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A New $\mathbb{Z}_3$-Graded Quantum Group

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Abstract. We introduce a $\mathbb{Z}_3$-graded version of the exterior (Grassmann) algebra with two generators. Using this object we obtain a new $\mathbb{Z}_3$-graded quantum group denoted by $\mathcal{O}(\widetilde{GL}_q(2))$ and discuss some of its properties.
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1. Introduction

The quantum plane [7] is a well known example of a quantum homogeneous space in quantum group theory. One specific approach to represent a quantum group is to introduce the quantum plane (and its dual). When there exists an appropriate set of noncommuting variables linearly spanning a representation space, the endomorphisms on that space preserving the noncommutative structure allow to set up a quantum group. The natural extension to $\mathbb{Z}_2$-graded space was introduced in [8]. The present work starts a $\mathbb{Z}_3$-graded version of the exterior plane, denoted by $\tilde{\mathbb{R}}_q^{0|2}$, where q is a cubic root of unity. In this case, of course, it will not go back to the original objects. The term "plane" is used as a formal title based upon its construction. Following Manin's approach to quantum group $GL(2)$ we see that there exists a $\mathbb{Z}_3$-graded (quantum) group acting on the $\mathbb{Z}_3$-graded exterior plane. A detailed discussion of this group is given in Sect. 3. In [2] Chung finds commutation relations between the elements of a $\mathbb{Z}_3$-graded quantum 2x2 matrix using the differential structure established on the quantum (1+1)-superplane. With a similar idea, in [1] the author obtains similar (but not identical) relations. However, all structures introduced in the present study are completely different from both [2] and [1] except for the matrix T.

2. $\mathbb{Z}_3$-graded planes

The aim of this section is to introduce the $\mathbb{Z}_3$-graded version of the exterior algebra and its dual. It is known that the Manin quantum plane is introduced as a q-deformation of the commutative plane in the sense that it becomes the classical

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plane when q is equal to 1. In our case, the parameter q is a cubic root of unity and there is no classical limit. To understand what this means, let's begin by recalling some facts about the exterior algebra.

2.1. $\mathbb{Z}_3$-gradation.

A $\mathbb{Z}_3$-graded vector space is a vector space V together with a decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}} \oplus V_{\bar{2}}$. Members of $V_{\bar{0}} \oplus V_{\bar{1}} \oplus V_{\bar{2}}$ are called homogeneous elements. The *grade* (or *degree*) of a homogenous element $v \in V_i$ is denoted by $\tau(v) = i$, $i \in \mathbb{Z}_3$. An element in $V_{\bar{0}}$ (resp. $V_{\bar{1}}$ and $V_{\bar{2}}$) is of degree 0 (resp. 1 and 2).

A $\mathbb{Z}_3$-graded algebra $\mathcal{A}$ is a $\mathbb{Z}_3$-graded vector space $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}} \oplus \mathcal{A}_{\bar{2}}$ which is also an associative algebra such that $\mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}$ or, equivalently, $\tau(\xi_1 \cdot \xi_2) = \tau(\xi_1) + \tau(\xi_2) \pmod 3$ for all homogeneous elements $\xi_1, \xi_2 \in \mathcal{A}$ i.e.

$$\begin{aligned}
\tau(a_0 \cdot a_1) &= \tau(a_0) + \tau(a_1) = 0 + 1 = 1\,(\text{mod}3) \quad &&\text{for} \quad a_0 \in \mathcal{A}_{\bar{0}}, a_1 \in \mathcal{A}_{\bar{1}},\\
\tau(a_1 \cdot a_2) &= \tau(a_1) + \tau(a_2) = 1 + 2 = 0\,(\text{mod}3) \quad &&\text{for} \quad a_1 \in \mathcal{A}_{\bar{1}}, a_2 \in \mathcal{A}_{\bar{2}}\\
\tau(a_2 \cdot a_2) &= \tau(a_2) + \tau(a_2) = 2 + 2 = 1\,(\text{mod}3) \quad &&\text{for} \quad a_2 \in \mathcal{A}_{\bar{2}}.
\end{aligned}$$

2.2. The algebra of functions on the $\mathbb{Z}_3$-graded exterior plane.

A possible way to generalize the $\mathbb{Z}_3$-graded exterior plane is to increase the power of nilpotency of its generators and to impose a $\mathbb{Z}_3$-graded commutation relation on the generators. We will assume that q is a cubic root of unity.

One usually needed to put the wedge product between the coordinates of exterior plane, but this does not matter in the $\mathbb{Z}_3$-graded case.

Definition 2.1. Let $O(\tilde{\mathbb{R}}_q^{0|2})$ be the algebra with the generators θ and φ obeying the relations

$$\theta \cdot \varphi = q^2 \varphi \cdot \theta, \quad \theta^3 = 0 = \varphi^3 \tag{1}$$

where the coordinates θ and φ are of grade 1 and 2, respectively. We call $O(\tilde{\mathbb{R}}_q^{0|2})$ the algebra of functions on the $\mathbb{Z}_3$-graded Grassmann plane $\tilde{\mathbb{R}}_q^{0|2}$.

Definition 2.2. The exterior algebra $\Lambda(\tilde{\mathbb{R}}_q^{0|2})$ of $\mathbb{Z}_3$-graded Grassmann plane is the algebra generated by elements ξ and x, and relation

$$\xi \cdot x = x \cdot \xi, \tag{2}$$

where the generators ξ, x are of degree 2, 0, respectively.

3. The $\mathbb{Z}_3$-graded (quantum) group

The algebraic group $SL(2,\mathbb{C})$ has the coordinate algebra $\mathcal{O}(SL(2,\mathbb{C}))$. As an algebra $\mathcal{O}(SL(2,\mathbb{C}))$ is the quotient of the commutative polynomial algebra $\mathbb{C}[a,b,c,d]$ by the two-sided ideal generated by the element $ad - bc - 1$ where the indeterminates a,b,c,d are the coordinate functions on $SL(2,\mathbb{C})$. Taking the group structure in $SL(2,\mathbb{C})$, we may encode it in terms of maps m (multiplication), η (identity) and S (inversion). Dualizing these maps to $\mathcal{O}(SL(2,\mathbb{C}))$, we get the corresponding co-maps called comultiplication Δ, counit ϵ, and antipode S, respectively. The axioms for the group structure of $SL(2,\mathbb{C})$, in terms of the maps, are then reversed giving us relations among the co-maps. Taking the natural axioms satisfied in $\mathcal{O}(SL(2,\mathbb{C}))$ by the maps m, η, Δ, ϵ and S, one obtains a Hopf algebra. The quantum group $\mathcal{O}_q(SL(2,\mathbb{C}))$ is a noncommutative deformation of $\mathcal{O}(SL(2,\mathbb{C}))$. General concepts related to quantum groups (Hopf algebras) can be found in the books of Klimyk and Schmüdgen [4] or Majid [5].

In this section, we will consider the 2x2 matrices acting on the $\mathbb{Z}_3$-graded Grassmann plane and will discuss the properties of such matrices. So, let a, β, γ, d be elements of an algebra $\mathcal{A}$ where the generators a and d are of degree 0, the generators γ and β are of degree 1 and 2, respectively. Let $\tilde{M}(2)$ be defined as the polynomial algebra $k[a,\beta,\gamma,d]$. It will sometimes be convenient and more illustrative to write a point (a,β,γ,d) of $\tilde{M}(2)$ in the matrix form

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = (t_{ij}). \tag{3}$$

We construct the $\mathbb{Z}_3$-graded matrix algebra $\tilde{M}(2)$ as follows: We divide the algebra $\tilde{M}(2)$ into three parts as $\tilde{M}(2) = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}} \oplus \mathcal{A}_{\bar{2}}$. In this case, if a matrix has the form

$$T_0 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad (\text{resp.} \quad T_1 = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}),$$

then it is an element of $\mathcal{A}_{\bar{0}}$ (resp. $\mathcal{A}_{\bar{1}}$, $\mathcal{A}_{\bar{2}}$) and is of grade 0 (resp. 1, 2). This gives a $\mathbb{Z}_3$-graded structure to the algebra of matrices, in the sense that $\tau(T_iT_j) = \tau(T_i) + \tau(T_j) \pmod 3$. It is easy to check that the product of two Z_3-graded matrices is also a $\mathbb{Z}_3$-graded matrix. As it can easily be shown that matrices of the form (3) form a group provided that $ad - \beta\gamma \neq 0$. We denote this group by $\widetilde{GL}(2)$.

3.1. The algebra $\mathcal{O}(\tilde{M}_q(2))$.

To determine a q-analogue of the algebra $\mathcal{O}(\tilde{M}(2))$, we will first obtain the commutation relations between the matrix elements of the matrix T.

If $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{Z}_3$-graded algebras, then their tensor product $\mathcal{A} \otimes \mathcal{B}$ is the $\mathbb{Z}_3$-graded algebra whose underlying space is a $\mathbb{Z}_3$-graded tensor product of $\mathcal{A}$ and $\mathcal{B}$. The following definition [6] gives the product rule for the tensor product of algebras.

Definition 3.1. If $\mathcal{A}$ is a $\mathbb{Z}_3$-graded algebra, then the product rule in the $\mathbb{Z}_3$-graded algebra $\mathcal{A} \otimes \mathcal{A}$ is defined by

$$(a_1 \otimes a_2)(a_3 \otimes a_4) = q^{\tau(a_2)\tau(a_3)} a_1a_3 \otimes a_2a_4 \tag{4}$$

where a_i's are homogeneous elements in the algebra $\mathcal{A}$.

Remark 3.2. It is well known that the matrix T given in (3) defines the linear transformation $T : \tilde{\mathbb{R}}_q^{0|2} \longrightarrow \tilde{\mathbb{R}}_q^{0|2}$. As a result of this we have $T\Theta = \Theta' \in \tilde{\mathbb{R}}_q^{0|2}$, where $\Theta = (\theta, \varphi)^t$. However, the relation $\alpha_1\alpha_2 = q^{\tau(\alpha_1)\tau(\alpha_2)}\alpha_2\alpha_1$ for all elements α_1 and α_2 in the $\mathbb{Z}_3$-graded algebra is inconsistent. Therefore, we will use the following transform while obtaining the commutation relations between the matrix elements of T.

Let a, β, γ, d be elements of the algebra $\mathcal{O}(\tilde{M}(2))$. We also assume that the generators a and d are of degree 0, the generators γ and β are of degree 1 and 2, respectively. Then we can change the coordinates of a vector in $\tilde{\mathbb{R}}_q^{0|2}$ as follows

$$\Theta' = \begin{pmatrix}\theta' \\ \varphi'\end{pmatrix} := \begin{pmatrix}a & \beta \\ \gamma & d\end{pmatrix} \dot{\otimes} \begin{pmatrix}\theta \\ \varphi\end{pmatrix}, \quad \Theta'' = \begin{pmatrix}\theta'' \\ \varphi''\end{pmatrix} := \begin{pmatrix}\theta & \varphi\end{pmatrix} \dot{\otimes} \begin{pmatrix}a & \beta \\ \gamma & d\end{pmatrix}. \tag{5}$$

So, we can give the following proposition that can be proved with straightforward computations.

Proposition 3.3. *The coordinates of Θ' and Θ'' satisfy (1) if and only if the generators a, β, γ, d fulfill the relations*

$$a\beta = \beta a, \qquad \beta\gamma = \gamma\beta, \qquad d\beta = \beta d, \tag{6}$$

$$a\gamma = q\gamma a, \qquad d\gamma = q^2\gamma d, \tag{7}$$

$$ad = da + (q-1)\beta\gamma, \tag{8}$$

where q is a cubic root of unity.

Remark 3.4. Unlike the usual quantum group [7], one interesting feature is that the element β belongs to the center of the algebra.

Definition 3.5. The $\mathbb{Z}_3$-graded algebra $\mathcal{O}(\tilde{M}_q(2))$ is the quotient of the free algebra $k\{a, \beta, \gamma, d\}$ by the two-sided ideal J_q generated by the six relations (6)-(8) of Proposition 3.2.

By relation (8), we have

$$\mathcal{D}_q := ad - q\beta\gamma = da - \beta\gamma. \tag{9}$$

This element of $\mathcal{O}(\tilde{M}_q(2))$ is called the $\mathbb{Z}_3$*-graded determinant.*

The proof of the following assertion is given by a direct computation using the relations (6)-(8).

Remark 3.6. The $\mathbb{Z}_3$-graded quantum determinant defined in (9) commutes with a, β, γ and d, so that the requirement $\mathcal{D}_q = 1$ is consistent.

Proposition 3.7. *Let T and T' be two matrices such that their matrix elements satisfy the relations (6)-(8). If all elements of T commute according to the rule (4) with all elements of T', then the elements of the matrix (tensor) product TT' obey the relations (6)-(8). We also have*

$$\mathcal{D}_q(T\dot{\otimes}T') = \mathcal{D}_q(T) \otimes \mathcal{D}_q(T').$$

Proof. Let the matrix (tensor) product of T with T' be

$$T\dot{\otimes}T' = \begin{pmatrix}a & \beta \\ \gamma & d\end{pmatrix} \dot{\otimes} \begin{pmatrix}a' & \beta' \\ \gamma' & d'\end{pmatrix} = \begin{pmatrix}X & Y \\ Z & W\end{pmatrix}.$$

Then using the relations (6)-(8) with (4) we get

$$\begin{aligned} XY &= (a \otimes a' + \beta \otimes \gamma')(a \otimes \beta' + \beta \otimes d') \\ &= a^2 \otimes a'\beta' + a\beta \otimes a'd' + \beta a \otimes \gamma'\beta' + q^2\beta^2 \otimes \gamma' d' \\ &= a^2 \otimes \beta' a' + a\beta \otimes d'a' + qa\beta \otimes \beta'\gamma' + \beta^2 \otimes d'\gamma' \\ YX &= a^2 \otimes \beta' a' + a\beta \otimes d'a' + qa\beta \otimes \beta'\gamma' + \beta^2 \otimes d'\gamma'. \end{aligned}$$

It can be similarly shown that relations $XZ = qZX$, $YZ = ZY$, etc., are satisfied. The proof of the last claim is as follows:

$$\begin{aligned} XW &= a\gamma \otimes a'\beta' + ad \otimes a'd' + q\beta\gamma \otimes \gamma'\beta' + \beta d \otimes \gamma' d' \\ YZ &= q^2 a\gamma \otimes \beta' a' + ad \otimes \beta'\gamma' + \beta\gamma \otimes d'a' + \beta d \otimes d'\gamma' \\ XW - qYZ &= ad \otimes (a'd' - q\beta'\gamma') - q\beta\gamma \otimes (d'a' - \gamma'\beta') \end{aligned}$$

and so $\mathcal{D}_q(T\dot{\otimes}T')$ reduces to $\mathcal{D}_q(T) \otimes \mathcal{D}_q(T')$. ■

3.2. Bialgebra structure on $\tilde{M}_q(2)$.

We now endow the algebra $\mathcal{O}(\tilde{M}_q(2))$ with a bialgebra structure. The comultiplication and the counit will be the same as in the usual quantum groups.

Proposition 3.8. (1) *There exist $\mathbb{Z}_3$-graded algebra homomorphisms*

$$\Delta : \mathcal{O}(\tilde{M}_q(2)) \longrightarrow \mathcal{O}(\tilde{M}_q(2)) \otimes \mathcal{O}(\tilde{M}_q(2)), \quad \epsilon : \mathcal{O}(\tilde{M}_q(2)) \longrightarrow \mathbb{C}$$

uniquely determined by

$$\begin{aligned} \Delta(a) &= a \otimes a + \beta \otimes \gamma, & \Delta(\beta) &= a \otimes \beta + \beta \otimes d, \\ \Delta(\gamma) &= \gamma \otimes a + d \otimes \gamma, & \Delta(d) &= \gamma \otimes \beta + d \otimes d, \\ \epsilon(a) &= 1 = \epsilon(d), & \epsilon(\beta) &= 0 = \epsilon(\gamma). \end{aligned}$$

(2) *With these maps, the algebra $\mathcal{O}(\tilde{M}_q(2))$ is a bialgebra which is neither commutative nor cocommutative.*
(3) *The quantum determinant $\mathcal{D}_q$ is a group-like element of $\mathcal{O}(\tilde{M}_q(2))$.*

Proof. (1) In order to prove that Δ and ϵ are algebra homomorphisms, it is sufficient to show that the relations (6)-(8) remain invariant under Δ and ϵ. As an example let us show that $\Delta(a\beta) = \Delta(\beta a)$:

$$\begin{aligned} \Delta(a\beta) &= \Delta(a)\Delta(\beta) = (a \otimes a + \beta \otimes \gamma)(a \otimes \beta + \beta \otimes d) \\ &= a^2 \otimes a\beta + a\beta \otimes ad + \beta a \otimes \gamma\beta + q^2\beta^2 \otimes \gamma d \\ &= a^2 \otimes \beta a + \beta a \otimes da + qa\beta \otimes \beta\gamma + \beta^2 \otimes d\gamma \\ \Delta(\beta a) &= \Delta(\beta)\Delta(a) = (a \otimes \beta + \beta \otimes d)(a \otimes a + \beta \otimes \gamma) \\ &= a^2 \otimes \beta a + qa\beta \otimes \beta\gamma + \beta a \otimes da + \beta^2 \otimes d\gamma. \end{aligned}$$

Analogously, one can prove the other relations. For ϵ it is completely analogous.

(2) It is not difficult to check that the comultiplication Δ is coassociative in the sense that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

and the counit ϵ has the property

$$m \circ (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = m \circ (\mathrm{id} \otimes \epsilon) \circ \Delta.$$

It follows that $\mathcal{O}(\tilde{M}_q(2))$ is indeed a bialgebra.
(3) To prove that the $\mathbb{Z}_3$-graded determinant $\mathcal{D}_q$ is group-like, it is sufficient to show that

$$\Delta(\mathcal{D}_q) = \mathcal{D}_q \otimes \mathcal{D}_q \quad \text{and} \quad \epsilon(\mathcal{D}_q) = 1.$$

Indeed, some computations give

$$\begin{aligned}
\Delta(\mathcal{D}_q) &= \Delta(a)\Delta(d) - q\Delta(\beta)\Delta(\gamma) \\
&= ad \otimes ad + q\beta\gamma \otimes \beta\gamma - qad \otimes \beta\gamma - q\beta\gamma \otimes da \\
&= ad \otimes (ad - q\beta\gamma) + q\beta\gamma \otimes (\beta\gamma - da) \\
&= \mathcal{D}_q \otimes \mathcal{D}_q
\end{aligned}$$

and $\epsilon(ad - q\beta\gamma) = \epsilon(a)\epsilon(d) - q\epsilon(\beta)\epsilon(\gamma) = 1$. ■

The bialgebra $\mathcal{O}(\tilde{M}_q(2))$ is called the coordinate algebra of the $\mathbb{Z}_3$-graded (quantum) matrix space $\tilde{M}_q(2)$.

3.3. The $\mathbb{Z}_3$-graded Hopf algebra $\mathcal{O}(\widetilde{GL}_q(2))$.

Using the quantum determinant $\mathcal{D}_q$ belonging to the algebra $\mathcal{O}(\tilde{M}_q(2))$, we can define a *new* Hopf algebra adding the inverse $\mathcal{D}_q^{-1}$ to $\mathcal{O}(\tilde{M}_q(2))$. Let $\mathcal{O}(\widetilde{GL}_q(2))$ be the quotient of the algebra $\mathcal{O}(\tilde{M}_q(2))$ by the two-sided ideal generated by the element $t\mathcal{D}_q - 1$. For short we write

$$\mathcal{O}(\widetilde{GL}_q(2)) := \mathcal{O}(\tilde{M}_q(2))[t]/\langle t\mathcal{D}_q - 1\rangle.$$

Then the algebra $\mathcal{O}(\widetilde{GL}_q(2))$ is again a bialgebra.

Lemma 3.9. *The elements of the matrix*

$$\tilde{T} = \begin{pmatrix} \tilde{a} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{d} \end{pmatrix} = \begin{pmatrix} d & -\beta \\ -q\gamma & a \end{pmatrix}$$

satisfy the defining relations of the algebra $\mathcal{O}(\widetilde{GL}_{q^2}(2))$ and thus $\mathcal{O}(\widetilde{GL}_{q^2}(2))$ is the opposite algebra of $\mathcal{O}(\widetilde{GL}_q(2))$.

Proof. The use of relations (6)-(8) imply

$$\begin{aligned}
&\tilde{a}\tilde{\beta} = \tilde{\beta}\tilde{a}, && \tilde{\beta}\tilde{\gamma} = \tilde{\gamma}\tilde{\beta}, && \tilde{\beta}\tilde{d} = \tilde{d}\tilde{\beta}, \\
&\tilde{a}\tilde{\gamma} = q^2\tilde{\gamma}\tilde{a}, && \tilde{\gamma}\tilde{d} = q^2\tilde{d}\tilde{\gamma}, && \\
&\tilde{a}\tilde{d} = \tilde{d}\tilde{a} + (1 - q^2)\tilde{\beta}\tilde{\gamma}, && &&
\end{aligned}$$

which are the defining relations of the algebra $\mathcal{O}(\widetilde{GL}_{q^2}(2))$. The second claim follows from the fact that $q^3 = 1$. ■

Proposition 3.10. *The bialgebra $\mathcal{O}(\widetilde{GL}_q(2))$ is a $\mathbb{Z}_3$-graded Hopf algebra. The antipode S of $\mathcal{O}(\widetilde{GL}_q(2))$ is given by*

$$S(a) = d\,\mathcal{D}_q^{-1}, \quad S(\beta) = -\beta\,\mathcal{D}_q^{-1}, \quad S(\gamma) = -q\gamma\,\mathcal{D}_q^{-1}, \quad S(d) = a\,\mathcal{D}_q^{-1}.$$

Proof. By Lemma 3.6, there exists an algebra anti-homomorphism S from $\mathcal{O}(\widetilde{GL}_q(2))$ to $\mathcal{O}(\widetilde{GL}_{q^2}(2))$ such that $S(a) = \tilde{a}$, etc. To prove that S is an antipode for $\mathcal{O}(\widetilde{GL}_q(2))$, we have to check the antipode axiom

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$$

for the generators. Checking the antipode axiom for the generators is equivalent to verify the following matrix equality

$$T\tilde{T}\mathcal{D}_q = \epsilon(T) = \tilde{T}T\mathcal{D}_q$$

which follows from $\mathcal{D}_q = ad - q\beta\gamma$ in $\mathcal{O}(\widetilde{GL}_q(2))$ with $S(T) = \mathcal{D}_q^{-1}\tilde{T} = T^{-1}$. The calculations are straightforward. ■

Definition 3.11. The $\mathbb{Z}_3$-graded Hopf algebra $\mathcal{O}(\widetilde{GL}_q(2))$ is called the coordinate algebra of the $\mathbb{Z}_3$-graded (quantum) group $\widetilde{GL}_q(2)$.

3.4. Coactions on the $\mathbb{Z}_3$-graded Grassmann plane.

In bialgebra terminology, Proposition 3.2 yields the following.

Proposition 3.12. *The algebra $\mathcal{O}(\tilde{\mathbb{R}}_q^{0|2})$ is a left and right comodule algebra of the bialgebra $\mathcal{O}(\tilde{M}_q(2))$ with left coaction δ_L and right coaction δ_R such that*

$$\delta_L(\theta) = a \otimes \theta + \beta \otimes \varphi, \qquad \delta_L(\varphi) = \gamma \otimes \theta + d \otimes \varphi, \tag{10}$$

$$\delta_R(\theta) = \theta \otimes a + \varphi \otimes \gamma, \qquad \delta_R(\varphi) = \theta \otimes \beta + \varphi \otimes d. \tag{11}$$

Proof. It is not difficult to verify that (10) and (11) define algebra homomorphisms δ_L from $O(\tilde{\mathbb{R}}_q^{0|2})$ to $\mathcal{O}(\tilde{M}_q(2)) \otimes \mathcal{O}(\tilde{\mathbb{R}}_q^{0|2})$ and δ_R from $\mathcal{O}(\tilde{\mathbb{R}}_q^{0|2})$ to $\mathcal{O}(\tilde{\mathbb{R}}_q^{0|2}) \otimes \mathcal{O}(\tilde{M}_q(2))$, respectively. It remains to be checked that δ_L and δ_R are coactions, i.e., the conditions

$$(\Delta \otimes \mathrm{id}) \circ \delta_L = (\mathrm{id} \otimes \delta_L) \circ \delta_L, \quad m \circ (\epsilon \otimes \mathrm{id}) \circ \delta_L = \mathrm{id} \tag{12}$$

and

$$(\mathrm{id} \otimes \Delta) \circ \delta_R = (\delta_R \otimes \mathrm{id}) \circ \delta_R, \quad m \circ (\mathrm{id} \otimes \epsilon) \circ \delta_R = \mathrm{id}$$

are satisfied. For example,

$$\begin{aligned}(\Delta \otimes \mathrm{id})\delta_L(\theta) &= (\Delta \otimes \mathrm{id})(a \otimes \theta + \beta \otimes \varphi) \\ &= (a \otimes a + \beta \otimes \gamma) \otimes \theta + (a \otimes \beta + \beta \otimes d) \otimes \varphi \\ &= a \otimes (a \otimes \theta + \beta \otimes \varphi) + \beta \otimes (\gamma \otimes \theta + d \otimes \varphi) \\ &= a \otimes \delta_L(\theta) + \beta \otimes \delta_L(\varphi) \\ &= (\mathrm{id} \otimes \delta_L)\delta_L(\theta)\end{aligned}$$

and

$$\begin{aligned} m \circ (\epsilon \otimes \mathrm{id})\delta_L(\theta) &= m(\epsilon \otimes \mathrm{id})(a \otimes \theta + \beta \otimes \varphi) \\ &= m(1 \otimes \theta + 0 \otimes \varphi) \\ &= \theta \end{aligned}$$

as expected. ■

Remark 3.13. In fact, there exists a left coaction of $\Lambda(\tilde{\mathbb{R}}_q^{0|2})$ on the second plane $(\xi \cdot x = x \cdot \xi)$, called a left comodule-$\Lambda(\tilde{\mathbb{R}}_q^{0|2})$ satisfying the conditions (12).

Remark 3.14. An easy computation shows that the ideal $(\vartheta := \theta \cdot \varphi - q^2 \varphi \cdot \theta)$ of $\tilde{\mathbb{R}}_q^{0|2}$ is a subcomodule of $\tilde{\mathbb{R}}_q^{0|2}$.

The proof is immediate: Indeed, since δ_L is an algebra map, it is only necessary to show that $\delta_L(\vartheta) = \mathcal{D}_q \otimes \vartheta$. Using relations (6)-(8) with (9) we get

$$\begin{aligned} \delta_L(\vartheta) &= \delta_L(\theta)\delta_L(\varphi) - q^2 \delta_L(\varphi)\delta_L(\theta) \\ &= qa\gamma \otimes \theta^2 + ad \otimes \theta\varphi + q^2 \beta\gamma \otimes \varphi\theta + \beta d \otimes \varphi^2 - q^2 \gamma a \otimes \theta^2 \\ &\quad - q\gamma\beta \otimes \theta\varphi - q^2 da \otimes \varphi\theta - d\beta \otimes \varphi^2 \\ &= (ad - q\beta\gamma) \otimes \theta\varphi - q^2(da - \beta\gamma) \otimes \varphi\theta = \mathcal{D}_q \otimes \vartheta \end{aligned}$$

as expected. ■

3.5. The Hopf algebra $\mathcal{O}(\widetilde{SL}_q(2))$.

We know that the two-sided ideal $\langle \mathcal{D}_q - 1 \rangle$ generated by the element $\mathcal{D}_q - 1$ is a biideal of $\mathcal{O}(\widetilde{M}_q(2))$ since the determinant $\mathcal{D}_q$ is group-like. So the quotient $\mathcal{O}(\widetilde{SL}_q(2)) := \mathcal{O}(\tilde{M}_q(2))/\langle \mathcal{D}_q - 1 \rangle$ is a bialgebra.

Remark 3.15. There exists a Hopf $\star$-algebra structure on the Hopf algebra $\mathcal{O}(\widetilde{SL}_q(2))$ such that

$$a^\star = a, \quad \beta^\star = \beta, \quad \gamma^\star = q^2 \gamma, \quad d^\star = d.$$

4. $\mathbb{Z}_3$-graded quantum algebra of $\widetilde{GL}_q(2)$

In this section, using the method of [3], we give an R-matrix formulation for the $\mathbb{Z}_3$-graded quantum group $\widetilde{GL}_q(2)$ and obtain a $\mathbb{Z}_3$-graded universal enveloping algebra $U_q(\widetilde{gl}(2))$.

4.1. The FRT construction for $\widetilde{GL}_q(2)$.

The R-matrix formulation (the FRT-relation $\hat{R}T_1T_2 = T_1T_2\hat{R}$) for the quantum matrix groups [3] can be considered as a compact matrix form of the commutation relations between the generators of an associative algebra.

The formulation for the $\mathbb{Z}_3$-graded quantum group $\widetilde{GL}_q(2)$ has the same form, but matrix tensor product includes additional q-factors related to $\mathbb{Z}_3$-grading. Two matrices A, B $(\tau(A_{ij}) = \tau(i) + \tau(j))$ are multiplied according to the rule

$$(A \otimes B)_{ij,kl} = q^{\tau(j)(\tau(i)+\tau(k))} A_{ik} B_{jl}.$$

Due to this fact $T_2 = I \otimes T$ has the same block-diagonal form as in the standard (ungraded) case while $T_1 = T \otimes I$ includes the additional factors q for graded elements standing at some of odd rows of blocks. For the $\mathbb{Z}_3$-graded quantum group $\widetilde{GL}_q(2)$ the R-matrix satisfying the $\mathbb{Z}_3$-graded Yang-Baxter equation has the form

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l0 & 1 & q-q^2 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} = (\hat{R}_{ij}^{kl})$$

where $\hat{R} = \underline{P}R$ and $\underline{P}$ denotes the $\mathbb{Z}_3$-graded permutation operator defined by $\underline{P}(a \otimes b) = q^{\tau(a)\tau(b)} b \otimes a$ on homogeneous elements. A simple calculation shows that this operator represents the 3rd-root of the permutation operator P with the action $P(a \otimes b) = b \otimes a$.

The condition for the matrices belonging to the $\mathbb{Z}_3$-graded quantum group $\widetilde{GL}_q(2)$ is given below without proof.

Proposition 4.1. *A 2x2-matrix T is a $\mathbb{Z}_3$-graded quantum matrix if and only if*

$$\hat{R} T_1 T_2 = T_1 T_2 \hat{R}$$

where matrix elements of T are $\mathbb{Z}_3$-graded.

4.2. A $\mathbb{Z}_3$-graded universal enveloping algebra $U_q(\widetilde{gl}(2))$.

The construction of the $\mathbb{Z}_3$-graded quantum algebra of $\widetilde{GL}_q(2)$ can be done analogous to the approach of the Leningrad school. The $\mathbb{Z}_3$-graded quantum algebra of $\widetilde{GL}_q(2)$ has four generators: U and V are of degree 0, X_- and X_+ are of degrees 1 and 2, respectively.

Proposition 4.2. *The generators of the $\mathbb{Z}_3$-graded quantum algebra satisfy the following relations*

$$UV = VU, \quad UX_\pm = q^{\pm 2} X_\pm U, \quad VX_\pm = q^{\mp 2} X_\pm V, \tag{13}$$

$$X_+ X_- - X_- X_+ = \frac{UV^{-1} - VU^{-1}}{q^2 - q}. \tag{14}$$

Proof. The generators U, V, $X_\pm$ can be written as elements of two 2x2 matrix as follows

$$L^+ = \begin{pmatrix} U & \lambda X_+ \\ 0 & V \end{pmatrix}, \quad L^- = \begin{pmatrix} U^{-1} & 0 \\ \lambda X_- & V^{-1} \end{pmatrix}$$

where $\lambda = q - q^2$. The matrices $L^\pm$ satisfy the following relations

$$R^+ L_1^\pm L_2^\pm = L_2^\pm L_1^\pm R^+, \tag{15}$$

where the matrix R^+ is defined by $R^+ = \underline{P}R\underline{P}$. The relations (13) follow from the relations (15). To obtain the relation (14) we use the relation

$$R^+ L_1^- L_2^+ = L_2^+ L_1^- R^+.$$

■

Proposition 4.3. *The coproduct of the generators is given by*

$$\Delta(L^\pm) = L^\pm \dot{\otimes} L^\pm.$$

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