Convergence of the Gutt Star Product

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Abstract.   In this work we consider the Gutt star product viewed as an associative deformation of the symmetric algebra $S^\bullet(\mathfrak{g})$ over a Lie algebra $\mathfrak{g}$ and discuss its continuity properties: we establish a locally convex topology on $S^\bullet(\mathfrak{g})$ such that the Gutt star product becomes continuous. Here we have to assume a mild technical condition on $\mathfrak{g}$: it has to be an *Asymptotic Estimate* Lie algebra. This condition is e.g. fulfilled automatically for all finite-dimensional Lie algebras. The resulting completion of the symmetric algebra can be described explicitly and yields not only a locally convex algebra but also the Hopf algebra structure maps inherited from the universal enveloping algebra are continuous. We show that all Hopf algebra structure maps depend analytically on the deformation parameter. The construction enjoys good functorial properties.
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1.   Introduction

Formal deformation quantization as introduced in [1] has reached a remarkable state where the existence and the classification of formal star products is by now understood very well: Kontsevich's formality theorem gives both, the general existence of formal star products on Poisson manifolds as well as their classification up to equivalence. In the symplectic case, earlier results gave the existence [14, 17] as well as the classification [4, 13, 26].

Beside the symplectic situation, the linear Poisson structures, i.e. the Kirillov-Kostant-Souriau bracket on the dual of a Lie algebra $\mathfrak{g}$, remained the only example which allowed a formal deformation quantization for a long time: the existence of a star product on $\mathfrak{g}^*$ is contained in the construction of Gutt [21].

The basic idea of the construction is rather simple. The algebra of polynomial functions on $\mathfrak{g}^*$ is isomorphic to the symmetric algebra $S^\bullet(\mathfrak{g})$ which is, as filtered vector space, isomorphic to the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ via the Poincaré-Birkhoff-Witt theorem. Taking the canonical PBW isomorphism, i.e. the total symmetrization map, one can pull-back the product of $\mathscr{U}(\mathfrak{g})$ to obtain a product on $S^\bullet(\mathfrak{g})$ and hence on $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$. Taking into account the degrees of the polynomials in the right way allows to plug in a deformation parameter z such that one ends up with a star product $\star_z$ on $\mathfrak{g}^*$. This star product has many remarkable features, one of them is that it converges for trivial reasons on the polynomial functions $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ for all z. This is clear by the very construction. From the quantization point of view, z is something like $i\hbar$ with $\hbar$ being Planck's constant.

In general, formal star products are very hard to control from an analytic point of view: not much is known about the convergence properties of the formal power series. The reason is rather simple: in a formal star product the number of involved derivatives of the two factors is typically equal or higher as the power of the deformation parameter: hence the classical Borel lemma allows to construct smooth functions for which the star product has radius of convergence zero. The interesting question is of course whether one can find a reasonable subalgebra where the star product has a nontrivial radius of convergence. Beside very few examples, related to the Weyl-Moyal product [33] and the Kähler structure on the Poincaré disk [2], not much is known in this direction. Alternatively, there are various approaches to strict deformation quantization based on integral formulas for the deformed product. The formal star product then arises as asymptotic expansion of the integrals. Here we refer to e.g. [30] as well as to [5, 7]. However, the usage of integral formulas bounds these approaches to finite dimensions whereas a direct investigation of the convergence of series may still be applicable in infinite dimensions as needed for (quantum) field-theoretic models.

In this paper we want to add yet another example where the convergence of a formal star product can be controlled in an efficient way: the Gutt star product on $\mathfrak{g}^*$.

In fact, our construction will work even in some infinite-dimensional cases. Therefore we need a slight reformulation to incorporate these situations, too: instead of the polynomial functions we focus on the symmetric algebra $S^\bullet(\mathfrak{g})$ of the Lie algebra. In finite dimensions this will make no difference but in infinite dimensions, $S^\bullet(\mathfrak{g})$ only injects into $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ but is strictly smaller. Since the Gutt star product on $S^\bullet(\mathfrak{g})$ can be constructed in any dimension, this seems to be a reasonable framework. Our basic idea is now to establish a locally convex topology on $S^\bullet(\mathfrak{g})$ in such a way that the Gutt star product becomes continuous. Then it automatically extends to the completion which we want to be as large as possible.

However, we want the completion to be small enough so that its elements are still functions on the (topological) dual $\mathfrak{g}'$ of $\mathfrak{g}$. This requires the evaluation functionals on points in $\mathfrak{g}'$ to be continuous. While in finite dimensions this will work for all Lie algebras equally well, in infinite dimensions we have to add some technical continuity properties on the Lie bracket of $\mathfrak{g}$.

In the following, we consider a locally convex Lie algebra $\mathfrak{g}$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, i.e. a real or complex locally convex topological vector space with a continuous Lie bracket. We focus on a honestly continuous Lie bracket instead of a separately continuous one throughout this work. This means that for every continuous seminorm p there exists another continuous seminorm q such that for all $\xi, \eta \in \mathfrak{g}$ one has

$$p([\xi, \eta]) \leq q(\xi)q(\eta). \tag{1}$$

For our study of the Gutt star product, this will not be enough, since we will have to control an arbitrarily high number of nested brackets without getting a new seminorm for each bracket. Thus, we will need an estimate which does not depend on the number of Lie brackets implied. This motivates the following definition, see also [10]:

Definition 1.1 (Asymptotic estimate algebra). Let $\mathscr{A}$ be a Hausdorff locally convex algebra (not necessarily associative) with $\cdot$ denoting the multiplication, and let p be a continuous seminorm.

i.) A continuous seminorm q is said to be an asymptotic estimate for p, if

$$p\left(w_n(x_1, \ldots, x_n)\right) \leq q(x_1) \cdots q(x_n) \tag{2}$$

for all words $w_n(x_1, \ldots, x_n)$ made out of $n - 1$ products of the elements $x_1, \ldots, x_n \in \mathscr{A}$ with arbitrary position of placing brackets.

ii.) A locally convex algebra is said to be an asymptotic estimate algebra (AE-algebra), if every continuous seminorm has an asymptotic estimate.

We are mainly interested in the case of an AE-Lie algebra but of course also associative AE-algebras are of interest. Given an associative AE-algebra its commutator Lie algebra is a AE-Lie algebra. Also, all finite-dimensional Lie algebras are AE-Lie algebras. More generally, all *locally multiplicatively convex* algebras are of this type, i.e. those where one finds a defining set of continuous seminorms p such that

$$p(x \cdot y) \leq p(x)p(y) \tag{3}$$

for all algebra elements x, y. Clearly, all finite-dimensional Lie algebras are locally multiplicatively convex.

In order to formulate our continuity results we have to specify the topology on $S_R^\bullet(\mathfrak{g})$. Here we rely on the results of [33], where a definition for a locally convex topology on the tensor algebra $T^\bullet(V)$ of a locally convex vector space V was given: given $R \in \mathbb{R}$, one can define the locally convex vector spaces $T_R^\bullet(V)$ and $S_R^\bullet(V)$, where $S^\bullet(V)$ denotes the symmetric tensor algebra viewed as subspace of the tensor algebra. The basic idea is to control the growth of tensor powers of a seminorm p^n applied to the homogeneous tensor parts of degree n by the R-th power of $n!$. There were also definitions given for the projective limits $T_{R-}^\bullet(V)$ and $S_{R-}^\bullet(V)$. We will define those spaces more precisely in Section 2. Using this topology on the symmetric algebra, we can now state the main results:

Theorem 1.2.   *Let $\mathfrak{g}$ be an AE-Lie algebra and let $R \geq 1$.*

  i.) The Gutt star product $\star_z$ is continuous with respect to the S_R-topology for every $z \in \mathbb{K}$.

 ii.) The completion $\widehat{S}_R^\bullet(\mathfrak{g})$ becomes a locally convex Hopf algebra with respect to the Gutt star product and the undeformed coproduct, antipode, and counit.

iii.) The Gutt star product is convergent as series in $z \in \mathbb{K}$.

iv.) The construction is functorial for continuous Lie algebra homomorphisms.

Theorem 1.3.   *Let $\mathfrak{g}$ be a nilpotent locally convex Lie algebra. Then the statement of Main Theorem 1.2 holds for all $R \geq 1$ and for the projective limit $R \to 1^-$.*

Remarkably, in the general case of Main Theorem 1.2, the completion will *not* contain the exponentials of elements in $\mathfrak{g}$: this is in some sense to be expected as otherwise the product of two such exponentials would again be defined as an element of the completion. This way, one would be able to reconstruct a convergent BCH series for all elements in $\mathfrak{g}$, which is known to be impossible. However, in the nilpotent case, we have the exponentials inside the completion. Of course, this matches well with the fact that the BCH series is non-problematic in this case.

Remark 1.4.   In the finite-dimensional case one can directly consider the universal enveloping algebra and establish topologies on it. The analogous topology to our S_R-topology has been used and investigated by [29] and [20]. From that point of view, our results can be seen as a generalization to the possibly infinite-dimensional case including the much more involved proofs for the AE Lie algebra case. The strategy in [20] is based very much on the fact that in finite-dimensions one has a Banach-Lie algebra. Moreover, the topology is obtained by a quotient procedure starting with the T_R-topology on the tensor algebra and proving that the ideal generated by the Lie relations is actually closed in order to give a Haus-

dorff topology on the universal enveloping algebra. What is missing is of course the continuity of the coefficient maps C_n of the Gutt star product since there is only the filtration of the universal enveloping algebra but not the grading of the symmetric algebra available. This makes the question on analytic dependence on a deformation parameter meaningless: at the time of [29] and [20] the notion of star products was not yet commonly known.

The paper is organized as follows: In Section 2 we first outline the construction of the locally convex topologies on the tensor algebra and the symmetric algebra according to [33]. Then we recall the basic construction of the Gutt star product and provide several equivalent descriptions. The most useful for our purposes is the one based on the Baker-Campbell-Hausdorff series [16]. Section 3 contains the heart of this work: we establish the continuity of the Gutt star product with respect to the S_R-topology for $R \geq 1$. The first approach works for general asymptotic estimate Lie algebras. We also include a second and easier proof which, however, works only in the locally multiplicatively convex case. Then we show that we in fact obtain an entire deformation enjoying good functorial properties with respect to continuous Lie algebra homomorphisms. Section 4 is devoted to the nilpotent case. Here we can improve the previous continuity statements to the projective limit $R \longrightarrow 1^-$ since the BCH series has significantly less terms in this situation. The completion will now include exponential functions. In Section 5 we show the continuity of the remaining Hopf algebra maps. This is now much simpler as they are the classical maps not depending on the deformation parameter z. Finally, Section 6 contains some open questions and an outlook on further research we want to pursue in the future. In Appendix A we have included algebraic proofs of the equivalence of various forms of the Gutt star product, statements which are folklore knowledge but hard to trace down in the literature.

2. Preliminaries

In this section we collect some preliminary results on the locally convex topologies we use as well as on the Gutt star product.

2.1. The Topologies on $\mathrm{T}^\bullet(V)$ and $\mathrm{S}^\bullet(V)$.

Let V be a locally convex vector space over $\mathbb{K}$ where $\mathbb{K}$ stands for either $\mathbb{R}$

or $\mathbb{C}$. We want to recall the definition of the locally convex topology on the tensor algebra from [33] and some of its consequences. We endow every tensor power $V^{\otimes n}$ with the π-topology: we will denote for a given (continuous) seminorm p its tensor power by $p^n = p^{\otimes n}$ for $n \geq 1$. For $n = 0$ we take p^0 to be the absolute value on the field $\mathbb{K}$. Then the π-topology on $V^{\otimes_\pi n}$ is obtained by taking all p^n for all continuous seminorms p.

In order to define the Gutt star product, we need the symmetric algebra over the underlying vector space. Recall that the symmetrization map

$$\mathscr{S}_n \colon V^{\otimes_\pi n} \longrightarrow V^{\otimes_\pi n}, \quad (v_1 \otimes \ldots \otimes v_n) \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \quad (4)$$

is continuous and we have for all $v \in V^{\otimes_\pi n}$ the estimate

$$p^n(\mathscr{S}_n(v)) \leq p^n(v). \quad (5)$$

Since $\mathscr{S}_n$ is idempotent, it turns out that each symmetric tensor power

$$\mathrm{S}_\pi^n(V) = \operatorname{im} \mathscr{S}_n = \ker(\mathsf{id} - \mathscr{S}_n) \subseteq V^{\otimes_\pi n} \quad (6)$$

is a closed subspace with respect to the π-topology. The symmetric tensor product is then given by

$$vw = \mathscr{S}_{n+m}(v \otimes w) \quad (7)$$

for $v \in \mathrm{S}_\pi^n(V)$ and $w \in \mathrm{S}_\pi^m(V)$. Since the tensor product obeys $p^{n+m}(v \otimes w) \leq p^n(v)p^m(w)$ we get the continuity

$$p^{n+m}(vw) = p^{n+m}(\mathscr{S}_{n+m}(v \otimes w)) \leq p^n(v)p^m(w) \quad (8)$$

of the symmetric tensor product as well. Then the symmetric algebra $\mathrm{S}^\bullet(V) = \bigoplus_{n=0}^\infty \mathrm{S}^n(V)$ becomes a commutative associative unital algebra.

We now want to set up a topology on $\mathrm{T}^\bullet(V)$ and $\mathrm{S}^\bullet(V)$, which yields the π-topology on each component such that the (symmetric) tensor product becomes continuous. We recall the following definition [33, Def. 3.5 and Def. 3.12]:

Definition 2.1 (T_R-, S_R-, and S_{R^-}-topology).   Let $R \in \mathbb{R}$.

i.) For every continuous seminorm p on V we define

$$p_R = \sum_{n=0}^\infty n!^R p^n \quad (9)$$

on the tensor algebra $\mathrm{T}^\bullet(V)$.

ii.) The locally convex topology arising from all such seminorms p_R is called the T_R-topology on $T^\bullet(V)$, which we denote by $T_R^\bullet(V)$ when equipped with this topology.

iii.) The induced topology on the subspace $S^\bullet(V) \subseteq T^\bullet(V)$ is called the S_R-topology, and we write $S_R^\bullet(V)$.

iv.) The S_{R^-}-topology is defined as the projective limit of the $S_{R-\epsilon}$-topologies for $\epsilon \longrightarrow 0$ and we set

$$S_{R^-}^\bullet(V) = \operatorname*{proj\,lim}_{\epsilon \longrightarrow 0} S_{R-\epsilon}^\bullet(V). \tag{10}$$

We now want to collect the most important results on the locally convex algebras $T_R^\bullet(V)$ and $S_R^\bullet(V)$ which we will later use. Proofs and more detailed explanations can be found in [33, Sect. 3 and Sect. 4].

Proposition 2.2. *Let $R' \geq R \geq 0$ and let q, p be continuous seminorms.*

i.) *We have $p_{R'} \geq p_R$ and if $q \geq p$ then $q_R \geq p_R$.*

ii.) *The (symmetric) tensor product is continuous and satisfies*

$$p_R(vw) \leq p_R(v \otimes w) \leq (2^R p)_R(v)(2^R p)_R(w). \tag{11}$$

iii.) *For all $n \in \mathbb{N}$ the induced topology on $T^n(V) \subseteq T_R^\bullet(V)$ and on $S^n(V) \subseteq S_R^\bullet(V)$ is the π-topology.*

iv.) *For all $n \in \mathbb{N}$ the projection and the inclusion maps π_n and ι_n*

$$T_R^\bullet(V) \xrightarrow{\pi_n} V^{\otimes_\pi n} \xrightarrow{\iota_n} T^\bullet(V) \quad and \quad S_R^\bullet(V) \xrightarrow{\pi_n} S_\pi^n(V) \xrightarrow{\iota_n} S_R^\bullet(V) \tag{12}$$

are continuous.

v.) *The completions $\widehat{T}_R^\bullet(V)$ of $T_R^\bullet(V)$ and $\widehat{S}_R^\bullet(V)$ of $S_R^\bullet(V)$ can be described explicitly as*

$$\widehat{T}_R^\bullet(V) = \left\{ v = \sum_{n=0}^\infty v_n \;\middle|\; p_R(v) < \infty, \text{ for all continuous } p \right\} \subseteq \prod_{n=0}^\infty V^{\hat{\otimes}_\pi n} \tag{13}$$

and

$$\widehat{S}_R^\bullet(V) = \left\{ v = \sum_{n=0}^\infty v_n \;\middle|\; p_R(v) < \infty, \text{ for all continuous } p \right\} \subseteq \prod_{n=0}^\infty \widehat{S}_\pi^n(V), \tag{14}$$

where the p_R are extended to the Cartesian product allowing the value $+\infty$.

vi.) If $R' > R$, then the topology on $\mathrm{T}^\bullet_{R'}(V)$ is strictly finer than the one on $\mathrm{T}^\bullet_R(V)$, the same holds for $\mathrm{S}^\bullet_{R'}(V)$ and $\mathrm{S}^\bullet_R(V)$. Therefore the completions for R' are smaller than the ones for R.

vii.) The inclusion maps $\widehat{\mathrm{T}}^\bullet_{R'}(V) \longrightarrow \widehat{\mathrm{T}}^\bullet_R(V)$ and $\widehat{\mathrm{S}}^\bullet_{R'}(\mathfrak{g}) \longrightarrow \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ are continuous.

viii.) The T_R-topology on $\mathrm{T}^\bullet_R(V)$ and the S_R-topology on $\mathrm{S}^\bullet_R(V)$ are locally multiplicatively convex with respect to the (symmetric) tensor product iff $R = 0$.

ix.) The space V is nuclear iff $\mathrm{S}^\bullet_R(V)$ is nuclear iff $\mathrm{T}^\bullet_R(V)$ is nuclear. In particular, this is the case if $\dim V < \infty$.

x.) The evaluation functionals $\delta_\varphi \colon \mathrm{S}^\bullet_R(V) \longrightarrow \mathbb{K}$ for $\varphi \in V'$ are continuous.

2.2. The Gutt Star Product.

Let us now briefly recall the basic construction of the Gutt star product according to [21]: originally, this was just an intermediate step to get a star product on the cotangent bundle of a Lie group G with Lie algebra $\mathfrak{g}$. However, the resulting star product on $\mathfrak{g}^*$ can be described entirely algebraic as follows: first we replace the polynomials on $\mathfrak{g}^*$ by the symmetric algebra $\mathrm{S}^\bullet(\mathfrak{g})$ which behaves better in infinite dimensions. Then we use the explicit PBW isomorphism $\mathfrak{q} \colon \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathscr{U}(\mathfrak{g})$ via

$$\mathfrak{q}_n \colon \mathrm{S}^n(\mathfrak{g}) \longrightarrow \mathscr{U}(\mathfrak{g}), \quad \xi_1 \cdots \xi_n \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \odot \cdots \odot \xi_{\sigma(n)}, \quad \mathfrak{q} = \sum_{n=0}^\infty \mathfrak{q}_n. \quad (15)$$

We always denote the multiplication in $\mathscr{U}(\mathfrak{g})$ by $\odot$ to avoid confusion. For $z \in \mathbb{K}$ the Gutt star product $\star_z$ is then given by the pull-back of the product of $\mathscr{U}(\mathfrak{g})$ together with a degree-dependent rescaling by z. In detail, one defines

$$x \star_z y = \sum_{n=0}^{k+\ell-1} z^n \pi_{k+\ell-n} \left(\mathfrak{q}^{-1}(\mathfrak{q}(x) \odot \mathfrak{q}(y)) \right) \quad (16)$$

for homogeneous $x \in \mathrm{S}^k(\mathfrak{g})$ and $y \in \mathrm{S}^\ell(\mathfrak{g})$ and extends this bilinearly to $\mathrm{S}^\bullet(\mathfrak{g})$, where π_r projects on the homogeneous part of degree r. Another approach to this star product is due to Drinfel'd who based the construction on the Baker-Campbell-Hausdorff series [16]:

Proposition 2.3. *Let $\mathfrak{g}$ be a Lie algebra. Then we have*

$$\exp(\xi) \star_z \exp(\eta) = \exp \left(\frac{1}{z} \mathrm{BCH}\, (z\xi, z\eta) \right), \quad (17)$$

where we consider the exponentials as formal power series in the variables ξ and η. Conversely, by differentiating the right hand side with respect to these variables, one can determine $\star_z$ completely.

This result seems to be well-known folklore. One can find proofs based on differential geometric arguments e.g. in [9, Lemma 10]. Since those arguments do not work in infinite dimensions any more, for convenience we give an entirely combinatorial proof in Appendix A. There is yet another way to define the Gutt star product, since one can take the universal enveloping algebra $\mathscr{U}(\mathfrak{g}_z)$ of $\mathfrak{g}$ with the Lie bracket rescaled by z: there, we have the relation

$$\xi \odot \eta - \eta \odot \xi - z[\xi, \eta] = 0. \tag{18}$$

We use again the Poincaré-Birkhoff-Witt isomorphism $\mathfrak{q}_z \colon \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathscr{U}(\mathfrak{g}_z)$ from Equation (15) and get the next, well-known result:

Proposition 2.4. *Let $\mathfrak{g}$ be a Lie algebra. Then we have for $x, y \in \mathrm{S}^\bullet(\mathfrak{g})$ and all $z \in \mathbb{K}$*

$$x \star_z y = \mathfrak{q}_z^{-1}\left(\mathfrak{q}_z(x) \odot_z \mathfrak{q}_z(y)\right). \tag{19}$$

Since we will use the isomorphism $\mathfrak{q}_z$, we also give a proof of this in Appendix A.

Remark 2.5 (Integral formula)**.** Note that there is also an integral formula for the Gutt star product, which of course only holds in finite dimensions. This approach can be found in Berezin's work [3, Formula (24)], for example. One can understand $\mathscr{U}(\mathfrak{g})$ as the distributions on the Lie Group G of $\mathfrak{g}$ with compact support near the unit element using the convolution as multiplication. One uses the exponential map to get a star product on $\mathfrak{g}^*$ from this. Since this formula uses the Fourier transform and the Baker-Campbell-Hausdorff series, one has to make sure that the Fourier transformed functions only have support in an area, where the exponential map is diffeomorphic and where the BCH series converges. For our purpose, the way via the formal series is more suitable since on one hand we can avoid those difficulties and on the other hand, our approach remains valid in infinite dimensions.

Most of our analysis of $\star_z$ is based on properties of the BCH series. To this end, we briefly recall the relevant facts and establish some notation. First of all, we set for $\xi, \eta \in \mathfrak{g}$

$$\mathrm{BCH}(\xi, \eta) = \sum_{n=1}^{\infty} \mathrm{BCH}_n(\xi, \eta) = \sum_{a,b=0}^{\infty} \mathrm{BCH}_{a,b}(\xi, \eta), \tag{20}$$

where $\mathrm{BCH}_n(\xi, \eta)$ gives all the BCH terms which have exactly n letters and therefore $n-1$ brackets. Moreover, $\mathrm{BCH}_{a,b}(\xi, \eta)$ stands for all BCH terms which contain exactly a times the letter ξ and b times the letter η. We hence have

$$\mathrm{BCH}_n(\xi, \eta) = \sum_{a+b=n} \mathrm{BCH}_{a,b}(\xi, \eta) \tag{21}$$

with $a, b \geq 1$ for $n > 1$ and $a, b \geq 0$ for $n = 1$. The difficulty with the BCH series is that there is no unique way to write $\mathrm{BCH}_{a,b}(\xi, \eta)$ since one can re-arrange terms by antisymmetry and Jacobi identity without changing the number of ξ's and η's. Luckily, we will need only estimates for $\mathrm{BCH}_{a,b}(\xi, \eta)$ later on. Before going into these details, we just mention the following well-known formula (e.g. see [15, part 2.8.12 (c)]) for the lowest order terms of the BCH series:

Lemma 2.6. *Let $\mathfrak{g}$ be a Lie algebra and $\xi, \eta \in \mathfrak{g}$. Then we can write the Baker-Campbell-Hausdorff series up to first order in η as*

$$\mathrm{BCH}(\xi, \eta) = \xi + \sum_{n=0}^{\infty} \frac{B_n^*}{n!} (\mathrm{ad}_\xi)^n (\eta) + \mathcal{O}(\eta^2), \tag{22}$$

where the Bernoulli numbers B_n^* are defined by the series

$$\frac{z}{1 - e^{-z}} = \sum_{n=0}^{\infty} \frac{B_n^*}{n!} z^n. \tag{23}$$

Recall that the series (23) converges absolutely for $|z| < 2\pi$ and that we have the (quite rough) estimate

$$|B_n^*| \leq n! \tag{24}$$

for the Bernoulli numbers.

2.3. A Formula for the Gutt Star Product.

In a next step, we develop some technical tools which will be useful for proving the continuity of the Gutt star product. First we write the Gutt star product for $x, y \in \mathrm{S}^\bullet(\mathfrak{g})$ as

$$x \star_z y = \sum_{n=0}^{\infty} z^n C_n(x, y) \tag{25}$$

with bilinear operators $C_n \colon \mathrm{S}^\bullet(\mathfrak{g}) \times \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g})$ as usual. Now, the main goal is to use Proposition 2.3 and the BCH series to obtain fairly explicit formulas for the contributions $C_n(x, y)$, up to the knowledge of the BCH series.

From the original work of Gutt [21, Prop. 1], see also [15, 2.8.12 (c)] as well as [27, Rem. 5.2.8] and [23, Eq. 2.23], we get the following formula for $\star_z$ whenever one factor, say the second, is linear:

Proposition 2.7. *Let* $z \in \mathbb{K}$.
i.) For all $\xi, \eta \in \mathfrak{g}$ *and* $k \in \mathbb{N}$ *we have*

$$\xi^k \star_z \eta = \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} \, (\mathrm{ad}_\xi)^j \, (\eta).$$ (26)

ii.) For all $k \in \mathbb{N}$ *and* $\xi_1, \ldots, \xi_k, \eta \in \mathfrak{g}$ *we have*

$$\xi_1 \cdots \xi_k \star_z \eta = \sum_{j=0}^{k} \frac{1}{k!} \binom{k}{j} z^j B_j^* \sum_{\sigma \in S_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k-j)} [\xi_{\sigma(k-j+1)}, [\ldots [\xi_{\sigma(k)}, \eta] \ldots]].$$ (27)

Proof. For convenience,we sketch the proof: the first part is essentially Lemma 2.6 together with Proposition 2.3. The second is then obtained by polarization from the first: Set

$$\Xi = \Xi(t_1, \ldots, t_k) = \sum_{j=1}^{k} t_j \xi_j$$

for parameters $t_1, \ldots, t_k \in \mathbb{R}$. Then we get by differentiating

$$\frac{\partial^k}{\partial t_1 \cdots \partial t_k} \Xi^k = k! \xi_1 \cdots \xi_k.$$

This formal differentiation in Equation (26) gives the result. ■

As a consequence, the previous proposition determines the explicit form of the bilinear operators C_n of the Gutt star product whenever one factor is linear. In particular, by associativity we have

$$\xi_1 \star_z \cdots \star_z \xi_k = \sum_{\substack{1 \le j \le k-1 \\ i_j \in \{0, \ldots, j\}}} z^{i_1 + \cdots + i_{k-1}} C_{i_{k-1}} \left(\cdots C_{i_2} \left(C_{i_1} \left(\xi_1, \xi_2 \right), \xi_3 \right), \ldots, \xi_k \right)$$ (28)

for $2 \le k \in \mathbb{N}$ and $\xi_1, \ldots, \xi_k \in \mathfrak{g}$. While this formula is easy to handle for small k, it becomes quite tedious in general.

For our continuity estimates we need to go beyond linear terms: both factors in the product have to be general. By differentiating the identity form Proposition 2.3, we get the next result.

Lemma 2.8. *Let $\mathfrak{g}$ be a Lie algebra.*

i.) Let $\xi, \eta \in \mathfrak{g}$. We have

$$\xi^k \star_z \eta^\ell = \sum_{n=0}^{k+\ell-1} z^n C_n \left(\xi^k, \eta^\ell \right), \tag{29}$$

with $C_0(\xi^k, \eta^\ell) = \xi^k \eta^\ell$ and

$$C_n \left(\xi^k, \eta^\ell \right) = \frac{k! \ell!}{(k+\ell-n)!} \sum_{\substack{a_1, b_1, \ldots, a_r, b_r \geq 0 \\ a_i + b_i \geq 1 \\ a_1 + \cdots + a_r = k \\ b_1 + \cdots + b_r = \ell}} \mathrm{BCH}_{a_1, b_1} \left(\xi, \eta \right) \cdots \mathrm{BCH}_{a_r, b_r} \left(\xi, \eta \right) \tag{30}$$

for $k, \ell \in \mathbb{N}$ and $n \geq 1$, where we set $r = k + \ell - n$ for abbreviation.

ii.) Denote by $\widetilde{\mathrm{BCH}}_{a,b} \left(\cdot ; \cdot \right)$ the unique $a + b$-linear map, symmetric in the first a and in the last b arguments, such that

$$\widetilde{\mathrm{BCH}}_{a,b} \left(\xi, \ldots, \xi; \eta, \ldots, \eta \right) = \mathrm{BCH}_{a,b} \left(\xi, \eta \right) \tag{31}$$

for $\xi, \eta \in \mathfrak{g}$. Then we have

$$C_n \left(\xi_1 \cdots \xi_k ; \eta_1 \cdots \eta_\ell \right) = \frac{1}{(k+\ell-n)!} \sum_{\sigma \in S_k, \tau \in S_\ell} \sum_{\substack{a_1, b_1, \ldots, a_r, b_r \geq 0 \\ a_i + b_i \geq 1 \\ a_1 + \cdots + a_r = k \\ b_1 + \cdots + b_r = \ell}} \tag{32}$$

$$\widetilde{\mathrm{BCH}}_{a_i, b_i} \left(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(a_1)} ; \eta_{\tau(1)}, \ldots, \eta_{\tau(b_1)} \right) \cdots$$

$$\widetilde{\mathrm{BCH}}_{a_r, b_r} \left(\xi_{\sigma(k-a_r+1)}, \ldots, \xi_{\sigma(k)} ; \eta_{\tau(\ell-b_r+1)}, \ldots, \eta_{\tau(\ell)} \right).$$

for $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_\ell \in \mathfrak{g}$.

Proof. We consider $z \neq 0$ since the claim is trivial for $z = 0$. Using Proposition 2.3 we get the star product of ξ^k and η^ℓ by differentiating

$$\xi^k \star_z \eta^\ell = \frac{\partial^k}{\partial t^k} \frac{\partial^\ell}{\partial s^\ell} \bigg|_{t,s=0} \exp \left(\frac{1}{z} \mathrm{BCH} \left(z t \xi, s z \eta \right) \right)$$

$$= \frac{\partial^k}{\partial t^k} \frac{\partial^\ell}{\partial s^\ell} \bigg|_{t,s=0} \sum_{r=0}^{\infty} \frac{1}{r!} \frac{1}{z^r} \left(\mathrm{BCH} \left(z t \xi, z s \eta \right) \right)^r$$

$$= \frac{\partial^k}{\partial t^k} \frac{\partial^\ell}{\partial s^\ell} \bigg|_{t,s=0} \sum_{r=0}^{\infty} \frac{1}{r!} \frac{1}{z^r} \left(\sum_{j=0}^{k+\ell} \mathrm{BCH}_j \left(z t \xi, z s \eta \right) \right)^r$$

$$= \sum_{r=0}^{k+\ell} \frac{1}{r!} \frac{z^{k+\ell}}{z^r} k!\ell! \sum_{\substack{a_1,b_1,\ldots,a_r,b_r \geq 0 \\ a_i+b_i \geq 1 \\ a_1+\cdots+a_r=k \\ b_1+\cdots+b_r=\ell}} \mathrm{BCH}_{a_i,b_i}(\xi,\eta) \cdots \mathrm{BCH}_{a_r,b_r}(\xi,\eta).$$

Here we have used that the k-th derivative by t at $t = 0$ gives $k!$ times the coefficient of ξ^k and analogously for the ℓ-th derivative by s. The bounds on the parameters $a_1,\ldots,a_r,b_1,\ldots,b_r$ and thereby the bounds on r originate from the fact that the BCH series has no constant term. Moreover, higher powers of the BCH series as $k+\ell$ will clearly not contain terms we need. The second formula is obtained by polarizing the first: We introduce again parameters $t_i, s_j \in \mathbb{R}$ with $i = 1,\ldots,k$ and $j = 1,\ldots,\ell$ setting

$$\Xi = \sum_{i=1}^{k} t_i \xi_i \quad \text{and} \quad \mathrm{H} = \sum_{j=1}^{\ell} s_j \eta_j.$$

Differentiating $C_n(\Xi, \mathrm{H})$ as in Formula (30) once by every parameter and dividing by $k!\ell!$, we get permutations of all ξ_i and all η_j and hence Equation (32). ∎

Remark 2.9. The importance of this formula is that we have reduced the complexity of $\star_z$ to the difficulties to compute the homogeneous parts of the BCH series. This is of course still a complicated and tedious problem but luckily we are only interested in estimating the terms $\mathrm{BCH}_{a,b}(\xi,\eta)$ instead of computing them explicitly.

3.   The Continuity of $\star_z$

The next step is finding continuity estimates for $\star_z$. All estimates which are done in the next three sections follow mostly the same scheme: We extend maps from $\mathrm{S}_R^\bullet(\mathfrak{g})$ to $\mathrm{T}_R^\bullet(\mathfrak{g})$ by using the symmetrization map $\mathscr{S}$ beforehand. Our first examples for this are the Gutt star product and the C_n-operators from Equation (25). Set

$$\star_z \colon \mathrm{T}_R^\bullet(\mathfrak{g}) \otimes \mathrm{T}_R^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}_R^\bullet(\mathfrak{g}) \quad \text{with} \quad \star_z = \star_z \circ (\mathscr{S} \otimes \mathscr{S}) \tag{33}$$

and analogously for the C_n. It is clear that all extended maps coincide with the original maps on $\mathrm{S}_R^\bullet(\mathfrak{g})$. Then, we use the AE-property for the seminorms (which is always valid for locally convex nilpotent Lie algebras), to estimate Lie brackets. Finally, we use a feature of the projective tensor product, in order to generalize statements about factorizing tensors to arbitrary ones. This is done once explicitly at the end of the proof of Proposition 3.2. We then just refer to this construction since it always works analogously.

3.1. A Direct Continuity Result.

For a word w in the two letters ξ and η we denote by $[w]$ the unique Lie bracket expression of this word, where we have nested the Lie brackets to the left in the sense that

$$[\xi\eta\eta\ldots] = [\ldots[[\xi,\eta],\eta],\ldots]. \tag{34}$$

Moreover, $|w|$ denotes the number of letters, i.e. the length of the word.

Lemma 3.1. *Let $\mathfrak{g}$ be a AE-Lie algebra, p a continuous seminorm, q an asymptotic estimate for it.*

i.) For $n \in \mathbb{N}$ there are numbers $g_w \in \mathbb{Q}$ such that for $\xi, \eta \in \mathfrak{g}$ one has

$$\mathrm{BCH}_n(\xi, \eta) = \sum_{|w|=n} \frac{g_w}{n}[w]. \tag{35}$$

ii.) The coefficients g_w can be chosen to fulfil the estimate

$$\sum_{|w|=n} \left|\frac{g_w}{n}\right| \leq \frac{2}{n}. \tag{36}$$

iii.) For every word w which consists of a times the letter ξ and b times the letter η, we have

$$p([w]) \leq q(\xi)^a q(\eta)^b. \tag{37}$$

iv.) Let $a, b \in \mathbb{N}$ and $\xi_1, \ldots, \xi_a, \eta_1, \ldots, \eta_b \in \mathfrak{g}$. We have the estimate

$$p\left(\widetilde{\mathrm{BCH}}_{a,b}(\xi_1, \ldots, \xi_a; \eta_1, \ldots, \eta_b)\right) \leq \frac{2}{a+b} q(\xi_1)\cdots q(\xi_a)q(\eta_1)\cdots q(\eta_b). \tag{38}$$

Proof. Goldberg found a form of writing the BCH series as a series of words in two letters X and Y with certain coefficients [19] called $c_X(s_1, \ldots, s_m)$ or $c_Y(s_1, \ldots, s_m)$ depending whether a word begins with the letter X or Y. Here, $c_X(s_1, \ldots, s_m)$ belongs to the word

$$X^{s_1}Y^{s_2}\ldots(X \text{ or } Y)^{s_m},$$

where the word ends with the letter X or Y if m is odd or even, respectively. In [31], Thompson put this into Lie bracket form and proved that using $g_w = c_X$ [or $g_w = c_Y$] one gets identity (35). In [32] Thompson put estimates on these coefficients and proved the estimate (36). The inequality (37) is due to the AE-property, which does not see the way how brackets are set but just counts the number of ξ's and η's in the whole expression. We use the notation $|w|_\xi$ for the

number of ξ's appearing in a word w and $|w|_\eta$ for the number of η's. Clearly, $|w| = |w|_\xi + |w|_\eta$. With (36) and the AE-property of $\mathfrak{g}$, we get

$$p\left(\widetilde{\mathrm{BCH}}_{a,b}(\xi_1,\ldots,\xi_a;\eta_1,\ldots,\eta_b)\right) \leq \sum_{\substack{|w|_\xi=a \\ |w|_\eta=b}} p\left(\frac{g_w}{a+b}[w]\right)$$

$$\leq \sum_{\substack{|w|_\xi=a \\ |w|_\eta=b}} \left|\frac{g_w}{a+b}\right| q(\xi_1)\cdots q(\xi_a)q(\eta_1)\cdots q(\xi_b)$$

$$\leq \frac{2}{a+b} q(\xi_1)\cdots q(\xi_a)q(\eta_1)\cdots q(\eta_b).$$

∎

In a next step, we want to approach the estimate via the formula

$$\xi_1\cdots\xi_k \star_z \eta_1\cdots\eta_\ell = \sum_{n=0}^{k+\ell-1} z^n C_n(\xi_1\cdots\xi_k,\eta_1\cdots\eta_\ell). \tag{39}$$

To shorten the very long expression from Equation (32), we occasionally abbreviate the summations by

$$C_n(\xi_1\cdots\xi_k,\eta_1\cdots\eta_\ell) = \frac{1}{r!}\sum_{\sigma,\tau}\sum_{a_i,b_j}\widetilde{\mathrm{BCH}}_{a_1,b_1}\left(\xi_{\sigma(i)};\eta_{\tau(j)}\right)\cdots\widetilde{\mathrm{BCH}}_{a_r,b_r}\left(\xi_{\sigma(i)};\eta_{\tau(j)}\right),$$

$$\tag{40}$$

meaning the summations as given in Lemma 2.8 and using $r = k+\ell-n$.

Proposition 3.2. *Let $\mathfrak{g}$ be an AE-Lie algebra, $R \geq 0$, p a continuous seminorm with an asymptotic estimate q, and $z \in \mathbb{K}$.*

i.) For $n \in \mathbb{N}$, the operator C_n is continuous and for all $x,y \in \mathrm{T}_R^\bullet(\mathfrak{g})$ we have the estimate

$$p_R(C_n(x,y)) \leq \frac{n!^{1-R}}{2\cdot 8^n}(32q)_R(x)(32q)_R(y). \tag{41}$$

ii.) For $R \geq 1$, the Gutt star product is continuous and for all $x,y \in \mathrm{T}_R^\bullet(\mathfrak{g})$ we have the estimate:

$$p_R(x \star_z y) \leq (cq)_R(x)(cq)_R(y) \tag{42}$$

with $c = 32(|z|+1)$. Hence, the estimate (42) holds on $\widehat{\mathrm{S}}_R^\bullet(\mathfrak{g})$ for all $z \in \mathbb{K}$, too.

Proof. Let us use $r = k+\ell-n$ as before and recall that the products are taken in the symmetric algebra. Then we can use Equation (30) from Lemma 2.8 and

put estimates on it. Let p be a continuous seminorm and let q be an asymptotic estimate for it. Then we get

$$
p_R\big(C_n(\xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_\ell)\big)
$$

$$
= p_R\bigg(\frac{1}{r!} \sum_{\sigma,\tau} \sum_{a_i,b_j} \widetilde{\mathrm{BCH}}_{a_1,b_1}\big(\xi_{\sigma(i)}; \eta_{\tau(j)}\big) \cdots \widetilde{\mathrm{BCH}}_{a_r,b_r}\big(\xi_{\sigma(i)}; \eta_{\tau(j)}\big)\bigg)
$$

$$
\overset{(a)}{\le} \frac{1}{r!} r!^R \sum_{\sigma,\tau} \sum_{a_i,b_j} p\left(\widetilde{\mathrm{BCH}}_{a_1,b_1}\big(\xi_{\sigma(i)}; \eta_{\tau(j)}\big)\right) \cdots p\left(\widetilde{\mathrm{BCH}}_{a_r,b_r}\big(\xi_{\sigma(i)}; \eta_{\tau(j)}\big)\right)
$$

$$
\overset{(b)}{\le} \frac{1}{r!^{1-R}} \sum_{\sigma,\tau} \sum_{a_i,b_j} \frac{2}{a_1+b_1} \cdots \frac{2}{a_r+b_r} q(\xi_1) \cdots q(\xi_k) q(\eta_1) \cdots q(\eta_\ell)
$$

$$
\overset{(c)}{\le} q(\xi_1) \cdots q(\xi_k) q(\eta_1) \cdots q(\eta_\ell) 2^r \frac{k!\ell!}{r!^{1-R}} \sum_{a_i,b_j} 1,
$$

where we just used the continuity estimate for the symmetric tensor product in (a), Lemma 3.1, $iv.$), in (b) and $\frac{2}{a_i+b_i} \le 2$ in (c). We estimate the number of terms in the sum and get

$$
\sum_{\substack{a_1,b_1,\ldots,a_r,b_r \ge 0 \\ a_i+b_i \ge 1 \\ a_1+\cdots+a_r=k \\ b_1+\cdots+b_r=\ell}} 1 \le \sum_{\substack{a_1,b_1,\ldots,a_r,b_r \ge 0 \\ a_1+b_1+\cdots+a_r+b_r=k+\ell}} 1 = \binom{k+\ell+2r-1}{k+\ell} \le 2^{3(k+\ell)-2n-1}.
$$

Using this estimate, we get

$$
p_R\big(C_n(\xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_\ell)\big)
$$

$$
\le q(\xi_1) \cdots q(\xi_k) q(\eta_1) \cdots q(\eta_\ell) 2^{k+\ell-n} \frac{k!\ell!}{(k+\ell-n)!^{1-R}} 2^{3(k+\ell)-2n-1}
$$

$$
= q_R(\xi_1 \otimes \cdots \otimes \xi_k) q_R(\eta_1 \otimes \cdots \otimes \eta_\ell) 2^{4(k+\ell)-3n-1} \left(\frac{k!\ell!n!}{(k+\ell-n)!n!}\right)^{1-R}
$$

$$
\le q_R(\xi_1 \otimes \cdots \otimes \xi_k) q_R(\eta_1 \otimes \cdots \otimes \eta_\ell) 2^{4(k+\ell)-3n-1} 2^{(1-R)(k+\ell)} n!^{1-R}
$$

$$
= \frac{n!^{1-R}}{2 \cdot 8^n} (32q)_R(\xi_1 \otimes \cdots \otimes \xi_k)(32q)_R(\eta_1 \otimes \cdots \otimes \eta_\ell).
$$

The estimate (41) is now proven on factorizing tensors. For general tensors $x, y \in \mathrm{T}_R^\bullet(\mathfrak{g})$, we use the following argument: let

$$
x = \sum_{m=0}^{k} x^{(m)} = \sum_{m=0}^{k} \sum_i x_i^{(m)}, \quad y = \sum_{n=0}^{\ell} y^{(n)} = \sum_{n=0}^{\ell} \sum_j y_j^{(n)},
$$

where the $x_i^{(m)}$ and the $y_j^{(n)}$ are factorizing tensors of homogeneous degrees m and n, respectively, with the maximal degree of x and y being k and ℓ, respectively. Hence we have

$$x_i^{(m)} = x_i^{(m),1} \otimes \cdots \otimes x_i^{(m),m}, \quad y_j^{(n)} = y_j^{(n),1} \otimes \cdots \otimes y_j^{(n),n}.$$

Now we get the following estimate:

$$
\begin{aligned}
p_R\left(C_n(x,y)\right) &= p_R\left(C_n\left(\left(\sum_{m=0}^k \sum_i x_i^{(m)}\right), \left(\sum_{n=0}^\ell \sum_j y_j^{(n)}\right)\right)\right) \\
&= p_R\left(\sum_{m=0}^k \sum_{n=0}^\ell \sum_i \sum_j C_n\left(x_i^{(m)}, y_j^{(n)}\right)\right) \\
&\leq \sum_{m=0}^k \sum_{n=0}^\ell \sum_i \sum_j \frac{n!^{1-R}}{2\cdot 8^n}(32q)_R\left(x_i^{(m)}\right)(32q)_R\left(y_j^{(n)}\right) \\
&= \frac{n!^{1-R}}{2\cdot 8^n}\left(\sum_{m=0}^k \sum_i (32q)_R\left(x_i^{(m)}\right)\right)\left(\sum_{n=0}^\ell \sum_j (32q)_R\left(y_j^{(n)}\right)\right) \\
&= \frac{n!^{1-R}}{2\cdot 8^n}\left(\sum_{m=0}^k 32^m m!^R \sum_i q\left(x_i^{(m),1}\right)\cdots q\left(x_i^{(m),m}\right)\right) \\
&\quad \cdot\left(\sum_{n=0}^\ell 32^n n!^R \sum_j q\left(y_j^{(n),1}\right)\cdots q\left(y_j^{(n),n}\right)\right).
\end{aligned}
$$

We have to take the infimum on both sides over all representations of the $x^{(m)}$ and the $y^{(n)}$. On the right hand side, we get for the x-terms

$$\inf\left\{\sum_i q\left(x_i^{(m),1}\right)\cdots q\left(x_i^{(m),m}\right) \,\middle|\, \sum_i x_i^{(m),1} \otimes \cdots \otimes x_i^{(m),m} = x^{(m)}\right\} = q^m\left(x^{(m)}\right).$$

The y-terms give in the same way $q^n\left(y^{(n)}\right)$. This is exactly the definition of the tensor power of a seminorm as needed for the projective tensor product. We can recollect the factorials and the coefficients and get

$$
\begin{aligned}
p_R\left(C_n(x,y)\right) &\leq \frac{n!^{1-R}}{2\cdot 8^n}\left(\sum_{m=0}^k 32^k k!^R \sum_i q^m\left(x_i^{(m)}\right)\right)\left(\sum_{n=0}^\ell 32^n n!^R \sum_j q^n\left(y_j^{(n)}\right)\right) \\
&= \frac{n!^{1-R}}{2\cdot 8^n}\left(\sum_{m=0}^k (32q)_R\left(x^{(m)}\right)\right)\left(\sum_{n=0}^\ell (32q)_R\left(y^{(n)}\right)\right)
\end{aligned}
$$

$$= \frac{n!^{1-R}}{2 \cdot 8^n} (32q)_R(x)(32q)_R(y),$$

which proves (41) on general tensors. For the second statement, let x and y be tensors of degree at most k and ℓ respectively. We have

$$
\begin{aligned}
p_R (x \star_z y) &= p_R \left(\sum_{n=0}^{k+\ell-1} z^n C_n(x,y) \right) \\
&\leq \sum_{n=0}^{k+\ell-1} p_R (z^n C_n(x,y)) \\
&\stackrel{(a)}{\leq} \sum_{n=0}^{k+\ell-1} \frac{|z|^n}{2 \cdot 8^n} n!^{1-R} (32q)_R(x)(32q)_R(y) \\
&\stackrel{(b)}{\leq} \frac{(|z|+1)^{k+\ell}}{2} (32q)_R(x)(32q)_R(y) \sum_{n=0}^{\infty} \frac{1}{8^n} \\
&\leq (32(|z|+1)q)_R(x)(32(|z|+1)q)_R(y),
\end{aligned}
\tag{43}
$$

by using (41) in (a) and $R \geq 1$ in (b). Since estimates on $S_R^\bullet(\mathfrak{g})$ also hold for the completion, the second part is done and hence the first part of our Main Theorem 1.2 is proven. $\blacksquare$

It is easy to see that we need at least $R \geq 1$ to get rid of the factorials which come up because of the combinatorics of the star product, but it is interesting to know that the Gutt star product really fails continuity, if $R < 1$:

Example 3.3. Let $0 \leq R < 1$ and $\mathfrak{g}$ be the Heisenberg algebra in three dimensions, i.e. the Lie algebra generated by the elements P, Q and E with the bracket $[P, Q] = E$ and all other brackets vanish. We impose on $\mathfrak{g}$ the ℓ^1-topology with the norm n and $n(P) = n(Q) = n(E) = 1$. Then we consider the sequences

$$a_k = \frac{P^k}{k!^{R+\epsilon}} \quad \text{and} \quad b_k = \frac{Q^k}{k!^{R+\epsilon}}$$

with $2\epsilon < 1 - R$. It is easy to see that

$$n_R(a_k) = n_R(b_k) = k!^{-\epsilon}$$

and hence we get the limit for any $c > 0$ by

$$\lim_{k \to \infty} (cn)_R(a_k) = \lim_{k \to \infty} (cn)_R(a_k) = 0$$

We want to show that there is no $c > 0$ such that

$$n_R(a_k \star_z b_k) \leq (cn)_R(a_k)(cn)_R(b_k).$$

In other words, $n_R(a_k \star_z b_k)$ grows faster than exponentially. But this is the case, since we can calculate the star product explicitly and see

$$
\begin{aligned}
n_R(a_k \star_z b_k) &= n_R\left(\sum_{j=0}^{k} \binom{k}{j}\binom{k}{j}j!\frac{1}{k!^{2R+2\epsilon}}P^{k-j}Q^{k-j}E^j\right) \\
&= \sum_{j=0}^{k}\frac{k!^2 j!(2k-j)!^R}{(k-j)!^2 j!^2 k!^{2R+2\epsilon}}\underbrace{n^{2k-j}(P^{k-j}Q^{k-j}E^j)}_{=1} \\
&= \sum_{j=0}^{k}\underbrace{\binom{k}{j}^2\binom{2k}{k}\binom{2k}{j}^{-1}}_{\geq 1}\frac{j!^{1-R}}{k!^{2\epsilon}} \\
&\geq \sum_{j=0}^{k}\frac{j!^{1-R}}{k!^{2\epsilon}} \\
&\geq k!^{1-R-2\epsilon}.
\end{aligned}
$$

Hence for every continuous seminorm p_R in the T_R-topology, we have

$$p_R(a_k \star_z b_k) \longrightarrow \infty \quad \text{and} \quad n_R(a_k), n_R(b_k) \longrightarrow 0,$$

so the star product is not continuous.

3.2. An Inductive Continuity Result.

As already mentioned, we also get continuity via Proposition 2.7 by imposing the submultiplicativity of the seminorms:

$$p([\xi, \eta]) \leq p(\xi)p(\eta). \tag{44}$$

This is fulfilled for a big class of Lie algebras but, for example, no longer for general nilpotent ones. In any case, it gives an alternative proof of the most important part of our Main Theorem and is therefore given here.

Lemma 3.4. *Let $\mathfrak{g}$ be a locally multiplicatively convex Lie algebra and $R \geq 1$. Then if $|z| < 2\pi$ or $R > 1$ there exists, for $x \in T^\bullet(\mathfrak{g}), \eta \in \mathfrak{g}$ of degree at most k and each continuous submultiplicative seminorm p, a constant $c_{z,R}$ only depending on z and R such that the following estimate holds:*

$$p_R(x \star_z \eta) \leq c_{z,R}(k+1)^R p_R(x)q(\eta) \tag{45}$$

Proof.   We have for $\xi_1, \ldots, \xi_k, \eta \in \mathfrak{g}$

$$p_R\left(\xi_1 \otimes \cdots \otimes \xi_k \star_z \eta\right)$$

$$= p_R\left(\sum_{n=0}^{k} \frac{B_n^* z^n}{n!(k-n)!} \sum_{\sigma \in S_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k-n)} \left(\mathrm{ad}_{\xi_{\sigma(k-n+1)}} \circ \cdots \circ \mathrm{ad}_{\xi_{\sigma(k)}}\right)(\eta)\right)$$

$$= \sum_{n=0}^{k} \frac{|B_n^*||z|^n}{n!(k-n)!}$$

$$\sum_{\sigma \in S_k} (k+1-n)!^R p^{k+1-n} \left(\xi_{\sigma(1)} \cdots \xi_{\sigma(k-n)} \left(\mathrm{ad}_{\xi_{\sigma(k-n+1)}} \circ \cdots \circ \mathrm{ad}_{\xi_{\sigma(k)}}\right)(\eta)\right)$$

$$\leq (k+1)^R \sum_{n=0}^{k} \frac{|B_n^*||z|^n}{n!}(k-n)!^{R-1} k! p(\xi_1) \cdots p(\xi_k) p(\eta)$$

$$= (k+1)^R \sum_{n=0}^{k} \frac{|B_n^*||z|^n}{n!^R}\left(\frac{(k-n)!n!}{k!}\right)^{R-1} p_R\left(\xi_1 \otimes \cdots \otimes \xi_k\right) p(\eta)$$

$$\leq (k+1)^R p_R\left(\xi_1 \otimes \cdots \otimes \xi_k\right) p(\eta) \sum_{n=0}^{k} \frac{|B_n^*||z|^n}{n!^R}.$$

Now if $|z| < 2\pi$ the sum can be estimated by extending it to a series which converges. So we get a constant $c_{z,R}$ depending on R and on z such that

$$p_R\left(\xi_1 \otimes \cdots \otimes \xi_k \star_z \eta\right) \leq (k+1)^R c_{z,R} p_R\left(\xi_1 \otimes \cdots \otimes \xi_k\right) p(\eta).$$

On the other hand, if $|z| \geq 2\pi$ and $R > 1$ we can estimate

$$p_R\left(\xi_1 \otimes \cdots \otimes \xi_k \star_z \eta\right) \leq (k+1)^R p_R\left(\xi_1 \otimes \cdots \otimes \xi_k\right) p(\eta) \left(\sum_{n=0}^{k} \frac{|B_n^*|}{n!}\right) \left(\sum_{n=0}^{k} \frac{|z|^n}{n!^{R-1}}\right).$$

Again, both series will converge and give constants depending only on z and R. Hence, we have the estimate on factorizing tensors and can extend this to generic tensors of degree at most k by taking the infimum as in the proof of Proposition 3.2.   ■

     In the following, we assume again that either $R > 1$ or $R \geq 1$ and $|z| < 2\pi$ in order the use Lemma 3.4. Now we can give a simpler proof of Proposition 3.2 for the case of a locally multiplicatively convex Lie algebra:

Alternative Proof of Proposition 3.2.   Assume that $\mathfrak{g}$ is now even locally multiplicatively convex. We want to replace η in the foregoing lemma by an

arbitrary tensor y of degree at most ℓ. Let $\eta_1, \ldots, \eta_\ell \in \mathfrak{g}$. On factorizing tensors we get

$$
\begin{aligned}
p_R \left(\xi_1 \otimes \cdots \otimes \xi_k \star \eta_1 \otimes \cdots \otimes \eta_\ell \right) &= p_R \left(\frac{1}{\ell!} \sum_{\tau \in S_\ell} \xi_1 \otimes \cdots \otimes \xi_k \star \eta_{\tau(1)} \star \cdots \star \eta_{\tau(\ell)} \right) \\
&\leq c_{z,R}(k+\ell)^R \frac{1}{\ell!} \sum_{\tau \in S_\ell} p_R \left(\xi_1 \otimes \cdots \otimes \xi_k \star \eta_{\tau(1)} \star \cdots \star \eta_{\tau(\ell-1)} \right) p \left(\eta_{\tau(\ell)} \right) \\
&\leq c_{z,R}^\ell ((k+\ell) \cdots (k+1))^R p_R \left(\xi_1 \otimes \cdots \otimes \xi_k \right) p(\eta_1) \cdots p(\eta_\ell) \\
&= c_{z,R}^\ell \left(\frac{(k+\ell)!}{k! \ell!} \right)^R p_R \left(\xi_1 \otimes \cdots \otimes \xi_k \right) p_R \left(\eta_1 \otimes \cdots \otimes \eta_\ell \right) \\
&\leq (2^R p)_R \left(\xi_1 \otimes \cdots \otimes \xi_k \right) (2^R c_{z,R} p)_R \left(\eta_1 \otimes \cdots \otimes \eta_\ell \right).
\end{aligned}
$$

Once again, we have the estimate on factorizing tensors via polarization and extend it via the infimum argument on the whole tensor algebra, since the estimate depends no longer on the degree of the tensors. ∎

While the above proof is of course much simpler, we had to invest a slightly stronger assumption compared to the AE Lie algebra case.

3.3. Dependence on the Formal Parameter.

Let us consider the completion $\widehat{S}_R^\bullet(\mathfrak{g})$ of the symmetric algebra with the Gutt star product $\star_z$. Exponentials do not belong to this completion, as we can show.

Proposition 3.5. Let $\xi \in \mathfrak{g}$ and $R \geq 1$, then $\exp(\xi) \notin \widehat{S}_R^\bullet(\mathfrak{g})$, where $\exp(\xi) = \sum_{n=0}^\infty \frac{\xi^n}{n!}$.

Proof. Take p a seminorm such that $p(\xi) \neq 0$. Then set $c = p(\xi)^{-1}$. For ξ^n the powers in the sense of either the usual tensor product, or the symmetric product or the star product are the same. So we have for $N \in \mathbb{N}$

$$
(cp)_R \left(\sum_{n=0}^N \frac{c^n}{n!} \xi^n \right) = \sum_{n=0}^N \frac{n!^R}{n!} c^n p_R \left(\xi^n \right) = \sum_{n=0}^N n!^{R-1} \geq N.
$$

Hence $(cp)_R(\exp(\xi))$ does not converge. ∎

Since the formal series converges to the star product on $\widehat{S}_R^\bullet(\mathfrak{g})$ and all the projections on the homogeneous components are continuous from Lemma 2.2, we can reinterpret the continuity result we found in Proposition 3.2.

Proposition 3.6. *Let $R \geq 1$, then for all $x, y \in \widehat{S}_R^\bullet(\mathfrak{g})$ the map*

$$\mathbb{K} \ni z \longmapsto x \star_z y \in \widehat{S}_R^\bullet(\mathfrak{g}) \tag{46}$$

is analytic with (absolutely convergent) Taylor expansion at $z = 0$ given by Equation (30). The collection of algebras $\left\{ \left(\widehat{S}_R^\bullet(\mathfrak{g}), \star_z \right) \right\}_{z \in \mathbb{K}}$ is an entire deformation of the completed symmetric tensor algebra $\widehat{S}_R^\bullet(\mathfrak{g})$.

Proof. The crucial point is that for $x, y \in \widehat{S}_R^\bullet(\mathfrak{g})$ and every continuous semi-norm p we have an asymptotic estimate q such that

$$
\begin{aligned}
p_R \left(x \star_z y \right) = p_R &\left(\sum_{n=0}^\infty z^n C_n(x, y) \right) \\
&= \sum_{n=0}^\infty |z|^n p_R(C_n(x, y)) \\
&\leq (16q)_R(x)(16q)_R(y) \sum_{n=0}^\infty \frac{|z|^n n!^{1-R}}{2 \cdot 8^n},
\end{aligned}
$$

where we used the fact that the estimate (41) extends to the completion. For $R > 1$, this map is clearly analytic and absolutely convergent for all $z \in \mathbb{K}$. If $R = 1$, then for every $M \geq 1$ we go back to homogeneous, factorizing tensors $x^{(k)}$ and $y^{(\ell)}$ of degree k and ℓ respectively, and have

$$
\begin{aligned}
M^n p_R \left(C_n \left(x^{(k)}, y^{(\ell)} \right) \right) &\leq \frac{M^n}{2 \cdot 8^n} (16q)_R \left(x^{(k)} \right) (16q)_R \left(y^{(\ell)} \right) \\
&\leq M^{k+\ell} (16q)_R \left(x^{(k)} \right) (16q)_R \left(y^{(\ell)} \right) \\
&= (16Mq)_R \left(x^{(k)} \right) (16Mq)_R \left(y^{(\ell)} \right),
\end{aligned}
$$

where we used that $0 \leq n \leq k + \ell - 1$. The infimum argument gives the estimate on all tensors $x, y \in T_R^\bullet(\mathfrak{g})$ and it extends to the completion such that

$$p_R \left(z^n C_n(x, y) \right) \leq (16Mq)_R(x)(16Mq)_R(y) \frac{|z|^n}{2 \cdot (8M)^n}$$

and hence

$$p_R(x \star_z y) \leq (16Mq)_R(x)(16Mq)_R(y) \sum_{n=0}^\infty \frac{|z|^n}{2 \cdot (8M)^n}.$$

So the power series converges for all $z \in \mathbb{K}$ with $|z| < 8M$ and converges uniformly if $|z| \leq cM$ for $c < 8$. But then, the right hand side of (46) converges on all open discs centred around $z = 0$, and it must therefore be entire. ∎

3.4. Functoriality and Representations.

Let $z \in \mathbb{K}$, $\mathscr{A}$ an associative, locally convex algebra and $\phi_z \colon \mathfrak{g}_z \longrightarrow \mathscr{A}$ a continuous Lie algebra homomorphism with respect to the z-scaled Lie bracket. Then we have the commuting diagram

$$
\begin{array}{ccc}
\mathscr{U}_R(\mathfrak{g}_z) & \xrightarrow{\;\;\mathfrak{q}_z^{-1}\;\;} & \mathrm{S}_R^\bullet(\mathfrak{g}) \\[2mm]
\Big\uparrow{\scriptstyle \iota_z} & \;\;\Phi_z & \Big\downarrow{\scriptstyle \widetilde{\Phi}_z} \\[2mm]
\mathfrak{g} & \xrightarrow[\;\;\phi_z\;\;]{} & \mathscr{A}
\end{array}
\tag{47}
$$

from the algebraic theory. A crucial question is now whether the algebra homomorphisms Φ_z and $\widetilde{\Phi}_z$ are continuous. This question is partly answered by the following result:

Proposition 3.7. *Let $\mathfrak{g}$ be an AE-Lie algebra, let $\mathscr{A}$ be an associative AE-algebra, and let $\phi_z \colon \mathfrak{g} \longrightarrow \mathscr{A}$ be a continuous Lie algebra homomorphism with respect to the z-scaled Lie bracket. If $R \geq 0$, then the induced algebra homomorphisms Φ_z and $\widetilde{\Phi}_z$ are continuous.*

Proof.   We define an extension of $\widetilde{\Phi}_z$ on the whole tensor algebra by

$$
\Psi_z \colon \mathrm{T}_R^\bullet(\mathfrak{g}) \longrightarrow \mathscr{A}, \quad \Psi = \widetilde{\Phi}_z \circ \mathscr{S} \,.
$$

It is clear that if Ψ_z is continuous on factorizing tensors, we get the continuity of $\widetilde{\Phi}_z$ and Φ_z via the infimum argument. So let p be a continuous seminorm on $\mathscr{A}$ with its asymptotic estimate q and $\xi_1, \ldots, \xi_n \in \mathfrak{g}$. Since ϕ_z is continuous, we find a continuous seminorm r on $\mathfrak{g}$ such that for all $\xi \in \mathfrak{g}$ we have $q(\phi_z(\xi)) \leq r(\xi)$. Then we have

$$
\begin{aligned}
p\left(\Psi_z\left(\xi_1 \otimes \cdots \otimes \xi_n\right)\right) &= p\left(\widetilde{\Phi}_z\left(\xi_1 \star_z \cdots \star_z \xi_n\right)\right) \\
&= p(\phi_z(\xi_1) \cdots \phi_z(\xi_n)) \\
&\leq q(\phi_z(\xi_1)) \cdots q(\phi_z(\xi_n)) \\
&\leq r(\xi_1) \cdots r(\xi_n) \\
&\leq r_R(\xi_1 \otimes \cdots \otimes \xi_n),
\end{aligned}
$$

where the last inequality is true for all $R \geq 0$.   ∎

Our construction fails to be universal since the universal enveloping algebra endowed with our topology is *not* AE for $R > 0$. This is can be seen as follows:

Example 3.8. Take $\xi \in \mathfrak{g}$, then we know that $\xi^{\otimes n} = \xi^{\star n} = \xi^n$ for $n \in \mathbb{N}$ where we set the deformation parameter to $z = 1$. Let $R > 0$ and let p be a continuous seminorm in $\mathfrak{g}$ then we find

$$p_R(\xi^n) = n!^R p(\xi)^n = \frac{n!^R}{c^n} q(\xi)^n \tag{48}$$

for $c = \frac{p(\xi)}{q(\xi)}$ for a different seminorm q with $q(\xi) \neq 0$. But since the $\frac{n!^R}{c^n}$ always diverges for $n \to \infty$ we do not get an asymptotic estimate for p_R.

Nevertheless, from Proposition 3.7 we get the following conclusion:

Corollary 3.9. *Let $R \geq 1$ and $\mathscr{U}_R(\mathfrak{g}_z)$ the universal enveloping algebra with rescaled Lie bracket of an AE-Lie algebra $\mathfrak{g}$, then for every continuous representation ϕ_z of $\mathfrak{g}_z$ into the bounded linear operators $\mathfrak{B}(V)$ on a Banach space V the induced homomorphism of associative algebras $\Phi_z \colon \mathscr{U}(\mathfrak{g}_z) \longrightarrow \mathfrak{B}(V)$ is continuous.*

Proof. This follows directly from Proposition 3.7 and $\mathfrak{B}(V)$ being a Banach algebra. ∎

Now let $\mathfrak{g}, \mathfrak{h}$ be two AE-Lie algebras. We know that a Lie algebra homomorphism lifts ϕ to a unital homomorphism of algebras Φ_z

$$
\begin{array}{ccc}
\mathrm{S}_R^\bullet(\mathfrak{g}) & \xrightarrow{\ \widetilde{\Phi}_z\ } & \mathrm{S}_R^\bullet(\mathfrak{h}) \\[1mm]
{\scriptstyle \mathfrak{q}_z^{-1}}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathfrak{q}_z^{-1}} \\[1mm]
\mathscr{U}_R(\mathfrak{g}_z) & \xrightarrow{\ \Phi_z\ } & \mathscr{U}_R(\mathfrak{h}_z) \\[1mm]
{\scriptstyle \iota_z}\Big\uparrow & & \Big\uparrow{\scriptstyle \iota_z} \\[1mm]
\mathfrak{g} & \xrightarrow[\ \phi\ ]{} & \mathfrak{h}
\end{array}
\tag{49}
$$

for all $z \in \mathbb{K}$ since in this case $\phi = \phi_z \colon \mathfrak{g}_z \longrightarrow \mathfrak{h}_z$ is a Lie algebra morphism for all $z \in \mathbb{K}$. If ϕ is a continuous Lie algebra homomorphism, we can ask if Φ_z will be continuous, too. The answer is yes and hence our construction is functorial. For the proof, we will need the next two lemmas.

Lemma 3.10. *Let $\mathfrak{g}$ be an AE-Lie algebra, $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in \mathfrak{g}$, $i_j \in \{0, \ldots, j\}$, $\forall_{j=1,\ldots,n-1}$ and denote $I = \sum_j i_j$. Then we have the formula*

$$z^{i_{n-1}} C_{i_{n-1}} \left(\ldots z^{i_2} C_{i_2} \left(z^{i_1} C_{i_1} \left(\xi_1, \xi_2 \right), \xi_3 \right) \ldots, \xi_n \right)$$

$$= z^I B^*_{i_{n-1}} \cdots B^*_{i_1} \frac{\binom{1}{i_1}}{1!} \frac{\binom{2-i_1}{i_2}}{(2-i_1)!} \cdots \frac{\binom{n-1-i_1-\cdots-i_{n-2}}{i_{n-1}}}{(n-1-i_1-\cdots-i_{n-2})!}$$

$$\sum_{\substack{\sigma_1 \in S_{2-i_1} \\ \cdots \\ \sigma_{n-1} \in S_{n-1-i_1-\cdots-i_{n-2}}}} [w_1] \cdots [w_{n-I}], \tag{50}$$

where the expressions $[w_i]$ denote nested Lie brackets in the ξ_i.

Proof. The proof is done by induction and follows directly from Formula (27) and the bilinearitiy of the C_n. $\blacksquare$

Lemma 3.11. *Let $\mathfrak{g}$ be an AE-Lie algebra, $R \geq 1$ and $z \in \mathbb{C}$. Then for p a continuous seminorm, q an asymptotic estimate for it, $n \in \mathbb{N}$, and all $\xi_1, \ldots, \xi_n \in \mathfrak{g}$ the following estimate*

$$p_R \left(\xi_1 \star_z \cdots \star_z \xi_n \right) \leq c^n n!^R q(\xi_1) \cdots q(\xi_n) \tag{51}$$

holds with $c = 8\mathrm{e}(|z| + 1)$.

Proof. For a continuous seminorm p we have

$$p_R \left(\xi_1 \star_z \cdots \star_z \xi_n \right)$$

$$= p_R \left(\sum_{\ell=0}^{n-1} \sum_{\substack{1 \leq j \leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} z^{i_{n-1}} C_{i_{n-1}} \left(\ldots z^{i_2} C_{i_2} \left(z^{i_1} C_{i_1} \left(\xi_1, \xi_2 \right), \xi_3 \right) \ldots, \xi_n \right) \right)$$

$$\leq \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1 \leq j \leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} p^{n-\ell} \left(z^{i_{n-1}} C_{i_{n-1}} \left(\ldots z^{i_2} C_{i_2} \left(z^{i_1} C_{i_1} \left(\xi_1, \xi_2 \right), \xi_3 \right) \ldots, \xi_n \right) \right)$$

$$\overset{(a)}{\leq} \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1 \leq j \leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} |z|^\ell |B^*_{i_1}| \cdots |B^*_{i_{n-1}}|$$

$$\cdot \binom{1}{i_1} \binom{2-i_1}{i_2} \cdots \binom{n-1-i_1-\cdots-i_{n-2}}{i_{n-1}} q(\xi_1) \cdots q(\xi_n)$$

$$\overset{(b)}{\leq} \sum_{\ell=0}^{n-1}(n-\ell)!^{R} \sum_{\substack{1\leq j\leq n-1 \\ i_j\in\{0,\dots,j\} \\ \sum_{j=1}^{n-1} i_j=\ell}} |z|^{i_{n-1}}\cdots|z|^{i_1} 1^{i_1} 2^{i_2}\cdots (n-1)^{i_{n-1}} q(\xi_1)\cdots q(\xi_n). \quad (52)$$

In (a), we used Lemma 3.10 and used the AE-property. Then the inverse factorials cancel with the sums over the permutations. In (b), we used the estimate

$$|B^*_{i_j}|\binom{j-i_1-\dots-i_{j-1}}{i_j} \leq i_j!\binom{j-i_1-\dots-i_{j-1}}{i_j} = \frac{(j-i_1-\dots-i_{j-1})!}{(j-i_1-\dots-i_j)!} \leq j^{i_j}$$

for all $j=1,\dots,n-1$. Now, we estimate the number of terms in the sum and get

$$\sum_{\ell=0}^{n-1} \sum_{\substack{1\leq j\leq n-1 \\ i_j\in\{0,\dots,j\} \\ \sum_{j=1}^{n-1} i_j=\ell}} 1 \overset{(a)}{\leq} \sum_{\ell=0}^{n-1}\binom{n-1+\ell-1}{\ell-1} \overset{(b)}{\leq} 2^{2n}.$$

In (a) the estimate for the big sum is that for every $j=1,\dots,n-1$ we have $i_j\in\{0,1,\dots,n-1\}$ and the sum of all the i_j is ℓ. If we forget about all other restrictions the number of summands equals the sum of ways to distribute ℓ items on $n-1$ places, which is given by $\binom{n-1+\ell-1}{\ell-1}$. In (b) we use

$$\binom{n-1+\ell-1}{\ell-1} \leq \binom{2n}{\ell-1}$$

with the binomial coefficient being zero for $\ell=0$. By using the fact that $|B^*_m|\leq m!$ for all $m\in\mathbb{N}$ and grouping together the powers of $|z|$, we get from Inequality (52)

$$p_R\left(\xi_1\star_z\cdots\star_z\xi_n\right) \overset{(a)}{\leq} \sum_{\ell=0}^{n-1}(n-\ell)!^R \sum_{\substack{1\leq j\leq n-1 \\ i_j\in\{0,\dots,j\} \\ \sum_{j=1}^{n-1} i_j=\ell}} |z|^\ell n^\ell q(\xi_1)\cdots q(\xi_n)$$

$$\overset{(b)}{\leq} \sum_{\ell=0}^{n-1}(n-\ell)!^R |z|^\ell \mathrm{e}^n 2^n \ell! q(\xi_1)\cdots q(\xi_n) \sum_{\substack{1\leq j\leq n-1 \\ i_j\in\{0,\dots,j\} \\ \sum_{j=1}^{n-1} i_j=\ell}} 1$$

$$\overset{(c)}{\leq} n!^R(|z|+1)^n(8\mathrm{e})^n q(\xi_1)\cdots q(\xi_n).$$

We used

$$1^{i_1}\cdots(n-1)^{i_{n-1}}\leq n^\ell,$$

since $\sum_{j=1}^{n-1} i_j=\ell$ in (a) and $n^\ell\leq \mathrm{e}^n\frac{n!}{(n-\ell)!}=\mathrm{e}^n\binom{n}{\ell}\ell!\leq \mathrm{e}^n 2^n\ell!$ in (b). The last step (c) is just $(n-\ell)!\ell!\leq n!$, the estimate for the sum, and $(|z|+1)\geq|z|$, which finishes the proof. ∎

Proposition 3.12. *Let $R \geq 1$, let $\mathfrak{g}, \mathfrak{h}$ be AE-Lie algebras and let $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ be a continuous homomorphism between them. Then it lifts to a continuous unital homomorphism of locally convex algebras $\Phi_z\colon \mathscr{U}_R(\mathfrak{g}_z) \longrightarrow \mathscr{U}_R(\mathfrak{h}_z)$ for all $z \in \mathbb{K}$.*

Proof. First, if $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ is continuous, then for every continuous seminorm q on $\mathfrak{h}$, we have a continuous seminorm r on $\mathfrak{g}$ such that for all $\xi \in \mathfrak{g}$

$$q\left(\phi(\xi)\right) \leq r(\xi).$$

Second, we define Ψ_z by

$$\Psi_z\colon \mathrm{T}_R^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}_R^\bullet(\mathfrak{h}), \quad \Psi_z = \widetilde{\Phi}_z \circ \mathscr{S}$$

as before. Clearly, Φ_z and $\widetilde{\Phi}_z$ will be continuous if Ψ_z is continuous. From this, we get for a seminorm p on $\mathfrak{h}$, an asymptotic estimate q for it, and $\xi_1, \ldots, \xi_n$

$$
\begin{aligned}
p_R\left(\Psi_z\left(\xi_1 \otimes \ldots \otimes \xi_n\right)\right) &= p_R\left(\phi\left(\xi_1\right) \star_z \ldots \star_z \phi\left(\xi_1\right)\right) \\
&\overset{(a)}{\leq} c^n n!^R q\left(\phi\left(\xi_1\right)\right) \ldots q\left(\phi\left(\xi_n\right)\right) \\
&\overset{(b)}{\leq} c^n n!^R r\left(\xi_1\right) \ldots r\left(\xi_n\right) \\
&= (cr)_R\left(\xi_1 \otimes \ldots \otimes \xi_n\right).
\end{aligned}
$$

Again, we use the infimum argument and we have the estimate on all tensors in $\mathrm{T}_R^\bullet(\mathfrak{g})$. It extends to the completion and the statement is proven. $\blacksquare$

4. Nilpotent Lie Algebras

Let us now consider nilpotent locally convex Lie algebras. Our results are still valid in this case but we can make some more observations. On one hand, things should not change drastically for nilpotent Lie algebras; in fact, Example 3.3 shows that even for nilpotent Lie algebras the star product is not continuous for $R < 1$. Thus, there is no reason to expect much larger algebras and completions. On the other hand, the Weyl algebra studied in [33] is a quotient of the Heisenberg algebra and has a continuous product for $R \geq \frac{1}{2}$. The quotient procedure must therefore have some influence on the estimates. Finally, the fact that exponentials are not in $\widehat{\mathrm{S}}_1^\bullet(\mathfrak{g})$ is not unexpected. If so, Equation (17) would mean that could give some sense to $\mathrm{BCH}\left(\xi, \eta\right)$ for all ξ, η from some arbitrary Lie algebra $\mathfrak{g}$, which would be surprising. In the nilpotent case, this is no longer the case since the Baker-Campbell-Hausdorff series converges globally. So it would be nice to have something more than $R \geq 1$.

4.1. A Projective Limit.

In the following we show that the Gutt star product is continuous in the projective limit $R \longrightarrow R^-$ that and the exponential actually belongs to the completion.

Proposition 4.1. *Let $\mathfrak{g}$ be a nilpotent locally convex Lie algebra with continuous Lie bracket and $N \in \mathbb{N}$ such that $N + 1$ Lie brackets vanish.*

i.) If $0 \le R < 1$, the C_n-operators are continuous and fulfil the estimate

$$p_R\left(C_n(x, y)\right) \le \frac{1}{2 \cdot 8^n}(32\mathrm{e}q)_{R+\epsilon}(x)(32\mathrm{e}q)_{R+\epsilon}(y), \qquad (53)$$

for all $x, y \in \mathrm{S}_R^\bullet(\mathfrak{g})$, where p is a continuous seminorm, q an asymptotic estimate for p, and $\epsilon = \frac{N-1}{N}(1 - R)$.

ii.) The Gutt star product $\star_z$ is continuous for the locally convex projective limit $\mathrm{S}_{1-}^\bullet(\mathfrak{g})$ and we have

$$p_R\left(x \star_z y\right) \le (cq)_{R+\epsilon}(x)(cq)_{R+\epsilon}(y) \qquad (54)$$

with $c = 32\mathrm{e}(|z| + 1)$. Hence it extends continuously to $\widehat{\mathrm{S}}_R^\bullet(\mathfrak{g})$, where it coincides with the formal series.

Proof. In this proof, we use $\star_z$ on the whole tensor algebra and compute the estimate for factorizing tensors $\xi_1 \otimes \cdots \otimes \xi_k$ and $\eta_1 \otimes \cdots \otimes \eta_\ell$ but in this case we get restrictions for the values of n. Recall that $C_n\left(\xi^{\otimes k}, \eta^{\otimes \ell}\right)$ has n brackets it has degree $k + \ell - n$. We get therefore

$$(k + \ell - n)N \ge k + \ell \quad \Longleftrightarrow \quad n \le (k + \ell)\frac{N-1}{N}.$$

For all n that violate this condition, $C_n\left(\xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_\ell\right) = 0$. Hence, we get bounds for $n!^{1-R}$ in Equation (41): set $\delta = \frac{N-1}{N}$ and also denote a factorial where we have non-integers, meaning the gamma function. We get

$$\begin{aligned}
n!^{1-R} &\le (\delta(k + \ell)!)^{1-R} \\
&\le (\delta(k + \ell))^{(1-R)\delta(k+\ell)} \\
&\le (k + \ell)^{(1-R)\delta(k+\ell)} \\
&= \left((k + \ell)^{(k+\ell)}\right)^{(1-R)\delta} \\
&\le \left(\mathrm{e}^{k+\ell}2^{k+\ell}k!\ell!\right)^{(1-R)\delta} \\
&= \left((2\mathrm{e})^{\delta(1-R)}\right)^{k+\ell} k!^\epsilon \ell!^\epsilon,
\end{aligned}$$

using $\epsilon = \delta(1-R)$. Then

$$p_R \left(C_n \left(\xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_\ell \right) \right)$$

$$\leq \frac{\left((2\mathrm{e})^{\delta(1-R)} \right)^{k+\ell} k! ^\epsilon \ell! ^\epsilon}{2 \cdot 8^n} (16q)_R \left(\xi_1 \otimes \cdots \otimes \xi_k \right) (16q)_R \left(\eta_1 \otimes \cdots \otimes \eta_\ell \right)$$

$$\leq \frac{1}{2 \cdot 8^n} (cq)_{R+\epsilon} \left(\xi_1 \otimes \cdots \otimes \xi_k \right) (cq)_{R+\epsilon} \left(\eta_1 \otimes \cdots \otimes \eta_\ell \right)$$

with $c = 16(2\mathrm{e})^{\delta(1-R)} \leq 32\mathrm{e}$. We then get the estimate on all tensors, extend it to the completion and get the estimate (53). Recall, that for every $R < 1$ we also have $R + \epsilon < 1$ with the $\epsilon = \delta(1-R)$ from above. Iterating this continuity estimate, we get arbitrarily close to 1 and it is not possible to repeat this process an arbitrary number of times and stop at some value strictly less than 1. The proof of Equation (54) is done analogously to the proof of part *ii.)* of Proposition 3.2. ∎

Remark 4.2. Assuming even submultiplicativity of the seminorms one ends up with the same result by using the easier formula (27).

Of course, the projective limit case gives us a slightly bigger completion. We immediately get the following result:

Corollary 4.3. *Let $\mathfrak{g}$ be a nilpotent, locally convex Lie algebra.*
 i.) For $\xi \in \mathfrak{g}$, the we have $\exp(\xi) \in \widehat{\mathrm{S}}^\bullet_{1^-}(\mathfrak{g})$.
 ii.) For $\xi, \eta \in \mathfrak{g}$ and $z \neq 0$ we have $\exp(\xi) \star_z \exp(\eta) = \exp\left(\frac{1}{z} \mathrm{BCH}\left(z\xi, z\eta \right) \right)$.
 iii.) For $s, t \in \mathbb{K}$ and $\xi \in \mathfrak{g}$ we have $\exp(t\xi) \star_z \exp(s\xi) = \exp((t+s)\xi)$.

Proof. For the first part, recall that the completion of the projective limit 1^- contains all those series $x = \sum_{n=0}^\infty x_n$ with $x_n \in \mathrm{S}^n(\mathfrak{g})$ such that

$$\sum_{n=0}^\infty p^n(x_n) n!^{1-\epsilon} < \infty$$

for all continuous seminorms p and all $\epsilon > 0$. This is the case for the exponential series of $t\xi$ for $t \in \mathbb{K}$ and $\xi \in \mathfrak{g}$. The second part follows from the fact that all the projections π_n onto the homogeneous subspaces S^n_π are continuous. The third part is then a direct consequence of the second. ∎

4.2. Functoriality and Representations.
In the nilpotent case we get the same results for the extension of maps from $\mathfrak{g}$ into associative AE-algebras and the same functorial properties in this case. We just need to adapt Lemma 3.11:

Lemma 4.4. *Let $\mathfrak{g}$ be locally convex nilpotent Lie algebra, $0 \leq R < 1$ and $z \in \mathbb{C}$. Then for a continuous seminorm p with an asymptotic estimate p, $n \in \mathbb{N}$ and all $\xi_1, \ldots, \xi_n \in \mathfrak{g}$ the following estimate*

$$p_R \left(\xi_1 \star_z \cdots \star_z \xi_n \right) \leq c^n n!^{R+\epsilon} q^n (\xi_1 \otimes \cdots \otimes \xi_n) \tag{55}$$

holds with $c = 16\mathrm{e}^2(|z| + 1)$ and $\epsilon = \frac{N-1}{N}(1 - R)$. The estimate is locally uniform in z.

Proof. Take $R < 1$ and consider the estimate (52) in the proof of Lemma 3.11. We know that, since we can have at most N brackets, also the values for ℓ are restricted to

$$\ell \leq \frac{N-1}{N}n = \delta n.$$

Using the same steps of the proof of Lemma 3.11 we get

$$p_R \left(\xi_1 \star_z \cdots \star_z \xi_n \right) \leq \sum_{\ell=0}^{\delta n} (n-\ell)!^R \sum_{\substack{1 \leq j \leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} |z|^\ell n^\ell q(\xi_1) \cdots q(\xi_n)$$

$$\leq \sum_{\ell=0}^{\delta n} (n-\ell)!^R \sum_{\substack{1 \leq j \leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} |z|^\ell (2\mathrm{e})^n \ell! q(\xi_1) \cdots q(\xi_n)$$

$$\leq (2\mathrm{e})^n (|z| + 1)^n q(\xi_1) \cdots q(\xi_n) \sum_{\ell=0}^{\delta n} (n-\ell)!^R \ell! \binom{n+\ell-2}{\ell-1}.$$

We have the inequality

$$\ell! = \ell!^R \ell!^{1-R} \leq \ell!^R \left((\delta n)^{\delta n} \right)^{1-R} \leq \ell!^R n^{\delta n(1-R)} \leq \ell!^R n!^{\delta(1-R)} \mathrm{e}^{\delta n(1-R)}$$

and together with $\ell!^R (n-\ell)!^R \leq n!^R$ this gives

$$p_R \left(\xi_1 \star_z \cdots \star_z \xi_n \right)$$

$$\leq (2\mathrm{e})^n (|z| + 1)^n n!^R n!^{\delta(1-R)} q(\xi_1) \cdots q(\xi_n) \sum_{\ell=0}^{\delta n} \binom{n+\ell-2}{\ell-1} \mathrm{e}^{\delta n(1-R)}$$

$$\leq (2\mathrm{e})^n (|z| + 1)^n \left(\mathrm{e}^{(1-R)\delta} \right)^n 4^n n!^{R+\epsilon} q(\xi_1) \cdots q(\xi_n),$$

with $\epsilon = \delta(1 - R)$. It is clear that for all $R < 1$ we have $R + \epsilon < 1$. To complete the proof we set

$$c = 8\mathrm{e}(|z| + 1)\mathrm{e}^{(1-R)\delta} \leq 16\mathrm{e}^2(|z| + 1),$$

and notice that the estimate is locally uniform in z, even though it is not uniform in z. ∎

Proposition 4.5. *Let $\mathfrak{g}$ be a nilpotent locally convex Lie algebra with continuous Lie bracket. Then the statements of Proposition 3.7, Corollary 3.9, and Proposition 3.12 hold for the projective limit $S_{1-}^\bullet(\mathfrak{g})$, too.*

4.3. The Link to the Weyl Algebra.

In this section we aim to discuss the link to the Weyl algebra from [33]. For simplicity, we consider the easiest case of the Weyl algebra with two generators. Recall that the Weyl algebra is a quotient of the enveloping algebra of the Heisenberg algebra $\mathfrak{h}$ which one gets from dividing out its center. So let $c \in \mathbb{K}$ and we have a projection

$$\pi \colon S_R^\bullet(\mathfrak{h}) \longrightarrow \mathcal{W}_R(\mathfrak{h}) = \left(\frac{S_R^\bullet(\mathfrak{h})}{\langle E - c\mathbf{1} \rangle} \right) \tag{56}$$

Proposition 4.6. *The projection π is continuous for $R \geq 0$.*

Proof. We extend π to the whole tensor algebra by symmetrizing beforehand. Let then p be a continuous seminorm on $\mathfrak{h}$, $k, \ell, m \in \mathbb{N}_0$. We have

$$
\begin{aligned}
p_R(\pi(Q^{\otimes k} \otimes P^{\otimes \ell} \otimes E^{\otimes m})) &= p_R(Q^k P^\ell c^m) \\
&= |c|^m (k + \ell)!^R p^{k+\ell}(Q^k P^\ell) \\
&\leq (|c| + 1)^{k+\ell+m} (k + \ell + m)!^R p(Q)^k p(P)^\ell p(E)^m \\
&= ((|c| + 1)p)_R(Q^{\otimes k} \otimes P^{\otimes \ell} \otimes E^{\otimes m}).
\end{aligned}
$$

Then we do the usual infimum argument and have the result on arbitrary tensors again. ∎

To establish the link to the continuity results of the Weyl algebra, we need $\pi \circ \star_z$ to be continuous for $R \geq \frac{1}{2}$.

Proposition 4.7. *Let $R \geq \frac{1}{2}$ and π the projection from Equation (56). Then the map $\pi \circ \star_z$ is continuous.*

Proof. Since we are in finite dimensions, we can choose a submultiplicative norm p with $p(Q) = p(P) = p(E)$ without restriction. Moreover, let $k, k', \ell, \ell', m, m' \in \mathbb{N}_0$. Then we have to get an estimate for $p_R\left(\pi \left(Q^k P^\ell E^m \star_z Q^{k'} P^{\ell'} E^{m'} \right) \right)$. If we calculate the star product explicitly, we see, that we only get Lie brackets where we have P's and Q's. Let $r = k + \ell + m$ and $s = k' + \ell' + m'$, then we can

actually simplify the calculations by

$$p_R\left(\pi(Q^k P^\ell E^m \star_z Q^{k'} P^{\ell'} E^{m'})\right)$$

$$= (p_R \circ \pi)\left(\sum_{n=0}^{r+s-1} z^n C_n(Q^k P^\ell E^m, Q^{k'} P^{\ell'} E^{m'})\right)$$

$$\leq \sum_{n=0}^{r+s-1} |z|^n (p_R \circ \pi)\left(C_n(Q^k P^\ell E^m, Q^{k'} P^{\ell'} E^{m'})\right)$$

$$\leq \sum_{n=0}^{r+s-1} |z|^n (p_R \circ \pi)\left(C_n(Q^r, P^s)\right)$$

$$= \sum_{n=0}^{r+s-1} |z|^n \frac{r!s!}{(r-n)!(s-n)!n!}(p_R \circ \pi)\left(Q^{r-n} P^{s-n} E^n\right)$$

$$= \sum_{n=0}^{r+s-1} |z|^n |c|^n \frac{r!s!}{(r-n)!(s-n)!n!} p_R\left(Q^{r-n} P^{s-n}\right)$$

$$\leq \sum_{n=0}^{r+s-1} |z|^n |c|^n \frac{r!s!}{(r-n)!(s-n)!n!} \frac{(r+s-2n)!^R}{r!^R s!^R} p_R\left(Q^{\otimes r}\right) p_R\left(P^{\otimes s}\right)$$

$$\leq \sum_{n=0}^{r+s-1} |z|^n |c|^n \binom{r}{n}\binom{s}{n} \frac{(r+s-2n)!^R n!}{r!^R s!^R} p_R\left(Q^{\otimes r}\right) p_R\left(P^{\otimes s}\right)$$

$$\leq \sum_{n=0}^{r+s-1} |z|^n |c|^n \binom{r}{n}\binom{s}{n} \frac{(r+s-2n)!^R n!}{r!^R s!^R} p_R\left(Q^{\otimes r}\right) p_R\left(P^{\otimes s}\right)$$

$$\stackrel{(a)}{\leq} \sum_{n=0}^{r+s-1} |z|^n |c|^n \binom{r}{n}\binom{s}{n}\binom{r+s}{s}^R \binom{r+s}{2n}^{-R} p_R\left(Q^{\otimes r}\right) p_R\left(P^{\otimes s}\right)$$

$$\leq \sum_{n=0}^{r+s-1} (|z|+1)^n (|c|+1)^n 4^{r+s} p_R\left(Q^{\otimes r}\right) p_R\left(P^{\otimes s}\right)$$

$$\stackrel{(b)}{\leq} \underbrace{(8(|z|+1)(|c|+1))^{r+s}}_{=\tilde{c}^{r+s}} p_R\left(Q^{\otimes r}\right) p_R\left(P^{\otimes s}\right)$$

$$= (\tilde{c}p)_R\left(Q^{\otimes r}\right)(\tilde{c}p)_R\left(P^{\otimes s}\right)$$

$$\stackrel{(c)}{=} (\tilde{c}p)_R\left(Q^{\otimes k} \otimes P^{\otimes \ell} \otimes E^{\otimes m}\right)(\tilde{c}p)_R\left(Q^{\otimes k'} \otimes P^{\otimes \ell'} \otimes E^{\otimes m'}\right),$$

where in we rearranged the factorials in (a) and used $R \geq \frac{1}{2}$. The estimates (b) are the standard binomial coefficient estimates. In (c) we used $p(Q) = p(P) = p(E)$.

Now we just use

$$\left(Q^{\otimes k} \otimes P^{\otimes \ell} \otimes E^{\otimes m}\right) \star_z \left(Q^{\otimes k'} \otimes P^{\otimes \ell'} \otimes E^{\otimes m'}\right) = Q^k P^\ell E^m \star_z Q^{k'} P^{\ell'} E^{m'}$$

and the infimum argument to expand this estimate to all tensors. This concludes the proof. ∎

5. The Hopf Algebra Structure

In a last step we investigate the continuity of the Hopf algebra structure maps on $\mathscr{U}_R(\mathfrak{g}_z)$. Pulling back the coproduct Δ_z and the antipode S_z from $\mathscr{U}_R(\mathfrak{g}_z)$ to $\mathrm{S}^\bullet(\mathfrak{g})$ we get a coproduct Δ and an antipode S with respect to $\star_z$. It is well-known that Δ and S are independent of z and coincide with the classical shuffle coproduct and antipode which on the symmetric algebra can be written as follows:

Lemma 5.1.   *For $\xi_1, \ldots, \xi_n \in \mathfrak{g}$ we have the identities*

$$S(\xi_1 \otimes \cdots \otimes \xi_n) = (-1)^n \xi_1 \cdots \xi_n \tag{57}$$

and

$$\Delta(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{I \subseteq \{1,\ldots,n\}} \xi_I \otimes \xi_1 \cdots \widehat{\xi_I} \cdots \xi_n \tag{58}$$

where ξ_I denotes the symmetric tensor product of all ξ_i with $i \in I$ and $\widehat{\xi_I}$ means that the ξ_i with $i \in I$ are left out.

We need a topology on the tensor product in (58), for which we take again the projective tensor product. The continuity of the two maps is now easy to prove. In fact, it does not refer to the Lie structure at all and holds in full generality:

Proposition 5.2.   *Let V be a locally convex vector space and let $R \geq 0$. For every continuous seminorm p and all $x \in \widehat{\mathrm{S}}^\bullet_R(V)$ the following estimates hold:*

$$p_R\left(S(x)\right) \leq p_R(x) \tag{59}$$

and

$$(p_R \otimes p_R)\left(\Delta(x)\right) \leq (2p)_R(x). \tag{60}$$

Proof.   We use the extension to the whole tensor algebra by symmetrizing beforehand. Inequality (59) is clear on factorizing tensors and extends to all tensors by the infimum argument. To get the estimate (60), we compute it on factorizing tensors:

$$(p_R \otimes p_R)\left(\Delta\left(\xi_1 \otimes \cdots \otimes \xi_n\right)\right)$$

$$= (p_R \otimes p_R) \left(\sum_{I \subseteq \{1,\dots,n\}} \xi_I \otimes \xi_1 \cdots \widehat{\xi_I} \cdots \xi_n \right)$$

$$\leq \sum_{I \subseteq \{1,\dots,n\}} |I|!^R (n - |I|)!^R p^{|I|} (\xi_I) \, p^{n-|I|} \left(\xi_1 \cdots \widehat{\xi_I} \cdots \xi_n \right)$$

$$\leq \sum_{I \subseteq \{1,\dots,n\}} |I|!^R (n - |I|)!^R p(\xi_1) \cdots p(\xi_n)$$

$$\leq \sum_{I \subseteq \{1,\dots,n\}} n!^R p(\xi_1) \cdots p(\xi_n)$$

$$= 2^n n!^R p(\xi_1) \cdots p(\xi_n)$$

$$= (2p)_R \left(\xi_1 \otimes \cdots \otimes \xi_n \right).$$

This extends to all tensors by the infimum argument and restricts to the symmetric algebra afterwards. ∎

Since the continuity of the unit and the counit $\epsilon = \pi_0$ is clear by the definition of our topology, we have the following result.

Proposition 5.3. *Let $\mathfrak{g}$ be an AE-Lie algebra and $z \in \mathbb{C}$. Then, if $R \geq 1$, $\widehat{\mathrm{S}}_R^\bullet(\mathfrak{g})$ is a topological Hopf algebra. The same holds for $\widehat{\mathrm{S}}_{1-}^\bullet(\mathfrak{g})$, if $\mathfrak{g}$ is a nilpotent locally convex Lie algebra with continuous Lie bracket.*

6. Outlook and Open Questions

The two main theorems allow us to make some observations, which also go beyond deformation quantization. We found a locally convex topology on the universal enveloping algebra and encountered a special class of AE(-Lie) algebras. It is worth looking at those two issues more closely.

6.1. Asymptotic Estimate Algebras.

The term asymptotic estimate has, to the best of our knowledge, first been used by Boseck, Czichowski and Rudolph in [10]. Their definition of an AE-algebra differs from ours since not just one but a whole series of asymptotic estimates has to exist for every seminorm. This series has to fulfil two technical properties. This is not the case in our definition, which is, in general, weaker.

Furthermore, in [18], Glöckner and Neeb use a property to which they referred as $(*)$ for associative algebras. This was then used in [8] by Bogfjellmo and Dahmen and Schmedig, who called it the GN-property. It is easy to see that it is equivalent to being AE.

As we already know, there are a lot of examples of these structures, like C^*-algebras, Banach algebras and more generally: locally multiplicatively convex

algebras. Recall that a locally convex algebra (either associative or Lie) is called *locally multiplicatively convex* if there exists a defining system of seminorms such that for all seminorms p of this system one has

$$p(x \cdot y) \leq p(x)p(y) \tag{61}$$

for all $x, y \in \mathscr{A}$. Clearly, this implies (2). The associative locally multiplicatively convex algebras have been discussed in detail by Michael [24]. Clearly, finite-dimensional (Lie) algebras are Banach and hence locally multiplicatively convex.

Also, all nilpotent locally convex (Lie) algebras belong to this category, since one just needs to take the maximum of a finite number of seminorms. So far, we did not find an example for an AE(-Lie) algebra, which was neither locally multiplicatively convex nor nilpotent and it seems to be a non-trivial question, whether such an algebra exists at all.

Are there AE(-Lie) algebras, which are not already locally multiplicatively convex or nilpotent?

In some special cases, however, we can give an answer to this question: every associative AE algebra admits an entire holomorphic calculus in the following sense: let $\mathscr{A}$ be a complete associative AE algebra and let

$$f \colon \mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function, then for every $x \in \mathscr{A}$ and every continuous seminorm p with asymptotic estimate q we have

$$p\left(f(x)\right) = p\left(\sum_{n=0}^{\infty} a_n x^n\right) \leq \sum_{n=0}^{\infty} |a_n| p\left(x^n\right) \leq \sum_{n=0}^{\infty} |a_n| q(x)^n < \infty. \tag{62}$$

Thus $f(x) \in \mathscr{A}$ is defined for every x and every entire function f, obeying the usual rules of a functional calculus. In this sense, $\mathscr{A}$ behaves very much like a locally multiplicatively convex algebra. If in addition, $\mathscr{A}$ is commutative and Fréchet, we get an answer to our question by using a result [25] due to Mitiagin, Rolewicz and Żelazko. They showed that an associative, commutative Fréchet algebra admitting an entire calculus *is* actually locally multiplicatively convex.

For non-commutative algebras, the situation is different. There are associative Fréchet algebras admitting an entire holomorphic calculus which are not locally multiplicatively convex. In [34], Żelazko gave an example of such an algebra. However, his example is also not AE.

In the Lie case, not much seems to be known about the relation between AE-Lie algebras and locally multiplicatively convex Lie algebras.

6.2. Another Topology on the Universal Enveloping Algebra.

We also imposed a locally convex topology on the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, since we can just pull back the topology on $S_R^\bullet(\mathfrak{g})$ with the Poincaré-Birkhoff-Witt isomorphism. It is not the only one on $\mathcal{U}(\mathfrak{g})$ which is reasonable: in [28] Schottenloher and Pflaum mention another locally convex topology in the case of a finite-dimensional Lie algebra, which we will call τ for now. They take the coarsest locally convex topology, such that all finite-dimensional representations of $\mathfrak{g}$ extend to continuous algebra homomorphisms. This topology is in fact even locally multiplicatively convex and therefore different from ours. It is easy to show, that our topology is finer.

Proposition 6.1. *Let $R \geq 1$ and $\mathscr{U}_R(\mathfrak{g})$ the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$, then the S_R-topology is strictly finer than τ.*

Proof. From Corollary 3.9 we know that if $\mathfrak{g}$ is finite-dimensional, all finite-dimensional representations will extend to continuous algebra homomorphisms. Since τ was the coarsest topology such that this is the case, the S_R-topology must be finer. It is strictly finer, since the S_R-topology does not have an entire calculus for $R \geq 1$ and can hence not be locally multiplicatively convex. ∎

For representations on infinite-dimensional Banach or Hilbert spaces, the statement is less important, since there one rarely has norm-continuous representations, but merely strongly continuous ones.

Remark 6.2. One could argue, that a topology which is locally multiplicatively convex is much more useful than a "just locally convex" one. Of course, submultiplicativity of the seminorms allows for more constructions like an entire calculus, but there are good reasons why the S_R-topology also has its advantages: first of all, it is also defined for infinite-dimensional Lie algebras with reasonable functorial properties. Second, the topology build upon representations does not allow to have a continuous deformation of $S^\bullet(\mathfrak{g})$, at least in a similar way: the canonical projections $\pi_n \colon S^\bullet(\mathfrak{g}) \longrightarrow S^n(\mathfrak{g})$ from Proposition 2.2 can be shown to be discontinuous with respect to τ. For $z = 1$, $\xi, \eta \in \mathfrak{g}$ and π_1 being the projection onto the Lie algebra itself, we get

$$\pi_1\left(\exp(\xi) \star \exp(\eta)\right) = \mathrm{BCH}\left(\xi, \eta\right), \tag{63}$$

if we assume the product and the projection to be convergent. With respect to τ, the exponential series $\exp(\xi)$ exists for all $\xi \in \mathfrak{g}$ and therefore the Baker-Campbell-Hausdorff series must exist for all $\xi, \eta \in \mathfrak{g}$. Since this is not the case in a generic Lie algebra, we get a contradiction.

6.3. Possible Applications and Generalizations.

First we note that it is just a matter of collecting signs to extend our results to super Lie algebras. Here the analysis part is robust enough as the signs will not alter the estimates. We do not formulate the details here.

More interesting is the application of our result to deformation quantization, say in finite dimensions for simplicity. While the Gutt star product itself may be seen as yet rather simple, it encodes quantizations of coadjoint orbits: from [11] one knows that the Gutt star product does not just restrict to coadjoint orbits in general, even though there are interesting exceptional cases. For these cases, we immediately get convergent star products on coadjoint orbits:

Theorem 6.3.   *Let G be a finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ and $\mathcal{O} \subseteq \mathfrak{g}^*$ a coadjoint orbit to which the Gutt star product is tangential. Then the Gutt star product restricts to a continuous star product on $\mathcal{O}$.*

Proof.   Here we use now that $S_R(\mathfrak{g})$ can be identified with the polynomial functions $\mathrm{Pol}(\mathfrak{g}^*)$ on $\mathfrak{g}^*$. Now since the evaluation functionals at points in $\mathfrak{g}^*$ are continuous by Proposition 2.2, *x.)*, the vanishing ideal of the orbit is a closed subspace of $S_R(\mathfrak{g})$. Since we assume that the Gutt star product is tangential, it is also a two-sided ideal with respect to $\star_z$. Thus, on the one hand, the quotient inherits the product and, on the other hand, it inherits a Hausdorff locally convex topology for which the product is continuous. We then can complete again to get a holomorphic deformation of those functions on the orbit which are obtained by restricting functions on $\mathfrak{g}^*$ in the completion of $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ with respect to the S_R-topology to the orbit.   ∎

However, even if the Gutt star product is not tangential, there is some hope to get a star product on coadjoint orbits for which the convergence can be controlled: in [6, Thm. 5.2] a construction of a deformed restriction map $\boldsymbol{\iota} = \iota + \cdots$ of functions on $\mathfrak{g}^*$ to functions on a coadjoint orbit $\iota \colon \mathcal{O} \longrightarrow \mathfrak{g}^*$ was given, provided the Lie algebra $\mathfrak{g}$ is compact and the orbit is regular. It will be left to a future project to investigate the behaviour of the deformed restriction map with respect to polynomial functions and their S_R-topology. Beside this fairly general construction, it seems also plausible to generalize the restriction procedure to particular other cases.

A.   Proofs of the Propositions 2.3 and 2.4

Since we used the results from Proposition 2.3 and Proposition 2.4, we need to give a proof that that they hold for any (also infinite-dimensional) Lie algebra $\mathfrak{g}$. Our analysis is based on a careful investigation of the BCH series which can be

found in the literature at many places, see e.g. [15, part 2.8.12] or [22, Chap. 5].

To avoid any possible confusion, we define three different star products.

Definition A.1.   Let $\mathfrak{g}$ be a Lie algebra and $z \in \mathbb{C}$. Then we define the Gutt star product $\star_z$ via (16), the deformed PBW star product $\widehat{\star}_z$ via (19) and the Drinfel'd star product $*_z$ via (17).

We need to prove that all of them are identical. Therefore, it is enough to show that they coincide on $\xi^n \star \eta$ for $\xi, \eta \in \mathfrak{g}$ and any $n \in \mathbb{N}$, since those terms generate the whole algebra. We get equality for $\xi_1 \ldots \xi_n \star \eta$ by polarization and then for $\xi_1 \ldots \xi_n \star \eta_1 \ldots \eta_\ell$ by iteration.

Recall that we have $\star_z = \widehat{\star}_z$ if $z = 1$ from the construction, since $\mathfrak{q} = \mathfrak{q}_1$. In Proposition 2.7 we have sketched the proof of a formula for $*_z$:

$$\xi^k *_z \eta = \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} (\mathrm{ad}_\xi)^j (\eta). \tag{64}$$

We first show $*_z = \widehat{\star}_z$.

Proposition A.2.   *Let $\mathfrak{g}$ be a Lie algebra, $k \in \mathbb{N}$ and $\xi, \eta \in \mathfrak{g}$. Then*

$$\xi^k \widehat{\star}_z \eta = \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} (\mathrm{ad}_\xi)^j (\eta). \tag{65}$$

Proof.   We need to show

$$\mathfrak{q}_z \left(\sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} (\mathrm{ad}_\xi)^j (\eta) \right) = \xi^k \cdot \eta$$

in $\mathscr{U}(\mathfrak{g}_z)$. We divide it into two lemmata.

Lemma A.3.   *Let $\xi, \eta \in \mathfrak{g}$ and $k \in \mathbb{N}$. Then we have*

$$\mathfrak{q}_z \left(\sum_{n=0}^{k} z^n \binom{k}{n} B_n^* \xi^{k-n} (\mathrm{ad}_\xi)^n (\eta) \right) = \sum_{s=0}^{k} \mathcal{K}(k,s) \xi^{k-s} \cdot \eta \cdot \xi^s$$

with

$$\mathcal{K}(k,s) = \frac{1}{k+1} \sum_{n=0}^{k} \binom{k+1}{n} B_n^* \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{\ell=0}^{k-n} \delta_{s, \ell+j}$$

Proof. We need the identities

$$\mathfrak{q}_z\left(\xi^k\eta\right) = \frac{1}{k+1}\sum_{j=0}^{k}\xi^{k-l}\cdot\eta\cdot\xi^k$$

and

$$\mathfrak{q}_z\left((z^n\,\mathrm{ad}_\xi)^k(\eta)\right) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\xi^{k-j}\cdot\eta\cdot\xi^j$$

which are easy to check. Since the map $\mathfrak{q}_z$ is linear, we can pull out the constants and get

$$\mathfrak{q}_z\left(\sum_{n=0}^{k}z^n\binom{k}{n}B_n^*\xi^{k-n}(\mathrm{ad}_\xi)^k(\eta)\right) = \sum_{n=0}^{k}\binom{k}{n}B_n^*\mathfrak{q}_z\left(z^n\xi^{k-n}(\mathrm{ad}_\xi)^k(\eta)\right).$$

Now we just have to use the two equalities to get

$$\sum_{n=0}^{k}\binom{k}{n}B_n^*\mathfrak{q}_z(z^n\xi^{k-n}(\mathrm{ad}_\xi)^k(\eta))$$

$$= \sum_{n=0}^{k}\binom{k}{n}\frac{B_n^*}{k-n+1}\sum_{\ell=0}^{k-n}\xi^{k-n-\ell}\cdot\left(\sum_{j=0}^{n}(-1)^j\binom{n}{j}\xi^{n-j}\cdot\eta\cdot\xi^j\right)\cdot\xi^\ell$$

$$= \frac{1}{k+1}\sum_{n=0}^{k}\binom{k+1}{n}B_n^*\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{\ell=0}^{k-n}\xi^{k-\ell-j}\cdot\eta\cdot\xi^{\ell+j}$$

and collect the terms for which we have $\ell + j = s$. ∎

Now we have to show the following statement:

Lemma A.4. *Let $\mathcal{K}(k,s)$ be defined as in Lemma A.3, then we have for all $k\in\mathbb{N}$*

$$\mathcal{K}(k,s) = \begin{cases} 1 & s=0 \\ 0 & else \end{cases}$$

Proof. We need some standard identities on binomial coefficients and the recursive definition of the Bernoulli numbers. Besides that, we have to use the equality

$$(-1)^k\sum_{j=0}^{k}\binom{k}{j}B_{m+j} = (-1)^m\sum_{i=0}^{m}\binom{m}{i}B_{k+i}$$

which was proven by Carlitz in [12]. We take $k \in \mathbb{N}$ and prove the lemma by induction over s. It is straightforward to see that $\mathcal{K}(k,0) = 1$. The induction starts at $s = 1$:

$$\mathcal{K}(k,1) = \underbrace{\frac{1}{k+1}}_{n=0} - \underbrace{kB_k^*}_{n=k} + \frac{1}{k+1} \sum_{n=1}^{k-1} \binom{k+1}{n} B_n^* \left(1 + (-1)\binom{n}{1}\right)$$

$$= \underbrace{\frac{1}{k+1} \sum_{n=0}^{k-1} \binom{k+1}{n} B_n^*}_{=1-B_k^*} - \frac{1}{k+1} \sum_{n=0}^{k} \binom{k+1}{n} nB_n^*$$

$$= 1 - B_k^* - k - 1 + \sum_{n=0}^{k-1} \binom{k}{n} B_n^* - B_k^*$$

$$= 0.$$

Now we can do the step $s \to s+1$. It is sufficient to prove $\mathcal{K}(k,s+1) - \mathcal{K}(k,s) = 0$. For this purpose, we rewrite $\mathcal{K}(k,s)$ and we find

$$\mathcal{K}(k,s) = \frac{1}{k+1} \left[1 + \sum_{n=k+1-s}^{s} \binom{k+1}{n} B_n (-1)^{k-s} \binom{n-1}{k-s} \right.$$

$$\left. + \sum_{n=s+1}^{k} \binom{k+1}{n} B_n \left((-1)^{k-s} \binom{n-1}{k-s} + (-1)^{n+s} \binom{n-1}{s} \right) \right].$$

We get rid of the factor in front by multiplying with $(k+1)$ and finally get after some combinatorial manipulations

$$(k+1)\left(\mathcal{K}(k,s) - \mathcal{K}(k,s+1)\right) = \sum_{n=k-s}^{k} \binom{k+1}{s+1}\binom{s+1}{n+s-k} B_n(-1)^{k-s}$$

$$+ \sum_{n=s+1}^{k} \binom{k+1}{s+1}\binom{k-s}{n-s-1} B_n(-1)^{n+s}.$$

Rewriting it and dividing by $\binom{k+1}{s+1}$ (which is not 0 for $s \le k$), two sums vanish by the Carlitz identity. We are left with

$$(-1)^{s+1}B_{k+1} - (-1)^{k-s}B_{k+1} = -(-1)^s B_{k+1}\left(1 + (-1)^k\right) = 0$$

since the bracket will be zero if k is odd and $B_{k+1} = 0$ if k is even. ∎

From those two lemmas, the proposition is clear. ∎

Now, we only have to prove $\star_z = \widehat{\star}_z$ for $z \neq 1$. But we know that

$$\xi^k \widehat{\star}_z \eta = \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} (\mathrm{ad}_\xi)^j (\eta) \tag{66}$$

and also

$$\xi^k \star_1 \eta = \mathfrak{q}^{-1} \left(\mathfrak{q} \left(\xi^k \right) \cdot \mathfrak{q}(\eta) \right) = \sum_{j=0}^{k} \binom{k}{j} B_j^* \xi^{k-j} (\mathrm{ad}_\xi)^j (\eta). \tag{67}$$

We finally get from Equation (16)

$$\xi^k \star_z \eta = \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} (\mathrm{ad}_\xi)^j (\eta). \tag{68}$$

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