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Local Lie Algebras and some Kac-Moody Algebras of Indefinite Type

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Abstract. We show a sufficient condition for an indefinite Kac-Moody algebra to admit minimal $\mathbb{Z}$-gradation with local part given by an integrable highest weight module. Examples of orbit Lie algebras arising from 2-fold affinization and Lorentzian Kac-Moody algebras are discussed.
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1. Introduction.

A local Lie algebra is a direct sum $L_{-1} \oplus L_0 \oplus L_1$ of vector spaces with anticommutative bilinear maps $L_i \times L_j \to L_{i+j}$ for $|i|, |j|, |i| + |j| \leq 1$, such that Jacobi identity holds for any defined $[x, [y, z]]$ (see [7, Ex. 1.8], [8, Definition 5]). For any $\mathbb{Z}$-graded Lie algebra $L = \oplus_i L_i$, $L_{-1} \oplus L_0 \oplus L_1$ is a local Lie algebra, called the local part of L. Note that, by [1, (1.1)], a direct sum $V \oplus \mathfrak{g} \oplus W$ is a local Lie algebra if and only if (i) $\mathfrak{g}$ is a Lie algebra, (ii) V and W are $\mathfrak{g}$-modules and (iii) there is a $\mathfrak{g}$-module homomorphism $\varphi\colon V \otimes W \to \mathfrak{g}$, where $\mathfrak{g}$ is viewed as a $\mathfrak{g}$-module via adjoint action.

For any local Lie algebra $V \oplus \mathfrak{g} \oplus W$, by Kac-Kantor's construction [1, 4, 8, 11], among all $\mathbb{Z}$-graded Lie algebras which have $V \oplus \mathfrak{g} \oplus W$ as their local part, the maximal one is given by $\mathcal{F} = \mathcal{F}^- \oplus (V \oplus \mathfrak{g} \oplus W) \oplus \mathcal{F}^+$, where $\mathcal{F}^+ = \oplus_{i \geq 1} \mathcal{F}_i$ (resp. $\mathcal{F}^- = \oplus_{i \leq -1} \mathcal{F}_i$) is the free Lie algebra generated by $\mathcal{F}_1 = W$ (resp. $\mathcal{F}_{-1} = V$), while the minimal one is $\widehat{\mathcal{F}} = \mathcal{F}/\mathcal{J}$, where $\mathcal{J}$ is the unique maximal ideal among homogeneous ideals of $\mathcal{F}$ which intersect $V \oplus \mathfrak{g} \oplus W$ trivially. So by [8, Definition 6] we may say that a Lie algebra has a minimal $\mathbb{Z}$-gradation if it is isomorphic to such an $\widehat{\mathcal{F}}$. That is, to show a Lie algebra $\widehat{\mathfrak{g}}$ admits a minimal $\mathbb{Z}$-gradation, it suffices to construct a suitable local Lie algebra $V \oplus \mathfrak{g} \oplus W$ and show the corresponding Kac-Kantor algebra $\widehat{\mathcal{F}}$ is isomorphic to $\widehat{\mathfrak{g}}$.

For a finite or affine Cartan matrix A and a fixed k, by [8, Propositions 5 and 14], the derived Kac-Moody algebra $\mathfrak{g}'(A)$ has a minimal $\mathbb{Z}$-gradations

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(standard gradation) with local part of the form $V \oplus \mathfrak{g}'(B) \oplus W$, where B is obtained from A by removing the k-th row and the k-th column (so B has finite type or is a disjoint union of finite types), and V (resp. W) is a finite-dimensional irreducible highest (resp. lowest) weight $\mathfrak{g}'(B)$-module. On the other hand, for indefinite Kac-Moody algebras, as far as we know, there is no unified local Lie algebra construction using standard gradations as in [8]. However, indefinite (derived) Kac-Moody algebras studied in [1, 2, 4, 9, 10, 12] do admit minimal $\mathbb{Z}$-gradations. For example, if $\mathfrak{g} = HA_n$ [4, 10] or HC_n [12], then $\mathfrak{g}$ can be realized as the minimal $\mathbb{Z}$-graded Lie algebra associated to a local Lie algebra given by the corresponding affine algebra and a level 1 integrable highest weight module. Such a local Lie algebra construction of $\mathfrak{g}$ makes possible to compute root multiplicities of $\mathfrak{g}$ via weight multiplicities of an integrable highest weight module.

In this paper we shall study local Lie algebras of the form $L(\lambda)^* \oplus \mathfrak{g} \oplus L(\lambda)$ and their corresponding minimal $\mathbb{Z}$-graded Lie algebras, where $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra and $L(\lambda)$ is an integrable highest weight $\mathfrak{g}$-module. This consideration is motivated by a class of indefinite Kac-Moody algebras arising from 2-fold affinization of Cartan matrices, which we review briefly as follows.

Given a Cartan matrix A of simply-laced type of rank $\ell \geq 2$, let $A^{[2]}$ be the 2-fold affinization of A with respect to the highest root [3]. Then $A^{[2]}$ is a generalized intersection matrix $A^{[2]}$ in the sense of Slodowy [15]. By using a canonical diagram automorphism $\boldsymbol{\tau}$ of the covering matrix of $A^{[2]}$ we get a generalized Cartan matrix $A^{[2],\boldsymbol{\tau}}$ of indefinite type (see (26) below). As far as we know $A^{[2],\boldsymbol{\tau}}$ firstly appeared in [14], used to study toroidal Weyl groups. Actually $A^{[2],\boldsymbol{\tau}}$ is a folding Cartan matrix in the sense of [6], and we call the Kac-Moody algebra $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ of $A^{[2],\boldsymbol{\tau}}$ an orbit Lie algebra as in [6].

A question arises whether $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ has a minimal $\mathbb{Z}$-gradation as in the cases of HA_n and HC_n. We study this question in a little more general setting. Given a pair $(\widehat{\mathfrak{g}}, \mathfrak{g})$ of symmetrizable Kac-Moody algebras, we prove that, under a condition (see (18) below, which forces $\widehat{\mathfrak{g}}$ to be indefinite, see Lemma 3.1) on their generalized Cartan matrices, $\widehat{\mathfrak{g}}$ has a minimal $\mathbb{Z}$-gradation with its local part being of the form $L(\lambda)^* \oplus \mathfrak{g} \oplus L(\lambda)$, where $L(\lambda)$ is an integrable highest weight $\mathfrak{g}$-module, and $L(\lambda)^*$ is its contragradient dual (see Theorem 3.4 below). Then we show that this result is applicable to $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ and Lorentzian Kac-Moody algebras.

The main difference of the method to establish Theorem 3.4 from those in [1, 2, 4, 8, 9, 10, 12] is that we use a complete description of all local Lie algebra structures on $L(\lambda)^* \oplus \mathfrak{g} \oplus L(\lambda)$. Recall that, for any Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-modules V, W, to obtain a local Lie algebra $V \oplus \mathfrak{g} \oplus W$ it suffices to construct a $\mathfrak{g}$-module homomorphism $\varphi\colon V \otimes W \to \mathfrak{g}$. In [1, §2, §3], [4], [12] such a φ is explicitly given by using an orthonormal basis of $\mathfrak{g}$. Here we show that, given a symmetrizable Kac-Moody algebra $\mathfrak{g}$ and any integrable highest weight $\mathfrak{g}$-module $L(\lambda)$, it holds that $\mathrm{Hom}_{\mathfrak{g}}(L(\lambda)^* \otimes L(\lambda), \mathfrak{g}) = \mathfrak{h}^\lambda$ (for definition see Lemma 2.3). It follows that any correspondence $v^* \otimes v \mapsto x \in \mathfrak{h}^\lambda$ determines uniquely a $\mathfrak{g}$-module homomorphism φ from $L(\lambda)^* \otimes L(\lambda)$ to $\mathfrak{g}$, where $v \in L(\lambda)$ (resp. $v^* \in L(\lambda)^*$) is a highest (resp. lowest)-weight vector such that $v^*(v) = 1$. Hence a local Lie algebra $L(\lambda)^* \oplus \mathfrak{g} \oplus L(\lambda)$ is obtained, and any local Lie algebra structure on $L(\lambda)^* \oplus \mathfrak{g} \oplus L(\lambda)$ arises in this way (Proposition 2.4). This result is used to construct the local Lie algebra $L(-\alpha_{n+1})^* \oplus \mathfrak{g} \oplus L(-\alpha_{n+1})$ (Corollary 3.2) and its Kac-Kantor algebra.

Unlike the cases of HA_n and HC_n, the minimal $\mathbb{Z}$-gradation of $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ is given by a level 4, not level 1, integrable highest weight module of the corresponding affine algebra. So it would be more complicated to use such a local Lie algebra construction to compute root multiplicities of $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$. However, we can describe some roots of $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ in term of representation theory of affine algebras (Corollary 4.4) by using its local Lie algebra construction. Furthermore we show that $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ is a subalgebra of the corresponding Lorentzian Kac-Moody algebra $\mathfrak{g}(A_\ell^H)$ by using Feingold and Nicolai's method [5], and hence is a subalgebra of the hyperbolic algebra E_{10} by a result of Viswanath [16].

All Lie algebras in this paper are over the complex number field $\mathbb{C}$. For any Lie algebra L, $U(L)$ denotes the universal enveloping algebra of L. All generalized Cartan matrices are assumed to be indecomposable. For any matrix X we denote the (i,j)-entry of X by $X(i,j)$.

2. Preliminary.

Let $\mathfrak{g} = \mathfrak{g}(C) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a Kac-Moody algebra associated to a generalized Cartan matrix $C = (C(i,j))_{n\times n}$ with a minimal realization $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\Pi = \{\alpha_1, \cdots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{h_1, \cdots, h_n\} \subset \mathfrak{h}$ are linearly independent subsets such that $\alpha_i(h_j) = C(i,j)$. Let $L(\lambda)$ be the integrable highest weight $\mathfrak{g}$-module associated to $\lambda \in \mathfrak{h}^*$. As is known, to describe local Lie algebra structures on $L(\lambda)^* \oplus \mathfrak{g} \oplus L(\lambda)$ it suffices to compute $\mathrm{Hom}_{\mathfrak{g}}(L(\lambda)^* \otimes L(\lambda), \mathfrak{g})$. In the following $M(\lambda)$ denotes the Verma module of $\mathfrak{g}$ associated to λ.

Consider the auxiliary Lie algebra $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(C)$, which is generated by e_i, f_i, $h \in \mathfrak{h}$ $(1 \le i \le n)$ subject to the following defining relations [7, §1.2]:

$$[\mathfrak{h}, \mathfrak{h}] = 0, \ [e_i, f_j] = \delta_{ij} h_i, \ [h, e_i] = \alpha_i(h) e_i, \ [h, f_i] = -\alpha_i(h) f_i. \tag{1}$$

Let $\tilde{\mathfrak{n}}_-$ (resp. $\tilde{\mathfrak{n}}_+$) be the Lie subalgebra of $\tilde{\mathfrak{g}}$ generated by $f_1, \cdots, f_n$ (resp. e_i, $\cdots$, e_n). Note that both $\tilde{\mathfrak{n}}_-$ and $\tilde{\mathfrak{n}}_+$ are free Lie algebras and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$. For $\lambda \in \mathfrak{h}^*$, the Verma module $\widetilde{M}(\lambda)$ of $\tilde{\mathfrak{g}}$ is the quotient of the universal enveloping algebra $U(\tilde{\mathfrak{g}})$ modulo the left ideal generated by $h - \lambda(h)$ $(h \in \mathfrak{h})$ and $\tilde{\mathfrak{n}}_+$. Let v be a nonzero highest-weight vector. Note that $\widetilde{M}(\lambda)$ is a free $U(\tilde{\mathfrak{n}}_-)$-module: $\widetilde{M}(\lambda) = U(\tilde{\mathfrak{n}}_-)v$, so $\widetilde{M}(\lambda)$ has a basis consisting of all elements of the form $f_J v = f_{i_1} f_{i_2} \cdots f_{i_s} v$, where $J = i_1 i_2 \cdots i_s$ is a word in $1 \le i_j \le n$. Dually, the finite dual (contragredient dual module) $\widetilde{M}(\lambda)^*$ is a lowest weight $\tilde{\mathfrak{g}}$-module with lowest weight $-\lambda$, which is the quotient of the $U(\tilde{\mathfrak{g}})$ modulo the left ideal generated by $h + \lambda(h)$ $(h \in \mathfrak{h})$ and $\tilde{\mathfrak{n}}_-$. Let v^* be a nonzero highest-weight vector such that $v^*(v) = 1$. Then $\widetilde{M}(\lambda)^*$ is a free $U(\tilde{\mathfrak{n}}_+)$-module: $\widetilde{M}(\lambda) = U(\tilde{\mathfrak{n}}_+)v^*$, so $\widetilde{M}(\lambda)$ has a basis consisting of all elements of the form $e_I v^* = e_{p_1} e_{p_2} \cdots e_{p_t} v^*$, where $I = p_1 p_2 \cdots p_t$ is a word in $1 \le p_j \le n$. We have the following

Lemma 2.1. *For any $x \in \mathfrak{h}$ there is a unique $\tilde{\mathfrak{g}}$-module homomorphism $\tilde{\varphi}$ from $\widetilde{M}(\lambda)^* \otimes \widetilde{M}(\lambda)$ to $\tilde{\mathfrak{g}}$ such that $\tilde{\varphi}(v^* \otimes v) = x$, and* $\mathrm{Hom}_{\tilde{\mathfrak{g}}}(\widetilde{M}(\lambda)^* \otimes \widetilde{M}(\lambda), \tilde{\mathfrak{g}}) \cong \mathfrak{h}$ *as vector spaces.*

Proof. Since both $\widetilde{M}(\lambda)^* \otimes \widetilde{M}(\lambda)$ and $\tilde{\mathfrak{g}}$ are weight $\tilde{\mathfrak{g}}$-modules, and $v^* \otimes v$ belongs to the 0-weight subspace of $\widetilde{M}(\lambda)^* \otimes \widetilde{M}(\lambda)$, the map $\varphi \mapsto \varphi(v^* \otimes v) \in \mathfrak{h}$ from $\mathrm{Hom}_{\tilde{\mathfrak{g}}}(\widetilde{M}(\lambda)^* \otimes \widetilde{M}(\lambda), \tilde{\mathfrak{g}})$ to $\mathfrak{h}$ is well defined.

It remains to show that, for any fixed $x \in \mathfrak{h}$ the correspondence $v^* \otimes v \mapsto x$ can be extended uniquely to a $\tilde{\varphi} \in \mathrm{Hom}_{\tilde{\mathfrak{g}}}(\widetilde{M}(\lambda)^* \otimes \widetilde{M}(\lambda), \tilde{\mathfrak{g}})$ such that $\tilde{\varphi}(v^* \otimes v) = x$. Since $e_I v^* \otimes f_J v$ for various words I, J form a basis of $\widetilde{M}(\lambda)^* \otimes \widetilde{M}(\lambda)$, it suffices to make induction on the lengths $|I|$ of I and $|J|$ of J to show that $\tilde{\varphi}(e_I v^* \otimes f_J v) \in \tilde{\mathfrak{g}}$ can be uniquely defined satisfying

(i) $[e_i, \tilde{\varphi}(e_{I'} v^* \otimes f_{J'} v)] = \tilde{\varphi}(e_i(e_{I'} v^* \otimes f_{J'} v))$,

(ii) $[f_j, \tilde{\varphi}(e_{I'} v^* \otimes f_{J'} v)] = \tilde{\varphi}(f_j(e_{I'} v^* \otimes f_{J'} v))$,

for all $1 \leq i, j \leq n$, $|I'| \leq |I|$, $|J'| \leq |J|$, and

(iii) $[h, \tilde{\varphi}(e_I v^* \otimes f_J v)] = \tilde{\varphi}(h(e_I v^* \otimes f_J v))$ for all $h \in \mathfrak{h}$.

Since $e_i v^* \otimes v = e_i(v^* \otimes v)$ and $v^* \otimes f_j v = f_j(v^* \otimes v)$, by (i) and (ii) there is only one way to define

$$\tilde{\varphi}(e_i v^* \otimes v) \triangleq [e_i, x] = -\alpha_i(x) e_i, \;\; \tilde{\varphi}(v^* \otimes f_j v) \triangleq [f_j, x] = \alpha_j(x) f_j, \tag{2}$$

and it's direct to see that (iii) is also satisfied: For any $h \in \mathfrak{h}$,

$$[h, \tilde{\varphi}(e_i v^* \otimes v)] = \tilde{\varphi}(h(e_i v^* \otimes v)), \;\; [h, \tilde{\varphi}(v^* \otimes f_j v)] = \tilde{\varphi}(h(v^* \otimes f_j v)). \tag{3}$$

Since in $U(\tilde{\mathfrak{g}})$ it holds that $e_i f_j = \delta_{ij} h_i + f_j e_i$ and $e_i v = 0$,

$$\begin{aligned} e_i v^* \otimes f_j v &= e_i(v^* \otimes f_j v) - v^* \otimes (e_i f_j v) = e_i(v^* \otimes f_j v) - \delta_{ij} v^* \otimes h_i v \\ &= e_i(v^* \otimes f_j v) - \delta_{ij} \lambda(h_i) v^* \otimes v, \end{aligned}$$

by (i) there is only one way to define

$$\tilde{\varphi}(e_i v^* \otimes f_j v) \triangleq [e_i, \tilde{\varphi}(v^* \otimes f_j v)] - \delta_{ij} \lambda(h_i) \tilde{\varphi}(v^* \otimes v). \tag{4}$$

Similarly, since

$$\begin{aligned} e_i v^* \otimes f_j v &= f_j(e_i v^* \otimes v) - f_j e_i v^* \otimes v = f_j(e_i v^* \otimes v) + \delta_{ij} h_i v^* \otimes v \\ &= f_j(e_i v^* \otimes v) - \delta_{ij} \lambda(h_i) v^* \otimes v, \end{aligned}$$

by (ii) there is only one way to define

$$\tilde{\varphi}(e_i v^* \otimes f_j v) \triangleq [f_j, \tilde{\varphi}(e_i v^* \otimes v)] - \delta_{ij} \lambda(h_i) \tilde{\varphi}(v^* \otimes v). \tag{5}$$

By (2) it's direct to check that definitions of $\tilde{\varphi}(e_i v^* \otimes f_j v)$ in (4) and (5) coincide. Moreover, for any $h \in \mathfrak{h}$, by (3) and (4) it follows that

$$\begin{aligned} [h,\, \tilde{\varphi}(e_i v^* \otimes f_j v)] &= [h,\, [e_i,\, \tilde{\varphi}(v^* \otimes f_j v)]] \\ &= \alpha_i(h)[e_i,\, \tilde{\varphi}(v^* \otimes f_j v)] + [e_i,\, \tilde{\varphi}(h(v^* \otimes f_j v))] \\ &= (\alpha_i - \alpha_j)(h)[e_i,\, \tilde{\varphi}(v^* \otimes f_j v)] \\ &= (\alpha_i - \alpha_j)(h)([e_i,\, \tilde{\varphi}(v^* \otimes f_j v)] - \delta_{ij} \lambda(h_i) \tilde{\varphi}(v^* \otimes v)) \\ &= (\alpha_i - \alpha_j)(h) \tilde{\varphi}(e_i v^* \otimes f_j v) \\ &= \tilde{\varphi}(h(e_i v^* \otimes f_j v)), \end{aligned}$$

which means that (iii) is also satisfied. So $\tilde{\varphi}(e_I v^* \otimes f_J v)$ satisfying (i)-(iii) is determined uniquely for $|I| \le 1$ and $|J| \le 1$.

Assume that $\tilde{\varphi}(e_{I'} v^* \otimes f_{J'} v)$ satisfying (i)-(iii) is uniquely defined for any $|I'| \le |I|$ and $|J'| \le |J|$. We show that $\tilde{\varphi}(e_i e_I v^* \otimes f_J v)$, $\tilde{\varphi}(e_I v^* \otimes f_j f_J v)$ and $\tilde{\varphi}(e_i e_I v^* \otimes f_j f_J v)$ satisfying (i)-(iii) can be defined uniquely. Note that in $U(\tilde{\mathfrak{g}})$

$$e_i f_J = f_J e_i + r_i(J), \quad f_j e_I = e_I f_j + r'_j(I), \tag{6}$$

where $r_i(J)$ (resp. $r'_j(I)$) is either in $U(\mathfrak{h})$ or in $U(\tilde{\mathfrak{n}}_-)$ (resp. $U(\tilde{\mathfrak{n}}_+)$) which is a linear combination of $f_{J'}$ (resp. $e_{I'}$) with $|J'| = |J| - 1$ (resp. $|I'| = |I| - 1$). Also, in $U(\tilde{\mathfrak{g}})$ we have, for any $h \in \mathfrak{h}$ and words I, J,

$$[h, e_I] = he_I - e_I h = \alpha(h) e_I, \ \ [h, f_J] = hf_J - f_J h = -\beta(h) f_J, \tag{7}$$

where $\alpha, \beta \in \mathfrak{h}^*$ are given by I and J respectively. By (6) and (7) it follows that

$$[h, r_i(J)] = (\alpha_i - \beta)(h) r_i(J), \quad [h, r'_j(I)] = (\alpha - \alpha_j)(h) r'_j(I), \quad h \in \mathfrak{h}. \tag{8}$$

By (6) and $e_i v = 0$,

$$\begin{aligned} e_i e_I v^* \otimes f_J v &= e_i(e_I v^* \otimes f_J v) - e_I v^* \otimes e_i f_J v \\ &= e_i(e_I v^* \otimes f_J v) - e_I v^* \otimes r_i(J) v. \end{aligned}$$

So, by (i) there is only one way to define

$$\tilde{\varphi}(e_i e_I v^* \otimes f_J v) \triangleq [e_i, \tilde{\varphi}(e_I v^* \otimes f_J v)] - \tilde{\varphi}(e_I v^* \otimes r_i(J) v), \tag{9}$$

which is unique by inductive hypothesis. By (7), (8) and (9), for any $h \in \mathfrak{h}$ it follows that

$$\begin{aligned} [h, \tilde{\varphi}(e_i e_I v^* \otimes f_J v)] &= [h, [e_i, \tilde{\varphi}(e_I v^* \otimes f_J v)]] - [h, \tilde{\varphi}(e_I v^* \otimes r_i(J) v)] \\ &= \alpha_i(h)[e_i, \tilde{\varphi}(e_I v^* \otimes f_J v)] + [e_i, [h, \tilde{\varphi}(e_I v^* \otimes f_J v)]] \\ &\quad -[h, \tilde{\varphi}(e_I v^* \otimes r_i(J) v)] \\ &= \alpha_i(h)[e_i, \tilde{\varphi}(e_I v^* \otimes f_J v)] + [e_i, \tilde{\varphi}(h(e_I v^* \otimes f_J v))] \\ &\quad -\tilde{\varphi}(h(e_I v^* \otimes r_i(J) v)) \ \text{((iii) of inductive hypothesis)} \\ &= (\alpha - \beta + \alpha_i)(h) \tilde{\varphi}(e_i e_I v^* \otimes f_J v) \\ &= \tilde{\varphi}(h(e_i e_I v^* \otimes f_J v)), \end{aligned}$$

which means that (iii) is satisfied for the definition in (9). Similarly, since

$$\begin{aligned} e_I v^* \otimes f_j f_J v &= f_j(e_I v^* \otimes f_J v) - f_j e_I v^* \otimes f_J v \\ &= f_j(e_I v^* \otimes f_J v) - r'_j(I) v^* \otimes f_J v, \end{aligned}$$

by (ii) there is only one way to define

$$\tilde{\varphi}(e_I v^* \otimes f_j f_J v) \triangleq [f_j, \tilde{\varphi}(e_I v^* \otimes f_J v)] - \tilde{\varphi}(r'_j(I) v^* \otimes f_J v), \tag{10}$$

which is unique by inductive hypothesis. By a similar argument it follows that, for any $h \in \mathfrak{h}$,

$$\begin{aligned} [h, \tilde{\varphi}(e_I v^* \otimes f_j f_J v)] &= (\alpha - \beta - \alpha_j)(h) \tilde{\varphi}(e_I v^* \otimes f_j f_J v) \\ &= \tilde{\varphi}(h(e_I v^* \otimes f_j f_J v)), \end{aligned}$$

which means that (iii) is satisfied for the definition in (10). Note that

$$\begin{aligned} e_ie_Iv^*\otimes f_jf_Jv &= e_i(e_Iv^*\otimes f_jf_Jv)-e_Iv^*\otimes e_if_jf_Jv \\ &= e_i(f_j(e_Iv^*\otimes f_Jv)-f_je_Iv^*\otimes f_Jv)-e_Iv^*\otimes \delta_{ij}h_if_Jv \\ &\quad -e_Iv^*\otimes f_je_if_Jv \\ &= (e_if_j)(e_Iv^*\otimes f_Jv)-e_i(f_je_Iv^*\otimes f_Jv)-e_Iv^*\otimes \delta_{ij}h_if_Jv \\ &\quad -e_Iv^*\otimes f_jr_i(J)v \\ &= (e_if_j)(e_Iv^*\otimes f_Jv)-e_i(r_j'(I)v^*\otimes f_Jv)-e_Iv^*\otimes \delta_{ij}h_if_Jv \\ &\quad -f_j(e_Iv^*\otimes r_i(J)v)-r_j'(I)v^*\otimes r_i(J)v. \end{aligned}$$

So, by (i) and (ii) there is only one way to define

$$\begin{aligned} \tilde\varphi(e_ie_Iv^*\otimes f_jf_Jv) \triangleq &[e_i,[f_j,\tilde\varphi(e_Iv^*\otimes f_Jv)]]-[e_i,\tilde\varphi(r_j'(I)v^*\otimes f_Jv)] \\ &-[f_j,\tilde\varphi(e_Iv^*\otimes r_i(J)v)]-\tilde\varphi(e_Iv^*\otimes \delta_{ij}h_if_Jv) \\ &-\tilde\varphi(r_j'(I)v^*\otimes r_i(J)v), \end{aligned} \tag{11}$$

which is unique by inductive hypothesis. Similarly,

$$\begin{aligned} e_ie_Iv^*\otimes f_jf_Jv &= f_j(e_ie_Iv^*\otimes f_Jv)-f_je_ie_Iv^*\otimes f_Jv \\ &= (f_je_i)(e_Iv^*\otimes f_Jv)-f_j(e_Iv^*\otimes r_i(J)v)+\delta_{ij}h_ie_Iv^*\otimes f_Jv \\ &\quad -e_i(r_j'(I)v^*\otimes f_Jv)-r_j'(I)v^*\otimes r_i(J)v. \end{aligned}$$

So, by (i) and (ii) there is only one way to define

$$\begin{aligned} \tilde\varphi(e_ie_Iv^*\otimes f_jf_Jv) \triangleq &[f_j,[e_i,\tilde\varphi(e_Iv^*\otimes f_Jv)]]-[e_i,\tilde\varphi(r_j'(I)v^*\otimes f_Jv)] \\ &-[f_j,\tilde\varphi(e_Iv^*\otimes r_i(J)v)]+\tilde\varphi(\delta_{ij}h_ie_Iv^*\otimes f_Jv) \\ &-\tilde\varphi(r_j'(I)v^*\otimes r_i(J)v), \end{aligned} \tag{12}$$

which is unique by inductive hypothesis. The difference of the right sides in (11) and (12) is

$$\begin{aligned} &[[e_i,f_j],\tilde\varphi(e_Iv^*\otimes f_Jv)]-\delta_{ij}\tilde\varphi(e_Iv^*\otimes h_if_Jv+h_ie_Iv^*\otimes f_Jv) \\ = &\ \delta_{ij}\tilde\varphi(h_i(e_Iv^*\otimes f_Jv)-e_Iv^*\otimes h_if_Jv-h_ie_Iv^*\otimes f_Jv)=0, \end{aligned}$$

which means that the definitions in (11) and (12) are equal. Thus $\tilde\varphi(e_ie_Iv^*\otimes f_jf_Jv)$ is well defined by (11), or equivalently, by (12). For any $h\in\mathfrak{h}$, by (11), or equivalently, (12) and (iii) of the inductive hypothesis, using similar argument as in the computation of $[h,\,\tilde\varphi(e_ie_Iv^*\otimes f_Jv)]$ before, we get

$$\begin{aligned} [h,\,\tilde\varphi(e_ie_Iv^*\otimes f_jf_Jv)] &= (\alpha-\beta+\alpha_i-\alpha_j)(h)\tilde\varphi(e_ie_Iv^*\otimes f_jf_Jv) \\ &= \tilde\varphi(h(e_ie_Iv^*\otimes f_jf_Jv)), \end{aligned}$$

which means that (iii) is satisfied for the definition in (11), or equivalently, (12). This completes the induction. ■

We denote again by v the highest-weight vector of the Verma module $M(\lambda)$ (resp. irreducible highest weight module $L(\lambda)$), and v^* the lowest-weight vector of the contragredient module $M(\lambda)^*$ (resp. $L(\lambda)^*$) such that $v^*(v)=1$.

We recall some definitions and facts given in [7]. Let $\mathfrak{r}$ be the maximal ideal of $\tilde{\mathfrak{g}}$ among the ideals which intersect $\mathfrak{h}$ trivially. Then $\mathfrak{r} = \mathfrak{r}_- \oplus \mathfrak{r}_+$ is a direct sum of ideals, where $\mathfrak{r}_\pm = \mathfrak{r} \cap \tilde{\mathfrak{n}}_\pm$ (cf. [7, Theorem 1.2]), and $\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{r}$ by definition. Note that $\mathfrak{n}_\pm = \tilde{\mathfrak{n}}_\pm/\mathfrak{r}_\pm$. For any $\lambda \in \mathfrak{h}^*$, we have $M(\lambda) \cong U(\mathfrak{n}_-)$ and $M(\lambda)^* \cong U(\mathfrak{n}_+)$ as vector spaces. Set

$$N(\lambda) = \left(U(\mathfrak{r}_+) \otimes \widetilde{M}(\lambda)\right) + \left(\widetilde{M}(\lambda)^* \otimes U(\mathfrak{r}_-)\right), \tag{13}$$

where $U(\mathfrak{r}_\pm)$ is the universal enveloping algebra of $\mathfrak{r}_\pm$. Then, under the identification of vector spaces $\widetilde{M}(\lambda)^* \leftrightarrow U(\tilde{\mathfrak{n}}_+)$ given by $e_I v^* \leftrightarrow e_I$ and $\widetilde{M}(\lambda) \leftrightarrow U(\tilde{\mathfrak{n}}_-)$ given by $f_J v \leftrightarrow f_J$, respectively, we have

$$M(\lambda)^* \otimes M(\lambda) = \frac{\widetilde{M}(\lambda)^* \otimes \widetilde{M}(\lambda)}{N(\lambda)}. \tag{14}$$

Lemma 2.2. *For any $x \in \mathfrak{h}$ there is a unique $\mathfrak{g}$-module homomorphism φ from $M(\lambda)^* \otimes M(\lambda)$ to $\mathfrak{g}$ such that $\varphi(v^* \otimes v) = x$, and $\mathrm{Hom}_{\mathfrak{g}}(M(\lambda)^* \otimes M(\lambda), \mathfrak{g}) \cong \mathfrak{h}$ as vector spaces.*

Proof. It suffices to show that the correspondence $v^* \otimes v \mapsto x$ can be uniquely extended to a $\mathfrak{g}$-module homomorphism from $M(\lambda)^* \otimes M(\lambda)$ to $\mathfrak{g}$.

For any $x \in \mathfrak{h}$, let $\tilde{\varphi} \in \mathrm{Hom}_{\tilde{\mathfrak{g}}}(\widetilde{M}(\lambda)^* \otimes \widetilde{M}(\lambda), \tilde{\mathfrak{g}})$ be the unique homomorphism given by Lemma 2.1 such that $\tilde{\varphi}(v^* \otimes v) = x$. For any $r \in \mathfrak{r}$, $\tilde{\varphi}(r(e_I v^* \otimes f_J v)) = [r, \tilde{\varphi}(e_I v^* \otimes f_J v)] \in \mathfrak{r}$ for any words I, J. So, by (13) and (14) it suffices to show that $\tilde{\varphi}(N(\lambda)) \subset \mathfrak{r}$.

For any $r \in \mathfrak{r}_+ \subset \tilde{\mathfrak{n}}_+$, $y \in \tilde{\mathfrak{n}}_-$, since $[r, y] \in \tilde{\mathfrak{n}}_+$, $ryv = [r, y]v + yrv = 0$. By this and the fact that $\tilde{\varphi} \in \mathrm{Hom}_{\tilde{\mathfrak{g}}}(\widetilde{M}(\lambda)^* \otimes \widetilde{M}(\lambda), \tilde{\mathfrak{g}})$ it follows that

$$\tilde{\varphi}(rv^* \otimes yv) = \tilde{\varphi}(r(v^* \otimes yv)) - \tilde{\varphi}(v^* \otimes ryv) = [r, \tilde{\varphi}(v^* \otimes yv)] \in \mathfrak{r}_+.$$

Moreover, for any $r, r' \in \mathfrak{r}_+$, by the same argument as above,

$$\tilde{\varphi}(rr'v^* \otimes yv) = \tilde{\varphi}(r(r'v^* \otimes yv)) - \tilde{\varphi}(r'v^* \otimes ryv) = [r, \tilde{\varphi}(r'v^* \otimes yv)] \in \mathfrak{r}_+.$$

Since $U(\mathfrak{r}_+)$ is generated by $\mathfrak{r}_+$, we have shown that $\tilde{\varphi}\left(U(\mathfrak{r}_+) \otimes \widetilde{M}(\lambda)\right) \subset \mathfrak{r}_+$. Similarly, we have $\tilde{\varphi}\left(\widetilde{M}(\lambda)^* \otimes U(\mathfrak{r}_-)\right) \subset \mathfrak{r}_-$. So, $\tilde{\varphi}(N(\lambda)) \subset \mathfrak{r}$, which means that $\tilde{\varphi}$ factors to a $\mathfrak{g}$-module homomorphism φ from $M(\lambda)^* \otimes M(\lambda)$ to $\mathfrak{g}$, extending the correspondence $v^* \otimes v \mapsto x \in \mathfrak{h}$. ∎

Assume further that $\mathfrak{g}$ is symmetrizable and λ is a dominant integral weight. Then the integrable highest weight $\mathfrak{g}$-module $L(\lambda)$ is given by (cf. [7, (10.4.6) or Corollary 10.4])

$$L(\lambda) = \frac{M(\lambda)}{\sum_{i=1}^{n} \left(U(\mathfrak{n}_-) f_i^{\lambda(h_i)+1} v\right)}.$$

Dually, the integrable lowest weight $\mathfrak{g}$-module $L(\lambda)^*$ is given by

$$L(\lambda)^* = \frac{M(\lambda)^*}{\sum_{i=1}^{n}\left(U(\mathfrak{n}_+)e_i^{\lambda(h_i)+1}v^*\right)}.$$

Let $J(\lambda)$ be the subspace

$$\left(\sum_{i=1}^{n}\left(U(\mathfrak{n}_+)e_i^{\lambda(h_i)+1}v^*\right)\otimes M(\lambda)\right) + \left(M(\lambda)^*\otimes \sum_{i=1}^{n}\left(U(\mathfrak{n}_-)f_i^{\lambda(h_i)+1}v\right)\right)$$

of $M(\lambda)^*\otimes M(\lambda)$. Then we have

$$L(\lambda)^*\otimes L(\lambda) = \frac{M(\lambda)^*\otimes M(\lambda)}{J(\lambda)}. \tag{15}$$

Lemma 2.3. *Assume that $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra and λ is a dominant integral weight. Let $\mathrm{t}(\lambda)$ be the set of j's such that $\lambda(h_j)=0$. Let v (resp. v^*) be a highest-weight vector of the integrable highest weight $\mathfrak{g}$-module $L(\lambda)$ (resp. the lowest-weight vector of the $\mathfrak{g}$-module $L(\lambda)^*$) such that $v^*(v)=1$. Set $\mathfrak{h}^\lambda = \cap_{j\in \mathrm{t}(\lambda)}\ker\alpha_j \subseteq \mathfrak{h}$. Then for any $x\in\mathfrak{h}^\lambda$ there is a unique $\mathfrak{g}$-module homomorphism φ from $L(\lambda)^*\otimes L(\lambda)$ to $\mathfrak{g}$ such that $\varphi(v^*\otimes v)=x$, and $\mathrm{Hom}_{\mathfrak{g}}(L(\lambda)^*\otimes L(\lambda),\,\mathfrak{g})\cong\mathfrak{h}^\lambda$ as vector spaces. (Here, if the set $\mathrm{t}(\lambda)$ is empty then $\mathfrak{h}^\lambda=\mathfrak{h}$.)*

Proof. For any $\varphi\in\mathrm{Hom}_{\mathfrak{g}}(L(\lambda)^*\otimes L(\lambda),\,\mathfrak{g})$, we claim that $\varphi(v^*\otimes v)\in\mathfrak{h}^\lambda$. Indeed, by (15), if $\varphi(v^*\otimes v)=x\in\mathfrak{h}$ and $\lambda(h_j)=0$ for some j, then $f_jv=0$ and hence

$$0=\varphi(v^*\otimes f_jv)=[f_j,\,\varphi(v^*\otimes v)]=[f_j,\,x]=\alpha_j(x)f_j, \tag{16}$$

which means that $x\in\mathfrak{h}^\lambda$, and the claim follows. So the map $\varphi\mapsto\varphi(v^*\otimes v)$ from $\mathrm{Hom}_{\mathfrak{g}}(L(\lambda)^*\otimes L(\lambda),\,\mathfrak{g})$ to $\mathfrak{h}^\lambda$ is well defined.

So, for any $x\in\mathfrak{h}^\lambda$ it suffices to show that the correspondence $v^*\otimes v\mapsto x$ can be uniquely extended to a $\mathfrak{g}$-module homomorphism from $L(\lambda)^*\otimes L(\lambda)$ to $\mathfrak{g}$. Let $\varphi\in\mathrm{Hom}_{\mathfrak{g}}(M(\lambda)^*\otimes M(\lambda),\,\mathfrak{g})$ be the unique $\mathfrak{g}$-module homomorphism such that $\varphi(v^*\otimes v)=x$ given by Lemma 2.2. By (15) it suffices to show that φ annihilates $J(\lambda)$. We make the following

Claim: $\varphi(e_i^{\lambda(h_i)+1}v^*\otimes y_-v)=0=\varphi(y_+v^*\otimes f_i^{\lambda(h_i)+1}v)$ for $1\le i\le n$, $y_\pm\in U(\mathfrak{n}_\pm)$.

Since $U(\mathfrak{n}_\pm)$ has a $\mathbb{Z}_{\ge 0}$-gradation given by the generators e_j (resp. f_j), without loss of generality we may assume that $y_\pm$ is homogenous and we make induction on the degree of y_- to prove the first identity, while the second is the same.

At first $\varphi(e_i v^* \otimes v) = \varphi(e_i(v^* \otimes v)) = [e_i, x] = -\alpha_i(x)e_i$. So, if $\lambda(h_i) = 0$ then $\alpha_i(x) = 0$ and hence $\varphi(e_i v^* \otimes v) = 0$ since $x \in \mathfrak{h}^\lambda$. Otherwise $\varphi(e_i^2 v^* \otimes v) = \varphi(e_i(e_i v^* \otimes v)) = -\alpha_i(x)[e_i, e_i] = 0$, and hence $\varphi(e_i^{\lambda(h_i)+1} v^* \otimes v) = 0$ as required.

Assume that $\varphi(e_i^{\lambda(h_i)+1} v^* \otimes y_- v)=0$. We show that $\varphi(e_i^{\lambda(h_i)+1} v^* \otimes f_j y_- v)=0$ for all $1 \leq j \leq n$. Recall that in $M(\lambda)^*$ it holds that for any integer $k \geq 1$,

$$f_j e_i^k v^* = \delta_{ij} k(\lambda(h_i) - (k-1)) e_i^{k-1} v^*, \text{ in particular, } f_j e_i^{\lambda(h_i)+1} v^* = 0, \tag{17}$$

which can be verified by using the facts that $f_j e_i = -\delta_{ij} h_i + e_i f_j$ and $f_j v^* = 0$.

So, by inductive hypothesis and (17) we have that

$$\begin{aligned}\varphi(e_i^{\lambda(h_i)+1} v^* \otimes f_j y_- v) &= \varphi(f_j(e_i^{\lambda(h_i)+1} v^* \otimes y_- v)) - \varphi(f_j e_i^{\lambda(h_i)+1} v^* \otimes y_- v) \\ &= [f_j, \varphi(e_i^{\lambda(h_i)+1} v^* \otimes y_- v)] - 0 = 0\end{aligned}$$

as required, and the claim follows.

Lastly, for any e_j and $y_- \in U(\mathfrak{n}_-)$, by the claim as above

$$\begin{aligned}\varphi(e_j e_i^{\lambda(h_i)+1} v^* \otimes y_- v) &= \varphi(e_j(e_i^{\lambda(h_i)+1} v^* \otimes y_- v)) - \varphi(e_i^{\lambda(h_i)+1} v^* \otimes e_j y_- v) \\ &= [e_j, \varphi(e_i^{\lambda(h_i)+1} v^* \otimes y_- v)] - 0 = 0.\end{aligned}$$

Since $U(\mathfrak{n}_+)$ is generated by e_j $(1 \leq j \leq n)$, by induction hypothesis and

$$\begin{aligned}\varphi(e_k e_J e_i^{\lambda(h_i)+1} v^* \otimes y_- v) &= \varphi(e_k(e_J e_i^{\lambda(h_i)+1} v^* \otimes y_- v)) \\ &\quad -\varphi(e_J e_i^{\lambda(h_i)+1} v^* \otimes e_k y_- v),\end{aligned}$$

where e_J is a word in e_t, $1 \leq t \leq n$, it follows that

$$\varphi\left(\sum_{i=1}^n \left(U(\mathfrak{n}_+) e_i^{\lambda(h_i)+1} v^*\right) \otimes M(\lambda)\right) = 0.$$

Similarly, $\varphi\left(M(\lambda)^* \otimes \sum_{i=1}^n \left(U(\mathfrak{n}_-) f_i^{\lambda(h_i)+1} v\right)\right) = 0$. So $\varphi(J(\lambda)) = 0$, and hence φ factors to a $\mathfrak{g}$-module homomorphism from $L(\lambda)^* \otimes L(\lambda)$ to $\mathfrak{g}$ extending the correspondence $v^* \otimes v \mapsto x \in \mathfrak{h}^\lambda$. ∎

We would like to point out that Lemma 2.1 holds for any $\lambda \in \mathfrak{h}^*$, while Lemma 2.3 is shown only for dominant integral weights λ, since our proof needs the presentation of the irreducible module $L(\lambda)$ given by [7, (10.4.6) or Corollary 10.4]. We reformulate Lemma 2.3 in terms of local Lie algebras as the following

Proposition 2.4. *Assume that $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra and λ is a dominant integral weight. Let the notation be as in Lemma 2.3. Then any $x \in \mathfrak{h}^\lambda$ determines uniquely a local Lie algebra $L(\lambda)^* \oplus \mathfrak{g} \oplus L(\lambda)$ via $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(L(\lambda)^* \otimes L(\lambda), \mathfrak{g})$ given by $\varphi(v^* \otimes v) = x$, and any local Lie algebra structure on $L(\lambda)^* \oplus \mathfrak{g} \oplus L(\lambda)$ arises in this way.*

A special case of Lemma 2.3 is the following

Example 2.5. Let $(\mathfrak{g} = \mathfrak{g}(C), \mathfrak{h}, \Pi, \Pi^\vee)$ be a quadruple associated to a symmetrizable generalized Cartan matrix C. For $\lambda = 0$ and the trivial $\mathfrak{g}$-module $L(0) = \mathbb{C}$, it holds that $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{C} \otimes \mathbb{C}, \mathfrak{g}) = \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, \mathfrak{g}) \cong Z(\mathfrak{g})$, the center of $\mathfrak{g}$. Note that $Z(\mathfrak{g}) = \cap_{i=1}^n \ker \alpha_i$ ([7, Proposition 1.6]), which is exactly $\mathfrak{h}^0$ here by definition. Thus the fact that $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, \mathfrak{g}) \cong Z(\mathfrak{g})$ coincides with Lemma 2.3.

As an application of Lemma 2.3 we have the following

Corollary 2.6. *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and λ be a dominant integral weight. Then the multiplicity of the adjoint module $\mathfrak{g}$ in $L(\lambda)^* \otimes L(\lambda)$ is $\dim \mathfrak{h}^\lambda$.*

Proof. Since the adjoint $\mathfrak{g}$-module is simple, by Weyl's complete reducibility theorem and Schur Lemma, the multiplicity of $\mathfrak{g}$ in $L(\lambda)^* \otimes L(\lambda)$ is equal to $\dim \mathrm{Hom}_{\mathfrak{g}}(L(\lambda)^* \otimes L(\lambda), \mathfrak{g})$, which is $\dim \mathfrak{h}^\lambda$ by Lemma 2.3. ∎

Example 2.7. Assume that $\mathfrak{g} = \mathfrak{sl}_2$ with standard basis $\{x, h, y\}$. For any integer $m = \lambda(h) > 0$, the dual module $V(m)^*$ of the corresponding simple $\mathfrak{g}$-module $V(m)$ is isomorphic to $V(m)$ itself. By the classical Clebsch-Gordan formula it follows that the multiplicity of $\mathfrak{g} \cong V(2)$ in $V(m) \otimes V(m)$ is 1, which also follows by Corollary 2.6, since $\mathfrak{h}^\lambda = \mathfrak{h}$ in this case.

3. The minimal gradation.

Assume that $n > 1$ is an integer and a pair $(\widehat{C}, C)$ of indecomposable generalized Cartan matrices satisfies the following condition:

$$\begin{cases} \widehat{C} \text{ is a symmetrizable of order } n+1; \\ C \text{ is the first } n \times n \text{ principal submatrix of } \widehat{C}; \\ \mathrm{rank}(\widehat{C}) = 2 + \mathrm{rank}(C). \end{cases} \tag{18}$$

Fix a minimal realization $(\mathfrak{h}, \widehat{\Pi} = \{\alpha_1, \cdots, \alpha_{n+1}\}, \widehat{\Pi}^\vee = \{h_1, \cdots, h_{n+1}\})$ of $\widehat{C}$. Let $\widehat{\mathfrak{g}} := \widehat{\mathfrak{g}}(\widehat{C})$ be the Kac-Moody algebra associated to $\widehat{C}$ with respect to this minimal realization, and its Chevalley generators are e_i, f_i $(1 \le i \le n+1)$, $h \in \mathfrak{h}$.

Lemma 3.1. *(1) $\widehat{C}$ has indefinite type.*

(2) The triple $(\mathfrak{h}, \Pi = \{\alpha_1, \cdots, \alpha_n\}, \Pi^\vee = \{h_1, \cdots, h_n\})$ is a minimal realization of C. Hence the subalgebra of $\widehat{\mathfrak{g}}$ generate by e_i, f_i $(1 \le i \le n)$, $\mathfrak{h}$, is the Kac-Moody algebra $\mathfrak{g} := \mathfrak{g}(C)$ associated to C, in particular, $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ have the same Cartan subalgebra $\mathfrak{h}$.

(3) $-\alpha_{n+1}$ is a dominant integral weight of $\mathfrak{g}$ and $\mathfrak{h}_{n+1} \in \mathfrak{h}^{-\alpha_{n+1}}$ (for definition see Lemma 2.3).

Proof. (1) Assume that $\widehat{C}$ has either finite or affine type. Then C is of finite type. By $\mathrm{rank}(C) = \mathrm{rank}(\widehat{C}) - 2$ it follows that $\mathrm{rank}(C)$ is either $n-1$ or $n-2$, which means that C can not be of finite type, a contradiction.

(2) It suffices to note that $\dim \mathfrak{h} = 2(n+1) - \operatorname{rank}(\widehat{C}) = 2n - \operatorname{rank}(C)$ holds if and only if $\operatorname{rank}(\widehat{C}) = 2 + \operatorname{rank}(C)$.

(3) For any $1 \le j \le n$, $-\alpha_{n+1}(h_j) = -\widehat{C}(n+1, j) \in \mathbb{Z}_{\ge 0}$, and $-\alpha_{n+1}(h_j) = -\widehat{C}(n+1, j) = 0$ if and only if $\alpha_j(h_{n+1}) = \widehat{C}(j, n+1) = 0$ by the definition of generalized Cartan matrices. ■

It follows that the irreducible highest weight $\mathfrak{g} = \mathfrak{g}(C)$-module $L(-\alpha_{n+1})$ is integrable, and so is the contragradient dual $L(-\alpha_{n+1})^*$ of $L(\alpha_{n+1})$. Moreover, it's easy to see that C is symmetrizable. Choose a highest-weight vector v of $L(-\alpha_{n+1})$ and a lowest-weight vector v^* of $L(-\alpha_{n+1})^*$ such that $v^*(v) = 1$. Then by Proposition 2.4 and Lemma 3.1 (3) we have the following

Corollary 3.2. *$L(-\alpha_{n+1})^* \oplus \mathfrak{g} \oplus L(-\alpha_{n+1})$ becomes a local Lie algebra via the $\mathfrak{g}$-module homomorphism φ from $L(-\alpha_{n+1})^* \otimes L(-\alpha_{n+1})$ to $\mathfrak{g}$ extending the correspondence $v^* \otimes v \mapsto h_{n+1}$.*

So we have the maximal $\mathbb{Z}$-graded Lie algebra $\mathcal{F} = \oplus_{i=-\infty}^{+\infty} \mathcal{F}_i$ with local part $L(-\alpha_{n+1})^* \oplus \mathfrak{g} \oplus L(-\alpha_{n+1})$ given as follows (see [8, Proposition 4]):

$$\begin{aligned} &\mathcal{F}_{-1} = L(-\alpha_{n+1})^*,\ \mathcal{F}_0 = \mathfrak{g},\ \mathcal{F}_1 = L(-\alpha_{n+1});\\ &\mathcal{F}^+ := \sum_{i\ge 1} \mathcal{F}_i \text{ (resp. } \mathcal{F}^- := \sum_{i \ge 1} \mathcal{F}_{-i}) \text{ is the free Lie algebra}\\ &\qquad\qquad \text{generated by } L(-\alpha_{n+1}) \text{ (resp. } L(-\alpha_{n+1})^*), \end{aligned} \tag{19}$$

and the Lie bracket of $\mathcal{F}$ is uniquely determined by the following relations via Leibniz rule:

$$\begin{aligned} &[a,u] = au = -[u,a],\ [a,w^*] = aw^* = -[w^*,a]:\ a \in \mathcal{F}_0, u \in \mathcal{F}_1, w^* \in \mathcal{F}_{-1},\\ &[u,w^*] = -[w^*,u] = \varphi(w^* \otimes u):\ u \in \mathcal{F}_1, w^* \in \mathcal{F}_{-1},\\ &[a,[u',u'']] = [[a,u'],u''] + [u',[a,u'']]:\ a \in \mathcal{F}_0, u', u'' \in \mathcal{F}^+,\\ &[a,[\eta',\eta'']] = [[a,\eta'],\eta''] + [\eta',[a,\eta'']]:\ a \in \mathcal{F}_0, \eta', \eta'' \in \mathcal{F}^-,\\ &[u,[\eta',\eta'']] = [[u,\eta'],\eta''] + [\eta',[u,\eta'']]:\ u \in \mathcal{F}^+, \eta', \eta'' \in \mathcal{F}^-,\\ &[\eta,[u',u'']] = [[\eta,u'],u''] + [u',[\eta,u'']]:\ u', u'' \in \mathcal{F}^+, \eta \in \mathcal{F}^-. \end{aligned} \tag{20}$$

We have the following

Lemma 3.3. *As a Lie algebra $\mathcal{F}$ is generated by e_i, v^*, f_i, v $(1 \le i \le n)$, $h \in \mathfrak{h}$.*

Proof. Similar to the proof of [1, Proposition 2.12]. ■

By (20) and the definition of $\varphi:\ L(-\alpha_{n+1})^* \otimes L(-\alpha_{n+1}) \to \mathfrak{g}$ which extends $v^* \otimes v \mapsto h_{n+1} \in \mathfrak{h}^{-\alpha_{n+1}}$, the following hold in $\mathcal{F}$: For any $h, h' \in \mathfrak{h}$,

$$\begin{aligned} &[h,e_i] = \alpha_i(h)e_i,\ [h,f_i] = -\alpha_i(h)f_i,\ [e_i,f_j] = \delta_{ij}h_i,\ 1 \le i,j \le n;\\ &[h,v^*] = hv^* = \alpha_{n+1}(h)v^*,\ [h,v] = hv = -\alpha_{n+1}(h)v,\ [h,h'] = 0,\\ &[e_i,v] = 0 = [f_i,v^*],\ [v^*,v] = h_{n+1},\ 1 \le i \le n. \end{aligned} \tag{21}$$

Let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(\widehat{C})$ be the auxiliary Lie algebra associated to $\widehat{C}$ (cf. (1)). By Lemma 3.3 and (21) there is a surjective Lie algebra homomorphism $\psi\colon\ \tilde{\mathfrak{g}} \to \mathcal{F}$ given by

$$\begin{aligned}&\psi(e_i) = e_i,\, \psi(f_i) = f_i,\ \psi(h) = h,\ 1 \le i \le n, h \in \mathfrak{h},\\ &\psi(e_{n+1}) = v^*,\, \psi(f_{n+1}) = v.\end{aligned} \tag{22}$$

By (21) and Lemma 3.3 $\mathcal{F}$ has the following weight subspace decomposition:

$$\begin{aligned}&\mathcal{F} = \oplus_{\alpha\in\mathfrak{h}^*}\mathcal{F}^\alpha,\ \mathcal{F}^\alpha = \{x \in \mathcal{F} \,|\, [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\},\\ &\alpha \in \widehat{Q} := \oplus_{i=1}^{n+1}\mathbb{Z}\alpha_i.\end{aligned} \tag{23}$$

By (22) ψ maps the root subspace $\tilde{\mathfrak{g}}^\alpha$ to $\mathcal{F}^\alpha$. In particular, since $\tilde{\mathfrak{g}}$ has triangular decomposition, so does $\mathcal{F}$, that is, $\mathcal{F}^\alpha \ne 0,\ \alpha \ne 0$ imply that either $\alpha > 0$ or $\alpha < 0$ in the usual sense. Moreover, $\mathcal{F}$ becomes a $\mathfrak{g}$-module via the adjoint action. For $k > 1$, let $\mathcal{J}_{\pm k}$ be the subspaces

$$\mathcal{J}_{\pm k} = \{x \in \mathcal{F}_{\pm k} \,|\, [y_1, \cdots, y_{k-1}, x] = 0 \text{ for all } y_1, \cdots, y_{k-1} \in \mathcal{F}_{\mp 1}\}. \tag{24}$$

Then $\mathcal{J}^+ := \sum_{k>1}\mathcal{J}_k$ (resp. $\mathcal{J}^- := \sum_{k>1}\mathcal{J}_{-k}$) is an ideal of $\mathcal{F}^+$ (resp. $\mathcal{F}^-$), and $\mathcal{J}^+ + \mathcal{J}^-$ is the largest graded ideal of $\mathcal{F}$ which intersects the local part trivially (see [1, Proposition 1.7] or [4, Proposition 4.2]). Thus we get Kac-Kantor algebra

$$\mathcal{L} = \mathcal{L}(\widehat{C}) := \mathcal{F}/\mathcal{J} = \mathcal{F}^+/\mathcal{J}^+ \oplus \mathcal{F}_0 \oplus \mathcal{F}^-/\mathcal{J}^- \tag{25}$$

with the local part being $L(-\alpha_{n+1})^* \oplus \mathfrak{g} \oplus L(-\alpha_{n+1})$. The following is the main result of this paper.

Theorem 3.4. *Let $(\widehat{C}, C)$ be a pair of generalized Cartan matrices satisfying* (18). *Let $\widehat{\mathfrak{g}}$ (resp. $\mathfrak{g}$) be the Kac-Moody algebra associated to $\widehat{C}$ (resp. C), and $\mathcal{L}$ be the $\mathbb{Z}$-graded Lie algebra given by* (25). *Then there is a Lie algebra isomorphism $\psi : \widehat{\mathfrak{g}} \to \mathcal{L}$ given by $e_i \mapsto e_i$, $f_i \mapsto f_i$ $(1 \le i \le n)$, $e_{n+1} \mapsto v^*$, $f_{n+1} \mapsto v$ and $h \mapsto h$. In particular, $\widehat{\mathfrak{g}}$ has a minimal $\mathbb{Z}$-gradation with local part $L(-\alpha_{n+1})^* \oplus \mathfrak{g} \oplus L(-\alpha_{n+1})$.*

Proof. At first we show that $\psi\colon\ \tilde{\mathfrak{g}} \to \mathcal{F}$ given by (22) factors through $\widehat{\mathfrak{g}}$ to $\mathcal{L}$. It suffices to show that, for any ideal $\mathcal{I}$ of the auxiliary Lie algebra $\tilde{\mathfrak{g}}$ such that $\mathcal{I} \cap \mathfrak{h} = 0$, it holds that $\psi(\mathcal{I}) \cap (L(\alpha_{n+1})^* \oplus \mathfrak{g} \oplus L(\alpha_{n+1})) = 0$.

Note that $\psi(\mathcal{I})$ is also an ideal of $\mathcal{F}$ since ψ is surjective, and elements from $L(\alpha_{n+1})^*$, $\mathfrak{g}$ and $L(\alpha_{n+1})$ respectively have different degrees in the $\widehat{Q}$-gradation of $\mathcal{F}$. So it suffice to show that $\psi(\mathcal{I}) \cap L(\alpha_{n+1})^* = \psi(\mathcal{I}) \cap \mathfrak{g} = \psi(\mathcal{I}) \cap L(\alpha_{n+1}) = 0$.

Suppose $\psi(\mathcal{I}) \cap L(\alpha_{n+1})^* \ne 0$. Then $\psi(\mathcal{I}) \cap L(\alpha_{n+1})^* = L(\alpha_{n+1})^*$ since $L(\alpha_{n+1})^*$ is an irreducible $\mathfrak{g}$-module and $\psi(\mathcal{I}) \cap L(\alpha_{n+1})^*$ is a submodule of $L(\alpha_{n+1})^*$. (Note that $\psi(\mathcal{I})$ is a $\mathfrak{g}$-module via adjoint action in $\mathcal{F}$.) It follows that, in particular, $v^* \in \psi(\mathcal{I})$ and hence $[v^*, v] = h_{n+1} \in \psi(\mathcal{I}) \cap \mathfrak{h} \ne 0$. Since ψ preserves $\widehat{Q}$-gradation, it follows that $\mathcal{I} \cap \mathfrak{h} \ne 0$, a contradiction. So $\psi(\mathcal{I}) \cap L(\alpha_{n+1})^* = 0$. Similarly, $\psi(\mathcal{I}) \cap L(\alpha_{n+1}) = 0$.

Suppose $\psi(\mathcal{I}) \cap \mathfrak{g} \neq 0$. Since $\psi(\mathcal{I}) \cap \mathfrak{g}$ is also an ideal of $\mathfrak{g}$, it follows that $\psi(\mathcal{I}) \cap \mathfrak{h} \neq 0$, which is impossible by the above argument.

Hence ψ factors to a Lie algebra homomorphism, denoted again by ψ, from $\widehat{\mathfrak{g}}$ to $\mathcal{L}$. Clearly ψ is surjective. Since $\psi|_{\mathfrak{h}}$ is an isomorphism, $\ker \psi \cap \mathfrak{h} = 0$ and hence $\ker \psi = 0$ by the definition of $\widehat{\mathfrak{g}}$, which means that ψ is injective. ■

4. Some orbit Lie algebras and Lorentzian Kac-Moody algebras.

Let A be an indecomposable Cartan matrix of ADE type of rank $\ell \geq 2$. Then the 2-fold affinization $A^{[2]}$ of A in [3] is a generalized intersection matrix in the sense Slodowy [15]. The covering matrix of $A^{[2]}$ defined by Slodowy has a diagram automorphism $\boldsymbol{\tau}$, which gives rise to the following generalized Cartan matrix

$$A^{[2],\boldsymbol{\tau}} = \begin{bmatrix} A & \tilde{\theta}^t & \tilde{\theta}^t \\ \tilde{\theta} & 2 & -2 \\ \tilde{\theta} & -2 & 2 \end{bmatrix}_{(\ell+2)\times(\ell+2)}, \quad \tilde{\theta} = (-(\mathring{\alpha}_1|\theta), \cdots, -(\mathring{\alpha}_\ell|\theta)) \in \mathbb{Z}^\ell, \tag{26}$$

where θ is the highest root of A, $\mathring{\alpha}_j$ $(1 \leq j \leq \ell)$ are simple roots of A, and $(-|-)$ is given by $(\mathring{\alpha}_i|\mathring{\alpha}_j) = A(i,j)$. For details see [13, Lemma 2.2]. The affine matrix $A^{[1]}$ of A is the first $(\ell+1) \times (\ell+1)$ principal submatrix of $A^{[2],\boldsymbol{\tau}}$. We have the following

Lemma 4.1. *The pair* $(A^{[2],\boldsymbol{\tau}}, A^{[1]})$ *satisfies the condition* (18).

Proof. Since $A^{[2],\boldsymbol{\tau}}$ is symmetric, it suffices to show that

$$\operatorname{rank}(A^{[2],\boldsymbol{\tau}}) = \operatorname{rank}(A^{[1]}) + 2 = \ell + 2,$$

which follows from the fact that $A^{[2],\boldsymbol{\tau}}$ has nonzero determinant. ■

Let us fix a minimal realization $(\mathfrak{h}^{\boldsymbol{\tau}}, \Pi^{\boldsymbol{\tau}\vee}, \Pi^{\boldsymbol{\tau}})$ of $A^{[2],\boldsymbol{\tau}}$ given by

$$\mathfrak{h}^{\boldsymbol{\tau}} = \oplus_{i=1}^{\ell+2}\mathbb{C}\boldsymbol{h}_i, \quad \Pi^{\boldsymbol{\tau}\vee} = \{\boldsymbol{h}_1, \cdots, \boldsymbol{h}_{\ell+2}\}, \quad \Pi^{\boldsymbol{\tau}} = \{\alpha_1, \cdots, \alpha_{\ell+2}\}. \tag{27}$$

For a construction of such a triple $(\mathfrak{h}^{\boldsymbol{\tau}}, \Pi^{\boldsymbol{\tau}\vee}, \Pi^{\boldsymbol{\tau}})$ see [13, Lemma 2.2]. Hence the triple

$$(\mathfrak{h}^{\boldsymbol{\tau}}, \ \{\boldsymbol{h}_1, \cdots, \boldsymbol{h}_{\ell+1}\}, \ \{\alpha_1, \cdots, \alpha_{\ell+1}\}) \tag{28}$$

is a minimal realization of $A^{[1]}$. Since $A^{[2],\boldsymbol{\tau}}$ is symmetric, with respect to the minimal realization given by (27), the Kac-Moody algebra $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ associated to $A^{[2],\boldsymbol{\tau}}$ has the following presentation [7, Theorem 9.11]:

$$\begin{aligned} &\text{Generators}: \boldsymbol{h} \in \mathfrak{h}^{\boldsymbol{\tau}}, \boldsymbol{e}_i, \boldsymbol{f}_i, \ 1 \leq i \leq \ell+2. \\ &\text{Relations}: \\ &[\boldsymbol{h}, \boldsymbol{h}'] = 0, \ [\boldsymbol{h}, \boldsymbol{e}_j] = \alpha_j(\boldsymbol{h})\boldsymbol{e}_j, \ [\boldsymbol{h}, \boldsymbol{f}_j] = -\alpha_j(\boldsymbol{h})\boldsymbol{f}_j, \ [\boldsymbol{e}_i, \boldsymbol{f}_j] = \delta_{ij}\boldsymbol{h}_i, \\ &\boldsymbol{h}, \boldsymbol{h}' \in \mathfrak{h}^{\boldsymbol{\tau}}, \ 1 \leq i, j \leq \ell+2, \\ &(\mathrm{ad}\boldsymbol{e}_i)^{1-A^{[2],\boldsymbol{\tau}}(i,j)}\boldsymbol{e}_j = (\mathrm{ad}\boldsymbol{f}_i)^{1-A^{[2],\boldsymbol{\tau}}(i,j)}\boldsymbol{f}_j = 0, \ 1 \leq i \neq j \leq \ell+2. \end{aligned} \tag{29}$$

Since $A^{[2],\boldsymbol{\tau}}$ is non-degenerate, $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ is a simple Lie algebra. The subalgebra of $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ generated by $\boldsymbol{e}_i$, $\boldsymbol{f}_i$, $1 \le i \le \ell+1$, $\mathfrak{h}^{\boldsymbol{\tau}}$ is the affine Kac-Moody algebra $\mathfrak{g}^{[1]} = \mathfrak{g}(A^{[1]})$ associated to $A^{[1]}$ with the minimal realization given by (28). By Lemma 3.1 (3) and Lemma 4.1 it follows that $-\alpha_{\ell+2}$ is a dominant integral weight of $\mathfrak{g}^{[1]}$. In fact, since $A^{[2],\boldsymbol{\tau}}$ is non-degenerate, $\{\boldsymbol{h}_j\}_{j=1}^{\ell+2}$ is a basis of $\mathfrak{h}^{\boldsymbol{\tau}}$, and hence $-\alpha_{\ell+2}$ is the unique dominant integral weight of $\mathfrak{g}^{[1]}$ defined by $-\alpha_{\ell+2}(\boldsymbol{h}_j) = -A^{[2],\boldsymbol{\tau}}(\ell+2, j)$, $1 \le j \le \ell+2$. We have the following

Corollary 4.2. *The $\mathfrak{g}^{[1]}$-module $L(-\alpha_{\ell+2})$ has level 4.*

Proof. It follows by a case-by-case check that $-\alpha_{\ell+2}(K) = 4$, where K is the canonical central element of $\mathfrak{g}^{[1]}$ given by [7, Table Aff 1]. ■

Choose a highest-weight vector $v \in L(-\alpha_{\ell+2})$ and a lowest-weight vector $v^* \in L(-\alpha_{\ell+2})^*$ such that $v^*(v) = 1$. By Corollary 3.2 we get the local Lie algebra $L(-\alpha_{\ell+2})^* \oplus \mathfrak{g}^{[1]} \oplus L(-\alpha_{\ell+2})$ with respect to the $\mathfrak{g}^{[1]}$–module homomorphism $\varphi\colon L(-\alpha_{\ell+2})^* \otimes L(-\alpha_{\ell+2}) \to \mathfrak{g}^{[1]}$ given by $v^* \otimes v \mapsto \boldsymbol{h}_{\ell+2} \in (\mathfrak{h}^{\boldsymbol{\tau}})^{-\alpha_{\ell+2}}$, and hence we have the minimal $\mathbb{Z}$-graded Lie algebra $\mathcal{L} = \mathcal{L}(A^{[2],\boldsymbol{\tau}})$ given by (25). By Lemma 4.1 and Theorem 3.4 we get the following

Corollary 4.3. *The Kac-Moody algebra $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ is isomorphic to the Lie algebra $\mathcal{L} = \mathcal{L}(A^{[2],\boldsymbol{\tau}})$ via $\boldsymbol{e}_i \mapsto \boldsymbol{e}_i$, $\boldsymbol{f}_i \mapsto \boldsymbol{f}_i$, $1 \le i \le \ell+1$, and $\boldsymbol{e}_{\ell+2} \mapsto v^*$, $\boldsymbol{f}_{\ell+2} \mapsto v$, $\boldsymbol{h} \mapsto \boldsymbol{h} \in \mathfrak{h}^{\boldsymbol{\tau}}$. In particular, $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ has a minimal $\mathbb{Z}$-gradation with local part being $L(-\alpha_{\ell+2})^* \oplus \mathfrak{g}^{[1]} \oplus L(-\alpha_{\ell+2})$.*

We have a direct application of Corollary 4.3 as follows. Let $W^{[1]}$ be the Weyl group of $A^{[1]}$ and $\mathcal{P}_+$ be the set of dominant integral weights of $\mathfrak{g}^{[1]}$. Recall the usual partial order $\le$ on $(\mathfrak{h}^{\boldsymbol{\tau}})^*$ is given by $\beta \le \beta'$ if and only if $\beta' - \beta \in \oplus_{i=1}^{\ell+1}\mathbb{Z}_{\ge 0}\alpha_i$.

Let $\triangle^-(1)$ be the set of roots of $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ of the form $-\alpha_{\ell+2} - \beta$, where $\beta \in \oplus_{i=1}^{\ell+1}\mathbb{Z}_{\ge 0}\alpha_i$. By Corollary 4.3 $-\alpha_{\ell+2} - \beta \in \triangle^-(1)$ if and only if $-\alpha_{\ell+2} - \beta$ is a weight of the integrable highest weight $\mathfrak{g}^{[1]}$-module $L(-\alpha_{\ell+2})$. So by [7, Proposition 12.5] it follows that $\triangle^-(1) = W^{[1]} \cdot \{\mu \in \mathcal{P}_+ : \mu \le -\alpha_{\ell+2}\}$.

The set $\triangle^-(1)$ can be written in terms of maximal weights of the $\mathfrak{g}^{[1]}$-module $L(-\alpha_{\ell+2})$. Recall that, for any integrable highest weight $\mathfrak{g}^{[1]}$-module $L(\Lambda)$, a weight λ is maximal if $\lambda + \delta$ is not a weight, where δ is the null root of $\mathfrak{g}^{[1]}$. Let $\max(\Lambda)$ be the set of maximal weights of $L(\Lambda)$ as in [7]. Then by Corollary 4.3 and [7, Proposition 12.5] we have the following

Corollary 4.4. $\triangle^-(1) = \bigcup_{\lambda \in \max(-\alpha_{\ell+2})} \{\lambda - n\delta \,|\, n \in \mathbb{Z}_{\ge 0}\}$ *(disjoint union).*

Following [7, §5.11] the Lorentzian, or (almost) hyperbolic, Cartan matrix X_ℓ^H associated to the Cartan matrix X_ℓ of finite type is given by

$$X_\ell^H = \begin{bmatrix} X_\ell^{[1]} & \gamma \\ \gamma^t & 2 \end{bmatrix}_{(\ell+2)\times(\ell+2)}, \tag{30}$$

where $X_\ell^{[1]}$ is the nontwisted affine matrix associated to X_ℓ, and γ^t=$(0,\cdots,0,-1)$. That is, the Dynkin diagram of X_ℓ^H is obtained form that of $X_\ell^{[1]}$ by adding a vertex labelled as $\ell+2$ joined with the vertex labelled $\ell+1$ by a single edge. Here we use notation in [7, §5.11] with a slight modification of labels; and X_ℓ is different the notation A as above, since X_ℓ may have an arbitrary finite type.

Consider the $(\ell+2)$-dimensional complex vector space $\mathfrak{h}^{\boldsymbol{\tau}}$ with a basis $\Pi^\vee := \Pi^{\boldsymbol{\tau},\vee} = \{\boldsymbol{h}_1,\cdots,\boldsymbol{h}_{\ell+2}\}$ as in (27). Define $\beta_i \in (\mathfrak{h}^{\boldsymbol{\tau}})^*$ by $\beta_i(\boldsymbol{h}_j) = X_\ell^H(i,j)$, $1\le i,j\le \ell+2$. Since X_ℓ^H is non-singular, β_i is uniquely defined. Then the triple

$$(\mathfrak{h}^{\boldsymbol{\tau}},\Pi=\{\beta_1,\cdots,\beta_{\ell+2}\},\Pi^\vee=\{\boldsymbol{h}_1,\cdots,\boldsymbol{h}_{\ell+2}\}) \tag{31}$$

is a minimal realization of X_ℓ^H. We have the following

Lemma 4.5. *The pair* $(X_\ell^H, X_\ell^{[1]})$ *satisfies the condition* (18).

Proof. Since $\det(X_\ell^H) = -\det(X_\ell) \neq 0$, it holds that $\operatorname{rank}(X_\ell^H) = \ell+2 = \operatorname{rank}(X_\ell^{[1]})+2$. So it remains to show that X_ℓ^H is symmetrizable. Choose a symmetrization $DX_\ell^{[1]} = B$ of $X_\ell^{[1]}$, where $D = \operatorname{diag}(d_1,\cdots,d_{\ell+1})$ is a diagonal matrix with $d_i>0$, and B is symmetric. Then it's direct to see that the $(\ell+2)\times(\ell+2)$ diagonal matrix $D' := \operatorname{diag}(d_1,\cdots,d_{\ell+1},d_{\ell+1})$ satifies that $D'X_\ell^H$ is symmetric. ■

Let $\mathfrak{g}(X_\ell^H)$ be the Lorentzian Kac-Moody algebra associated to X_ℓ^H given by (30) with Chevalley generators e_i, f_i $(1\le i\le i\le \ell+2)$, $h\in\mathfrak{h}^{\boldsymbol{\tau}}$. With respect to the minimal realization of X_ℓ^H given by (31), the triple

$$(\mathfrak{h}^{\boldsymbol{\tau}},\{\beta_1,\cdots,\beta_{\ell+1}\},\{\boldsymbol{h}_1,\cdots,\boldsymbol{h}_{\ell+1}\})$$

is a minimal realization of the affine matrix $X_\ell^{[1]}$, and hence the subalgebra of $\mathfrak{g}(X_\ell^H)$ generated by e_i, f_i $(1\le i\le \ell+1)$, $\mathfrak{h}\in\mathfrak{h}^{\boldsymbol{\tau}}$ is the affine algebra $\mathfrak{g}(X_\ell^{[1]})$. By Lemma 3.1 (3) and Lemma 4.1 it follows that $-\beta_{\ell+2}$ is a dominant integral weight of $\mathfrak{g}(A_\ell^{[1]})$, and $\boldsymbol{h}_{\ell+2}\in(\mathfrak{h}^{\boldsymbol{\tau}})^{-\beta_{\ell+2}}$. Moreover, by [7, Table Aff 1] the integrable highest $\mathfrak{g}(X_\ell^{[1]})$-module $L(-\beta_{\ell+2})$ has level 1.

Choose a highest-weight vector v (resp. lowest-weight vector v^*) of $L(-\beta_{\ell+2})$ (resp. $L(-\beta_{\ell+2})^*$) such that $v^*(v)=1$. By Corollary 3.2 we get the local Lie algebra $L(-\beta_{\ell+2})^*\oplus\mathfrak{g}(X_\ell^{[1]})\oplus L(-\beta_{\ell+2})$ with respect to the $\mathfrak{g}(X_\ell^{[1]})$-module homomorphism φ: $L(-\beta_{\ell+2})^*\otimes L(-\beta_{\ell+2})\to\mathfrak{g}(X_\ell^{[1]})$ given by $v^*\otimes v\mapsto \boldsymbol{h}_{\ell+2}\in(\mathfrak{h}^{\boldsymbol{\tau}})^{-\beta_{\ell+2}}$, and hence we have the minimal $\mathbb{Z}$-graded Lie algebra $\mathcal{L}=\mathcal{L}(X_\ell^H)$ given by (25). So by Theorem 3.4 we have the following

Corollary 4.6. *There is a Lie algebra isomorphism* $\psi:\mathfrak{g}(X_\ell^H)\to\mathcal{L}=\mathcal{L}(X_\ell^H)$ *given by* $e_i\mapsto e_i$, $f_i\mapsto f_i$ $(1\le i\le \ell+1)$, $e_{\ell+2}\mapsto v^*$, $f_{\ell+2}\mapsto v$ *and* $h\mapsto h$. *In particular,* $\mathfrak{g}(X_\ell^H)$ *has a minimal* $\mathbb{Z}$*-gradation with local part being* $L(-\beta_{\ell+2})^*\oplus\mathfrak{g}(X_\ell^{[1]})\oplus L(-\beta_{\ell+2})$.

Remark 4.7. The case $X_\ell = \mathrm{A}_\ell$ is due to Feingold and Frenkel [4], Kang and Melville [10]; the case $X_\ell=\mathrm{C}_\ell$ is due to Klima and Misra [12].

Now we assume that $X_\ell = A$ is of ADE type. Then we have the Lorentzian Cartan matrix A_ℓ^H given by (30). We proceed to show that $\mathfrak{g}(A^{[2],\tau})$ is isomorphic to a subalgebra of the Lorentzian Kac-Moody algebra $\mathfrak{g}(A_\ell^H)$ following Feingold-Nicolai method in [5].

Identify the affine algebra $\mathfrak{g}^{[1]} = \mathfrak{g}(A^{[1]})$ with $\mathfrak{g}(A_\ell^{[1]})$, the subalgebra of $\mathfrak{g}(A_\ell^H)$ generated by e_i, f_i $(1 \leq i \leq \ell+1)$, $\boldsymbol{h} \in \mathfrak{h}^\tau$. Then the null root δ (in terms of β_i, $1 \leq i \leq \ell+1$) of $\mathfrak{g}^{[1]}$ is an isotropic imaginary root of $\mathfrak{g}(A_\ell^H)$. Note that $\beta_j(\boldsymbol{h}_{\ell+2}) = -\delta_{j,\ell+1}$ for $1 \leq j \leq \ell+1$, $\delta(\boldsymbol{h}_{\ell+2}) = -1$. We have the following

Lemma 4.8. *For any positive integer k set $\gamma_k = \beta_{\ell+1} + (k+1)\beta_{\ell+2} + k\delta$. Then*

(i) *γ_k is a positive real root of $\mathfrak{g}(A_\ell^H)$.*

(ii) *$\gamma_k - \beta_j$ is not a root of $\mathfrak{g}(A_\ell^H)$ for all $1 \leq j \leq \ell+1$.*

Proof. (i) Let $\mathrm{r}_{\ell+2}$ be the fundamental reflection of $(\mathfrak{h}^\tau)^*$ with respect to $\beta_{\ell+2}$. Since $\beta_{\ell+1} + k\delta$ is a real positive root of $\mathfrak{g}(A_\ell^H)$, the statement follows by

$$\mathrm{r}_{\ell+2}(\beta_{\ell+1} + k\delta) = \beta_{\ell+1} + \beta_{\ell+2} + k\delta + k\beta_{\ell+2} = \gamma_k.$$

(ii) Let $(-|-)$ be the bilinear form on $(\mathfrak{h}^\tau)^*$ induced by the standard invariant bilinear form, denoted again by $(-|-)$, on $\mathfrak{g}(A_\ell^H)$, which is, by non-singularity of A_ℓ^H, uniquely determined by $(\boldsymbol{h}_i|\boldsymbol{h}_j) = A_\ell^H(i,j)$ [7]. Then for any $1 \leq j \leq \ell+1$ it holds that

$$(\gamma_k|\beta_j) = \begin{cases} 2-(k+1) = 1-k \leq 0 & \text{if } j = \ell+1, \\ (\beta_{\ell+1}|\beta_j) = A_\ell^H(\ell+1,j) \leq 0 & \text{if } 1 \leq j \leq \ell. \end{cases}$$

So $(\gamma_k - \beta_j|\gamma_k - \beta_j) = (\gamma_k|\gamma_k) + (\beta_j|\beta_j) - 2(\gamma_k|\beta_j) = 4 - 2(\gamma_k|\beta_j) \geq 4$, which means that $\gamma_k - \beta_j$ can not be a root of $\mathfrak{g}(A_\ell^H)$. ■

Consider $\gamma_3 = \beta_{\ell+1} + 3\delta + 4\beta_{\ell+2}$. Choose an $\mathfrak{sl}_2$-triple $(x_{\gamma_3}, y_{\gamma_3}, \boldsymbol{h}_{\gamma_3})$ in $\mathfrak{g}(A_\ell^H)$ associated to γ_3, that is, x_{γ_3} belongs to the root subspace $\mathfrak{g}(A_\ell^H)_{\gamma_3}$, y_{γ_3} belongs to the root subspace $\mathfrak{g}(A_\ell^H)_{-\gamma_3}$ and $\boldsymbol{h}_{\gamma_3}$ belongs to $\mathfrak{h}^\tau$ satisfying that $[\boldsymbol{h}_{\gamma_3}, x_{\gamma_3}] = 2x_{\gamma_3}$, $[\boldsymbol{h}_{\gamma_3}, y_{\gamma_3}] = -2y_{\gamma_3}$, $[x_{\gamma_3}, y_{\gamma_3}] = \boldsymbol{h}_{\gamma_3}$, or equivalently, $(x_{\gamma_3}|y_{\gamma_3}) = 1$, $\boldsymbol{h}_{\gamma_3} = \nu^{-1}(\gamma_3)$, where $\nu : \mathfrak{h}^\tau \to (\mathfrak{h}^\tau)^*$ is the canonical linear map induced by $(-|-)$ [7, §2.1]. Then we have the following

Lemma 4.9. *The triple $(\mathfrak{h}^\tau, \{\beta_1, \cdots, \beta_{\ell+1}, \gamma_3\}, \{\boldsymbol{h}_1, \cdots, \boldsymbol{h}_{\ell+1}, \boldsymbol{h}_{\gamma_3}\})$ is a minimal realization of $A^{[2],\tau}$.*

Proof. By definition $\gamma_3(\boldsymbol{h}_{\gamma_3}) = \gamma_3(\nu^{-1}(\gamma_3)) = (\gamma_3|\gamma_3) = 2$. For any $1 \leq j \leq \ell+1$, it holds that

$$\begin{aligned} \beta_j(\boldsymbol{h}_{\gamma_3}) &= \beta_j(\nu^{-1}(\gamma_3)) = (\beta_j|\gamma_3) \\ &= \begin{cases} 2-4 = -2 & \text{if } j = \ell+1, \\ (\beta_j|\beta_{\ell+1}) = \beta_j(\boldsymbol{h}_{\ell+1}) = A^{[1]}_{j,\ell+1} & \text{if } 1 \leq j \leq \ell. \end{cases} \end{aligned} \tag{32}$$

Clearly, $\beta_j(\boldsymbol{h}_i) = A^{[1]}_{j,i}$ for $1 \leq i, j, \leq \ell + 1$. So, if we set $\alpha'_i = \beta_i$, $\boldsymbol{h}'_i = \boldsymbol{h}_i$ for $1 \leq i \leq \ell+1$ and $\alpha'_{\ell+2} = \gamma_3$, $\boldsymbol{h}'_{\ell+2} = \boldsymbol{h}_{\gamma_3}$, then by (26) the matrix $(\alpha'_i(\boldsymbol{h}'_j))_{1\leq i,j\leq \ell+2}$ is $A^{[2],\boldsymbol{\tau}}$ as required. Lastly, since $\beta_1, \cdots, \beta_{\ell+1}$, $\gamma_3 = \beta_{\ell+1} + 3\delta + 4\beta_{\ell+2}$ are linearly independent in $(\mathfrak{h}^{\boldsymbol{\tau}})^*$, and $A^{[2],\boldsymbol{\tau}}$ is non-singular, $\boldsymbol{h}_1, \cdots, \boldsymbol{h}_{\ell+1}, \boldsymbol{h}_{\gamma_3}$ are also linearly independent in $\mathfrak{h}^{\boldsymbol{\tau}}$. ■

We have the following

Proposition 4.10. *Let A be a Cartan matrix of* ADE *type with rank $\ell \geq 2$. Then the orbit Kac-Moody algebra $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ is isomorphic to a subalgebra of the Lorentzian Kac-Moody algebra $\mathfrak{g}(A^H_\ell)$. More precisely, let $\mathfrak{T}$ be the subalgebra of $\mathfrak{g}(A^H_\ell)$ generated by e_i, f_i $(1 \leq i \leq \ell+1)$, x_{γ_3}, y_{γ_3}, $\mathfrak{h}^{\boldsymbol{\tau}}$. Then there is an isomorphism $\chi: \mathfrak{g}(A^{[2],\boldsymbol{\tau}}) \to \mathfrak{T}$ given by $\boldsymbol{e}_i \mapsto e_i$, $\boldsymbol{f}_i \mapsto f_i$, $\boldsymbol{h}_i \mapsto \boldsymbol{h}_i$ $(1 \leq i \leq \ell+1)$, $\boldsymbol{e}_{\ell+2} \mapsto x_{\gamma_3}$, $\boldsymbol{f}_{\ell+2} \mapsto y_{\gamma_3}$, $\boldsymbol{h}_{\ell+2} \mapsto \boldsymbol{h}_{\gamma_3}$.*

Proof. It follows by [5, Theorem 3.1] and Lemma 4.8, Lemma 4.9. ■

Remark 4.11. By a result of S. Viswanath [16, Theorem 1], any simply-laced almost hyperbolic (Lorentzian) Kac-Moody algebra is isomorphic to a subalgebra of the hyperbolic Kac-Moody algebra E_{10}. So, by Proposition 4.10 the orbit Lie algebra $\mathfrak{g}(A^{[2],\boldsymbol{\tau}})$ is isomorphic to a subalgebra of E_{10}.

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