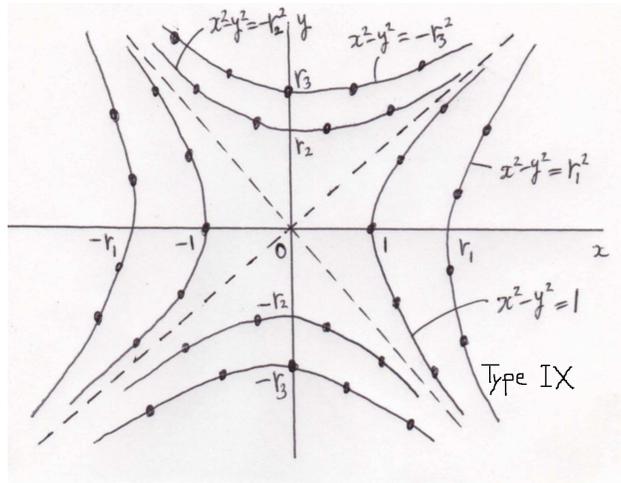
Finally, suppose that $(\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^{\theta/2}_{-r_2}) \cong (\Delta^{\theta'} \sqcup \Delta_{r_1'} \sqcup \Delta^{\theta'/2}_{-r_2'})$ via φ. Since an isomorphism of root systems keeps the ratio of lengths, we have $\varphi(\Delta^\theta) = \Delta^{\theta'}$, and $r_1 = r_1'$ and $r_2 = r_2'$. Also, by Theorem 7.1, we get $\theta = \theta'$. ∎

Example 14.2. Suppose that $m = 142$. If $(s_1, s_2) = (3, 1)$ or $(6, 1)$, then
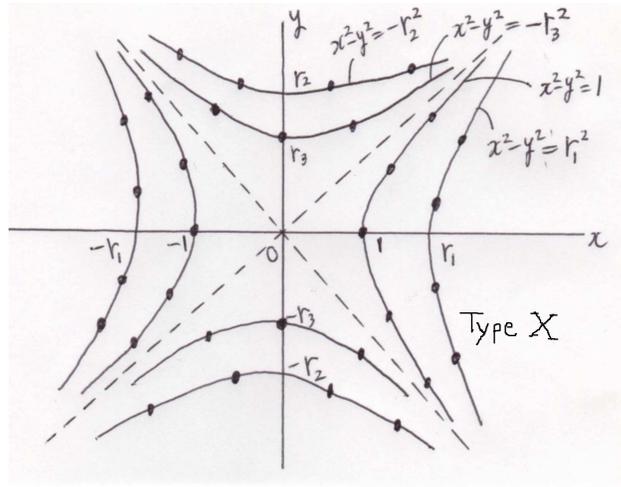
$$s_1 \mid m + 2, \quad s_1 < \sqrt{m+2}, \quad s_2 \mid m - 2 \quad \text{and} \quad s_1 \mid s_2\sqrt{m+2} \mid s_1(m-2)$$

hold. However, $(s_1, s_2) = (1, 1)$ does not satisfy $s_2\sqrt{m+2} \mid s_1(m-2)$. Also, $(s_1, s_2) = (9, 1)$ satisfies $s_2\sqrt{m+2} \mid s_1(m-2)$, but not $s_1 \mid s_2\sqrt{m+2}$.

15. Type IX and X: $\Delta = \Delta(\theta, r_1, r_2, r_3)$

Recall the root systems $\Delta(\theta, r_1, r_2, r_3)$ of Type IX and X (see (16)).

Type X

Theorem 15.1. *For Type IX (resp. Type X), we hjave $r_1 = \frac{\sqrt{m+2}}{s_1} > 1$, $r_2 = \frac{\sqrt{m-2}}{s_2} \geq 1$ and $r_3 = \frac{\sqrt{m^2-4}}{s_3} > 1$ with $r_2 < r_3$ (resp. $r_3 < r_2$), where s_1, s_2 and s_3 are positive integers such that $s_1 \mid m+2$, $s_2 \mid m-2$ and $s_3 \mid m^2-4$ with extra conditions*

$$s_1 \mid s_3 \mid (m-2)s_1 \quad and \quad s_2 \mid s_3 \mid (m+2)s_2.$$

($r_3 > 1$ for both types, by Corollary 5.6(ii).)

Conversely, such a set $\Delta(\theta, r_1, r_2, r_3)$ determined by θ, r_i and s_i is always a reduced hyperbolic root system of rank 2.

Moreover, $\Delta(\theta, r_1, r_2, r_3) \not\cong \Delta(\theta', r_1', r_2', r_3')$ if $\theta \neq \theta'$, $r_1 \neq r_1'$, $r_2 \neq r_2'$ or $r_3 \neq r_3'$.

In particular, Type IX and Type X are not isomorphic.

Proof. Type IX and X contain Type I, III, IV and VII, and so we have $s_1 \mid m+2$, $s_2 \mid m-2$, $s_3 \mid m^2-4$ and $s_1 \mid s_3 \mid (m-2)s_1$. We note that

$$\langle r_2\beta_{\frac{\theta}{2}}, r_3\beta_0\rangle = -\frac{2r_2}{r_3}\cosh\frac{\theta}{2} = -2\frac{\sqrt{m-2}}{s_2}\frac{s_3}{\sqrt{m^2-4}}\frac{\sqrt{m+2}}{2} = \frac{s_3}{s_2} \in \mathbb{Z} \qquad (24)$$

$$\langle r_3\beta_0, r_2\beta_{\frac{\theta}{2}}\rangle = -\frac{2r_3}{r_2}\cosh\frac{\theta}{2} = -2\frac{\sqrt{m^2-4}}{s_3}\frac{s_2}{\sqrt{m-2}}\frac{\sqrt{m+2}}{2} = -\frac{s_2(m+2)}{s_3} \in \mathbb{Z}.$$

Hence we have shown the first statement.

Conversely, to show that $\Delta(\theta, r_1, r_2, r_3)$ is a hyperbolic root system of rank 2, we only need to check (A3), (A4) and (A5) since Δ^θ already spans $\mathbb{R}^2$, and since $\Delta^\theta \sqcup \Delta_{r_1} \subset P$ and $\Delta_{-r_2} \sqcup \Delta_{-r_3}^\theta \subset N$. We already checked (A3) for pairs from $\Delta^\theta \sqcup \Delta_{r_1}$ in Type II, from $\Delta^\theta \sqcup \Delta_{-r_2}$ in Type III and from $\Delta^\theta \sqcup \Delta_{-r_3}^\theta$ in Type IV. Also, we checked pairs from $\Delta_{r_1} \sqcup \Delta_{-r_i}$ and $\Delta_{r_1} \sqcup \Delta_{-r_i}^\theta$ ($i = 2, 3$) in Type VI and VII. Thus we need to check (A3) for $\langle r_2\beta_{k\theta+\frac{\theta}{2}}, r_3\beta_{l\theta}\rangle$ and $\langle r_3\beta_{k\theta}, r_2\beta_{l\theta+\frac{\theta}{2}}\rangle$ for $k, l \in \mathbb{Z}$, which are equal to $\frac{2r_2}{r_3}\cosh(k-l+\frac{1}{2})\theta$ and $\frac{2r_3}{r_2}\cosh(k-l-\frac{1}{2})\theta$, respectively. But these follow from (24) above and Lemma 2.2. (A4) follows from Lemma 3.5. (A5) is clear since $\Delta^\theta \sqcup \Delta_{r_1}$, $\Delta^\theta \sqcup \Delta_{-r_2}$ and $\Delta^\theta \sqcup \Delta_{-r_3}^\theta$ are irreducible. Thus $\Delta(\theta, r_1, r_2, r_3)$ is a reduced hyperbolic root system of rank 2.

Finally, suppose that $\Delta(\theta, r_1, r_2, r_3) \cong \Delta(\theta', r_1', r_2', r_3')$ via φ. Since an isomorphism of root systems keeps the ratio of lengths, we have $\varphi(\Delta^\theta) = \Delta^{\theta'}$,

and $r_1 = r_1'$, $r_2 = r_2'$ and $r_3 = r_3'$ (even if $r_2 = 1$). Also, by Theorem 7.1, we get $\theta = \theta'$. The last statement will be proved in Theorem 16.2. ∎

Remark 15.2. Note that $s_1 < \sqrt{m+2}$, $s_2 \le \sqrt{m-2}$ and $s_3 < \sqrt{m^2-4}$. There are some (m, s_1, s_2, s_3) satisfying $s_1 \mid m+2$, $s_2 \mid m-2$, $s_3 \mid m^2-4$, $s_1 \mid s_3 \mid (m-2)s_1$ and $s_2 \mid s_3 \mid (m+2)s_2$ even if m is fixed.

For example, if $m = 10$, then $1 \le s_1, s_2 \le 3$ and $1 \le s_3 \le 9$, and $(s_1, s_2, s_3) = (1,1,1)$, $(1,1,2)$, $(1,2,2)$, $(2,2,2)$, $(1,2,4)$, $(2,2,4)$, $(3,2,6)$, $(1,1,4)$, $(2,1,2)$, $(2,1,4)$, $(3,1,3)$, $(3,1,6)$, $(1,2,8)$ or $(2,2,8)$ satisfies all the conditions above. Also, we need to have

$$1 \le r_2 < r_3 \iff s_3 < s_2\sqrt{m+2} \le \sqrt{m^2-4} \quad \text{for Type IX}$$
$$\text{or} \quad 1 < r_3 < r_2 \iff s_2\sqrt{m+2} < s_3 < \sqrt{m^2-4} \quad \text{for Type X.}$$

Thus, when $m = 10$, the first six, $(1,1,1)$, $(1,1,2)$, $(2,2,2)$, $(1,2,4)$, $(2,2,4)$ and $(3,2,6)$ are of Type IX, and the last five, $(1,1,4)$, $(1,2,2)$, $(2,1,2)$, $(2,1,4)$, $(3,1,3)$, $(3,1,6)$, $(1,2,8)$ and $(2,2,8)$ are of Type X.

16. Classification

We have proved so far:

Theorem 16.1. *A reduced hyperbolic root system Δ of rank 2 is isomorphic to one of the following:*

1. *Type I: $\Delta^\theta = \{\pm\alpha_{k\theta} \mid k \in \mathbb{Z}\}$ for $2\cosh\theta \in \mathbb{Z}_{\ge 3}$;*

2. *Type II: $\Delta^\theta \sqcup \Delta_r$, where $\Delta_r = \{\pm r\alpha_{k\theta+\frac{\theta}{2}} \mid k \in \mathbb{Z}\}$;*

3. *Type III: $\Delta^\theta \sqcup \Delta_{-r}$, where $\Delta_{-r} = \{\pm r\beta_{k\theta+\frac{\theta}{2}} \mid k \in \mathbb{Z}\}$;*

4. *Type IV: $\Delta^\theta \sqcup \Delta^\theta_{-r}$, where $\Delta^\theta_{-r} = \{\pm r\beta_{k\theta} \mid k \in \mathbb{Z}\}$;*

5. *Type V: $\Delta^\theta \sqcup \Delta^{\theta/2}_{-r}$, where $\Delta^{\theta/2}_{-r} = \{\pm r\beta_{\frac{k\theta}{2}} \mid k \in \mathbb{Z}\}$;*

6. *Type VI: $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2}$;*

7. *Type VII: $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^\theta_{-r_2}$;*

8. *Type VIII: $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta^{\theta/2}_{-r_2}$;*

9. *Type IX: $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2} \sqcup \Delta^\theta_{-r_3}$ $(r_2 < r_3)$;*

10. *Type X: $\Delta^\theta \sqcup \Delta_{r_1} \sqcup \Delta_{-r_2} \sqcup \Delta^\theta_{-r_3}$ $(r_2 > r_3)$.*

(We have omitted the precise information for $r, r_1, r_2, r_3 \in \mathbb{R}_{>0}$.)

Since an isomorphism preserves the ratio of roots, we have $\ell(\Delta) = \ell(\Delta')$ if $\Delta \cong \Delta'$. Thus, Type I, Type II, Type III-V, Type VI-VIII and Type IX-X are not isomorphic. In fact, we have:

Theorem 16.2. *Any two from Type I to Type X are not isomorphic.*

Proof.   It is enough to show that

(1) Type III, IV and V are not isomorphic;

(2) Type VI, VII and VIII are not isomorphic;

(3) Type IX and X are not isomorphic.

Let us call two roots α and β an *orthogonal pair* if $(\alpha, \beta) = 0$. Also, we write $\langle \Delta, \Delta \rangle$ for the set $\{ \langle \alpha, \beta \rangle \mid \alpha, \beta \in \Delta \}$. Thus if $\Delta \cong \Delta'$, then $\langle \Delta, \Delta \rangle = \langle \Delta', \Delta' \rangle$.

     For (1), there is no orthogonal pair in Type III. Hence it is enough to show that IV and V are not isomorphic. But each $\langle \Delta, \Delta \rangle$ is different for IV and V. Thus (1) is shown.

     For (2), considering $\langle \Delta, \Delta \rangle$ of each type, Type VIII is not isomorphic to Type VI or VII. So, suppose that VI $\cong$ VII. Then $\Delta^\theta \sqcup \Delta_{-r} \cong \Delta^{\theta'} \sqcup \Delta^{\theta'}_{-r'}$. But the left-hand side is of Type III and the right-hand side is of Type IV, and hence impossible by (1) above. Hence VI $\not\cong$ VII.

     For (3), suppose that IX $\cong$ X. Since an isomorphism of root systems keeps the ratio of lengths, we have $\Delta^\theta \sqcup \Delta_{-r} \cong \Delta^{\theta'} \sqcup \Delta^{\theta'}_{-r'}$. Thus by the same reason as in (2), we obtain a contradiction.   ■

17.   Weyl groups

We call the group $W := \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$ generated by the reflections σ_α for all $\alpha \in \Delta$ the *Weyl group* of Δ.

     Recall the property $\tau \sigma_\alpha \tau^{-1} = \sigma_{\tau(\alpha)}$ for a reflection σ_α and an isometry τ in general (not only in a euclidean space). In particular, we have $\sigma_\beta \sigma_\alpha \sigma_\beta = \sigma_{\sigma_\beta(\alpha)}$.

     Since the lengths of roots do not effect the Weyl group, most of the Weyl groups of the root systems above are isomorphic. For example, the Weyl groups of Type VIII, IX and X are clearly isomorphic. In fact, we can say more:

Theorem 17.1. *There are only two Weyl groups for hyperbolic root systems of rank 2. Namely, the Weyl groups of Type I, II and III are isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}_2$ (an infinite dihedral group), and the rest of types are isomorphic to $(\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$.*

Proof.   Note that $\sigma_{\alpha_{-\theta}} = \sigma_{\alpha_0} \sigma_{\alpha_\theta} \sigma_{\alpha_0}$, $\sigma_{\alpha_{-2\theta}} = \sigma_{\alpha_{-\theta}} \sigma_{\alpha_0} \sigma_{\alpha_{-\theta}}$, $\sigma_{\alpha_{-3\theta}} = \sigma_{\alpha_{-\theta}} \sigma_{\alpha_\theta} \sigma_{\alpha_{-\theta}}$ and $\sigma_{\alpha_{\pm k\theta}} = \sigma_{\alpha_0} \sigma_{\alpha_{\mp k\theta}} \sigma_{\alpha_0}$ (see Lemma 3.5). Thus, $W = \langle \sigma_{\alpha_0}, \sigma_{\alpha_\theta} \rangle$ for Type I. Also,

$$h := \sigma_{\alpha_\theta} \sigma_{\alpha_0} = \begin{pmatrix} -\cosh 2\theta & \sinh 2\theta \\ -\sinh 2\theta & \cosh 2\theta \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{pmatrix}$$

is the hyperbolic rotation which has an infinite order, and h satisfies the relation $\sigma_{\alpha_0} h \sigma_{\alpha_0} = h^{-1}$. Hence one can easily see that $W = \langle h, \sigma_{\alpha_\theta} \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}_2$.

     For Type II, note that $\sigma_{\alpha_{-\theta/2}} = \sigma_{\alpha_0} \sigma_{\alpha_{\theta/2}} \sigma_{\alpha_0}$, $\sigma_{\alpha_{-\theta}} = \sigma_{\alpha_{-\theta/2}} \sigma_{\alpha_0} \sigma_{\alpha_{-\theta/2}}$ and $\sigma_{\alpha_{-3\theta/2}} = \sigma_{\alpha_{-\theta/2}} \sigma_{\alpha_{\theta/2}} \sigma_{\alpha_{-\theta/2}}$. Thus, $W = \langle \sigma_{\alpha_0}, \sigma_{\alpha_{\theta/2}} \rangle$. Also,

$$h := \sigma_{\alpha_{\theta/2}} \sigma_{\alpha_0} = \begin{pmatrix} -\cosh \theta & \sinh \theta \\ -\sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}$$

is the hyperbolic rotation with $\sigma_{\alpha_0} h \sigma_{\alpha_0} = h^{-1}$. Hence $W = \langle h, \sigma_{\alpha_\theta} \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}_2$ again.

For Type III, note that $\sigma_{\beta_{-\theta/2}} = \sigma_{\alpha_0} \sigma_{\beta_{\theta/2}} \sigma_{\alpha_0}$, $\sigma_{\beta_{-\theta}} = \sigma_{\beta_{-\theta/2}} \sigma_{\alpha_0} \sigma_{\beta_{-\theta/2}}$ and $\sigma_{\beta_{-3\theta/2}} = \sigma_{\beta_{-\theta/2}} \sigma_{\beta_{\theta/2}} \sigma_{\beta_{-\theta/2}}$. Thus $W = \langle \sigma_{\alpha_0}, \sigma_{\beta_{\theta/2}} \rangle$. Also,

$$h := \sigma_{\beta_{\theta/2}} \sigma_{\alpha_0} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}.$$

is the hyperbolic rotation with $\sigma_{\alpha_0} h \sigma_{\alpha_0} = h^{-1}$. Hence $W = \langle h, \sigma_{\alpha_\theta} \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}_2$ again.

For the rest of types, we note that W contains $\sigma_{\alpha_0} \sigma_{\beta_0} = -I$. Thus we get, by Lemma 3.5, $W = \langle \sigma_{\alpha_0}, \sigma_{\alpha_\theta}, -I \rangle$ for Type IV, $W = \langle \sigma_{\beta_0}, \sigma_{\beta_{\theta/2}}, -I \rangle$ for Type V and $W = \langle \sigma_{\alpha_0}, \sigma_{\alpha_{\theta/2}}, -I \rangle$ for Types VI-X. As above, it is easily seen that each of the Weyl groups is isomorphic to $(\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$. ∎

Example 17.2. We fix $\theta = \log(3 + 2\sqrt{2})$. Then $\cosh\theta = 3$ and $\sinh\theta = 2\sqrt{2}$. (Note that $(3,2)$ is the fundamental solution of the Pell's equation $x^2 - 2y^2 = 1$.) Let $A := \sigma_{\alpha_0} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Since $2\theta = \log(3 + 2\sqrt{2})^2 = \log(17 + 12\sqrt{2})$, we have $B := \sigma_{\alpha_\theta} \sigma_{\alpha_0} = \begin{pmatrix} 17 & 12\sqrt{2} \\ 12\sqrt{2} & 17 \end{pmatrix}$. (Note that $(17, 12)$ is the next solution of the fundamental one.) Thus the Weyl group of Type I is equal to $\langle A, B \rangle$.

Let $C := \sigma_{\alpha_{\theta/2}} \sigma_{\alpha_0} = \begin{pmatrix} 3 & 2\sqrt{2} \\ 2\sqrt{2} & 3 \end{pmatrix}$. Then $W = \langle A, C \rangle$ for Type II and $W = \langle A, -C \rangle$ for Type III.

Note that $-I = -BABA$ and $\sigma_{\beta_0} = -\sigma_{\alpha_0} = -A$. Thus $W = \langle A, \pm B \rangle$ for Type IV and $W = \langle A, \pm C \rangle$ for Types V-X.

Remark 17.3. We note that $\mathbb{Z} \rtimes \mathbb{Z}_2$ and $(\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$ are both Coxeter groups. (They are not isomorphic since the latter has a nontrivial center.) In fact, $\mathbb{Z} \rtimes \mathbb{Z}_2$ is isomorphic to the Coxeter group determined by the graph:

$$\overset{\infty}{\circ\!\!-\!\!-\!\!-\!\!\circ}$$
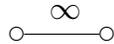
which is the same as the affine reflection group of type $\tilde{A}_1$ (see [H, p.96]), and $(\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$ is isomorphic to the Coxeter group determined by the graph:

$$\overset{\infty}{\circ\!\!-\!\!-\!\!-\!\!\circ} \qquad \circ$$
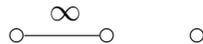
Remark 17.4. We can consider the set $\Delta^\theta = \{\pm\alpha_{k\theta} \mid k \in \mathbb{Z}\}$ for $\theta \in \mathbb{R}_{>0}$ without our assumption $2\cosh\theta \in \mathbb{Z}_{\geq 3}$ ($\Leftrightarrow 2\cosh\theta \in \mathbb{Z}$). This Δ^θ still satisfies the axioms of root systems except the crystallographic condition (A3). The Weyl group of Δ^θ is still isomorphic to the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}_2$, by the same

proof above. Note that the hyperbolic distance between $\alpha_{k\theta}$ and $\alpha_{(k+1)\theta}$ is defined to be θ. So we can say that Δ^θ is the set of vertices of a hyperbolic regular ∞-polygon in the hyperbolic plane $\mathbb{R}^2$ (not the upper half plane, the Klein disc nor the Poincaré disc).

Note that the classical irreducible root system of rank 2 having the same length is the root system of type A_2, which is the set of vertices of a regular hexagon. What is the speciality of a regular hexagon among the euclidean regular n-gons? That is $2\cos\theta \in \mathbb{Z}$ in the regular n-gon $\{e^{ik\theta} \mid k \in \mathbb{Z}\}$ in $\mathbb{C}$, where $\theta = 2\pi/n$ (and a point symmetry to exclude the regular triangle).

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