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Global Integration of Leibniz Algebras

Martin Bordemann and Friedrich Wagemann

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Abstract. In this article, we present a global integration procedure of any real finite-dimensional Leibniz algebra into a Lie rack which reduces in the particular case of a Lie algebra to the ordinary connected simply connected Lie group. The construction is not functorial.
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1. Introduction

All manifolds considered in this manuscript are assumed to be Hausdorff, finite dimensional and second countable.

A *pointed rack* is a pointed set (X,e) together with a binary operation $\rhd : X\times X\to X$ such that for all $x\in X$, the map $y\mapsto x\rhd y$ is bijective and such that for all $x,y,z\in X$, the self-distributivity and unit relations

$$x\rhd(y\rhd z) \,=\, (x\rhd y)\rhd(x\rhd z),\quad e\rhd x=x,\quad \text{and}\quad x\rhd e=e$$

are satisfied. All our racks will be left racks. Imitating the notion of a Lie group, the smooth version of a pointed rack is called *Lie rack*. One main example of a Lie rack is the *conjugation Lie rack* underlying a Lie group G, i.e. $X=G$ and $x\rhd y=xyx^{-1}$.

An important class of examples of racks are the so-called *augmented racks*, see [4]. An augmented rack is the data of a group G, a G-set X and a map $p:X\to G$ such that for all $x\in X$ and all $g\in G$,

$$p(g\cdot x) \,=\, gp(x)g^{-1}.$$

The set X becomes then a rack by setting $x\rhd y := p(x)\cdot y$.

Lie racks are intimately related to *Leibniz algebras* $\mathfrak{h}$. Recall that a (left) Leibniz algebra is a vector space $\mathfrak{h}$ with a bilinear bracket $[,]:\mathfrak{h}\otimes\mathfrak{h}\to\mathfrak{h}$ such that for all $X,Y,Z\in\mathfrak{h}$, $[X,-]$ acts as a derivation:

$$[X,[Y,Z]] \,=\, [[X,Y],Z]+[Y,[X,Z]].$$

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Indeed, Kinyon showed in [7] that the tangent space at $e \in H$ of a Lie rack H carries a natural structure of a Leibniz algebra, generalizing the relation between a Lie group and its tangent Lie algebra. Conversely, every (finite-dimensional real or complex) Leibniz algebra $\mathfrak{h}$ may be integrated into a Lie rack (with underlying manifold $\mathfrak{h}$) using the rack product

$$X \rhd Y := e^{\mathrm{ad}_X}(Y), \tag{1}$$

noting that the exponential of the inner derivation ad_X for each $X \in \mathfrak{h}$ is an automorphism. Although the assignment $\big(\mathfrak{h}, [\ ,\ ]\big) \to \big(\mathfrak{h}, 0, \rhd\big)$ is functorial since morphisms of Leibniz algebras are easily seen to go to morphisms of pointed Lie racks, the restriction to the category of all Lie algebras would not give the usual integration as a Lie group.

The purpose of the present paper is to give an integration procedure which integrates real, finite-dimensional Leibniz algebras into Lie racks in such a way that the restriction to Lie algebras gives the conjugation rack underlying the simply connected Lie group corresponding to a (real, finite-dimensional) Lie algebra.

This problem has been encountered by J.-L. Loday in 1993 [9] in the search of quantifying the lack of periodicity in algebraic K-Theory. Several attempts and constructions have been published since then. In 2010, Simon Covez [2] solves in his thesis the local integration problem by constructing a local Lie rack integrating a given (real, finite dimensional) Leibniz algebra in such a way that in the case of Lie algebras, one obtains the conjugation Lie rack underlying the usual (simply connected) Lie group integrating it.

In this article, we construct in Section 3 a (global) Lie rack integrating a given (real, finite-dimensional) Leibniz algebra $\mathfrak{h}$ in such a way that in the case of a Lie algebra, the construction yields the conjugation rack underlying the usual (simply connected) Lie group integrating it. More precisely, we work with augmented Leibniz algebras, i.e. Leibniz algebras $\mathfrak{h}$ with an action of a Lie algebra $\mathfrak{g}$ by derivations and an equivariant map $p : \mathfrak{h} \to \mathfrak{g}$ to $\mathfrak{g}$. We integrate the quotient Lie algebra $p(\mathfrak{h}) =: \mathfrak{g}'$ into a Lie group G' and integrate its action on $\mathfrak{h}$ such that the resulting (global) augmented Lie rack is an affine bundle over G' with typical fiber $\mathfrak{z} := \mathrm{Ker}(p)$. This is the content of our main theorem, Theorem 3.2.

The construction relies essentially on the existence of an open neighbourhood in the Lie group G' on which the exponential is a diffeomorphism and which is invariant under (the connected component of the identity) $\mathrm{Aut}_0(G')$. It is well known that such a neighbourhood exists, see e.g. [6]. On the other hand, it is the use of this neighbourhood and of the associated bump function which renders our construction non-functorial. For the moment, we do not know whether there exists a functorial construction of a Lie rack integrating a given Leibniz algebra (such that in the special case of a Lie algebra, we get back the conjugation rack underlying the usual simply connected Lie group).

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2. Augmented Leibniz algebras and Lie racks

2.1. (Augmented) Leibniz algebras.

All modules will be considered over $\mathbb{R}$ and all unadorned tensor products will always be over $\mathbb{R}$.

Recall that a (real) *Leibniz algebra* is an $\mathbb{R}$-vector space $\mathfrak{h}$ equipped with a linear map $[\ ,\ ] : \mathfrak{h}\otimes\mathfrak{h}\to\mathfrak{h}$, written $x\otimes y\mapsto [x,y]$ such that the *left Leibniz identity* holds for all $x,y,z\in\mathfrak{h}$

$$\big[x,[y,z]\big] = \big[[x,y],z\big] + \big[y,[x,z]\big] \tag{2}$$

A morphism of Leibniz algebras $\Phi:\mathfrak{h}\to\mathfrak{h}'$ is a $\mathbb{R}$-linear map preserving brackets, i.e. for all $x,y\in\mathfrak{h}$ we have $\Phi\big([x,y]\big) = \big[\Phi(x),\Phi(y)\big]'$. Recall first that each (real) Lie algebra is a (real) Leibniz algebra giving rise to a functor $\mathfrak{i}$ from the category of all Lie algebras, $\mathbb{R}\mathbf{LieAlg}$, to the category of all Leibniz algebras, $\mathbb{R}\mathbf{Leib}$.

Furthermore, recall that each Leibniz algebra has two canonical subspaces

$$\begin{aligned} Q(\mathfrak{h}) &:= \Big\{x\in\mathfrak{h}\mid \exists\ N\in\mathbb{N}\setminus\{0\},\ \exists\ \lambda_1,\dots,\lambda_N\in\mathbb{R},\ \exists\ x_1,\dots,x_N \\ &\qquad\qquad \text{such that } x=\sum_{r=1}^{N}\lambda_r[x_r,x_r]\Big\}, \qquad (3)\\ \mathfrak{z}(\mathfrak{h}) &:= \Big\{x\in\mathfrak{h}\mid \forall\ y\in\mathfrak{h}:\ [x,y]=0\Big\}. \qquad (4)\end{aligned}$$

It is well-known and not hard to deduce from the Leibniz identity that both $Q(\mathfrak{h})$ and $\mathfrak{z}(\mathfrak{h})$ are two-sided abelian ideals of $(\mathfrak{h},[\ ,\ ])$, that $Q(\mathfrak{h})\subset\mathfrak{z}(\mathfrak{h})$, and that the quotient Leibniz algebras

$$\overline{\mathfrak{h}} := \mathfrak{h}/Q(\mathfrak{h}) \quad \text{and} \quad \mathfrak{h}/\mathfrak{z}(\mathfrak{h}) \tag{5}$$

are Lie algebras. Since the ideal $Q(\mathfrak{h})$ is clearly mapped into the ideal $Q(\mathfrak{h}')$ by any morphism of Leibniz algebras $\mathfrak{h}\to\mathfrak{h}'$ (which is a priori not the case for $\mathfrak{z}(\mathfrak{h})$!), there is an obvious functor $\mathfrak{h}\to\overline{\mathfrak{h}}$ from the category of all Leibniz algebras to the category of all Lie algebras. It is not hard to see and not important for the sequel that the functor $\mathfrak{h}\to\overline{\mathfrak{h}}$ is a left adjoint functor of the inclusion functor of the category of all Lie algebras in the category of all Leibniz algebras whence the former is a reflective subcategory of the latter, see e.g. [10, p.91] for definitions.

It is easy to observe that in both cases of the above Lie algebras, $\overline{\mathfrak{h}}$ and $\mathfrak{h}/\mathfrak{z}(\mathfrak{h})$, there is the following structure:

Definition 2.1. A quintuple $\big(\mathfrak{h},p,\mathfrak{g},[\ ,\ ]_{\mathfrak{g}},\dot\rho\big)$ is called a $\mathfrak{g}$-*augmented Leibniz algebra* iff the following holds:

1. $\big(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}}\big)$ is a (real) Lie algebra.
2. $\mathfrak{h}$ is an $\mathbb{R}$-vector space which is a left $\mathfrak{g}$-module via the $\mathbb{R}$-linear map $\dot\rho:\mathfrak{g}\otimes\mathfrak{h}\to\mathfrak{h}$ written $\dot\rho_\xi(x)=\xi.x$ for all $\xi\in\mathfrak{g}$ and $x\in\mathfrak{h}$.

3. $p : \mathfrak{h} \to \mathfrak{g}$ is a $\mathbb{R}$-linear morphism of $\mathfrak{g}$-modules, i.e. for all $\xi \in \mathfrak{g}$ and $x \in \mathfrak{h}$

$$p(\xi.x) = \big[\xi, p(x)\big]_{\mathfrak{g}}. \tag{6}$$

A *morphism of augmented Leibniz algebras* $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot{\rho}\big) \to \big(\mathfrak{h}', p', \mathfrak{g}', [\ ,\ ]'_{\mathfrak{g}}, \dot{\rho}'\big)$ is a pair (Φ, ϕ) of $\mathbb{R}$-linear maps where $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a morphism of Lie algebras, $\Phi : \mathfrak{h} \to \mathfrak{h}$ is a morphism of Lie algebra modules over ϕ, i.e. for all $x \in \mathfrak{h}$ and $\xi \in \mathfrak{g}$

$$\Phi(\xi.x) = \phi(\xi).\Phi(x), \tag{7}$$

and

$$p' \circ \Phi = \phi \circ p. \tag{8}$$

The following properties are immediate from the definitions:

Proposition 2.2. *Let* $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot{\rho}\big)$ *be an augmented Leibniz algebra. Define the following bracket on* $\mathfrak{h}$*:*

$$[x, y]_{\mathfrak{h}} := p(x).y. \tag{9}$$

1. $\big(\mathfrak{h}, [,]_{\mathfrak{h}}\big)$ *is a Leibniz algebra on which* $\mathfrak{g}$ *acts by derivations. If* (Φ, ϕ) *is a morphism of augmented Leibniz algebras, then* Φ *is a morphism of Leibniz algebras.*

2. *The kernel of* p, $\mathrm{Ker}(p)$*, is a* $\mathfrak{g}$*-invariant two-sided abelian ideal of* $\mathfrak{h}$ *satisfying* $Q(\mathfrak{h}) \subset \mathrm{Ker}(p) \subset \mathfrak{z}(\mathfrak{h})$*.*

3. *The image of* p, $\mathrm{Im}(p)$*, is an ideal of the Lie algebra* $\mathfrak{g}$*.*

Proof. We just check the Leibniz identity: Let $x, y, z \in \mathfrak{h}$, then, writing $[\ ,\ ]_{\mathfrak{h}} = [\ ,\ ]$,

$$\begin{aligned}
\big[x, [y, z]\big] &= p(x).\big(p(y).z\big) = p(x).\big(p(y).z\big) - p(y).\big(p(x).z\big) + p(y).\big(p(x).z\big) \\
&= \big[p(x), p(y)\big]_{\mathfrak{g}}.z + \big[y, [x, z]\big] \overset{(6)}{=} p\big(p(x).y\big).z + \big[y, [x, z]\big] \\
&= \big[[x, y], z\big] + \big[y, [x, z]\big].
\end{aligned}$$

■

It follows that the class of all augmented Leibniz algebras forms a category $\mathbb{R}$**LeibA**, and there is an obvious forgetful functor from $\mathbb{R}$**LeibA** to $\mathbb{R}$**Leib** associating to $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot{\rho}\big)$ the Leibniz algebra $\big(\mathfrak{h}, [\ ,\ ]_{\mathfrak{h}}\big)$ where the Leibniz bracket $[\ ,\ ]_{\mathfrak{h}}$ is defined in equation (9).

On the other hand, there is a functor from $\mathbb{R}$**Leib** to $\mathbb{R}$**LeibA** associating to each Leibniz algebra $\big(\mathfrak{h}, [\ ,\ ]\big)$ the augmented Leibniz algebra $\big(\mathfrak{h}, p, \bar{\mathfrak{h}}, [\ ,\ ]_{\bar{\mathfrak{h}}}, \mathrm{ad}'\big)$, where $p : \mathfrak{h} \to \bar{\mathfrak{h}}$ is the canonical projection and the representation ad' of the Lie algebra $\bar{\mathfrak{h}}$ on the Leibniz algebra $\mathfrak{h}$ is defined by (for all $x, y \in \mathfrak{h}$)

$$\mathrm{ad}'_{p(x)}(y) := \mathrm{ad}_x(y) = [x, y]. \tag{10}$$

2.2. (Augmented) Lie racks.

Recall that a *pointed manifold* is a pair (M,e) where M is a differentiable manifold and e is a fixed element of M. Morphisms of pointed manifolds are base point preserving smooth maps.

A *Lie rack* is a pointed manifold (M,e) equipped with a smooth map $\mathbf{m} : M\times M \to M$ of pointed manifolds (i.e. $\mathbf{m}(e,e) = e$) such that $\mathbf{m}(x,-) : M\to M$ is a diffeomorphism for all $x\in M$ and satisfying the following identities for all $x,y,z\in M$ (where the standard notation is $\mathbf{m}(x,y) = x\rhd y$)

$$\begin{aligned} e\rhd x &= x, && (11)\\ x\rhd e &= e, && (12)\\ x\rhd(y\rhd z) &= (x\rhd y)\rhd(x\rhd z) && (13)\end{aligned}$$

The last condition (13) is called the *self distributivity condition.* A morphism of Lie racks $\phi : (M,e,\mathbf{m}) \to (M',e',\mathbf{m}')$ is a map of pointed manifolds satisfying for all $x,y\in M$ the condition $\phi(x\rhd y) = \phi(x)\rhd'\phi(y)$. The class of all Lie racks forms a category called **LieRack**. Note that every pointed differentiable manifold (M,e) carries a *trivial Lie rack structure* defined for all $x,y\in M$ by

$$x\rhd_0 y := y, \tag{14}$$

and this assignment is functorial.
Moreover, any Lie group G becomes a Lie rack upon setting for all $g,g'\in G$

$$g\rhd g' := gg'g^{-1}, \tag{15}$$

again defining a functor from the category of Lie groups to the category of all Lie racks. Examples of racks which are not the conjugation rack underlying a group abound: First of all, every conjugation class and every union of conjugation classes in a group (defining an immersed submanifold) in a Lie group is a Lie rack. Then, any Lie rack $(M,e,\rhd)$ can be *gauged* by any smooth map $f:(M,e)\to(M,e)$ of pointed manifolds satisfying for all $x,y\in M$

$$f(x\rhd y) = x\rhd f(y).$$

A straight-forward computation shows that the pointed manifold (M,e) equipped with the *gauged* multiplication $\rhd_f$ defined by

$$x\rhd_f y := f(x)\rhd y$$

is a Lie rack $(M,e,\rhd_f)$. We refer for more exotic examples to [4]. The following relation to Leibniz algebras is due to M. Kinyon [7]:

Proposition 2.3. *Let $(M,e,\mathbf{m})$ be a Lie rack and $\mathfrak{h} = T_eM$. Define the following bracket $[\ ,\ ]$ on $\mathfrak{h}$ by*

$$[x,y] = \frac{\partial}{\partial t}T_eL_{a(t)}(y)\bigg|_{t=0} \tag{16}$$

where $t\mapsto a(t)$ is any smooth curve defined on an open real interval containing 0 satisfying $a(0)=e$ and $(da/dt)(0) = x\in\mathfrak{h}$. Then we have the following

1. $(\mathfrak{h}, [\ ,\ ])$ *is a real Leibniz algebra.*
2. *Let* $\phi : (M, e, \mathbf{m}) \to (M', e', \mathbf{m}')$ *be a morphism of Lie racks. Then* $T_e\phi : \mathfrak{h} \to \mathfrak{h}'$ *is a morphism of Leibniz algebras.*

Proof. **1.** see the proof of Theorem 3.4 in [7]. **2.** Since ϕ maps e to e' its tangent map $T_e\phi$ maps T_eM to $T_{e'}M'$. We get for all $x, y \in \mathfrak{h} = T_eM$ where $t \mapsto a(t)$ is a smooth curve in M with $a(0) = e$ and $(da/dt)(0) = x$:

$$\begin{aligned} T_e\phi\big([x,y]\big) &= T_e\phi\left(\frac{\partial}{\partial t}T_eL_{a(t)}(y)\Big|_{t=0}\right) = \frac{\partial}{\partial t}\Big(T_e\big(\phi \circ L_{a(t)}\big)(y)\Big)\Big|_{t=0} \\ &= \frac{\partial}{\partial t}\Big(T_e\big(L'_{\phi(a(t))} \circ \phi\big)(y)\Big)\Big|_{t=0} = \frac{\partial}{\partial t}T_{e'}L'_{\phi(a(t))}\Big|_{t=0}\big(T_e\phi(y)\big) \\ &= \big[T_e\phi(x), T_e\phi(y)\big]. \end{aligned}$$

■

Let $\mathbb{R}\mathbf{Leib}_{fd}$ denote the category of all finite-dimensional real Leibniz algebras. The preceding proposition shows that there is a functor $T_*\mathcal{R} : \mathbf{LieRack} \to \mathbb{R}\mathbf{Leib}_{fd}$ which associates to any Lie rack $(M, e, \rhd)$ its tangent space $T_*\mathcal{R}(M) := T_eM$ at the distinguished point $e \in M$ equipped with the Leibniz bracket equation (16).

Furthermore, recall that an *augmented Lie rack* (see [4]) (M, ϕ, G, ℓ) consists of a pointed differentiable manifold (M, e_M), of a Lie group G, of a smooth map $\phi : M \to G$ (of pointed manifolds), and of a smooth left G-action $\ell : G \times M \to M$ (written $(g, x) \mapsto \ell(g, x) = \ell_g(x) = gx$) such that for all $g \in G$, $x \in M$

$$g e_M = e_M, \tag{17}$$

$$\phi(gx) = g\phi(x)g^{-1}. \tag{18}$$

It is a routine check that the multiplication $\rhd$ on M defined for all $x, y \in M$ by

$$x \rhd y := \ell_{\phi(x)}(y) \tag{19}$$

satisfies all the axioms (11), (12), and (13) of a Lie rack, thus making $(M, e_M, \rhd)$ into a Lie rack such that the map ϕ is a morphism of Lie racks, i.e. for all $x, y \in M$

$$\phi(x \rhd y) = \phi(x)\phi(y)\phi(x)^{-1}. \tag{20}$$

A morphism $(\Psi, \psi) : (M, \phi, G, \ell) \to (M', \phi', G', \ell')$ of augmented Lie racks is a pair of maps of pointed differentiable manifolds $\Psi : M \to M'$ and $\psi : G \to G'$ such that ψ is homomorphism of Lie groups and such that for all $g \in G$

$$\phi' \circ \Psi = \psi \circ \phi, \tag{21}$$

$$\Psi \circ \ell_g = \ell'_{\psi(g)} \circ \Psi. \tag{22}$$

The class of all augmented Lie racks thus forms a category **LieRackA** with the obvious forgetful functor $F : \mathbf{LieRackA} \to \mathbf{LieRack}$. Note that the trivial Lie rack structure of a pointed manifold (M, e) comes from an augmented Lie rack over the trivial Lie group $G = \{e\}$.

3. Global integration of Leibniz algebras

3.1. The main theorem.

Instead of integrating a Leibniz algebra $\mathfrak{h}$, we will rather integrate an augmented Leibniz algebra. Note that this does not restrict generality, as any Leibniz algebra $\mathfrak{h}$ gives rise to (several) augmented Leibniz algebras.

Let $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot{\rho}\big)$ be an augmented Leibniz algebra, and denote $\mathrm{Im}(p) =: \mathfrak{g}'$, $\mathrm{Ker}(p) =: \mathfrak{z}$ and by G resp. G' the connected, simply connected Lie groups associated to $\mathfrak{g}$ and $\mathfrak{g}'$.

One important ingredient in our main theorem is a smooth pointed equivariant map $\mathsf{s} : G' \to \mathfrak{g}'$ whose tangent map at $e_{G'} \in G'$ is the identity. Such maps do exist:

Lemma 3.1. *There is a smooth map $\mathsf{s} : G' \to \mathfrak{g}'$ having the following properties:*

$$\mathsf{s}(e_{G'}) = 0, \tag{23}$$

$$T_{e_{G'}}\mathsf{s} = \mathrm{id}_{\mathfrak{g}'}, \tag{24}$$

$$\forall\ g \in G,\ \forall\ g' \in G' : \quad \mathsf{s}\big(\mathrm{I}'_g(g')\big) = \mathrm{A}'_g\big(\mathsf{s}(g')\big), \tag{25}$$

i.e. the equivariance of s w.r.t. the conjugation action of G on G' and the adjoint action of G on $\mathfrak{g}'$ (see the next section).

With these preparations, the main theorem of our article reads:

Theorem 3.2. *Consider $p : \mathfrak{h} \to \mathfrak{g}'$ as an affine bundle over $\mathfrak{g}'$ with typical fibre $\mathfrak{z}$, and form the pull-back fibre bundle*

$$M := \mathsf{s}^*\mathfrak{h} \tag{26}$$

over G' w.r.t. any smooth pointed equivariant map $\mathsf{s} : G' \to \mathfrak{g}'$ whose tangent map at $e_{G'} \in G'$ is the identity. Then there is a canonical G-action ℓ on M such that $\big((M, (0, e_{G'}), \iota \circ \phi, G, \ell\big)$ is an augmented Lie rack with the following properties:

(a) *The induced Leibniz algebra structure on the tangent space $T_{(0,e_{G'})}M$ is isomorphic to $\big(\mathfrak{h}, [\ ,\ ]_{\mathfrak{h}}\big)$.*

(b) *In the particular case $\mathfrak{g} = \mathfrak{g}'$ and $\mathfrak{z} = \{0\}$, the above construction reduces to the usual conjugation Lie rack on $G = G'$.*

Remark 3.3. Let us comment on the relation of our theorem to the local integration of Leibniz algebras in [2]. The starting point in Covez' article is to view a Leibniz algebra $\mathfrak{h}$ as an abelian extension (in the category of Leibniz algebras) of the Lie algebra $\mathfrak{h}\,/\,\mathfrak{z}(\mathfrak{h})$ by the antisymmetric module $\mathfrak{z}(\mathfrak{h})$. The Lie algebra integrates into a connected, simply connected Lie group G and therefore the action integrates also, thus the main task is to integrate the Leibniz 2-cocycle characterizing this extension. The main idea is then to see the 2-cocycle as a 1-cocycle with values in 1-cocycles, and to integrate it in two steps. This is done using a neighborhood of 1 in the Lie group G where the exponential is a diffeomorphism (by van-Est's method using paths in G). The integration procedure for cocycles works only locally, and the outcome is thus a local Lie rack.

Let us emphasize that our method is completely different from Covez' method. In contrast to Covez' integration procedure, the main point is here to integrate *augmented* objects into *augmented* objects. This permits to phrase all objects in terms of Lie objects and representations which integrate by the usual Lie Theorems, and thus gets rid of the integration of the cocycle (which is done implicitely). It should be possible to recover the formula for the integration of the cocycle from chart changes in the affine bundle.

3.2. Proof of the main theorem and Lemma 3.1 .

As the main assertion of the theorem concerns a G-action on M, let us prepare the proof by a careful analysis of the actions which come into play.

For the augmented Leibniz algebra $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot{\rho}\big)$ we denoted by $\mathfrak{g}'$ the Lie ideal $p(\mathfrak{h})$ of $\mathfrak{g}$, and let $\mathfrak{z} := \mathrm{Ker}(p)$ which –we recall– is a two-sided ideal of the Leibniz algebra $\mathfrak{h}$ lying in the left centre of $\mathfrak{h}$. Let furthermore G (resp. G') be a connected simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$ (resp. $\mathfrak{g}'$). Since G is connected and simply connected, its adjoint represention Ad_G preserves the ideal $\mathfrak{g}'$ of its Lie algebra $\mathfrak{g}$, whence there is a Lie group homomorphism $g \mapsto \mathrm{A}'_g$ of G into $\mathrm{Aut}_0(\mathfrak{g}')$, the component of the identity of the Lie group of all automorphisms of the Lie algebra $\mathfrak{g}'$. Since this latter Lie group is well known to be isomorphic to $\mathrm{Aut}_0(G')$, the connected component of the identity of the topological group of all Lie group automorphisms of G' (which also is a Lie group), see e.g. [5] for details, there is a unique Lie group homomorphism

$$\mathrm{I}' : G \to \mathrm{Aut}_0(G') : g \mapsto \big(g' \mapsto \mathrm{I}'_g(g')\big)$$

such that $T_{e_G}\mathrm{I}'_g = \mathrm{A}'_g$ for all $g \in G$. Moreover, the injection $\mathfrak{g}' \to \mathfrak{g}$ induces a unique immersive Lie group homomorphism $\iota : G' \to G$ whose image is an analytic normal subgroup of G whence G acts on $\iota(G')$ by conjugations, and we have for all $g \in G$ and all $g' \in G'$

$$(\iota \circ \mathrm{I}'_g)(g') = g\iota(g')g^{-1}.$$

Next, let $\rho : G \times \mathfrak{h} \to \mathfrak{h}$ be the unique representation of G on $\mathfrak{h}$ such that for all $\xi \in \mathfrak{g}$ and $x \in \mathfrak{h}$

$$\frac{d}{dt}\rho_{\exp(t\xi)}(x)\bigg|_{t=0} = \dot{\rho}_\xi(x). \tag{27}$$

We get for all $g \in G$ and $x \in \mathfrak{h}$:

$$p\big(\rho_g(x)\big) = \mathrm{A}'_g\big(p(x)\big). \tag{28}$$

Proof of Theorem 3.2: The pull-back fibre bundle

$$M := \mathsf{s}^*\mathfrak{h} = \big\{(x, g') \in \mathfrak{h} \times G' \mid p(x) = \mathsf{s}(g')\big\} \stackrel{\phi}{\to} G' \tag{29}$$

over G' has $(0, e_{G'})$ as its distinguished point. Since $p : \mathfrak{h} \to \mathfrak{g}'$ is a surjective linear map, it is a surjective submersion whose fibre over $0 \in \mathfrak{g}'$ is equal to $\mathrm{Ker}(p) = \mathfrak{z}$, and whose fibre over any $\zeta \in \mathfrak{g}'$ is the affine subspace $p^{-1}(\{\zeta\})$ of $\mathfrak{h}$. Choosing any vector space complement $\mathfrak{b}$ to $\mathfrak{z}$ in $\mathfrak{h}$ leads to differential geometric trivialization over the global chart domain $\mathfrak{g}'$ of $\mathfrak{g}'$. Hence $p : \mathfrak{h} \to \mathfrak{g}'$ is a fibre bundle over $\mathfrak{g}'$ with typical fibre $\mathfrak{z}$, and therefore the pull-back $\mathsf{s}^*\mathfrak{h} = M$ is a well-defined fibre bundle over G' with typical fibre $\mathfrak{z}$. Recall that the projection $\phi : M \to G'$ is given by the restriction of the projection on the second factor $\mathfrak{h} \times G' \to G'$ to the submanifold $M \subset \mathfrak{h} \times G'$.

Since $\mathsf{s}(e_{G'}) = 0 = p(0)$ it follows that the point $(0, e_{G'})$ is in M, and clearly $\phi(0, e_{G'}) = e_{G'}$.

There is a canonical diagonal G-action $\hat{\ell}$ on $\mathfrak{h} \times G'$ defined by

$$\hat{\ell}_g(x, g') = \big(\rho_g(x), \mathrm{I}'_g(g')\big).$$

For any $(x, g') \in M$, we have by definition $p(x) = \mathsf{s}(g')$. Therefore we get (using also the equivariance of s and p)

$$\mathsf{s}\big(\mathrm{I}'_g(g')\big) = \mathrm{A}'_g\big(\mathsf{s}(g')\big) = \mathrm{A}'_g\big(p(x)\big) = p\big(\rho_g(x)\big).$$

This proves that for any $(x, g') \in M$, $\hat{\ell}_g(x, g') \in M$. Thus $\hat{\ell}$ restricts to a well-defined G-action ℓ on M. Clearly $\ell_g(0, e_{G'}) = (0, e_{G'})$. Moreover, for any $(x, g') \in M$ and $g \in G$ we have

$$\begin{aligned}(\iota \circ \phi)\big(\ell_g(x, g')\big) &= \iota\Big(\phi\big(\rho_g(x), \mathrm{I}'_g(g')\big)\Big) = \iota\big(\mathrm{I}'_g(g')\big) \\ &= g\iota(g')g^{-1} = g\big((\iota \circ \phi)(x, g')\big)g^{-1},\end{aligned}$$

showing that $\big((M, (0, e_{G'}), \iota \circ \phi, G, \ell\big)$ is an augmented Lie rack.

Property (a): According to the definition of the pull-back, the tangent space of M at $(e_{G'}, 0)$ is given by all pairs $(x, \zeta) \in \mathfrak{h} \times \mathfrak{g}' = T_0\mathfrak{h} \times T_{e_{G'}}G'$ such that

$$T_{e_{G'}}\mathsf{s}(\zeta) = p(x), \quad \text{hence } \zeta = p(x)$$

because of $T_{e_{G'}}\mathsf{s} = \mathrm{id}_{\mathfrak{g}'}$. It follows that the linear map $\theta_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h} \times \mathfrak{g}'$ given by

$$\theta_{\mathfrak{h}}(x) = \big(x, p(x)\big)$$

is an isomorphism of the vector space $\mathfrak{h}$ onto the tangent space $T_{(e_{G'},0)}M$. We get for all $g \in G$ and for all $y \in \mathfrak{h}$

$$T_{(e_{G'},0)}\ell_g\big(y, p(y)\big) = \frac{d}{dt}\Big(\rho_g(ty), \mathrm{I}'_g\big(\exp\big(tp(y)\big)\big)\Big)\Big|_{t=0} = \Big(\rho_g(y), \mathrm{A}'_g\big(p(y)\big)\Big).$$

Now for all $x, y \in \mathfrak{h}$, we get for the Leibniz bracket on the tangent space $T_{(e_{G'},0)}M$ of the augmented Lie rack $\big((M, (0, e_{G'}), \iota \circ \phi, G, \ell\big)$

$$\begin{aligned}\big[\theta_{\mathfrak{h}}(x), \theta_{\mathfrak{h}}(y)\big] &= \Big[\big((x, p(x)\big), \big(y, p(y)\big)\Big] \\ &= \frac{d}{dt}T_{(e_{G'},0)}\ell_{\exp\big(tp(x)\big)}\big(y, p(y)\big)\Big|_{t=0} \\ &= \frac{d}{dt}\Big(\rho_{\exp\big(tp(x)\big)}(y), \mathrm{A}'_{\exp\big(tp(x)\big)}\big(p(y)\big)\Big)\Big|_{t=0} \\ &= \Big([x, y]_{\mathfrak{h}}, \big[p(x), p(y)\big]_{\mathfrak{g}'}\Big) = \Big([x, y]_{\mathfrak{h}}, p\big([x, y]_{\mathfrak{h}}\big)\Big) = \theta_{\mathfrak{h}}\big([x, y]_{\mathfrak{h}}\big).\end{aligned}$$

showing that the induced Leibniz structure from the augmented Lie rack is isomorphic with the Leibniz bracket $[\ ,\ ]_{\mathfrak{h}}$ on $\mathfrak{h}$.

Property (b) is immediate. ■

Proof of Lemma 3.1: For any strictly positive real number τ, let $S_{\tau\mathbf{i}}$ be the open strip

$$S_{\tau\mathbf{i}} := \{z = \alpha + \mathbf{i}\beta \in \mathbb{C} \mid |\beta| < \tau\}. \tag{30}$$

Furthermore, we set

$$\mathcal{U}_{\tau\mathbf{i}} := \{\xi \in \mathfrak{g}' \mid \text{all the eigenvalues of } \mathrm{ad}_\xi \text{ lie in } S_{\tau\mathbf{i}}\}, \tag{31}$$

and recall for any $\xi \in \mathfrak{g}'$ the definition of its adjoint representation $\mathrm{ad}_\xi : \eta \mapsto [\xi, \eta]$ for all $\eta \in \mathfrak{g}'$. Furthermore, set

$$\mathcal{V}_{\tau\mathbf{i}} := \exp\big(\mathcal{U}_{\tau\mathbf{i}}\big) \subset G'. \tag{32}$$

We now claim the following assertions:

1. The subset $\mathcal{U}_{\tau\mathbf{i}}$ is an open $\mathrm{Aut}_0(\mathfrak{g}')$-invariant neighbourhood of $0 \in \mathfrak{g}'$ such that for all $\xi \in \mathcal{U}_{\tau\mathbf{i}}$ and for all η in the nilradical of $\mathfrak{g}'$, the element $\xi + \eta$ still lies in $\mathcal{U}_{\tau\mathbf{i}}$.
2. The restriction of the exponential map to $\mathcal{U}_{\tau\mathbf{i}}$ is a diffeomorphism onto $\mathcal{V}_{\tau\mathbf{i}}$ which is an open $\mathrm{Aut}_0(G')$-invariant neighbourhood of the unit element $e_{G'} \in G'$.
3. Let τ' be any real number such that $0 < \tau' < \tau \leq \pi$. Then there is a smooth $\mathrm{Aut}_0(\mathfrak{g}')$-invariant real-valued function $\gamma : \mathfrak{g}' \to \mathbb{R}$ such that
   (a) $\gamma(\mathfrak{g}') \subset [0, 1]$,
   (b) For all $\xi \in \mathcal{U}_{\tau'\mathbf{i}}$, we have $\gamma(\xi) = 1$,
   (c) The support of γ is contained in $\mathcal{U}_{\tau\mathbf{i}}$.

The first and second parts are well-known and can be recovered from similar assertions like e.g. Lemma 2.5 in [6] (see also the remarks on the bottom of p. 325 *loc. cit.*). The main idea is the use of the map $\tilde{\chi} : \mathfrak{g}' \to \mathbb{C}[\lambda]^1_n \cong \mathbb{C}^n$, where $\mathbb{C}[\lambda]^1_n$ denotes the space of unitary degree n polynomials in λ with complex coefficients, which is defined for all $\xi \in \mathfrak{g}'$ by

$$\tilde{\chi}(\xi)(\lambda) := \chi\big(\mathrm{ad}_\xi\big)(\lambda)$$

where we have written χ for the characteristic polynomial. Thus the set

$$\mathcal{U}_{\tau\mathbf{i}} = \tilde{\chi}^{-1}\big(\mathcal{T}(S_{\tau\mathbf{i}}^{\times n})\big)$$

is an open $\mathrm{Aut}_0(\mathfrak{g}')$-invariant subset of $\mathfrak{g}'$. Here $\mathcal{T} : \mathbb{C}^n \to \mathbb{C}[\lambda]^1_n$ is the continuous, closed and open map between $\mathbb{C}^n$ and $\mathbb{C}[\lambda]^1_n$ given by $(z_1, \ldots, z_n) \mapsto (\lambda - z_1) \ldots (\lambda - z_n)$ which projects onto a (well-known, but not unsurprising) homeomorphism $T : \mathbb{C}^n / S_n \to \mathbb{C}[\lambda]^1_n$. The invariance of $\tilde{\chi}$ under translation by elements ξ in the nilradical stems from the fact that sufficient powers of ad_ξ are zero and thus their traces do not contribute to $\tilde{\chi}$. In the second part of the statement, one uses the formula for the derivative of the exponential function and the Inverse Function Theorem. Injectivity can be shown using a beautiful argument of Lazard-Tits [8, Section 2.3]. The reasoning for the last part is also standard.

By the above Assertions 1. and 2., there are two open neighbourhoods $\mathcal{U}'_{(\pi/2)\mathbf{i}} \subset \mathcal{U}'_{\pi\mathbf{i}}$ of $0 \in \mathfrak{g}'$ which are both $\mathrm{Aut}_0(\mathfrak{g}')$-invariant and on which the restriction of the exponential map $\exp_{G'}$ is a diffeomorphism onto the $\mathrm{Aut}_0(G')$-invariant open neighbourhoods $\mathcal{V}'_{(\pi/2)\mathbf{i}} \subset \mathcal{V}'_{\pi\mathbf{i}}$ of the unit element $e_{G'}$ of G'. Moreover, by Assertion 3., there is an $\mathrm{Aut}_0(\mathfrak{g}')$-invariant bump function $\gamma' : \mathfrak{g}' \to [0,1]$ whose support is in $\mathcal{U}'_{\pi\mathbf{i}}$ and which is equal to 1 on $\mathcal{U}'_{(\pi/2)\mathbf{i}}$. Let us define the following map $\mathsf{s} : G' \to \mathfrak{g}'$ by

$$\mathsf{s}(g') := \begin{cases} \gamma'\big(\exp_{G'}^{-1}(g')\big)\exp_{G'}^{-1}(g') & \forall\, g' \in \mathcal{V}'_{\pi\mathbf{i}} \\ 0 & \forall\, g' \notin \mathcal{V}'_{\pi\mathbf{i}} \end{cases} \tag{33}$$

It is clear that s is a well-defined smooth map $G' \to \mathfrak{g}'$. Moreover $\mathsf{s}(e_{G'}) = 0$ by the properties of the exponential map, and for all $\zeta \in \mathfrak{g}' = T_{e_{G'}}G'$

$$\begin{aligned} T_{e_{G'}}\mathsf{s}(\zeta) &= \frac{d}{ds}\Big(\gamma'\big(\exp_{G'}^{-1}(\exp_{G'}(s\zeta))\big)\exp_{G'}^{-1}(\exp_{G'}(s\zeta))\Big)\,|_{s=0} \\ &= \frac{d}{ds}\,(s\zeta)\,|_{s=0} = \zeta \end{aligned}$$

because the bump function γ' is constant equal to 1 near 0. Hence $T_{e_{G'}}\mathsf{s} = \mathrm{id}_{\mathfrak{g}'}$. Finally, since for each $g \in G$ and $\zeta \in \mathfrak{g}'$

$$\mathrm{I}'_g\big(\exp_{G'}(\zeta)\big) = \exp_{G'}\big(\mathrm{A}'_g(\zeta)\big)$$

and since γ' is invariant under the action $g \mapsto \mathrm{A}'_g$ of G on $\mathfrak{g}'$ we get for all $g \in G$ and $g' \in G'$

$$\mathsf{s}\big(\mathrm{I}'_g(g')\big) = \mathrm{A}'_g\big(\mathsf{s}(g')\big),$$

proving the lemma. ∎

3.3. A counterexample.

Let us conclude with an example showing that the neighborhoods $\mathcal{U}_{\tau\mathbf{i}}$ do not depend functorially on the Lie algebra $\mathfrak{g}$. Namely, given a strictly positive real number τ, we give an example of a homomorphism of Lie algebras $f : \mathfrak{g} \to \mathfrak{g}'$ such that $F : G \to G'$ does not send the neighborhood $\mathcal{V}_{\tau\mathbf{i}} \subset G$ to the neighborhood $\mathcal{V}'_{\tau\mathbf{i}} \subset G'$ where G and G' are the simply connected Lie groups corresponding to $\mathfrak{g}$ and $\mathfrak{g}'$ respectively, F is the Lie group homomorphism existing by Lie's Second Theorem and

$$\mathcal{V}_{\tau\mathbf{i}} := \exp\big(\mathcal{U}_{\tau\mathbf{i}}\big) \subset G.$$

In order to construct such an example, observe that for the abelian Lie algebra $\mathfrak{g} = \mathbb{R}^n$, we have that $G = \mathbb{R}^n$ is the additive group and $\mathcal{U}_{\tau\mathbf{i}} = \mathfrak{g}$ for any strictly positive real τ. Furthermore, the exponential map is the identity of $\mathbb{R}^n$ and $\mathcal{V}_{\tau\mathbf{i}} = G$ is equally true for all strictly positive real τ. Thus for abelian Lie group/Lie algebra, the neighborhoods are in some sense maximal.

On the other hand, consider $\mathfrak{g}' := \mathfrak{su}(2,\mathbb{C})$, the Lie algebra of skewhermitian matrices of trace zero. It is well-known (see e.g. [3] Section 1.2B) that the adjoint representation of $SU(2,\mathbb{C})$ on $\mathfrak{su}(2,\mathbb{C})$ is given by rotations, namely for $a, b \in \mathbb{C}$, a matrix

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

acts by rotation with angle

$$\alpha := \left| \frac{\arccos(2(\operatorname{Re} a)^2 - 1)}{\sqrt{1 - (\operatorname{Re} a)^2}} \right| \sqrt{(\operatorname{Im} a)^2 + (\operatorname{Re} b)^2 + (\operatorname{Im} b)^2}.$$

The eigenvalues of the infinitesimal generator x of this rotation matrix are 0 and $\pm \mathbf{i}\alpha$. Thus x is an element of the neighborhood $\mathcal{U}'_{\tau\mathbf{i}}$ if and only if $\alpha \leq \tau$.

Now let us construct the counter-example: Let $\tau > 0$ be fixed in advance. Let $n = 1$ and $f : \mathbb{R} \to \mathfrak{su}(2,\mathbb{C})$ be given by sending $1 \in \mathbb{R}$ to some $x \in \mathfrak{su}(2,\mathbb{C})$ such that the above angle α is strictly bigger than τ. The homomorphism f integrates to a 1-parameter subgroup in $SU(2,\mathbb{C})$, i.e. a Lie group homomorphism $F : \mathbb{R} \to SU(2,\mathbb{C})$. By construction, F does not send $\mathcal{V}_{\tau\mathbf{i}} = \mathbb{R}$ into $\mathcal{V}'_{\tau\mathbf{i}}$.

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Martin Bordemann
Laboratoire de Mathématiques
Informatique et Applications
Université de Haute Alsace
4-6 rue des Frères Lumière
68093 Mulhouse, France
Martin.Bordemann@uha.fr

Friedrich Wagemann
Laboratoire de Mathématiques Jean
Leray
Université de Nantes
2, rue de la Houssinière
44322 Nantes cedex 3, France
Friedrich.Wagemann@univ-nantes.fr

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