Local Coefficient Matrices and the Metaplectic Correspondence

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Abstract.   The local coefficients of a principal series representation of a meta-plectic group $\widetilde{G}$ are defined in terms of the action of the standard intertwining operator on a canonical basis of the space of Whittaker functionals. By analyzing the nonsingularity of local coefficient matrices, we prove that for a certain class of unramified principal series representations of the metaplectic group, the local metaplectic correspondence preserves irreducibility.
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Introduction

In 1973, Shimura [22] described a correspondence between cusp forms of weight $\frac{1}{2}k$, where $k \geq 3$ is an odd integer, and modular forms of weight $k - 1$. This correspondence was reformulated by Gelbart and Piatetski-Shapiro (see [9], [10], and [11]) in terms of the representation theory of 2-fold metaplectic coverings of GL_2. Beginning in the 1980's, attention was placed on generalizing Shimura's correspondence within its representation-theoretic setting. Flicker [8] generalized the correspondence to n-fold covering groups of GL_2 and the works of Kazhdan and Patterson [14], Kazhdan and Flicker [12], and later, Suzuki [23] and Mezo [17], [18], sought a complete correspondence between representations on n-fold metaplectic coverings of GL_r and their nonmetaplectic counterparts. Here, we focus on a generalization of the local metaplectic correspondence, relating n-fold metaplectic coverings of GL_r to their d-fold counterparts, where d is a divisor of n.

For a certain class of unramified principal series representations, the first author [5] showed that when $r = 2$ and n has absolute value 1 in the local field, the local generalized metaplectic correspondence preserves irreducibility. This was achieved via a careful consideration of the action of the standard intertwining operator on a canonical basis of the finite-dimensional space of Whittaker functionals.

This article continues this investigation for arbitrary values of r.

The construction of the n-fold (c-twisted) metaplectic group $\widetilde{\mathrm{GL}}_r^{(c)}(\mathbb{F})$ is briefly revisited in Section 1 for the purpose of establishing the notations that will be used throughout the paper. Section 2 contains a description of the unramified principal series representations $(\pi_s, V_s)^{(n,r)}$ that will be the focus of our investigation. For each w in the Weyl subgroup $W \subset \mathrm{GL}_r(\mathbb{F})$, the local coefficients $\tau_w(\pi_s, \eta, \eta')$ are defined in Section 3. They are the metaplectic analogues of the local coefficients defined by Shahidi in [20] and [21]. In the case of the long element $w_0 \in W$, we arrange the local coefficients $\tau_{w_0}(\pi_s, \eta, \eta')$ into an $n^r \times n^r$ matrix $M_{w_0}^{(n,r)}(\pi_s)$, and describe an intimate relationship between the determinant of this matrix and the irreducibility of $(\pi_s, V_s)^{(n,r)}$.

In [5], the determinant of $M_{w_0}^{(n,2)}(\pi_s)$ was computed explicitly. When $r > 2$, the direct computation of the determinant of $M_{w_0}^{(n,r)}(\pi_s)$ proves to be much more challenging. Instead, we use an indirect method to compute

$$\det(M_{w_0}^{(n,r)}(\pi_{-s}) \cdot M_{w_0}^{(n,r)}(\pi_s)),$$

which is sufficient for determining the irreducibility of our representations and the main results in Section 4.

The outcome of our investigation is an explicit description (in Theorem 3.1) of the nonsingularity of the local coefficient matrices. The local metaplectic correspondence is given in Theorem 3.3, where it is shown that whenever d is a divisor of n and $|n|_{\mathbb{F}} = 1$,

$$(\pi_s, V_s)^{(n,r)} \text{ is irreducible} \iff (\pi_{sn/d}, V_{sn/d})^{(d,r)} \text{ is irreducible.}$$

Our analysis leads to another implication. We show in Corollary 3.4 that if $r' > r \geq 2$ and $|n|_{\mathbb{F}} = 1$, then

$$(\pi_s, V_s)^{(n,r')} \text{ is irreducible} \implies (\pi_s, V_s)^{(n,r)} \text{ is irreducible .}$$

It should be noted that the correspondence contained here is a local result, and although it has not been considered in our investigation, it may extend to the global setting. Such an extension is not trivial as our results assume $|n|_{\mathbb{F}} = 1$. Also, we focus on a particular class of representations in the unramified principal series, but it should be possible to apply these techniques to other metaplectic representations in order to achieve similar criteria for their irreducibility.

1. Metaplectic Groups

In this section, we recapitulate the construction of a metaplectic group and the main properties of 2-cocycles that allow us to work in this setting. Most of our notations and the systematic approach to the construction of metaplectic cocycles follow closely Section 1 of [2]. These topics are covered here for the sake of completeness and to establish the notations and tools to be used throughout the rest of the paper. It is also hoped that this brief exposition will give readers unfamiliar with the general theory of metaplectic groups the background they need to understand the motivation behind this article.

Let $\mathbb{F}$ denote a nonarchimedean local field containing μ_n, the full group of nth roots of unity. We will fix an embedding $\epsilon : \mu_n \hookrightarrow \mathbb{C}^\times$, identifying μ_n with the group of nth roots of unity in $\mathbb{C}^\times$. For any such field, let $\upsilon : \mathbb{F}^\times \to \mathbb{Z}$ be a normalized valuation on $\mathbb{F}$. As usual, υ is extended to all of $\mathbb{F}$ by setting $\upsilon(0) = \infty$ (with the convention that $\infty + z = \infty = z + \infty$ for all $z \in \mathbb{Z}$). Let $\mathcal{O}$ denote the ring of integers $\{x \in \mathbb{F} \mid \upsilon(x) \geq 0\}$ in $\mathbb{F}$ and $\mathfrak{p}$ the maximal ideal $\{x \in \mathbb{F} \mid \upsilon(x) \geq 1\}$ in $\mathcal{O}$. The notation ϖ will denote a uniformizing element. Let $q = p^f$ be the order of the residue field $\mathcal{O}/\mathfrak{p}$. Then the absolute value for $\mathbb{F}$ is given by $|x|_v = q^{-\upsilon(x)}$ and for each $s \in \mathbb{C}$, the local ζ-factor is $\zeta_v(s) = (1-q^{-s})^{-1}$.

Recall that the local n^{th}-order Hilbert symbol over $\mathbb{F}$ is a map

$$(\cdot, \cdot)_\mathbb{F} : \mathbb{F}^\times \times \mathbb{F}^\times \longrightarrow \mu_n$$

that satisfies the following properties:

$$
\begin{aligned}
&(aa', b)_\mathbb{F} = (a, b)_\mathbb{F} \, (a', b)_\mathbb{F}, \\
&(a, b)_\mathbb{F} \, (b, a)_\mathbb{F} = 1, \\
&(a, -a)_\mathbb{F} = (a, 1 - a)_\mathbb{F} = 1, \\
&\{x \in \mathbb{F}^\times \mid (x, y)_\mathbb{F} = 1, \text{ for all } y \in \mathbb{F}^\times\} = \mathbb{F}^{\times n},
\end{aligned}
$$

where $\mathbb{F}^{\times n} := \{x \in \mathbb{F}^\times \mid x = y^n, \text{ for some } y \in \mathbb{F}^\times\}$ (see [24], Chapter VIII sec. 5 for details). Throughout, we will assume that the Hilbert symbol is unramified:

$$(x, y)_\mathbb{F} = 1, \qquad \text{for all } x, y \in \mathcal{O}^\times.$$

This condition is equivalent to $|n|_\mathbb{F} = 1$. These basic properties of the Hilbert symbol will account for most of the 2-cocycle computations that will come later.

For every positive integer r and every $c \in \mathbb{Z}/n\mathbb{Z}$, define the n-fold (c-twisted) metaplectic covering $\widetilde{\mathrm{GL}}_r^{(c)}(\mathbb{F})$ of $\mathrm{GL}_r(\mathbb{F})$ to be a central extension of $\mathrm{GL}_r(\mathbb{F})$ by μ_n:

$$1 \longrightarrow \mu_n \overset{\mathbf{i}}{\longrightarrow} \widetilde{\mathrm{GL}}_r^{(c)}(\mathbb{F}) \overset{\mathbf{p}}{\longrightarrow} \mathrm{GL}_r(\mathbb{F}) \longrightarrow 1. \tag{1}$$

As a set, the elements of $\widetilde{\mathrm{GL}}_r^{(c)}(\mathbb{F})$ are of the form (g, ζ) where $g \in \mathrm{GL}_r(\mathbb{F})$ and $\zeta \in \mu_n$. Multiplication is given by

$$(g, \zeta)(g', \zeta') = (gg', \ \zeta\zeta' \ \sigma_r^{(c)}(g, g'))$$

where $\sigma_r^{(c)}$ is a nontrivial 2-cocycle in $Z^2\big(\widetilde{\mathrm{GL}}_r^{(c)}(\mathbb{F}); \mu_n\big)$ that satisfies the cocycle relation

$$\sigma_r^{(c)}(g, g')\sigma_r^{(c)}(gg', g'') = \sigma_r^{(c)}(g, g'g'')\sigma_r^{(c)}(g', g''),$$

for all $g, g', g'' \in \mathrm{GL}_r(\mathbb{F})$ (which is equivalent to associativity of multiplication in $\widetilde{\mathrm{GL}}_r^{(c)}(\mathbb{F})$). The twists of $\sigma_r := \sigma_r^{(0)}$ satisfy

$$\sigma_r^{(c)}(g, g') = \sigma_r(g, g')(\det g, \det g')_\mathbb{F}^c.$$

In the short exact sequence (1), the projection map $\mathbf{p} : \widetilde{\mathrm{GL}}_r^{(c)}(\mathbb{F}) \longrightarrow \mathrm{GL}_r(\mathbb{F})$ is given by $(g, \zeta) \mapsto g$ and the inclusion map $\mathbf{i} : \mu_n \longrightarrow \mathrm{GL}_r(\mathbb{F})$ is given by $\zeta \mapsto (1, \zeta)$. We will also need the preferred section $\mathbf{s} : \mathrm{GL}_r(\mathbb{F}) \longrightarrow \widetilde{\mathrm{GL}}_r^{(c)}(\mathbb{F})$ where $g \mapsto (g, 1)$.

The cocycle σ_2 was first described by Kubota [15] and is defined by

$$\sigma_2(g, g') = \left(\frac{X(gg')}{X(g)}, \frac{X(gg')}{X(g')\det(g)} \right)_{\mathbb{F}}$$

where

$$X\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0 \end{cases}.$$

The cocycle σ_r that will be used in this paper was constructed in [3] and agrees with the Kubota cocycle when $r = 2$. When $r = 1$, $\mathrm{GL}_1(\mathbb{F}) = \mathbb{F}^\times$, σ_1 is trivial, and the twists are given by $\sigma_1^{(c)}(g, g') = (g, g')_{\mathbb{F}}^c$ (see Corollary 8 in Section 3 of [3]).

To help in the evaluation of σ_r when $r > 2$, we define a standard Levi subgroup M of $GL_r(\mathbb{F})$ to be a group whose elements are of the form

$$\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}$$

where $g_i \in GL_{r_i}(\mathbb{F})$, $r_i \geq 1$, and $r = r_1 + r_2 + \cdots + r_k$. The following theorem concerning the block-compatibility of the cocycle σ_r was proved by Banks, Levy, and Sepanski (Theorem 11 in Section 3 of [3]).

Theorem 1.1.   *For every standard Levi subgroup of* $\mathrm{GL}_r(\mathbb{F})$, *the cocycle* σ_r *satisfies the following block formula:*

$$\sigma_r \left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}, \begin{pmatrix} g_1' & & \\ & \ddots & \\ & & g_k' \end{pmatrix} \right) = \prod_{i=1}^{k} \sigma_{r_i}(g_i, g_i') \prod_{1 \leq i < j \leq k} (\det g_i, \det g_j')_{\mathbb{F}}.$$

We will use the notation T to denote the subgroup of $\mathrm{GL}_r(\mathbb{F})$ consisting of diagonal matrices. If $\alpha = (i, j) \in \Phi$ denotes a root of $\mathrm{GL}_r(\mathbb{F})$ relative to T ($1 \leq i, j \leq r$ and $i \neq j$), then let

$$t^\alpha = t_i/t_j, \qquad \text{for all } t = \mathrm{diag}(t_i).$$

The set of positive roots ($i < j$) will be denoted Φ^+ and the set of negative roots ($i > j$) will be denoted Φ^-. We will use N to denote the unipotent radical of B, the Borel subgroup of $\mathrm{GL}_r(\mathbb{F})$.

Now assume that $r \geq 2$, and consider the $r - 1$ embeddings of $\mathrm{GL}_2(\mathbb{F})$ along the diagonal of $\mathrm{GL}_r(\mathbb{F})$:

$$\iota_i : \mathrm{GL}_2(\mathbb{F}) \hookrightarrow \mathrm{GL}_r(\mathbb{F}), \quad g \mapsto \begin{pmatrix} I_{i-1} & & \\ & g & \\ & & I_{r-1-i} \end{pmatrix}, \quad \text{for all } g \in \mathrm{GL}_2(\mathbb{F}).$$

For each i, let G_i denote the image of ι_i. The group $\mathrm{GL}_r(\mathbb{F})$ is then generated by the subgroups

$$\{G_i \mid 1 \leq i \leq r-1\}.$$

Define

$$w_i = \iota_i \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \quad \text{and} \quad s_i = \iota_i \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

and let $\ell(w)$ denote the length of an element w in the Weyl subgroup W. In other words, $\ell = \ell(w)$ is the smallest positive integer such that w can be written as a product of ℓ simple reflections: $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$. Let $\eta(w)$ denote the corresponding element $\eta(w) = w_{i_1} w_{i_2} \cdots w_{i_\ell}$ ($w \mapsto \eta(w)$ is well-defined by Lemma 6.2 of [16]). The group W acts on T (and on the set of roots) by conjugation:

$$t^{(w\alpha)} = (t^w)^\alpha = (w^{-1}tw)^\alpha.$$

All of the cocycle calculations contained in this paper are computed using σ_1, σ_2, Theorem 1.1, the properties of the Hilbert symbol given at the beginning of this section, and the properties of σ_r described in Theorem 7 of Section 3 of [3].

2. Unramified Principal Series Representations

In this section, we will describe a class of unramified principal series representations (induced from an irreducible representation of a Borel subgroup). Our description will require several important subgroups of $\widetilde{G} = \widetilde{\mathrm{GL}}_r^{(c)}(\mathbb{F})$. If H is any subgroup of $G = \mathrm{GL}_r(\mathbb{F})$, we will use the notation $\widetilde{H} = \mathbf{p}^{-1}(H)$ to denote the full metaplectic preimage of H. The groups $\widetilde{T}$ and $\widetilde{B}$ are formed in this manner. Although these groups depend upon the twist c, it will often be suppressed in our notation.

It is easily checked that the 2-cocycle $\sigma_r^{(c)}$ is trivial on N. Thus, $\widetilde{G}$ splits over N via the preferred section $\mathbf{s}$. We will denote its image by $N^* = \mathbf{s}(N)$. The group $\widetilde{G}$ also splits over the maximal compact subgroup $K = \mathrm{GL}_2(\mathcal{O})$ of G (see [19]). Let

$$\mathbf{k} : K \longrightarrow \widetilde{G}$$

denote the canonical splitting. We will not describe $\mathbf{k}$ here as it will be enough to note that $\mathbf{k}$ satisfies the relations

$$\mathbf{k}|_{T \cap K} = \mathbf{s}|_{T \cap K}, \quad \mathbf{k}|_W = \mathbf{s}|_W, \quad \text{and} \quad \mathbf{k}|_{N \cap K} = \mathbf{s}|_{N \cap K}$$

(see Proposition 0.1.3 of [13]). Set $K^* := \mathbf{k}(K)$ and for every $m \geq 0$, define $K_m^* := \mathbf{k}(K_m)$ where

$$K_m := \{k \in K \mid k \equiv I \pmod{\mathfrak{p}^m}\}.$$

The collection $\{K_m^* \mid m \geq 0\}$ forms a basis of open, compact neighborhoods of the identity in $\widetilde{G}$. These groups lead to the decomposition

$$\widetilde{G} = \widetilde{B}K^* = \widetilde{T}N^*K^*.$$

The principal series representations of $\widetilde{G} = \widetilde{\mathrm{GL}}_r^{(c)}(\mathbb{F})$ that we describe in this section are formed in a similar manner to the representations of $\widetilde{\mathrm{GL}}_2^{(c)}(\mathbb{F})$ that were the focus of [5]. We begin with a quasicharacter on the center of $\widetilde{T}$. As was noted in Section 1.1 of [13], the center of $\widetilde{T}$ is given by $\widetilde{T}^n Z(\widetilde{B})$ where

$$\widetilde{T}^n := \mathbf{p}^{-1}(T^n) = \mathbf{p}^{-1}\left\{ \left. \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix} \right| t_1, \ldots, t_r \in \mathbb{F}^{\times n} \right\}$$

and the center of $\widetilde{B}$ is given by

$$Z(\widetilde{B}) = \mathbf{p}^{-1}\left\{ \left. \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} \right| \lambda^{r-1+2rc} \in \mathbb{F}^{\times n} \right\}$$

by Proposition 0.1.1 of [13].

Let δ_B denote the modular quasicharacter of the Borel subgroup B. For any $s \in \mathbb{C}$, we will consider δ^s to be the quasicharacter δ_B^s on $\widetilde{T}^n Z(\widetilde{B})$ that is genuine. In other words,

$$\delta^s\left(\left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix}, \zeta\right)\right) = \zeta(|t_1|_{\mathbb{F}}^{r-1} |t_2|_{\mathbb{F}}^{r-3} \cdots |t_r|_{\mathbb{F}}^{1-r})^s.$$

This quasicharacter can be extended to a maximal abelian subgroup $\widetilde{T}_*$ of $\widetilde{T}$ in the obvious way. We use the maximal abelian subgroup

$$\widetilde{T}_* = (\widetilde{T} \cap K^*)\widetilde{T}^n Z(\widetilde{B})$$

and extend the quasicharacter δ to $\widetilde{B}_* := \widetilde{T}_* N^*$ by making it trivial on N^*.

For any $s \in \mathbb{C}$ and any unramified quasicharacter

$$\chi : \mathbb{F}^\times \longrightarrow \mathbb{C}^\times,$$

define the (normalized) induced representation

$$(\pi_s, V_s)^{(n,r)} := \mathrm{Ind}_{\widetilde{B}_*}^{\widetilde{G}}((\chi \circ \det \circ \mathbf{p}) \cdot \delta^s)$$

where V_s consists of all locally constant functions $f : \widetilde{G} \longrightarrow \mathbb{C}$ that satisfy

$$f(bg) = \chi(\det(\mathbf{p}(b))) \cdot \delta^s(b) \cdot \delta_B^{\frac{1}{2}}(\mathbf{p}(b)) \cdot f(g)$$

for all $b \in \widetilde{B}_*$ and $g \in \widetilde{G}$ and π_s acts by right translation on V_s:

$$(\pi_s(g)f)(g') = f(g'g).$$

The Stone-von Neumann Theorem (see [6] or [7]) guarantees that the isomorphism class of a representation constructed in this way depends only on the central

quasicharacter $(\chi \circ \det \circ \mathbf{p}) \cdot \delta^s$, not on the choice of maximal abelian subgroup nor on the quasicharacter's extension.

Now let $w \in W$, $\widetilde{w} := \mathbf{s}(w)$, and suppose that that $\omega : \widetilde{T}_* \longrightarrow \mathbb{C}^\times$ is an unramified quasicharacter. Then define ${}^w\omega(t) : \widetilde{T}_* \longrightarrow \mathbb{C}^\times$ by ${}^w\omega(t) := \omega(\widetilde{w}^{-1}t\widetilde{w})$. Such a character can then be extended to $\widetilde{B}_*$ by making it trivial on N^* (as was described above in the case of δ). Using this notation, we form

$$({}^w\pi_s, {}^wV_s)^{(n,r)} := \mathrm{Ind}_{\widetilde{B}_*}^{\widetilde{G}}({}^w((\chi \circ \det \circ \mathbf{p}) \cdot \delta^s))$$

and consider the standard intertwining operator $\mathbb{I}_w : V_s \longrightarrow {}^wV_s$ given by the absolutely convergent integral

$$(\mathbb{I}_w f)(g) = \int\limits_{n \in N^*(w)} f(\widetilde{w}^{-1}ng) \, dn$$

for all $f \in V_s$ and $g \in \widetilde{G}$. Here, $N^*(w)$ is the subgroup of N^* corresponding to the roots $\alpha \in \Phi^+$ such that $w^{-1}\alpha \in \Phi^-$ and dn is the unique Haar measure that satisfies $\mathrm{Vol}(N^* \cap K^*; dn) = 1$.

By Proposition 1.2.3 of [13], if $w_1, w_2 \in W$ satisfy $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, then

$$\mathbb{I}_{w_1w_2} : V_s \longrightarrow {}^{w_1w_2}V_s$$

is given by the composition

$$\mathbb{I}_{w_1w_2}f = \mathbb{I}_{w_1}(\mathbb{I}_{w_2}f).$$

This allows us to reduce the evaluation of an intertwining operator to simple reflections in W.

In general, a principal series representation of a metaplectic group does not have a unique Whittaker model. However, it does have a finite-dimensional space of Whittaker functionals. Fix a nontrivial additive unramified (unitary) character $\psi_\mathbb{F} : \mathbb{F} \longrightarrow \mathbb{C}^\times$ and extend it to a multiplicative character $\psi : N^* \longrightarrow \mathbb{C}^\times$ by

$$\psi(\widetilde{n}) := \psi_\mathbb{F}\left(\sum_{1 \le i < r} n_{i,i+1}\right),$$

where $\widetilde{n} := \mathbf{s}(n) \in N^*$ and $n_{i,i+i}$ is the entry in the $(i, i+1)^{th}$ position of $n \in N$. A ψ-Whittaker functional is a functional λ in the dual space V_s' of V_s that satisfies

$$\lambda(\pi_s(n)f) = \psi(n)\lambda(f) \tag{2}$$

for all $n \in N^*$ and $f \in V_s$.

For $\eta \in \widetilde{T}$, let $\lambda_\eta \in V_s'$ be given by

$$\lambda_\eta f = \int\limits_{n \in N^*} f(\eta\widetilde{w}_0 n) \, \overline{\psi}(n) \, dn. \tag{3}$$

Kazhdan and Patterson ([13], Lemma 1.2.3) showed that the integral (3) is absolutely convergent for $\mathrm{Re}(s) > 0$ and extends holomorphically to all of $\mathbb{C}$. One can check that λ_η satisfies (2) and that

$$\{\lambda_\eta \mid \eta \in \widetilde{T}_* \backslash \widetilde{T}\}$$

is a basis for $\mathrm{Wh}(\psi, V_s)$, the space of ψ-Whittaker functionals of V_s. By Lemma 1.3.2 of [13],

$$\dim(\mathrm{Wh}(\psi, V_s)) = \mathrm{Card}(\widetilde{T}_* \backslash \widetilde{T}) = n^r.$$

Let $\mathfrak{f} \in (\mathbb{Z}/n\mathbb{Z})^r$, where $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_r)$ (we will always use least residues of $\mathbb{Z}/n\mathbb{Z}$ as coset representatives), and write $\mathbf{s}(\varpi^{\mathfrak{f}}) := \mathbf{s}(\mathrm{diag}(\varpi^{\mathfrak{f}_i}))$. Then we will use the set

$$\Omega := \left\{ \mathbf{s}(\varpi^{\mathfrak{f}}) \mid \mathfrak{f} \in (\mathbb{Z}/n\mathbb{Z})^r \right\}$$

as coset representatives of $\widetilde{T}_* \backslash \widetilde{T}$.

In the next section, the element

$$\bar{\mathfrak{f}} = (0, 1, \ldots, r-1) \in (\mathbb{Z}/n\mathbb{Z})^r$$

will play an important role in the evaluation of local coefficients. The corresponding element in Ω will be denoted $\bar{\eta} = \mathbf{s}(\varpi^{\bar{\mathfrak{f}}})$. If $w \in W$, then let $w[\mathfrak{f}]$ be the element of $(\mathbb{Z}/n\mathbb{Z})^r$ given by

$$\varpi^{w[\mathfrak{f}]} := w \varpi^{\mathfrak{f} - \bar{\mathfrak{f}}} w^{-1} \varpi^{\bar{\mathfrak{f}}}.$$

This defines a group action of W on $(\mathbb{Z}/n\mathbb{Z})^r$. For two elements $\eta = \mathbf{s}(\varpi^{\mathfrak{f}})$ and $\eta' = \mathbf{s}(\varpi^{\mathfrak{f}'})$ in $\widetilde{T}_* \backslash \widetilde{T}$, we will write $\eta \sim \eta'$ if and only if either $\mathfrak{f}' \equiv \mathfrak{f} \pmod{n}$ or $\mathfrak{f}' \equiv s[\mathfrak{f}] \pmod{n}$ for some simple reflection $s \in W$. A straight forward computation shows that for all simple reflections, $\eta \sim \bar{\eta}$ implies that $\bar{\mathfrak{f}} \equiv \mathfrak{f} \pmod{n}$.

3. Nonsingularity of Local Coefficient Matrices

The local coefficients of $(\pi_s, V_s)^{(n,r)}$ are defined by considering the action of the standard intertwining operators on the space of Whittaker functionals. Using Kazhdan and Patterson's canonical basis for $\mathrm{Wh}(V_s)$ described in the previous section, the local coefficient $\tau_w(\pi_s, \eta, \eta')$ is defined by

$$\lambda_\eta \mathbb{I}_w = \sum_{\eta' \in \widetilde{T}_* \backslash \widetilde{T}} \tau_w(\pi_s, \eta, \eta') \lambda_{\eta'}. \tag{4}$$

Whenever $w, w' \in W$ satisfy $\ell(ww') = \ell(w) + \ell(w')$, we have

$$\tau_{ww'}(\pi_s, \eta, \eta') = \sum_{\eta'' \in \widetilde{T}_* \backslash \widetilde{T}} \tau_w({}^{w'}\pi_s, \eta, \eta'') \tau_{w'}(\pi_s, \eta'', \eta'), \tag{5}$$

thus reducing local coefficient calculations to the case of simple reflections.

To evaluate the local coefficients, we follow [13], Section 1.3, starting with an appropriate set of "test" functions in V_s. Let

$$f_{\eta'}(g) = \begin{cases} \chi(\det(\mathbf{p}(b))) \cdot \delta^s(b) \cdot \delta_B^{\frac{1}{2}}(\mathbf{p}(b)) & \text{if } g = b\eta'lw_0, \ b \in \widetilde{B}_*, \ l \in L \\ 0 & \text{otherwise} \end{cases}$$

where L is an open, compact subgroup of $\widetilde{G}$ that is taken to be sufficiently small. As in [13], we let $f_{\eta'}^{(m)}$ denote the test function satisfying

- $\text{Supp}(f_{\eta'}^{(m)}) = \widetilde{B}_*\eta'w_0K_m^* = \widetilde{B}_*\eta'w_0(K_m^+)^*$, where $K_m^+ = K_m \cap N$,
- $f_{\eta'}^{(m)}$ is right K_m^*-invariant,
- $f_{\eta'}^{(m)}(w_0) = 1$.

By Lemma 1.3.1 of [13], we have

$$\lambda_\eta f_{\eta'}^{(m)} = 0 \quad \Longleftrightarrow \quad \eta \neq \eta',$$

and for such functions, (4) gives

$$\tau_w(\pi_s, \eta, \eta') = \frac{\lambda_\eta \mathbb{I}_w f_{\eta'}^{(m)}}{\lambda_{\eta'} f_{\eta'}^{(m)}},$$

resulting in the following commutative diagram.

$$\begin{array}{ccc} V_s & \xrightarrow{\ \mathbb{I}_w\ } & {}^wV_s \\ \lambda_{\eta'} \downarrow & & \downarrow \lambda_\eta \\ \mathbb{C} & \xrightarrow[\tau_w(\pi_s,\eta,\eta')]{} & \mathbb{C} \end{array}$$

For all principal series representations $(\pi(\omega), V(\omega))$ (induced from a genuine unramified quasicharacter $\omega : \widetilde{T}_* \longrightarrow \mathbb{C}^\times$ that has been extended to $\widetilde{B}_*$ by making it trivial on N^*), it was shown in Section 1.3 of [13] that

$$\tau_s(\pi(\omega), \eta, \eta') = 0 \quad \text{whenever} \quad \eta \nsim \eta'.$$

In particular, this applies to π_s as well as ${}^w\pi_s$ for all $w \in W$.

Now we define the local coefficient matrix

$$M_w^{(n,r)}(\pi_s) := (\tau_w(\pi_s, \eta, \eta'))$$

by placing $\tau_w(\pi_s, \eta, \eta')$ in the $(n^{r-1}\mathfrak{f}_1 + n^{r-2}\mathfrak{f}_2 + \cdots n\mathfrak{f}_{r-1} + \mathfrak{f}_r + 1)^{th}$ row and $(n^{r-1}\mathfrak{f}'_1 + n^{r-2}\mathfrak{f}'_2 + \cdots n\mathfrak{f}'_{r-1} + \mathfrak{f}'_r + 1)^{th}$ column. The matrix $M_w^{(n,r)}(\pi_s)$ can be interpreted as a linear transformation $\mathbb{C}^{n^r} \longrightarrow \mathbb{C}^{n^r}$. It was shown in [1] that

$$V_s = \text{Span}\langle f_{\eta'} \rangle \oplus \text{Ker}\langle \lambda_{\eta'} \rangle.$$

Using the linearity of the Whittaker functionals, we obtain the following commutative diagram.

$$
\begin{array}{ccc}
V_s & \xrightarrow{\;\mathbb{I}_w\;} & {}^w V_s \\
\downarrow & & \downarrow \\
\mathbb{C}^{n^r} & \xrightarrow[M_w^{(n,r)}(\pi_s)]{} & \mathbb{C}^{n^r}
\end{array}
$$

Considering the long element $w_0 \in W$, we have

$$
\begin{array}{ccccc}
V_s & \xrightarrow{\;\mathbb{I}_{w_0}\;} & V_{-s} & \xrightarrow{\;\mathbb{I}_{w_0}\;} & V_s \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{C}^{n^r} & \xrightarrow[M_{w_0}^{(n,r)}(\pi_s)]{} & \mathbb{C}^{n^r} & \xrightarrow[M_{w_0}^{(n,r)}(\pi_{-s})]{} & \mathbb{C}^{n^r}
\end{array} \tag{6}
$$

where $\mathbb{I}_{w_0} \circ \mathbb{I}_{w_0} : V_s \longrightarrow V_s$ is an intertwining operator. By Proposition 1.2.2 of [13],

$$
\mathbb{I}_{w_0} \circ \mathbb{I}_{w_0} = c^{(n,r)}(s) \cdot I : V_s \longrightarrow V_s,
$$

where $c^{(n,r)}(s)$ is a constant and I is the identity map. From the diagram (6), it follows that

$$
M_{w_0}^{(n,r)}(\pi_{-s}) \cdot M_{w_0}^{(n,r)}(\pi_s) = c^{(n,r)}(s) \cdot I_{n^r},
$$

where I_{n^r} is the identity matrix in $\mathrm{GL}_{n^r}(\mathbb{C})$. Now we turn our attention to the evaluation of $c^{(n,r)}(s)$.

Theorem 3.1.   *If $r \geq 2$ and $|n|_{\mathbb{F}} = 1$, then*

$$
c^{(n,r)}(s) = \left(\frac{\zeta_v(2ns)\zeta_v(-2ns)}{\zeta_v(1-2ns)\zeta_v(1+2ns)} \right)^{\frac{r^2-4r+3}{2}+\varepsilon} \left(\frac{\zeta_v(4ns)\zeta_v(-4ns)}{\zeta_v(1-4ns)\zeta_v(1+4ns)} \right)^{\frac{r-1}{2}-\varepsilon}
$$

$$
\cdot \prod_{i=1}^{r-1} \frac{\zeta_v(2ins)\zeta_v(-2ins)}{\zeta_v(1-2ins)\zeta_v(1+2ins)}
$$

where

$$
\varepsilon = \left\{ \begin{array}{ll} \frac{1}{2} & \text{if } r \text{ even} \\ 0 & \text{if } r \text{ odd.} \end{array} \right.
$$

Proof.   We have seen that

$$
M_{w_0}^{(n,r)}(\pi_{-s}) \cdot M_{w_0}^{(n,r)}(\pi_s) = c^{(n,r)}(s) \cdot I_{n^r}.
$$

In [5], the determinant of $M_{w_0}^{(n,r)}(\pi_s)$ was explicitly computed for $r = 2$. However, when $r > 2$, the computation proves to be much more challenging. Rather than compute the determinant in a straightforward manner, we take advantage of the structure of the local coefficient matrices and the abundance of 0 entries. We begin by decomposing the long element as

$$
w_0 = w_{r-1} \cdots w_2 w_1 \quad \text{where} \quad w_j = s_1 s_2 \cdots s_j.
$$

This allows for the decomposition

$$M_{w_0}^{(n,r)}(\pi_s) = M_{w_{r-1}}^{(n,r)}({}^{w_{r-2}\cdots w_2 w_1}\pi_s) \cdots M_{w_2}^{(n,r)}({}^{w_1}\pi_s) \cdot M_{w_1}^{(n,r)}(\pi_s),$$

where each $M_w^{(n,r)}({}^{w'}\pi_s)$ can be decomposed into local coefficient matrices associated with simple reflections. At the end of Section 2, it was noted that, $\eta \sim \overline{\eta}$ implies that $\overline{\mathfrak{f}} \equiv \mathfrak{f}(\mathrm{mod}\, n)$. Thus, for each simple reflection $s_i \in W$ $(1 \le i < r)$, the entry $\tau_{s_i}({}^w\pi_s, \overline{\eta}, \overline{\eta})$ is the only nonzero entry in its row and column in $M_{s_i}^{(n,r)}({}^w\pi_s)$. This leads to the following decomposition of $\tau_{w_0}(\pi_s, \overline{\eta}, \overline{\eta})$:

$$\tau_{w_0}(\pi_s, \overline{\eta}, \overline{\eta}) = \tau_{w_1}(\pi_s, \overline{\eta}, \overline{\eta}) \cdot \prod_{j=2}^{r-1} \tau_{w_j}({}^{w_{j-1}\cdots w_2 w_1}\pi_s, \overline{\eta}, \overline{\eta}),$$

where

$$\tau_{w_j}({}^{w_{j-1}\cdots w_2 w_1}\pi_s, \overline{\eta}, \overline{\eta}) = \tau_{s_j}({}^{w_{j-1}\cdots w_2 w_1}\pi_s, \overline{\eta}, \overline{\eta})$$
$$\cdot \prod_{k=1}^{j-1} \tau_{s_{j-k}}({}^{s_{j+1-k}\cdots s_{j-1}s_j w_{j-1}\cdots w_2 w_1}\pi_s, \overline{\eta}, \overline{\eta}).$$

Each of the local coefficients in this product can be computed following Section 1.3 of [13]. Since this process is fairly standard, we omit the details of the computation. We have

$$\tau_{s_k}({}^{w_{k-1}\cdots w_2 w_1}\pi_s, \overline{\eta}, \overline{\eta}) = \epsilon_k(-1,-1)_{\mathbb{F}}^c \frac{\zeta_v(2nks\epsilon_k)}{\zeta_v(1 - 2nks)},$$

and if $1 \le i < k$, then

$$\tau_{s_i}({}^{s_{i+1}\cdots s_{k-1}s_k w_{k-1}\cdots w_2 w_1}\pi_s, \overline{\eta}, \overline{\eta}) = \begin{cases} \epsilon_i(-1,-1)_{\mathbb{F}}^c \frac{\zeta_v(-4ns\epsilon_i)}{\zeta_v(1-4ns)} & \text{if } k \text{ is even, } i = \frac{k}{2} \\ \\ \epsilon_i(-1,-1)_{\mathbb{F}}^c \frac{\zeta_v(-2ns\epsilon_i)}{\zeta_v(1-2ns)} & \text{otherwise,} \end{cases}$$

where

$$\epsilon_i = \begin{cases} 1 & \text{if } \mathfrak{f}_{i+1} \equiv 0(\mathrm{mod}\, n) \\ -1 & \text{if } \mathfrak{f}_{i+1} \not\equiv 0(\mathrm{mod}\, n). \end{cases}$$

By carefully multiplying out each of the simple reflection local coefficients and using the property

$$\tau_{w_0}({}^{w_0}\pi_s, \overline{\eta}, \overline{\eta}) \cdot \tau_{w_0}(\pi_s, \overline{\eta}, \overline{\eta}) = c^{(n,r)}(s),$$

we conclude the statement of the theorem. ∎

In [5], it was argued that whenever d is any divisor of n and $|n|_{\mathbb{F}} = 1$,

$$(\pi_s, V_s)^{(n,2)} \text{ is irreducible} \iff (\pi_{sn/d}, V_{sn/d})^{(d,2)} \text{ is irreducible.}$$

This result will be generalized to arbitrary values of $r \ge 2$ in this section and is dependent on the following Lemma. Unfortunately, there was a mistake in [5], so the following Lemma corrects the mistake for $r = 2$, in addition to handling the cases $r > 2$.

Lemma 3.2.   *The principal series representation* $(\pi_s V_s)^{(n,r)}$ *is irreducible if and only if*

$$det(M_{w_0}^{(n,r)}(\pi_{-s}) \cdot M_{w_0}^{(n,r)}(\pi_s)) \neq 0$$

(equivalently, if and only if $c^{(n,r)}(s) \neq 0$ *).*

Proof.   First consider the case in which $c^{(n,r)}(s) \neq 0$. From Theorem 3.1, it follows that $2ins \neq 1$ for $i = 1, \ldots, r-1$. Hence, by Corollary 1.2.8 of [13], it follows that $(\pi_s, V_s)^{(n,r)}$ is irreducible. Now suppose that $c^{(n,r)}(s) = 0$. Then $\text{Ker}(\mathbb{I}_{w_0}) \neq \{0\}$. Now we follow the approach used in the beginning of the proof of Theorem 1.2.9 of [13]. Recall that $\mathbb{I}_{w_0} : V_s \longrightarrow V_{-s}$ is defined by

$$(\mathbb{I}_{w_0} f)(g) = \int_{n \in N^*(w_0)} f(\tilde{w}_0^{-1} ng) \, dn.$$

Let L be a sufficiently small compact open subgroup of $\widetilde{G}$ and let f be the right L-invariant function in V_s with support $\widetilde{B}_* L$ such that $f(1) = 1$. If we let $g = w_0$, the integral converges and by choosing L sufficiently small, one has $\mathbb{I}_{w_0} f(w_0) \neq 0$. Hence, $\text{Ker}(\mathbb{I}_{w_0}) \neq V_s$, giving a proper nontrivial $\widetilde{G}$-stable subspace of V_s. Thus, (π_s, V_s) is reducible.   ∎

The main result of this section (and of this paper) is the following theorem, which follows from a comparison of the zeros of $c^{(n,r)}(s)$ over different values of n and r.

Theorem 3.3.   *If d is any divisor of n and $|n|_{\mathbb{F}} = 1$, then $(\pi_s, V_s)^{(n,r)}$ is irreducible if and only if $(\pi_{sn/d}, V_{sn/d})^{(d,r)}$ is irreducible.*

Proof.   In Lemma 3.2, it was shown that $(\pi_s, V_s)^{(n,r)}$ is irreducible if and only if $c^{(n,r)}(s) \neq 0$. The zeros of $c^{(n,r)}(s)$ are easily determined from Theorem 3.1. We see that $c^{(n,r)}(s) = 0$ if and only if s has any of the values

$$\pm \left(\frac{1}{2n} + \frac{\pi i k}{n \log q} \right), \pm \frac{1}{2} \left(\frac{1}{2n} + \frac{\pi i k}{n \log q} \right), \ldots, \pm \frac{1}{r-1} \left(\frac{1}{2n} + \frac{\pi i k}{n \log q} \right).$$

A comparison of these zeros with the corresponding zeros of $c^{(d,r)}(sn/d)$ results in the correspondence described in Theorem 3.3   ∎

Corollary 3.4.   *If $r' > r \geq 2$, $|n|_{\mathbb{F}} = 1$, and $(\pi_s, V_s)^{(n,r')}$ is irreducible, then $(\pi_s, V_s)^{(n,r)}$ is irreducible.*

Proof.   This corollary follows from a comparison of the zeros of $c^{(n,r')}(s)$ and $c^{(n,r)}(s)$ that were given in the proof of Theorem 3.3.   ∎

At the present time, only nonarchimedean local fields in which $|n|_{\mathbb{F}} \neq 1$ have been considered. However, the techniques contained in this article should be applicable to other metaplectic representations than just the ones considered here. We leave such investigations for future study, with the eventual goal of further developing the complete (local and global) metaplectic correspondence using local coefficient matrices.

References

[1] Banks, W., "Exceptional Representations on the Metaplectic Group," Dissertation, Stanford University, 1994.

[2] Banks, W., D. Bump, and D. Lieman, *Whittaker-Fourier Coefficients of Metaplectic Eisenstein Series,* Compos. Math. **135** (2003), 153–178.

[3] Banks, W., J. Levy, and M. Sepanski, *Block-Compatible Metaplectic Cocycles,* J. Reine Angew. Math. **507** (1999), 131–163.

[4] Berstein, I., and A. Zelevinski, *Induced Representations of Reductive p-adic Groups,* Ann. Scient. Ec. Norm. Sup. **10** (1977), 441–472.

[5] Budden, M., *Local Coefficient Matrices of Metaplectic Groups,* J. Lie Theory **16** (2006), 239–249.

[6] Bump, D., "Automorphic Forms and Representations," Cambridge University Press, New York, N.Y., 1997.

[7] Bump, D., and D. Ginzburg, *Symmetric Square L-Functions on $GL(r)$,* Ann. Math. **136** (1992), 137–205.

[8] Flicker, Y., *Automorphic Forms on Covering Groups of $GL(2)$,* Invent. Math. **57** (1980), 119–182.

[9] Gelbart, S., "Weil's Representation and the Spectrum of the Metaplectic Group," Springer Lecture Notes **530**, Springer-Verlag, Berlin etc., 1976.

[10] Gelbart, S., and I. Piatetski-Shapiro, *Distinguished Representations and Modular Forms of Half-Integral Weight,* Invent. Math. **59** (1980), 145–188.

[11] —, "On Shimura's Correspondence for Modular Forms of Half-Integral Weight," in: Proceedings, Colloquium on Automorphic Forms, Representation Theory, and Arithmetic, Bombay (1979), Tata Institute of Fundamental Research Studies in Mathematics #10, Springer-Verlag, Berlin, 1981.

[12] Kazhdan, D., and Y. Flicker, *Metaplectic Correspndence,* Publ. Math. IHES **64** (1986), 53–110.

[13] Kazhdan, D., and S. J. Patterson, *Metaplectic Forms,* Publ. Math. IHES **59** (1984), 35–142.

[14] —, *Towards a Generalized Shimura Correspondence,* Adv. in Math. **60** (1986), 161–234.

[15] Kubota, T., "Automorphic Forms and Reciprocity in a Number Field," Kiyokuniya Book Store, Kyoto University, Japan, 1969.

[16] Matsumoto, H., *Sur les Sous-Groupes Arithmétiques des Groupes Semi-Simples Déployés,* Ann. Scient. Éc. Norm. Sup. **2** (1969), 1–62.

[17] Mezo, P., *Comparisons of General Linear Groups and Their Metaplectic Coverings I,* Canad. J. Math. **54** (2002), 92–137.

[18] —, *Comparisons of General Linear Groups and Their Metaplectic Coverings II,* Represent. Theory **5** (2001), 524–580.

[19] Moore, C., *Group Extensions of p-adic and Adelic Linear Groups,* Publ. Math. IHES **35** (1968), 157–222.

[20] Shahidi, F., *On Certain L-functions,* Amer. J. Math. **103** (1981), 297–355.

[21] —, *Local Coefficients and Normalization of Intertwining Operators for $GL(n)$,* Compos. Math. **48** (1983), 271–295.

[22] Shimura, G., *On Modular Forms of Half-Integral Weight,* Ann. Math. **97** (1973), 440–481.

[23] Suzuki, T., *Distinguished Representations of Metaplectic Groups,* Amer. J. Math. **120** (1998), 723–755.

[24] Weil, A., "Basic Number Theory," 2nd edition, Springer-Verlag, New York, N.Y., 1973.

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