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On the Kernel of the Maximal Flat Radon Transform on Symmetric Spaces of Compact Type

Eric L. Grinberg and Steven Glenn Jackson

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Abstract. Let M be a Riemannian globally symmetric space of compact type, M' its set of maximal flat totally geodesic tori, and $\mathrm{Ad}(M)$ its adjoint space. We show that the kernel of the maximal flat Radon transform $\tau\colon L^2(M) \to L^2(M')$ is precisely the orthogonal complement of the image of the pullback map $L^2(\mathrm{Ad}(M)) \to L^2(M)$. In particular, we show that the maximal flat Radon transform is injective if and only if M coincides with its adjoint space.
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1. Introduction

As part of his 1911 doctoral dissertation (published as [Fun13]), Paul Funk considered the problem of recovering a function f on the sphere S^2 from its integrals over great circles. He showed that precisely the even part $f^+(x) = \frac{1}{2}(f(x) + f(-x))$ of f can be recovered from these integrals, while the odd part is annihilated by them. Thus was born the Funk transform—integration over closed geodesics in a Riemannian manifold.

Funk's original motivation was a problem in differential geometry suggested to him by his advisor David Hilbert: the study of surfaces all of whose geodesics are closed. The sphere is the obvious example of such a surface, and Darboux around 1884 had given a necessary condition for other surfaces of revolution to possess this property.[1] In 1892, Tannery had constructed an explicit, though singular, example ([Tan92]), and in 1901 Otto Zoll, another of Hilbert's students, had constructed

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[1]It is difficult now to assign an exact date to Darboux's condition. In [Tan92], Tannery writes: *"J'y ai été conduit en étudiant la note XV de la Mécanique de Despayrons, où M. Darboux a donné une règle pour trouver les surfaces de révolution admettant des lignes géodésiques fermées."* However, the authors of the present paper were unable to obtain this work; indeed, we could locate no reference to it but this and an entry in the June 24, 1905 issue of the *Publisher's Circular* dating it to 1884–85. The earliest publication of Darboux's result still readily obtainable seems to be [Dar93].

smooth examples (published as [Zol03]). Funk's dissertation sought to identify smooth deformations of the standard metric on S^2, all of whose geodesics remain closed at each stage of the deformation.

Funk's result on integration over great circles implied that no such deformation could exist if the deformed metrics were required to remain even (nowadays we say that the standard metric on $\mathbb{RP}^2$ admits no such deformations). Conversely, given any odd function h, Funk tried to construct by power series a conformal deformation of the standard metric with initial derivative h, but was unable to prove convergence of his series. Convergence of such deformations was finally proved by Guilleman in 1976 ([Gui76]). Thus, for S^2 and $\mathbb{RP}^2$ at least, the kernel of the Funk transform controls the "Zoll rigidity" of the manifold.[2] These results were extended to S^n and $\mathbb{RP}^n$, for arbitrary n, by Michel ([Mic73]) and Weinstein (see [Bes78, p. 120]).

Around this time, similar integral transforms arose in several related contexts. A smooth family $\{g_t\}$ of Riemannian metrics on a compact manifold M is said to be *isospectral* if the Laplace spectrum of (M, g_t) is independent of t. Such families are plentiful: given any local one-parameter group of diffeomorphisms $\phi_t : M \to M$ and any metric g on M, one obtains an isospectral family by setting g_t equal to the pullback of g along ϕ_t. Examples of this type are trivial in the sense that (M, g_t) is isometric to (M, g) for every t. One says that (M, g) is *spectrally rigid* if every isospectral deformation is trivial in this sense, and *spectrally rigid to first order* if every isospectral deformation agrees with a trivial one to first order in the deformation parameter t.

In the case of Riemannian globally symmetric spaces of compact type, results of Guillemin in [Gui79] implied that spectral rigidity is related to a certain natural "thickening" of the Funk transform. To be specific, let us refer to any integral transform obtained by integrating over a fixed class of totally geodesic submanifolds of fixed dimension as a "Radon transform." (Thus the Funk transform is the Radon transform associated with compact submanifolds of dimension one.) For the case where M is a Riemannian globally symmetric space of compact type, Guillemin's results implied that the initial derivative of any isospectral deformation must lie in the kernel of the Radon transform associated with maximal totally geodesic flat tori. One says that M is *Guillemin rigid* if the Lie derivatives of the metric (i.e. the initial derivatives of trivial deformations) fill the entire kernel of the maximal flat Radon transform for symmetric two-forms.[3] Then Guillemin rigidity is a sufficient condition for first-order spectral rigidity. Since the kernel of the Funk transform is contained in the kernel of the maximal flat Radon transform, an observation of Michel ([Mic73, Proposition 2.2.4]) implies that Guillemin rigidity is also a sufficient condition for first-order Zoll rigidity of P_l-metrics.

Each Riemannian globally symmetric space M of compact type admits

[2] To be precise, let us say that g is a *P_l-metric* if all its geodesics are periodic with least period l; that $\{g_t\}$ is a *P_l-deformation* of g_0 if g_t is a P_l-metric for all t; that the deformation $\{g_t\}$ is *trivial* if (M, g_t) is isometric to (M, g_0) for all t; and that g is *P_l-rigid* (or *Zoll rigid*) if every P_l-deformation of g is trivial. Then Funk's results showed that $\mathbb{RP}^2$ is Zoll rigid, while Guillemin's convergence argument showed that every odd function gives rise to a P_l-deformation of S^2.

[3] See [GG04] for the definition of the Radon transform on symmetric p-forms.

a unique *adjoint form* (or *universal covered space*): another symmetric space admitting M as a Riemannian cover, and not itself properly covering any other symmetric space. (For example, the adjoint form of the sphere S^n is the real projective space $\mathbb{RP}^n$.) In [Gri92], Grinberg conjectured that the maximal flat Radon transform for functions is injective if and only if M coincides with its adjoint form.

In a monumental study ([GG04]), Gasqui and Goldschmidt determined precisely which Grassmannians (over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$) are Guillemin rigid. Aware of Grinberg's conjecture, they observed that their criterion is equivalent to the requirement that M coincide with its adjoint form, and went on to prove that, if Grinberg's conjecture is true, then the only candidates for Guillemin rigidity are the irreducible adjoint spaces.

In the present paper, we prove a refinement of Grinberg's conjecture: the kernel of the maximal flat Radon transform consists precisely of those functions orthogonal to all functions pulled back from the adjoint form (Theorem 6.1). Applied to the sphere S^2, this recovers Funk's result of 1911: the kernel of the Funk transform consists of those functions orthogonal to all functions pulled back from $\mathbb{RP}^2$ (i.e. the functions orthogonal to all even functions, namely the odd functions). Our proof is based on elementary representation theory of the isometry group of M, and treats all Riemannian globally symmetric spaces of compact type uniformly; there is no case-by-case analysis.

The organization of the paper is as follows: in Section 2 we fix notation and gather basic tools; in Section 3 we show that the maximal flat Radon transform can be viewed as a Reynolds operator for a certain group of isometries of M; in Section 4 we relate this Reynolds operator to highest weight theory; in Section 5 we relate the resulting highest weight theory to the extraction of the adjoint form; and in Section 6 we use our accumulated results to prove Grinberg's conjecture.

2. Preliminaries

In this section, we fix notation and gather some useful results from the literature.

Symmetric spaces and isometry groups Let M be a Riemannian globally symmetric space of compact type, and let G be the identity component of the isometry group of M. Then G is a compact semisimple Lie group ([Hel78, Lemma IV.3.2, Definitions V.1, and Definition V.4]). Denote its Lie algebra by $\mathfrak{g}_0$, and put $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0$.

Fix a point $p \in M$, let σ be the geodesic symmetry of M at p, let K be the isotropy group of p in G, and let K_0 be the identity component of K. Define $\theta : G \to G$ by $\theta(g) = \sigma g \sigma^{-1}$; then θ is an involution of G. Letting K_θ be the set of fixed points of θ, we have $K_0 \subseteq K \subseteq K_\theta$ ([Hel78, Theorem IV.3.3]). These three groups share a common Lie algebra $\mathfrak{k}_0$ (namely the set of fixed points of $d\theta$), whose complexification we denote by $\mathfrak{k}$.

Since G is transitive on M ([Hel78, Theorem IV.3.3]), we have

$$M \simeq G/K.$$

The adjoint form Let Z be the center of G, and define

$$K_Z = \{g \in G \,|\, \theta(g)^{-1}g \in Z\}.$$

Evidently this is a subgroup of G containing K. Since G is semisimple, Z is finite, so K has finite index in K_Z.

Let $\mathrm{Ad}(G) = G/Z$ denote the adjoint group of G. Since θ stabilizes Z, we have an induced involution on $\mathrm{Ad}(G)$, whose fixed point set we denote by $\mathrm{Ad}(G)^\theta$. One shows easily that K_Z is the pre-image of $\mathrm{Ad}(G)^\theta$ under the natural projection $G \to \mathrm{Ad}(G)$; hence the space

$$\mathrm{Ad}(M) = G/K_Z \simeq \mathrm{Ad}(G)/\mathrm{Ad}(G)^\theta$$

is again a Riemannian globally symmetric space of compact type; we refer to it as the *adjoint form* of M.

Since K has finite index in K_Z, the natural map $\pi : M \to \mathrm{Ad}(M)$ is a covering map. In fact, $\mathrm{Ad}(M)$ is minimal in the isogeny class of M, by which we mean that it is covered by every Riemannian globally symmetric space whose universal cover is isometric to that of M ([Hel78, Corollary VII.9.3]).

Representative functions Let H be any closed subgroup of G. Since G and H are both compact, they are both unimodular and the homogeneous space G/H admits a unique normalized left Haar measure $\mu_{G/H}$ ([Loo53, Theorem 29E, Lemma 30A, and Theorem 33D]).

Define an action of G on $L^2(G/H)$ by the formula $(gf)(x) = f(g^{-1}x)$. Since the inner product $(f_1, f_2) = \int f_1\overline{f_2}\, d\mu_{G/H}$ is invariant under this action, $L^2(G/H)$ is a unitary (hence continuous) representation of G.

Let $C(G/H)$ denote the algebra of continuous functions on G/H. Let $\mathcal{R}(G/H)$ denote the set of G-finite vectors in $C(G/H)$. The elements of $\mathcal{R}(G/H)$ are called *representative functions* on G/H. Since G is compact, we have the inclusions

$$\mathcal{R}(G/H) \subseteq C(G/H) \subseteq L^2(G/H).$$

By the Peter-Weyl Theorem ([BtD85, Theorem III.5.7]), the algebra $\mathcal{R}(G/H)$ is dense in $L^2(G/H)$.

In the case where H is the trivial subgroup, we can say more. Define an action of $G \times G$ on $L^2(G)$ by the formula $((g_1, g_2)f)(x) = f(g_1^{-1}xg_2)$. Then (again since G is unimodular) this action is also unitary. Let e denote the identity element of G, and let LG and RG denote the groups $G \times \{e\}$ and $\{e\} \times G$, respectively. Then $f \in L^2(G)$ is LG-finite if and only if it is RG-finite ([BtD85, Proposition III.1.2]). Moreover, [BtD85, Proposition III.1.5] gives the $G \times G$-module decomposition

$$\mathcal{R}(G) \simeq \bigoplus_{V \in \mathrm{Irr}(G)} V \otimes V^*$$

where $\mathrm{Irr}(G)$ denotes the set of isomorphism classes of irreducible representations of G. (Since G is compact, it follows from the Peter-Weyl Theorem that all of its irreducible representations are finite-dimensional.)

Returning to the case of arbitrary closed H, we note that the natural projection $G \to G/H$ induces an injection $\mathcal{R}(G/H) \to \mathcal{R}(G)$ intertwining the (left) action of G. Identifying $\mathcal{R}(G/H)$ with its image under this injection, we shall regard $\mathcal{R}(G/H)$ as a subalgebra of $\mathcal{R}(G)$. From [BtD85, Example III.6.3] we have

$$\mathcal{R}(G/H) \simeq \bigoplus_{V \in \mathrm{Irr}(G)} V \otimes (V^*)^H$$

where $(V^*)^H$ denotes the space of H-invariant vectors in V^*. More generally, whenever $H_1 \subseteq H_2$ are both closed subgroups of G, we regard $\mathcal{R}(G/H_2)$ as a subalgebra of $\mathcal{R}(G/H_1)$ (and hence also a subalgebra of $\mathcal{R}(G)$).

One says that H is a *spherical subgroup* of G if $\dim\left((V^*)^H\right) \le 1$ for all $V \in \mathrm{Irr}(G)$. By passing to the complexified Lie algebra $\mathfrak{k}$ and using the proof of [GW09, Theorem 12.3.12], one sees that K_0 is spherical. Hence K and K_Z are also spherical.

The torus transform Let T be a maximal flat totally geodesic torus of M containing p. Let $\mathfrak{a}_0$ be the tangent space to T at p, and let $\mathfrak{p}_0 \subset \mathfrak{g}_0$ be the -1-eigenspace of $d\theta$. By [Hel78, Proposition V.6.1], the differential of the map $g \mapsto gp$ identifies $\mathfrak{a}_0$ with a maximal abelian subspace of $\mathfrak{p}_0$.

By [Hel78, Theorem V.6.2], G is transitive on the set M' of all maximal flat totally geodesic tori in M. Let L be the stabilizer of T in G. Then M' is in bijective correspondence with the homogeneous space G/L, with which we shall now identify it.

Let $A = \exp(\mathfrak{a}_0)$ be the Lie subgroup of G with Lie algebra $\mathfrak{a}_0$, and let $\overline{A}$ be the closure of A. Then $\overline{A}$ is a compact abelian subgroup of G whose elements satisfy the equation $\theta(a) = a^{-1}$. It follows that the Lie algebra of $\overline{A}$ is an abelian subspace of $\mathfrak{p}_0$ containing $\mathfrak{a}_0$. But $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}_0$, so we have equality, and A is in fact a closed subgroup of G (hence a torus).

Since T is totally geodesic, it follows from [Hel78, Theorem IV.3.3] that the geodesics of T passing through p are precisely the orbits of p under one-parameter subgroups of A. But every point of T is joined to p by some such geodesic, so it follows that A acts transitively on T. Evidently the full stabilizer L is also transitive on T, so we can write

$$T \simeq L/(L \cap K) \simeq A/(A \cap K).$$

The normalized Haar measure $\mu_{L/(L\cap K)}$ is an A-invariant measure on T, so the uniqueness of such measures forces $\mu_{L/(L\cap K)} = \mu_{A/(A\cap K)}$. For brevity, let us denote this measure by μ_T, and define a map $\tau_0 : \mathcal{R}(M) \to \mathcal{R}(M')$ by the formula

$$(\tau_0(f))(gT) = \int_{x \in T} f(gx)\, d\mu_T(x).$$

That $\tau_0(f)$ depends only on the torus gT and not on g itself follows from the L-invariance of μ_T; that it depends continuously on gT follows from the compactness of G and T; and that it is G-finite follows from the fact that τ_0 intertwines the left action of G.

In the sequel we shall show that $\|\tau_0(f)\|_2 \leq \|f\|_2$ (Theorem 3.2), from which it will follow that τ_0 admits a unique bounded extension $\tau : L^2(M) \to L^2(M')$. This extension is known in the literature as the *maximal flat Radon transform* on $L^2(M)$; we shall call it the *torus transform* for short.

Weights and representations Let $\mathfrak{h}_0$ be any maximal abelian subspace of $\mathfrak{g}_0$ containing $\mathfrak{a}_0$. Then by [Hel78, Lemma VI.3.2], $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, and it is stabilized by the action of θ. Denote by $\mathfrak{t}_0$ be the set of θ-fixed points in $\mathfrak{h}_0$. Since $\mathfrak{a}_0$ is maximal among abelian subspaces of $\mathfrak{p}_0$, we have $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$. Let H denote the maximal torus of G with Lie algebra $\mathfrak{h}_0$.

Next let $\mathfrak{a}$, $\mathfrak{t}$, and $\mathfrak{h}$ denote the complexifications of $\mathfrak{a}_0$, $\mathfrak{t}_0$, and $\mathfrak{h}_0$, respectively. The complexification of $d\theta$ stabilizes $\mathfrak{a}$, $\mathfrak{t}$, and $\mathfrak{h}$. From now on we shall use the same symbol θ to denote the involution of G, its differential, the complexification of its differential, and the induced involutions on the duals of all the vector spaces on which these act.

Let $\Delta \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and let $\Sigma \subset \mathfrak{a}^*$ be the restricted root system of the pair $(\mathfrak{g}, \mathfrak{k})$ (i.e. the set of non-zero restrictions of members of Δ to $\mathfrak{a}$). Choose a positive system Δ^+ for Δ such that the non-zero restrictions of its members form a positive system Σ^+ for Σ. Let $\Pi \subseteq \Delta^+$ and $\Pi' \subseteq \Sigma^+$ be the resulting sets of simple roots.

Denote by B the Killing form of $\mathfrak{g}$. Since B has non-degenerate restriction to $\mathfrak{h}$, it defines an isomorphism $\mathfrak{h} \to \mathfrak{h}^*$, by means of which we transfer B to an inner product $(\cdot,\cdot)$ on $\mathfrak{h}^*$. Then B is negative definite on $\mathfrak{h}_0$, but $(\cdot,\cdot)$ is positive definite on $\mathbb{R}\Delta$, the real span of Δ ([Hel78, Proposition II.6.6, Corollary II.6.7, and Theorem III.4.4]). For any $\alpha, \beta \in \mathbb{R}\Delta$ with $\beta \neq 0$, define

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}.$$

Since the restriction of B to $\mathfrak{a}$ is also non-degenerate, we can use the same procedure to define an inner product on $\mathfrak{a}^*$. With this inner product, $\mathfrak{a}^*$ is naturally isometric to $\operatorname{ann}(\mathfrak{t}) \subseteq \mathfrak{h}^*$, with which we shall now identify it.

Let W denote the normalizer of $\mathfrak{h}_0$ in G, modulo its centralizer. Then W acts faithfully on $\mathfrak{h}_0$, hence also on $\mathfrak{h}$ and its dual. One finds that W is generated by the root reflections $\{s_\alpha \mid \alpha \in \Delta\}$; hence W stabilizes $\mathbb{R}\Delta$. We say that a weight $\lambda \in \mathbb{R}\Delta$ is *dominant* if $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Pi$. Each element of $\mathbb{R}\Delta$ is W-conjugate to a unique dominant weight ([Hum72, Theorem 10.3 and Lemma 10.3B]).

Similarly, let $W_\mathfrak{a}$ denote the normalizer of $\mathfrak{a}_0$ in K, modulo its centralizer. Then $W_\mathfrak{a}$ acts faithfully on $\mathfrak{a}_0$ and hence also on $\mathfrak{a}$ and its dual. It is generated by the root reflections $\{s_\alpha \mid \alpha \in \Sigma\}$, so it stabilizes $\mathbb{R}\Sigma$. A restricted weight $\nu \in \mathbb{R}\Sigma$ is *dominant* if $(\nu, \alpha) \geq 0$ for all $\alpha \in \Pi'$. Each element of $\mathbb{R}\Sigma$ is $W_\mathfrak{a}$-conjugate to a unique dominant restricted weight ([Hel78, Corollary VII.2.13 and Theorem VII.2.22]).

We say that a weight $\lambda \in \mathbb{R}\Delta$ is *algebraically integral* if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Pi$. A weight $\omega \in \mathbb{R}\Delta$ is dominant and algebraically integral if and only if it is the highest weight of some finite-dimensional irreducible representation $V(\omega)$ of $\mathfrak{g}$. In this case every weight of $V(\omega)$ is algebraically integral ([Kna02, Theorem 5.5]).

We say that a weight $\lambda \in \mathbb{R}\Delta$ is *analytically integral* if it is the complexified differential of a character of H. In this case it is harmless to use the same symbol λ to denote the character of which it is the differential, and to say that λ is *trivial* on a subset $S \subseteq H$ if $\lambda(s) = 1$ for every $s \in S$. Every analytically integral weight is algebraically integral. A weight $\omega \in \mathbb{R}\Delta$ is dominant and analytically integral if and only if it is the highest weight of the differential of some irreducible representation of G, which we shall also denote by $V(\omega)$. In this case every weight of $V(\omega)$ is analytically integral ([Kna02, Proposition 4.59, Lemma 5.106, and Theorem 5.110]).

3. Properties of the torus transform

In this section we prove that τ_0 can be extended to a bounded operator on $L^2(M)$, and that it coincides with the restriction of a Reynolds operator on $\mathcal{R}(G)$.

Recall that $\mathcal{R}(M)$ can be identified with the algebra of right K-invariants in $\mathcal{R}(G)$, and $\mathcal{R}(M')$ can be identified with the algebra of right L-invariants. We have

Theorem 3.1. *Let $R_A : \mathcal{R}(G) \to \mathcal{R}(G)$ be the operator orthogonally projecting $\mathcal{R}(G)$ onto its space of right A-invariants, namely*

$$(R_A(f))(x) = \int_{a\in A} f(xa)\, d\mu_A(a)$$

where μ_A denotes normalized Haar measure on A. Then for any $f \in \mathcal{R}(M)$, we have

$$\tau_0(f) = R_A(f).$$

Proof. The torus T is isomorphic to $A/(A\cap K)$. The positive linear functional $I : C(A) \to \mathbb{C}$ defined by

$$I(F) = \int_{x(A\cap K)\in A/(A\cap K)} \left(\int_{y\in(A\cap K)} F(xy)\, d\mu_{A\cap K}(y) \right) d\mu_{A/(A\cap K)}(x)$$

is A-invariant, preserves monotone limits, and satisfies $I(1) = 1$; consequently I coincides with Haar integration over A ([Loo53, Definition 12A and Theorem 29D]). If F is right K-invariant, then the inner integral is just $F(x)$ (this depends only on the coset $x(A\cap K)$) so for $f \in \mathcal{R}(M)$ we have

$$\begin{aligned}(\tau_0(f))(g) &= \int_{x(A\cap K)\in A/(A\cap K)} f(gx)\, d\mu_{A/(A\cap K)}(x) \\ &= \int_{a\in A} f(ga)\, d\mu_A(a) \\ &= (R_A(f))(g).\end{aligned}$$ ∎

Corollary 3.2. *For any $f \in \mathcal{R}(M)$, we have $\|\tau_0(f)\|_2 \le \|f\|_2$.*

Proof. As above, the positive linear functional $J : C(G) \to \mathbb{C}$ defined by

$$J(F) = \int_{xK \in G/K} \left(\int_{y \in K} F(xy)\, d\mu_K(y) \right) d\mu_{G/K}(x)$$

coincides with Haar integration on G. It follows immediately that the injection $\mathcal{R}(M) \to \mathcal{R}(G)$ preserves L^2-norms. Similarly, the injection $\mathcal{R}(M') \to \mathcal{R}(G)$ preserves L^2-norms. But the identification of $\mathcal{R}(M)$ and $\mathcal{R}(M')$ with their images in $L^2(G)$ sends τ_0 to the restriction of an orthogonal projection operator. ∎

4. Supports of $\mathfrak{k}$-invariants

Let ω be an analytically integral dominant weight. Since K_0 is a spherical subgroup of G, the irreducible G-module $V(\omega)$ admits at most a one-dimensional space of $\mathfrak{k}$-invariants. Let v^ω be a non-zero $\mathfrak{k}$-invariant in $V(\omega)$, if one exists, and zero otherwise. Write

$$v^\omega = \sum_\lambda v^\omega_\lambda$$

where v^ω_λ is a weight vector of weight λ.

Definition 4.1. The *support* of ω is the set $\operatorname{supp}(\omega) = \{\lambda \,|\, v^\omega_\lambda \neq 0\}$.

In this section, we give a necessary and sufficient condition for a weight λ of $V(\omega)$ to lie in $\operatorname{supp}(\omega)$ (Theorem 4.11). We begin with a lemma.

Lemma 4.2. *Let α be a positive root. If $\theta(\alpha) \neq \alpha$ then $-\theta(\alpha)$ is also a positive root. On the other hand, if $\theta(\alpha) = \alpha$ then the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are both contained in $\mathfrak{k}$.*

Proof. This is [Hel78] Lemma VI.3.3. ∎

We shall also need a few facts concerning a certain partial order on $\mathfrak{a}^*$ (which, recall, we have identified with $\operatorname{ann}(\mathfrak{t})) \subseteq \mathfrak{h}^*$).

Definition 4.3. For $\lambda, \mu \in \mathfrak{a}^* \simeq \operatorname{ann} \mathfrak{t} \subseteq \mathfrak{h}^*$, define $\lambda \preceq \mu$ if and only if $\mu - \lambda$ lies in the non-negative span of Π'.

Lemma 4.4. *Suppose $\lambda \in \mathfrak{a}^*$, and let μ be the unique dominant $W_\mathfrak{a}$-conjugate of λ. Then $\lambda \preceq \mu$.*

Proof. The proof is essentially the same as the argument given in [Hel78, Theorem VII.2.22]. Let ν be any maximal element in the $W_\mathfrak{a}$-orbit of λ. Then for any $\alpha \in \Pi'$ we have $\nu - s_\alpha(\nu) = \langle \nu, \alpha \rangle \alpha$. Maximality of ν now forces $\langle \nu, \alpha \rangle \geq 0$. Since α was arbitrary in Π' this shows that ν is dominant, and hence $\nu = \mu$. But now the $W_\mathfrak{a}$-orbit is a finite poset with *unique* maximal element μ; hence μ is also a largest element and the lemma is proved. ∎

We now need a pair of lattices in $\operatorname{ann}(\mathfrak{t}) \subseteq \mathfrak{h}^*$.

Definition 4.5. Let Λ denote the set $\{\nu - \theta(\nu)\}$ where ν runs over the root lattice, and let $\widehat{\Lambda}$ denote the set $\{\mu - \theta(\mu)\}$ where μ runs over the lattice of algebraically integral weights.

Lemma 4.6. *Both* Λ *and* $\widehat{\Lambda}$ *are discrete additive subgroups of* $\operatorname{ann}(\mathfrak{t})$ *that span* $\operatorname{ann}(\mathfrak{t})$*.*

Proof. Both are obviously additive subgroups. The span of Λ (resp. $\widehat{\Lambda}$) is all of $\operatorname{ann}(\mathfrak{t})$ because the span of the root lattice (resp. the weight lattice) is all of $\mathfrak{h}^*$, and the map $\mu \mapsto (\mu - \theta(\mu))$ is a surjective linear map of $\mathfrak{h}^*$ onto $\operatorname{ann}(\mathfrak{t})$. Finally, Λ (resp. $\widehat{\Lambda}$) is discrete because it is a subgroup of the root lattice (resp. the weight lattice). ∎

Lemma 4.7. *Both* Λ *and* $\widehat{\Lambda}$ *are stable under the action of* W_a*.*

Proof. The group $W_{\mathfrak{a}}$ is a quotient of the centralizer of θ in W ([Hel78, Theorem VII.8.10]). But this centralizer evidently stabilizes both Λ and $\widehat{\Lambda}$. ∎

Lemma 4.8. *Suppose that* $V(\omega)$ *admits a non-zero* $\mathfrak{k}$*-invariant. Then* $\omega \in \operatorname{supp}(\omega)$*.*

Proof. Let λ be any element of $\operatorname{supp}(\omega)$, maximal with respect to $\preceq$. If $\lambda = \omega$ then we are finished; otherwise, since λ is not the highest weight of $V(\omega)$, we can find some positive root α such that $X_\alpha v^\omega_\lambda \neq 0$ (where X_α is some non-zero element of the root space $\mathfrak{g}_\alpha$). If α is fixed by θ, then Theorem 4.2 gives $X_\alpha \in \mathfrak{k}$, contradicting the $\mathfrak{k}$-invariance of v^ω. Consequently $\theta(\alpha) \neq \alpha$. But v^ω must be annihilated by $X_\alpha + \theta(X_\alpha) \in \mathfrak{k}$, so we have

$$\theta(X_\alpha)(v^\omega_{\lambda+\alpha-\theta(\alpha)}) = -X_\alpha v^\omega_\lambda,$$

forcing $\lambda + \alpha - \theta(\alpha) \in \operatorname{supp}(\omega)$. Again by Theorem 4.2, $-\theta(\alpha)$ is also a positive root, so $\lambda \prec \lambda + \alpha - \theta(\alpha)$, contradicting the maximality of λ. ∎

Lemma 4.9. *Suppose that* $V(\omega)$ *admits a non-zero* $\mathfrak{k}$*-invariant. Then* $\omega \in \widehat{\Lambda}$*.*

Proof. It suffices to consider the case in which G is simply connected. Consider the set

$$F_0 = \{a \in A \,|\, a^2 = e\}$$

where e denotes the identity element of G. This is a subset of K^θ. Indeed, let S denote the set of all $s \in \mathfrak{a}$ such that $\mu(s) \in \pi i\mathbb{Z}$ for every algebraically integral weight μ; then $F_0 = \exp(S)$, so F_0 is actually a subset of K_0. Then, since $\omega \in \operatorname{supp}(\omega)$, it follows that F_0 acts trivially on v^ω_ω. Thus, for any $s \in S$ (i.e. any $s \in \mathfrak{a}$ such that $\mu(s) \in 2\pi i\mathbb{Z}$ for every $\mu \in \widehat{\Lambda}$) we must also have $\omega(s) \in 2\pi i\mathbb{Z}$.

Next, choose a generating set $\{\mu_1, \ldots, \mu_k\}$ for $\widehat{\Lambda}$ such that $\{\mu_1, \ldots, \mu_k\}$ is a basis for $\operatorname{ann}(\mathfrak{t})$. (This is possible by Theorem 4.6.) Then $\omega \in \operatorname{supp}(\omega)$ forces $\omega \in \operatorname{ann}(\mathfrak{t})$, so we can write

$$\omega = c_1\mu_1 + \cdots + c_k\mu_k$$

for some scalars $c_1, \dots, c_k$. Let $b_1, \dots, b_k$ be the basis for $\mathfrak{a}$ dual to $\mu_1, \dots, \mu_k$, and put $a_j = 2\pi i b_j$, so that $a_1, \dots, a_k \in S$. Then for each j, we have $\omega(a_j) = 2\pi i c_j \in 2\pi i\mathbb{Z}$, showing that each c_j is an integer. ∎

Lemma 4.10. *Suppose that $\lambda \in \widehat{\Lambda}$, and $w \in W_{\mathfrak{a}}$. Then $\lambda - w(\lambda) \in \Lambda$.*

Proof. Choose an algebraically integral weight μ with $\lambda = \mu - \theta(\mu)$. Since $\Lambda \subseteq \widehat{\Lambda}$ and $W_{\mathfrak{a}}$ is generated by restricted root reflections, it suffices to prove the result in the case where $w = s_{\alpha|_{\mathfrak{a}}}$ for some root α with $\theta(\alpha) \neq \alpha$.

There are two cases to consider. Suppose first that $\alpha - \theta(\alpha)$ is proportional to some root β. Then

$$\begin{aligned} \lambda - w(\lambda) &= \langle \lambda, \beta \rangle \beta \\ &= 2 \langle \mu, \beta \rangle \beta \\ &= \langle \mu, \beta \rangle (\beta - \theta(\beta)) \end{aligned}$$

and, since μ is algebraically integral, this lies in Λ.

On the other hand, suppose that $\alpha - \theta(\alpha)$ is not proportional to any root. Then certainly $\theta(\alpha)$ is not proportional to α. But since exactly one of $\{\alpha, \theta(\alpha)\}$ is positive, we see that neither $\alpha - \theta(\alpha)$ nor $\alpha + \theta(\alpha)$ is a root, so [Hum72, Lemma 9.4] forces $(\alpha, \theta(\alpha)) = 0$. Then

$$\begin{aligned} \lambda - w(\lambda) &= 2\frac{(\lambda, \alpha - \theta(\alpha))}{(\alpha - \theta(\alpha), \alpha - \theta(\alpha))}(\alpha - \theta(\alpha)) \\ &= \frac{(\lambda, \alpha - \theta(\alpha))}{(\alpha, \alpha)}(\alpha - \theta(\alpha)) \\ &= 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)}(\alpha - \theta(\alpha)) \end{aligned}$$

and, since λ itself is algebraically integral, this again belongs to Λ. ∎

At last we have

Theorem 4.11. *Suppose that $V(\omega)$ admits a non-zero $\mathfrak{k}$-invariant, and let λ be any weight of $V(\omega)$. Then $\lambda \in \operatorname{supp}(\omega)$ if and only if $\theta(\lambda) = -\lambda$ and $\omega - \lambda$ lies in the lattice Λ.*

Proof. First suppose that $\lambda \in \operatorname{supp}(\omega)$. Since $\mathfrak{t} \subseteq \mathfrak{k}$ and v^ω is $\mathfrak{k}$-invariant, we have for any $t \in \mathfrak{t}$

$$0 = tv^\omega = \sum_\mu \mu(t) v^\omega_\mu$$

from which it follows that $\lambda(t) = 0$. Then, for any $x \in \mathfrak{h}$, write $x = x_+ + x_-$ with $x^+ \in \mathfrak{t}$ and $x_- \in \mathfrak{a}$. We have

$$\begin{aligned} (\theta(\lambda))(x) &= \lambda(\theta(x_+ + x_-)) = \lambda(x_+ - x_-) \\ &= \lambda(x_-) = \lambda(-x_+ - x_-) = (-\lambda)(x) \end{aligned}$$

so that $\theta(\lambda) = -\lambda$. The proof that $\omega - \lambda$ belongs to the lattice Λ is by downward induction with respect to the partial order $\preceq$. In the base case, $\omega - \lambda = 0$ certainly belongs to Λ. Otherwise λ is not a highest weight of $V(\omega)$, so by the argument given in the proof of Theorem 4.8, there is some positive root α with $\lambda + \alpha - \theta(\alpha) \in \operatorname{supp}(\omega)$, and the result follows immediately by induction.

On the other hand, suppose that $\theta(\lambda) = -\lambda$ and $\omega - \lambda$ belongs to Λ. We shall prove that $\lambda \in \operatorname{supp}(\omega)$, again by downward induction with respect to $\preceq$. The base case is Theorem 4.8. For the general case, recall that $W_{\mathfrak{a}}$ denotes the Weyl group of the pair (G, K); in particular, each of its elements is a coset of an element of K. It follows that $W_{\mathfrak{a}}$ stabilizes the set $\{\mu|_{\mathfrak{a}}\}_{\mu \in \operatorname{supp}\omega}$, so we may assume without loss of generality that $\lambda|_{\mathfrak{a}}$ is dominant. But this implies that λ itself is a dominant weight of $\mathfrak{g}$. To see this, begin by noting that $\lambda \in \operatorname{ann}(\mathfrak{t})$ since $\theta(\lambda) = -\lambda$. Then for any positive root α we can write $\alpha = \alpha_+ + \alpha_-$ with $\alpha_+ \in \operatorname{ann}(\mathfrak{a})$ and $\alpha_- \in \operatorname{ann}(\mathfrak{t})$, giving

$$(\lambda, \alpha) = (\lambda, \alpha_-) = (\lambda|_{\mathfrak{a}}, \alpha|_{\mathfrak{a}}) \geq 0$$

and proving that λ is dominant. Now write

$$\omega - \lambda = \sum_{\alpha \in \Delta^+} k_\alpha(\alpha - \theta(\alpha))$$

with $k_\alpha \in \mathbb{Z}$. Since ω is the *highest* weight of $V(\omega)$ we may take all k_α non-negative. Evidently we may also take $k_\alpha = 0$ whenever α is θ- fixed. Now choose $\alpha \in \Delta^+$ with $k_\alpha > 0$. Let λ' be the unique dominant $W_{\mathfrak{a}}$-conjugate of $\lambda+\alpha-\theta(\alpha)$. By Theorem 4.4, we have $\lambda \preceq \lambda'$. Evidently $\theta(\lambda') = -\lambda'$. Choose $w \in W_{\mathfrak{a}}$ with

$$w(\lambda + \alpha - \theta(\alpha)) = \lambda'.$$

We have

$$\begin{aligned} \omega - \lambda' &= \omega - w(\lambda - \omega + \omega + \alpha - \theta(\alpha)) \\ &= \omega - w(\omega) + w(\omega - \lambda) - w(\alpha - \theta(\alpha)) \end{aligned}$$

which, by Lemmas 4.7 and 4.10, lies in Λ. Then by induction, $\lambda' \in \operatorname{supp}(\omega)$, so also $\lambda + \alpha - \theta(\alpha) \in \operatorname{supp}(\omega)$.

Since $\theta(\alpha) \neq \alpha$, it follows from the Cauchy-Schwartz inequality that $(\theta(\alpha), \alpha) < |\alpha|^2$. Hence

$$\begin{aligned} (\lambda + \alpha - \theta(\alpha), \alpha) &= (\lambda, \alpha) + |\alpha|^2 - (\theta(\alpha), \alpha) \\ &> (\lambda, \alpha) \\ &\geq 0. \end{aligned}$$

from which it follows that

$$X_{-\alpha} v^\omega_{\lambda+\alpha-\theta(\alpha)} \neq 0.$$

Now since v^ω is $\mathfrak{k}$-invariant, we must have

$$X_{-\alpha} v^\omega = -\theta(X_{-\alpha}) v^\omega.$$

Extracting the component of each side of weight $\lambda - \theta(\alpha)$ gives

$$X_{-\alpha} v^\omega_{\lambda+\alpha-\theta(\alpha)} = -\theta(X_{-\alpha}) v^\omega_\lambda$$

forcing $v^\omega_\lambda \neq 0$ and hence $\lambda \in \operatorname{supp}(\omega)$. ∎

5. Representative functions on the adjoint form

In this section we prove Theorem 5.6 characterizing the image of the injection $\mathcal{R}(\mathrm{Ad}(M)) \to \mathcal{R}(M)$. We begin with a trivial modification of [KR71], Proposition 1:

Lemma 5.1. *Let F be the group $\{a \in A | a^2 \in Z\}$. Then $K_Z = FK_0$.*

Proof. Suppose first that $a \in F$. We have $\theta(a)^{-1}a = a^2 \in Z$, so $F \subseteq K_Z$. Now K_0 is also the connected component of K_Z, so F normalizes it and hence FK_0 is a group with $FK_0 \subseteq K_Z$. On the other hand, suppose $g \in K_Z$. Using the K_0AK_0 decomposition for G ([Hel78, Theorem V.6.7]), we can write $g = k_1ak_2$ for $k_i \in K_0$ and $a \in A$. Then $z = \theta(g)^{-1}g = k_2^{-1}ak_1^{-1}k_1ak_2 = k_2^{-1}a^2k_2$ lies in the center of G. But then $a^2 = k_2zk_2^{-1} = z$, showing that $a \in F$. Thus $g = k_1ak_2$ lies in FK_0. ∎

Lemma 5.2. *The group F coincides with the set $\{\exp(x) \,|\, x \in \mathfrak{a}$ and $(\alpha - \theta(\alpha))(x) \in 2\pi i\mathbb{Z}, \forall \alpha \in \Pi\}$.*

Proof. Evidently F coincides with $\{\exp(x) \,|\, x \in \mathfrak{a}$ and $\alpha(2x) \in 2\pi i\mathbb{Z}, \forall \alpha \in \Pi\}$. But $(\theta(\alpha))(x) = -\alpha(x)$. ∎

This leads immediately to

Corollary 5.3. *Let $x_1, \ldots, x_{\dim(\mathfrak{a})}$ be the basis for $\mathfrak{a}$ dual to Π' and put $S = \{\exp(\pi i x_1), \ldots, \exp(\pi i x_{\dim(\mathfrak{a})})\}$. Then F is generated by S.*

Lemma 5.4. *Let λ be an analytically integral weight annihilating $\mathfrak{t}$. Then λ is trivial on F if and only if λ belongs to the lattice Λ.*

Proof. Suppose first that λ is trivial on F. In the notation of Theorem 5.3, we must have $\lambda(x_i) \in 2\mathbb{Z}$. Let $\{\alpha_i\}$ be a set of positive roots restricting to Π'. Put $\mu = \sum_i (\frac{\lambda(x_i)}{2}(\alpha_i - \theta(\alpha_i)))$. Then λ agrees with μ on both $\mathfrak{t}$ and $\mathfrak{a}$, so in fact $\lambda = \mu$.

On the other hand, suppose that λ belongs to Λ. It follows immediately that $\lambda(x_i) \in 2\mathbb{Z}$ for all i, and hence that the corresponding weight of G is trivial on F. ∎

Theorem 5.5. *Let $V(\omega)$ be an irreducible representation of G containing a non-zero $\mathfrak{k}$-invariant. Then $V(\omega)$ contains a non-zero K_Z-invariant if and only if $\omega \in \Lambda$.*

Proof. Since both K_Z and K_0 are spherical subgroups of G, we need only determine when v^ω is also K_Z-invariant. This happens if and only if v^ω is F-invariant, which happens if and only if all the weights in $\mathrm{supp}(\omega)$ are trivial on F. By Theorem 5.4, this happens if and only if every member of $\mathrm{supp}(\omega)$ belongs to Λ, and by Theorem 4.11 this happens if and only if ω itself belongs to Λ. ∎

Corollary 5.6. *The image of the injection* $\mathcal{R}(\mathrm{Ad}(M)) \to \mathcal{R}(M)$ *induced by the covering map consists precisely of those* $V(\omega)$ *contained in* $\mathcal{R}(M)$ *such that* ω^**, the highest weight of the dual of* $V(\omega)$*, belongs to* Λ*.*

Proof. The ring $\mathcal{R}(M)$ consists of those $V(\omega)$ such that $V(\omega)^*$ contains a non-zero K-invariant, and the ring $\mathcal{R}(\mathrm{Ad}\, M)$ consists of those $V(\omega)$ such that $V(\omega)^*$ contains a non-zero K_Z-invariant. ∎

6. The main result

Theorem 6.1. *The kernel of the torus transform* τ *is the orthogonal complement of the image of the embedding* $L^2(\mathrm{Ad}(M)) \to L^2(M)$ *induced by the covering map.*

Proof. Since $\mathcal{R}(M)$ is dense in $L^2(M)$, the kernel of τ is the closure of the kernel of τ_o. Since A is connected, Theorem 3.1 implies that $\ker \tau_0$ consists of all those $V(\omega)$ such that $\mathrm{supp}(\omega^*)$ contains no weight restricting to zero on $\mathfrak{a}$. But by Theorem 4.11, these are precisely the $V(\omega)$ such that $\omega^* \notin \Lambda$, and by Theorem 5.6 these are precisely the $V(\omega)$ not in the image of the injection $\mathcal{R}(\mathrm{Ad}(M)) \to \mathcal{R}(M)$ induced by the covering map. The result follows by taking closures. ∎

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Eric L. Grinberg
Department of Mathematics
University of Massachusetts
100 Morrissey Boulevard
Boston, MA 02125, USA
eric.grinberg@umb.edu

Steven Glenn Jackson
Department of Mathematics
University of Massachusetts
100 Morrissey Boulevard
Boston, MA 02125, USA
jackson@math.umb.edu

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