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Compact Elements in Connected Lie Groups

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Abstract. We prove that the set of compact elements in the group extension of the 3D Heisenberg group by $SO(2)$ (so-called oscillator group) is not dense. We also give a new proof of the following criterion: The set of compact elements of a connected Lie group G is dense in G if and only if every Cartan subgroup of G is compact.
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1. Introduction

If G is a locally compact group and $g \in G$, then g is said to be compact in G if g is an element of a compact subgroup of G. Thus g is a compact element iff the closure of the subgroup generated by g is compact. If G is a discrete group, then $g \in G$ is compact if and only if g is an element of finite order. We denote by $\Omega(G)$ the set of all compact elements of G and by $\overline{\Omega(G)}$ its closure in G.

Question. *Is any closed subgroup H of a given locally compact group G with $H \subset \Omega(G)$ compact?*

It is easily that in general the answer to this question is in the negative. In the special case of connected groups this problem was studied by C. Moore [13], V. Platonov and A. Zalesskii [17], Yu. Merzlyakov [12], and D. Djoković [6]. C. Moore established that for the group $G = GL_n(\mathbb{R})$ the answer to the question is positive; V. Platonov and A. Zalesskii and independently Yu. Merzlyakov proved this for the group $G = GL_n(\mathbb{C})$; D. Djoković extended these results to arbitrary connected Lie groups.

As was shown in [17, 12], the condition $H \subset \Omega(G)$ cannot be relaxed to $H \subset \overline{\Omega(G)}$. Nevertheless, in [16] V. Platonov observed that if a connected semisimple Lie group G coincides with $\overline{\Omega(G)}$, then G is compact. In [6] D. Djoković has proved that if the intersection $S \cap \overline{\Omega(G)}$ of a connected semisimple subgroup S of a connected Lie group G is a neighbourhood of the identity 1 in S, then S is compact. Here and throughout, 1 stands for the identity element in a group. The condition that 1 lies in the interior of $S \cap \overline{\Omega(G)}$ cannot be relaxed to just

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$S \cap \overline{\Omega(G)}$ having a non-empty interior. For instance, in the group $G = SL_2(\mathbb{R})$ (see Example 2.3 below) both $\Omega(G)$ and its complement $G \setminus \Omega(G)$ have non-empty interiors.

The following two characterizations of denseness of $\Omega(G)$ were first found by the author in [7].

Theorem 1.1. *Let G be a connected Lie group whose compact elements except the identity lie outside the center of G. Then $\Omega(G)$ is dense in G if and only if*
(i) G is a semidirect product $V \cdot S$ of a connected, simply connected nilpotent normal subgroup V and a connected compact subgroup S of G and
(ii) there is $s \in S$, such that $C_V(s) = \{1\}$.

Theorem 1.2. *The following conditions on connected locally compact group G are equivalent:*
(i) $\Omega(G)$ is dense in G;
(ii) $\overline{\Omega(G)}$ is a neighbourhood of 1;
(iii) $G = VS$, where V is a connected nilpotent normal subgroup, S is a compact subgroup of G, and $C_G(g)$ is compact for some $g \in G$.

Different proofs of the same results have been given later by T.-S. Wu [20, Theorems 2.7, 2.8].

The following two theorems show that topological size of $\Omega(G)$ depends on the number of compact Cartan subgroups of G.

Theorem 1.3. [11, Theorem 1] *Let G be a connected Lie group. Then the following are equivalent:*
(i) $\Omega(G)$ has a non-empty interior;
(ii) $\overline{\Omega(G)}$ has a non-empty interior;
(iii) there is a compact Cartan subgroup of G.

Theorem 1.4. *Let G be a connected Lie group. Then*
(i) [11, Theorem 2] *$\Omega(G)$ is dense if and only if every Cartan subgroup of G is compact;*
(ii) [11, Theorem 3] *The interiors of both $\Omega(G)$ and its complement $G \setminus \Omega(G)$ are non-void if and only if G has both compact and noncompact Cartan subgroups;*
(iii) [11, Theorem 4] *If the semisimple part of G is compact, then one of the sets $\Omega(G)$ or $G \setminus \Omega(G)$ is nowhere dense in G.*

In [18] S. Wang posed the following

Question. *Does there exists a noncompact connected Lie group with a dense subgroup consisting of compact elements only.*

Remark 1.5. The first example of a noncompact connected Lie group with a dense subgroup consisting of compact elements only was constructed by H.Bass in [1, p. 19], who attributed the above question to I. Kaplansky.

The following statement proved in [10, Theorem 2] gives an answer to the question. Furthermore, this theorem provides a complete description of noncompact connected Lie groups with a dense subgroup consisting of compact elements only.

Theorem 1.6. *Let G be a connected Lie group coinciding with the closure of its commutator subgroup and let the center of G have no non-trivial compact elements. Then the following are equivalent:*
(i) $\Omega(G)$ is dense in G;
(ii) G contains a dense subgroup consisting of compact elements only;
(iii) $G = V \cdot S$, where V is a nilpotent connected, simply connected normal subgroup and S is a connected compact semisimple subgroup of G whose center $Z(S)$ acts by conjugations regularly (i.e. without fixed points) on V.

The implication $(iii) \Rightarrow (ii)$ is the hardest part of the proof of the theorem. For its proof it was required to demonstrate that the compact semisimple group S has a dense free subgroup all of whose non-identity elements act on V without non-trivial fixed points. More exactly in [10, Theorem 4], making use of Tits' alternative we proved the following Theorem from which the implication follows.

Theorem 1.7. *A connected compact semisimple Lie group G contains a free dense subgroup F such that every non-identity element of F generates a dense subgroup in a maximal torus of G.*

The existence of such free subgroups in the group $SO(n)$ is closely related to the Banach-Tarski paradoxical decompositions of the n-sphere in $\mathbb{R}^n$, $n \geq 3$. The question on the existence of free groups with this property was posed by J. Mycielski in [14]. Theorems 1.6 and 1.7 have been announced in [8]. Partial proofs were published in [9] and full versions in [10]. We note that the answer to Mycielki's question and the result analogous to Theorem 1.7 on free subgroups were also found independently by P. Deligne and D. Sullivan [5] and A. Borel [2].

This article is organized as follows. In section 2 we determine the sets $\Omega(G)$ and their closures for four simple examples of connected Lie groups G.

In section 3 we find the set of compact elements of the oscillator group G. This set is nowhere dense in G, furthermore every Cartan subgroup of G is not compact. Therefore, Corollary 6.8 and Example 6.9 in the paper [21] by M. Wüstner are incorrect.

In section 4 we give a new proof of part (i) of Theorem 1.4, based on Theorem 1.1. In our opinion this proof is much more transparent than the one given in [11, Theorem 2]

2. Simplest Examples

Let G be a connected Lie group. The set $H = \Omega(G)$ is closed in G if and only if H is a compact subgroup of G. A model example of this situation is the multiplicative group $G = \mathbb{C}^\star$ of complex numbers. In this group compact elements are complex numbers that lie on the unit circle, i.e. $\Omega(G)$ in this case is the one-dimensional torus $\mathbb{T}$.

In this section we consider four examples of connected Lie groups that actually exhaust all the possibilities available in the general case.

We will need the following definition. Let V be a group, H be a subgroup of $\mathrm{Aut}\,(V)$. We define on the set $G = V \times H$ the operation of multiplication

$$ug \cdot vh = ug(v) \cdot hg,\ u, v \in V,\ g, h \in H.$$

Clearly, G with this multiplication is a group, it is called a semidirect product of V and H and is denoted by $G = V \cdot H$.

Example 2.1. Let G be the group of rigid isometries of the Euclidean plane. Then $G = \mathbb{R}^2 \cdot \mathbb{T}$, where $\mathbb{R}^2$ is the group of translations and $\mathbb{T}$ is the group of rotations of the plane around the origin. An isometry $g \in G$, $g \neq 1$, is compact in G if and only if g is a rotation around some point. It follows that

$$\Omega(G) = \{vt |\, v \in \mathbb{R}^2,\, t \in \mathbb{T},\, t \neq 1\} \cup \{1\}$$

and $G = \overline{\Omega(G)}$.

We note that every Cartan subgroup of G is isomorphic to $\mathbb{T}$.

Example 2.2. Consider the group of rigid isometries of $\mathbb{R}^3$. Then $G = \mathbb{R}^3 \cdot SO(3)$, where $\mathbb{R}^3$ is the group of all translations and $SO(3)$ is the group of all rotations of $\mathbb{R}^3$ around the origin. As in Example 2.1, an isometry $g \in G$, $g \neq 1$, is compact in G if and only if g is a rotation about some axis. An isometry $g = vh$, $v \in \mathbb{R}^3$, $h \in SO(3)$, $h \neq 1$, is a rotation if and only if $v \in \mathrm{Im}\,(h-1)$. We shall use this fact to determine $\Omega(G)$. We can view h as an orthogonal 3×3 matrix. Denote by $(h-1)^*$ the adjugate matrix of $h-1$. Since $(h-1)^*(h-1) = 0$, we have $(h-1)^* v^T = 0$ for all $v \in \mathbb{R}^3$, $v \in \mathrm{Im}\,(h-1)$. Let $f : G \to \mathbb{R}$ be defined by

$$f(vh) = ||(h-1)^* v^T||^2$$

for all $g = vh \in G$. Clearly, the map f is polynomial and not identically zero on G. Furthermore, if $g = vh$ is compact in G then $f(g) = 0$. Thus $\Omega(G)$ is nowhere dense in G.

Finally we observe that the identity component of a Cartan subgroup of G is isomorphic to $\mathbb{R} \times \mathbb{T}$.

Example 2.3. Consider the group $G = SL_2(\mathbb{R})$ of 2×2 matrices with determinant 1. A matrix $a \in SL_2(\mathbb{R})$ is a compact element in G if and only if $|\mathrm{tr}\,(a)| < 2$ or $a = \pm 1$, so

$$\Omega(G) = \{a \mid |\mathrm{tr}\, a| < 2\} \cup \{\pm 1\} \text{ and } \overline{\Omega(G)} = \{a \mid |\mathrm{tr}(a)| \leq 2\}.$$

It follows from this that both $\Omega(G)$ and $G \setminus \Omega(G)$ contain non-void open subsets.

The identity component of a Cartan subgroup of G is isomorphic either $\mathbb{R}$ or $\mathbb{T}$.

Example 2.4. Let $G = SL_3(\mathbb{R})$. If a matrix $a \in G$ is a compact element in G, then $\det(a-1) = 0$. As in Example 2.2 this means that $\Omega(G)$ is nowhere dense in G and hence $G \setminus \overline{\Omega(G)}$ is dense and open in G.

The identity component of a Cartan subgroup of G is isomorphic either $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{T}$.

Remark 2.5. We can give a complete list of connected simple Lie groups both with compact Cartan subgroups and without; e.g. $SL_2(\mathbb{R})$ and $SL_3(\mathbb{R})$ respectively. As more examples we mention the groups $PSL_n(\mathbb{R})$ $(n > 2)$, $SU^*(2n)$ with no compact Cartan subgroups and the noncompact groups $SU(p,q)$, $SO(p,q)$ $(p+q = 2n+1)$ that have compact Cartan subgroups.

3. The Oscillator Group

Another example related to the sets of compact elements is provided by the oscillator group.

Consider the following representation of the first Heisenberg group H. Let $H = \mathbb{R}^3$ be endowed with the following operation of multiplication

$$(u,z)(u',z') = \left(u+u', z+z'+\frac{1}{2}\det(u,u')\right),$$

where $u, u' \in \mathbb{R}^2$, $z, z' \in \mathbb{R}$ and $\det(u,u')$ is the determinant of the 2×2 matrix made by the rows u, u'. Clearly, H with this multiplication is a group, whose identity being (0,0,0) and the inverse being $(x,y,z)^{-1} = (-x,-y,-z)$, the subgroup $Z = \{(0,0,z) \mid z \in \mathbb{R}\}$ being the center of H.

If

$$u = (x,y) \text{ and } \varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then we set $\varphi(u) = (ax+by, cx+dy)$. Let $\varphi \in SL_2(\mathbb{R})$ be a matrix with $\det(\varphi) = 1$. It can easily be verified that the map

$$\widehat{\varphi} : H \to H, \ (u,z) \mapsto (\varphi(u), z)$$

is an automorphism of H and the map $\varphi \mapsto \widehat{\varphi}$ is an isomorphic embedding of the group $SL_2(\mathbb{R})$ into the group $\operatorname{Aut}(H)$ with every element in Z being a fixed point of $\widehat{\varphi}$. In what follows we identify $\widehat{\varphi}$ and φ. Consider the group $G = H \cdot SO(2)$. The elements of G are multiplied as follows:

$$(h,\varphi)(h',\varphi') = (h\varphi(h'), \varphi\varphi'), \ h, h' \in H, \ \varphi, \varphi' \in SO(2).$$

For further computations we rewrite this in a more convenient form

$$(u,z,\varphi)(u',z',\varphi') = \left(u+\varphi(u'), z+z'+\frac{1}{2}\det(u,\varphi(u')), \varphi\varphi'\right),$$

where $u, u' \in \mathbb{R}^2$, $z, z' \in \mathbb{R}$, and $\varphi, \varphi' \in SO(2)$. The identity of G is the row $(\mathbf{0}, 0, \mathbf{1})$, where $\mathbf{0}$ is the zero row in $\mathbb{R}^2$, $0 \in \mathbb{R}$, and $\mathbf{1}$ is the identity of $SO(2)$. We then have

$$(u,z,\varphi)^{-1} = \left(-\varphi^{-1}(u), -z, \varphi^{-1}\right)$$

and $Z = \{(\mathbf{0}, z, \mathbf{1}) \mid z \in \mathbb{R}\} = Z(G)$, where $Z(G)$ is the center of G. Clearly, G is a soluble connected Lie group, called the oscillator group. The Lie algebra of G is given by the basis of four vectors x, y, z, t and Lie brackets

$$[x,y] = z,\ [t,x] = y,\ [t,y] = -x,\ [x,z] = [y,z] = [t,z] = 0.$$

Theorem 3.1. *Let G be the oscillator group. Then*
(i) $\Omega(G)$ is nowhere dense in G;
(ii) $Z = Z(G) \subset \overline{\Omega(G)}$ and $\Omega(G/Z)$ is dense in G/Z;
(iii) every Cartan subgroup of G is connected and isomorphic to $\mathbb{R} \times \mathbb{T}$.

Proof. We prove part (i). It is easily seen that

$$(v,0,\mathbf{1})^{-1}(u,z,\varphi)(v,0,\mathbf{1})$$
$$= \left(u + \varphi(v) - v, z + \frac{1}{2}(\det(u,v) - \det(v,\varphi(v)) + \det(u,\varphi(v))), \varphi\right) \tag{1}$$

for all $u, v \in \mathbb{R}^2,\ z \in \mathbb{R},\ \varphi \in SO(2)$.

Assuming $\varphi \neq \mathbf{1}$, $\varphi - \mathbf{1}$ is invertible in $GL_2(\mathbb{R})$. Substituting $-(\varphi - \mathbf{1})^{-1}(u)$ for v in (1) we find that the first component in the right hand side of (1) vanishes. To compute the second component we consider the following three determinants:

$$\det(u,v) = \det(u, -(\varphi-\mathbf{1})^{-1}(u)) = \frac{\det(u,\varphi(u))}{\det(\varphi-\mathbf{1})},$$

$$\det(v,\varphi(v)) = \det(-(\varphi-\mathbf{1})^{-1}(u), -\varphi(\varphi-\mathbf{1})^{-1}(u)) = \frac{\det(u,\varphi(u))}{\det(\varphi-\mathbf{1})},$$

$$\det(u,\varphi(v)) = \det(u, -\varphi(\varphi-\mathbf{1})^{-1}(u)) = \frac{\det(u,\varphi(u))}{\det(\varphi-\mathbf{1})}.$$

Thus with $v = -(\varphi-\mathbf{1})^{-1}(u)$, we have

$$(v,0,\mathbf{1})^{-1}(u,z,\varphi)(v,0,\mathbf{1}) = \left(\mathbf{0}, z + \frac{\det(u,\varphi(u))}{2\det(\varphi-\mathbf{1})}, \varphi\right). \tag{2}$$

We apply formula (2) to determine $\Omega(G)$. Clearly, if two elements of G are conjugate, then they are either both compact or both noncompact in G. We observe that all non-identity elements $(u, z, \mathbf{1})$ are noncompact in G and the elements $(\mathbf{0}, z, \varphi)$, for $z \neq 0$, are noncompact as well. On the other hand, for all $\varphi \in SO(2)$ the elements $(\mathbf{0}, 0, \varphi)$ are compact in G. It then follows from (2) that an element (u, z, φ) with $\varphi \neq \mathbf{1}$ is compact in G if and only if the following equality holds

$$2z\det(\varphi-\mathbf{1}) + \det(u,\varphi(u)) = 0. \tag{3}$$

By setting

$$u = (x,y) \text{ and } \varphi_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

(3) can be rewritten as

$$4z\sin\frac{t}{2} - (x^2+y^2)\cos\frac{t}{2} = 0. \tag{4}$$

We say in this case that the operator φ_t corresponds to parameter t, $t \in \mathbb{R}$. If $g = (x, y, z, \varphi_t) \in G$ and $\varphi_t \neq \mathbf{1}$, then by (2), g is a conjugate of $g' = (0, 0, z', \varphi_t)$, see (1). Therefore the element $g = (x, y, z, \varphi_t)$ is compact in G if and only if the point (x, y, z) lies on the paraboloid given by equation (4).

Let M be the solution set to equation (3). Since M is a zero set of a polynomial function, M is closed and nowhere dense in G. Since $\Omega(G) \subset M$, we conclude that $\overline{\Omega(G)} \subset M$. Thus $\Omega(G)$ is nowhere dense in G.

Now we prove part (ii) of Theorem 3.1. Indeed, it follows from (2) that each element of $(G/Z) \setminus (H/Z)$ is compact in G. Hence $\Omega(G/Z)$ is dense in G/Z. We shall show the inclusion $Z \subset \overline{\Omega(G)}$. Let $z \in \mathbb{R}$ be a positive real and let n be a positive integer. We consider $x_n = y_n = 1/n$ and $t_n = \cot^{-1}(2zn^2)$. Then $2zn^2 \to \infty$ and $t_n \to 0$, as $n \to \infty$, so

$$g_n = (x_n, y_n, z, \varphi_n) \to (0, 0, z, \mathbf{1})$$

as $n \to \infty$. Here φ_n is the operator corresponding to t_n. From (4) it follows that g_n is compact for all n. We thus have proved that $(0, 0, z, \mathbf{1}) \in \overline{\Omega(G)}$ for all $z > 0$. It then follows that the same holds for all $z < 0$. Thus $Z \subset \overline{\Omega(G)}$.

It is clear that every Cartan subgroup of G is connected and isomorphic to $\mathbb{R} \times \mathbb{T}$. This proves part (iii) of Theorem 3.1. ∎

In conclusion we emphasize once again that the set of compact elements of the oscillator group gives a counterexample to [21, Corollary 6.8 and Example 6.9].

4. New Proof of Theorem 1.4(i)

In this section we give a new proof of Theorem 1.4(i) based on Theorem 1.1. We first recall some properties of Cartan subgroups; the reader is referred to the texts [3] and [4], see also [15].

Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a Cartan subalgebra if $\mathfrak{h}$ is nilpotent and the normalizer of $\mathfrak{h}$ coincides with $\mathfrak{h}$. Every Lie algebra contains a Cartan subalgebra. All Cartan subalgebras have the same dimension and nilpotency class [3, VII, § 3, n° 3, Theorem 2]. The dimension of the Cartan subalgebras of $\mathfrak{g}$ is called the rank of the Lie algebra $\mathfrak{g}$ denoted $\operatorname{rank} \mathfrak{g}$.

A subgroup H of a Lie group G is called a Cartan subgroup if H is a maximal nilpotent subgroup of G and every normal subgroup of finite index in H is of finite index in its normalizer in G [4].

Let G be a connected Lie group and let $\mathfrak{g}$ be its Lie algebra. A Cartan subgroup of G is closed in G and its tangent algebra is a Cartan subalgebra of $\mathfrak{g}$. To every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ there corresponds a unique Cartan subgroup H of G for which $\mathfrak{h}$ is the tangent algebra, [4, VI, § 5, Theorem 5 and Corollary]. In particular, if H, H' is any pair of Cartan subgroups of G and H_0, H_0' are their identity components respectively, then H_0 and H_0' have a common dimension and nilpotency class.

Let $a \in G$. We denote by $\mathfrak{g}^1(a)$ the weight space of 1, i.e.

$$\mathfrak{g}^1(a) = \{u \in \mathfrak{g} \mid (Ad(a) - 1)^n(u) = 0\}, \text{ where } n > \dim G.$$

An element $a \in G$ is called *regular* if there is a neighbourhood U of a such that $\dim \mathfrak{g}^1(b) = \dim \mathfrak{g}^1(a)$ for each $b \in U$, [3, VII, § 4, n° 2, Definition 2]. The set of all regular elements of G is denoted by reg (G). This set reg (G) is a dense open subset of the group G [3, VII, § 4, n° 1, Proposition 1]. An element $a \in G$ is regular if and only if $\mathfrak{g}^1(a)$ is a Cartan subalgebra of $\mathfrak{g}$. For each Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ there exists a regular element a in G such that $\mathfrak{h} = \mathfrak{g}^1(a)$ [3, VII, § 4, n° 3, Propositions 7, 8].

Lemma 4.1. *Let G be a connected Lie group. Every regular element of G lies in the normalizer of the identity component of a Cartan subgroup.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G and let a be a regular element in G. Then $\mathfrak{g}^1(a)$ is a Cartan subalgebra of $\mathfrak{g}$. There exists a unique Cartan subgroup H of G for which $\mathfrak{g}^1(a)$ is the tangent algebra. Let H_0 be the identity component of H. Since the subalgebra $\mathfrak{g}^1(a)$ is invariant under Ad (a), we have $a \in N_G(H_0)$, so Lemma 4.1 follows. ■

Lemma 4.2. *Let H be a Cartan subgroup of a connected Lie group G and let H_0 be the identity component of H. Then the group H is compact if and only if H_0 is compact.*

For a proof of this statement we refer to [11, Lemma 6]. A different proof can be found in [19, Proposition 8] which appeared a bit later.

The Proof of Theorem 1.4(i). Let G be a connected Lie group. We shall show that $G = \overline{\Omega(G)}$ if and only if every Cartan subgroup of G is compact.

We show the 'only if' part. Let $G = \overline{\Omega(G)}$. We denote by $Z(G)$ the center of G. First suppose $Z(G)$ doesn't contain a compact element (except the identity). It follows from Theorem 1.1 that $G = V \cdot S$ where V is a connected, simply connected nilpotent normal subgroup of G and S is a connected compact subgroup of G such that $S \cap V = \{1\}$ and $C_V(s_0) = \{1\}$ for some $s_0 \in S$.

Let T be a Cartan subgroup of S containing s_0; see [4, VI, § 5, Proposition 5]. Suppose there exists $g \in G$ such that $g^{-1}Tg = T$. Then $s_0 g = gs'$ for some $s' \in T$. Let $g = vh$, where $v \in V$ and $h \in S$. Then $s_0 vh = vhs'$ and $s_0 v s_0^{-1}(s_0 h) = v(hs')$. Since $S \cap V = \{1\}$, and V is a normal subgroup of G, we have $s_0 v s_0^{-1} = v$. Combining this with $C_V(s_0) = \{1\}$ yields $v = 1$ and $g \in S$. We thus have $N_G(T) \subset N_S(T)$, whence $N_G(T) = N_S(T)$. This implies that T is a Cartan subgroup of G. Since any two Cartan subgroups in the compact connected Lie group S are conjugate [4, VI, § 5, Proposition 4], every Cartan subgroup of S is a Cartan subgroup of G. Note also that every Cartan subgroup of S is connected and abelian and hence is a maximal torus of S. Let H be a Cartan subgroup of G. If $h \in H$, then there exists a unique pair of elements $s \in S$, $v \in V$, such that $h = vs$. Let $\pi : H \to S$ be the projection well defined by

$\pi(h) = s$ for all $h \in H$. Clearly, π is a continuous homomorphism of H to S. We observed that G has abelian Cartan subgroups, hence the identity component H_0 of H is abelian, so $\pi(H_0)$ and $\overline{\pi(H_0)}$ are abelian connected subgroups of S. Let T be a Cartan subgroup of S such that $\overline{\pi(H_0)} \subset T$ [4, VI, § 5, Proposition 3]. Clearly $H_0 \subset V \cdot T$. Since $V \cdot T$ is a soluble Lie group, the subgroups T and H_0 are conjugate in $V \cdot T$ [4, VI, § 4, Proposition 19]. Therefore the group H_0 is compact, whence by Lemma 4.2 H itself is compact. We have thus proved that every Cartan subgroup of G is compact in this case.

Now suppose $Z(G)$ contains a compact element. Consider the set K of all compact elements in $Z(G)$. Then K is a normal compact subgroup of G. Clearly, an element $x \in G$ is compact in G if and only if xK is compact in the factor group G/K. It follows that the center of G/K doesn't contain compact elements (except the identity) and $\overline{\Omega(G/K)} = G/K$. It follows that every Cartan subgroup of G/K is compact and hence every Cartan subgroup of G is compact as well. We thus proved the 'only if' part.

Conversely, let G be a connected Lie group whose Cartan subgroups are all compact. If H is a Cartan subgroup of G and H_0 is the identity component of H, then the index $|H : H_0|$ is finite. Hence $|N_G(H_0) : H| < \infty$, so the subgroup $N_G(H_0)$ is compact.

We denote by M the union of the normalizers of the identity components of all Cartan subgroups of G. Then $M \subset \Omega(G)$. On the other hand, by Lemma 4.1 every regular element of G is contained in the normalizer of the identity component of a Cartan subgroup. Hence every regular element of G is compact. Therefore $\operatorname{reg}(G) \subset M \subset \Omega(G)$. Since the set $\operatorname{reg}(G)$ is dense in G (see [3, VII, § 4, n° 1, Proposition 1]), so is $\Omega(G)$ and we have $\overline{\Omega(G)} = G$. This completes the proof of part (i) of Theorem 1.4. ∎

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