

Topological Frobenius Reciprocity for Representations of Nilpotent Groups and Motion Groups

Robert J. Archbold and Eberhard Kaniuth

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Abstract. Let G be a locally compact group and H a closed subgroup of G , and let π and τ be irreducible representations of G and H , respectively. If G is compact then, by the classical Frobenius reciprocity theorem, π is contained in the induced representation $\text{ind}_H^G \tau$ if and only if $\pi|_H$ contains τ . Topological Frobenius properties, which a general locally compact group may or may not satisfy, are obtained by replacing containment by weak containment of representations. We investigate the ‘if’ and the ‘only if’ assertions for nilpotent locally compact groups and for motion groups.

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Introduction

Let G be a locally compact group and H a closed subgroup of G , and suppose that π and τ are irreducible unitary representations of G and H , respectively. If G is compact, then according to the classical Frobenius reciprocity theorem, the restriction $\pi|_H$ contains τ as a subrepresentation if and only if π is contained in the induced representation $\text{ind}_H^G \tau$.

Fell [10] was the first to introduce and study topological (or weak) Frobenius reciprocity properties which are obtained by replacing containment of representations by weak containment. Thus, a locally compact group G is said to have the topological or weak Frobenius property (FP) if for any closed subgroup H of G and irreducible representations π of G and τ of H , π is weakly contained in $\text{ind}_H^G \tau$ if and only if $\pi|_H$ weakly contains τ . It is reasonable to treat separately the “if” and the “only if” parts of (FP) which are referred to as (FP1) and (FP2), respectively. Several authors have studied these topological Frobenius properties: compare [3], [5], [9], [11], [14], [20], [21], [26], [28], [34], [35] and [36]. Since for general locally compact groups both properties have turned out to be fairly restrictive, it is of interest to find criteria for a fixed π to satisfy any one of these properties.

In this paper, we focus on three different classes of nilpotent locally compact groups on the one hand and on motion groups on the other. Firstly, if G is a compactly generated nilpotent locally compact group and G^c denotes the compact normal subgroup consisting of all compact elements in G , we prove that (FP) holds for G (if and) only if G/G^c is abelian (Theorem 2.3). Next, in Theorem 3.3, we establish property (FP1) for nilpotent groups with small invariant neighbourhoods of the identity (that is, nilpotent SIN-groups). Thirdly, suppose that G is a compact-free nilpotent SIN-group. Then an irreducible representation π of G satisfies (FP2) if and only if π is weakly equivalent to the representation induced by $\pi|_{Z(G)}$, where $Z(G)$ denotes the centre of G (Theorem 4.7). Finally, in Section 5 we investigate properties (FP1) and (FP2) for irreducible representations of semidirect products of abelian groups with compact groups, especially the classical motion groups $\mathbb{R}^N \rtimes SO(N)$. All these results generalize or complement earlier ones; the details are pointed out in the corresponding sections.

1. Preliminaries

Let G be a locally compact group. If π and ρ are representations of G , $\pi \otimes \rho$ will denote the (inner) tensor product of π and ρ and $\bar{\pi}$ the contragredient representation of π . For a closed subgroup H of G and representations τ of H and π of G , we denote by $\text{ind}_H^G \tau$ the representation of G induced by τ and by $\pi|_H$ the restriction of π to H . In this situation the formula

$$\pi \otimes \text{ind}_H^G \tau = \text{ind}_H^G (\pi|_H \otimes \tau)$$

holds. We shall use the same letter, say π , to denote a unitary representation of G and the corresponding $*$ -representation of the group C^* -algebra $C^*(G)$, and $\ker \pi$ denotes the kernel of π in $C^*(G)$. If S and T are sets of unitary representations of G , then S is *weakly contained in* T ($S \prec T$) if

$$\bigcap_{\sigma \in S} \ker \sigma \supseteq \bigcap_{\tau \in T} \ker \tau,$$

and S and T are said to be *weakly equivalent* ($S \sim T$) if $S \prec T$ and $T \prec S$. Then, if π is a representation of G , $\pi \prec T$ if and only if every positive definite function associated with π , can be approximated uniformly on compact subsets of G by sums of positive definite functions associated with representations in T . Weak containment is preserved by forming (inner) tensor products and restricting and inducing representations [10], [11]. More precisely, let H be a closed subgroup of G . Then

- if S and T are sets of representations of G such that $S \prec T$, then $S|_H \prec T|_H$;
- if S and T are sets of representations of H such that $S \prec T$, then

$$\text{ind}_H^G S = \{\text{ind}_H^G \sigma : \sigma \in S\} \prec \text{ind}_H^G T = \{\text{ind}_H^G \tau : \tau \in T\};$$

- if S, S', T and T' are sets of representations of G such that $S \prec S'$ and $T \prec T'$, then $S \otimes S' \prec T \otimes T'$.

Even more generally, the processes of inducing and restricting representations are continuous when the set of all pairs (H, π) , where H is a closed subgroup of G and $\pi \in \text{Rep}(H)$, is endowed with Fell's subgroup representation topology [11] (compare also [30, Section 5.4]). Here $\text{Rep}(H)$ denotes the set of equivalence classes of unitary representations of H of dimension less than or equal to the cardinality of $C^*(H)$.

Apart from this, we shall frequently use in this paper that if G is amenable and π is an arbitrary representation of G , then $1_G \prec \pi \otimes \bar{\pi}$ [4, Theorem 2] and $\pi \prec \text{ind}_H^G(\pi|_H)$ for every closed subgroup H of G [16, Theorem 5.1]. Actually, both of these latter properties characterize amenable groups since $1_G \prec \text{ind}_{\{e\}}^G 1_{\{e\}} = \lambda_G$, the left regular representation of G , implies amenability of G (see [15]).

The dual space \widehat{G} of G is the set of equivalence classes of irreducible unitary representations of G , and the primitive ideal space $\text{Prim}(C^*(G))$ consists of all ideals $\ker \pi$, $\pi \in \widehat{G}$. $\text{Prim}(C^*(G))$ carries the hull-kernel topology and \widehat{G} the inverse image of this topology with respect to the canonical mapping $\pi \rightarrow \ker \pi$ from \widehat{G} onto $\text{Prim}(C^*(G))$. Finally, for any representation π of G , the support of π is the closed subset $\text{supp } \pi = \{\rho \in \widehat{G} : \rho \prec \pi\}$ of \widehat{G} . By a character, we always mean a continuous homomorphism of G into the circle group \mathbb{T} .

Let N be a closed normal subgroup of G . Then G acts on representations π of N by $x \cdot \pi(y) = \pi(x^{-1}yx)$, $x \in G, y \in N$. The G -orbit of π under this action will be denoted $G(\pi)$ and $G_\pi = \{x \in G : x \cdot \pi = \pi\}$ is the stabilizer of π . The subgroup G_π need not be closed in G , but it is closed if, for instance, the singleton $\{\pi\}$ is closed in \widehat{N} .

Let G be a locally compact group and $\pi \in \widehat{G}$. We say that π satisfies the *topological Frobenius property*

- (FP1) if for any closed subgroup H and $\tau \in \widehat{H}$, $\pi|_H \succ \tau$ implies that $\pi \prec \text{ind}_H^G \tau$;
- (FP2) if for any closed subgroup H and $\tau \in \widehat{H}$, $\pi \prec \text{ind}_H^G \tau$ implies that $\pi|_H \succ \tau$;
- (FP) if both (FP1) and (FP2) hold for π .

The group G is said to have any of these properties when all irreducible representations do so.

It is not known whether any of the properties (FP1), (FP2) and (FP) is inherited by closed subgroups. However, there is a weakened version of (FP), the so-called *restricted Frobenius property* (RFP), which is obtained by restricting π to be the trivial representation 1_G and which has the advantage of being inherited by closed subgroups and by quotient groups [21, Lemma 2.1]. Actually, for several classes of locally compact groups (RFP) already implies (FP).

We mention a few results concerning (FP).

(1) (FP) holds for any locally compact group with relatively compact conjugacy classes [25, Theorems 2.5 and 2.6], in particular, it holds for abelian locally compact groups [11].

(2) If G is a connected and simply connected Lie group, then (FP) holds for G (if and) only if G is a direct product of a vector group and a compact semisimple group [9, Theorem 1].

(3) If G is a locally compact group with small invariant neighbourhoods of the identity (a so-called SIN-group), then (FP) holds for G (if and) only if G is a group with relatively compact conjugacy classes [21, Corollary 3.13]. Alternatively, this can be deduced from (1), [9, Theorem 3] and the fact that SIN-groups have an open subgroup of the form $V \times K$, where V is a vector group and K is a compact group [17, Theorem 2.13].

Lemma 1.1. *Let G be amenable and π an irreducible representation of G . Suppose that for any representation ρ of G , $1_G \prec \bar{\pi} \otimes \rho$ implies that $\pi \prec \rho$. Then (FP1) holds for π .*

Proof. Since closed subgroups of amenable groups are amenable, $1_H \prec \bar{\tau} \otimes \tau$ for every closed subgroup H of G and $\tau \in \widehat{H}$. The statement now follows from Lemma 1.1 of [25]. ■

The tensor product property in Lemma 1.1 is often easier to verify than property (FP1) directly (see [25], [26] and Section 3).

Lemma 1.2. *Let G be any locally compact group.*

- (i) *If G is amenable and π is a finite dimensional irreducible representation of G , then (FP1) holds for π .*
- (ii) *If (FP1) holds for the trivial representation of G , then G is amenable.*
- (iii) *Let $\pi \in \widehat{G}$ and suppose that (FP1) holds for π . Then $\{\ker \pi\}$ is closed in $\text{Prim}(C^*(G))$ (i.e., $\ker \pi$ is a maximal ideal of $C^*(G)$).*

Proof. (i) Let H be a closed subgroup of G and let τ be any representation of H such that $\pi|_H \succ \tau$. Then, since τ is finite dimensional, $\bar{\tau} \otimes \tau \succ 1_H$ and therefore, using amenability of G and that weak containment is preserved by forming tensor products of representations and inducing representations,

$$1_G \prec \text{ind}_H^G 1_H \prec \text{ind}_H^G(\bar{\tau} \otimes \tau) \prec \text{ind}_H^G(\bar{\pi}|_H \otimes \tau) = \bar{\pi} \otimes \text{ind}_H^G \tau.$$

Since π is finite dimensional, Lemma 4 of [10] implies that $\pi \prec \text{ind}_H^G \tau$.

(ii) Take $H = \{e\}$ and $\tau = 1_{\{e\}}$. If (FP1) holds for 1_G , then $1_G \prec \text{ind}_{\{e\}}^G 1_{\{e\}}$. This implies that G is amenable.

(iii) Just take $H = G$ and a $\tau \in \widehat{G}$ such that $\tau \prec \pi$, but $\tau \not\prec \pi$. ■

2. Property (FP) for nilpotent groups

Let G be a nilpotent locally compact group. We collect from [13] a number of structural results which will be used in the sequel. Firstly, G^c , the set of all compact elements of G , is a closed normal subgroup of G [13, Corollary 3.5.1 and Lemma 3.8]. The group G is said to be compact-free if $G^c = \{e\}$. When G is compactly generated, then every closed subgroup of G is again compactly generated and G^c is compact [13, Lemma 9.2 and Theorem 9.7]. From this latter

fact it can easily be deduced that G/G^c is compact-free. In particular, $(G/G^c)_0$, the connected component of the identity of G/G^c , is a simply connected nilpotent Lie group [13, Theorem 4.2]. In addition, G_0G^c is open in G and G/G_0G^c is torsion-free [13, Theorem 8.3]. Since G_0G^c/G^c is connected and open in G/G^c , $G_0G^c/G^c = (G/G^c)_0$. Hence G/G^c is a Lie group. When G is discrete, G^c is just the subgroup consisting of all elements of finite order which is usually denoted G^t and called the torsion subgroup of G .

In this section, we characterize the compactly generated nilpotent groups which satisfy property (FP1). In preparation for this, we need the following two lemmas.

Lemma 2.1. *Let G be a semidirect product $G = N \rtimes A$, where*

- (i) *A is an infinite cyclic group;*
- (ii) *N is a compact-free second countable locally compact abelian group;*
- (iii) *The quotient group $G/Z(G)$ is torsion-free.*

Then either G is abelian or $\text{ind}_A^G \alpha \succ \widehat{A}$ for any character α of A .

Proof. Note first that since G is second countable, for each $\pi \in \widehat{G}$ there exists $\chi \in \widehat{N}$ such that $\pi|_N \sim A(\chi) = G(\chi)$ [14, Theorem 2.1]. Let α be any character of A and identify A with \mathbb{Z} . Since G is amenable $(\text{ind}_A^G \alpha)|_N \sim \widehat{N}$ by [11, Theorem 5.3]. It follows that

$$X = \{\chi \in \widehat{N} : A(\chi) \sim \pi|_N \text{ for some } \pi \in \text{supp}(\text{ind}_A^G \alpha)\}$$

is dense in \widehat{N} . Suppose now that there are sequences $(\pi_n)_n$ in $\text{supp}(\text{ind}_A^G \alpha)$ and $(\chi_n)_n$ in \widehat{N} such that $\pi_n|_N \sim A(\chi_n)$ for each n , $\chi_n \rightarrow 1_N$ as $n \rightarrow \infty$ and $A_{\chi_n} = G_{\chi_n} \cap A = q_n\mathbb{Z}$, where $q_n \in \mathbb{N}$ and $q_n \rightarrow \infty$. For each n , there exists $\gamma_n \in \widehat{q_n\mathbb{Z}}$ such that

$$\pi_n = \text{ind}_{N \rtimes q_n\mathbb{Z}}^G(\chi_n \times \gamma_n).$$

Choose $\sigma_n \in \widehat{\mathbb{Z}}$ with $\sigma_n|_{q_n\mathbb{Z}} = \gamma_n$, $n \in \mathbb{N}$. Since $\widehat{\mathbb{Z}}$ is compact, after passing to a subsequence we can assume that $\sigma_n \rightarrow \sigma$ for some $\sigma \in \widehat{\mathbb{Z}}$. Let $\mathcal{K}(\mathbb{Z})$ denote the family of all subgroups of \mathbb{Z} , equipped with Fell's compact-open topology [?]. Then $q_n\mathbb{Z} \rightarrow \{0\}$ in $\mathcal{K}(\mathbb{Z})$ and therefore

$$(N \rtimes q_n\mathbb{Z}, \chi_n \times \gamma_n) \rightarrow (N, 1_N)$$

in Fell's subgroup representation topology [11]. By continuity of inducing [11, Theorem 4.2], it follows that

$$\pi_n = \text{ind}_{N \rtimes q_n\mathbb{Z}}^G(\chi_n \times \gamma_n) \rightarrow \text{ind}_N^G 1_N.$$

Since $\text{ind}_N^G 1_N \sim \widehat{A}$ and $\pi_n \prec \text{ind}_A^G \alpha$ for all n , the proof is complete under the above hypothesis.

If no sequences as above exist, then there exists an open neighbourhood V of 1_N in \widehat{N} with the following property: There exists $m \in \mathbb{N}$ such that for each $\chi \in V \cap X$ we have $A_\chi = q_\chi \mathbb{Z}$ with $q_\chi \leq m$. Then, putting $q = m!$, $A_\chi \supseteq q\mathbb{Z}$ for all $\chi \in V \cap X$. Since $X \cap V$ is dense in V , it follows that $q\mathbb{Z} \subseteq A_\chi$ for every $\chi \in V$ and hence for every χ in the open subgroup of \widehat{N} generated by V . However, this subgroup equals \widehat{N} because \widehat{N} , being the dual group of a compact-free abelian group, is connected. Consequently, $q\mathbb{Z}$ acts trivially on \widehat{N} and therefore also on N . In other words, $q\mathbb{Z} \subseteq Z(G)$. But then G must be abelian. In fact, denoting by $\varphi : G \rightarrow G/Z(G)$ the quotient homomorphism, $\varphi(\mathbb{Z})$ is a finite subgroup of $G/Z(G)$. However, since $G/Z(G)$ is torsion-free by hypothesis, we get $\mathbb{Z} \subseteq Z(G)$ and hence $G = N \rtimes \mathbb{Z}$ is abelian. ■

For any group G , we denote by G_f the normal subgroup consisting of all elements with finite conjugacy class, that is, all $x \in G$ such that the centralizer $C_G(x)$ of x has finite index in G . If $G_f = G$, then G is called an FC-group.

Lemma 2.2. *Let G be a compact-free, compactly generated nilpotent locally compact group. If (RFP) holds for G , then G is abelian.*

Proof. Note first that G_0 is a connected and simply connected nilpotent Lie group which satisfies property (RFP) and therefore has to be abelian by [35, Theorem] (for a simpler proof see [30, Theorem 6.33]), actually a vector group, $V = \mathbb{R}^k$ say.

Since V is open in G , G/V is a finitely generated, torsion-free nilpotent discrete group satisfying (RFP). Then, by [20, Theorem 2.7], G/V is an FC-group. Since a finitely generated FC-group has a finite commutator subgroup [38, Theorem 4.32] and G/V is torsion-free, it is abelian and hence isomorphic to \mathbb{Z}^d for some $d \in \mathbb{N}_0$. We are going to prove by induction on d that, whenever G is a compactly generated nilpotent locally compact group with property (RFP) and G contains an open normal vector group V such that $G/V = \mathbb{Z}^d$, for some $d \in \mathbb{N}_0$, then G is abelian.

The case $d = 0$ being clear, suppose that $d \geq 1$ and that the statement is true whenever $G/V = \mathbb{Z}^{d-1}$. Choose a subgroup N of G containing V such that $N/V = \mathbb{Z}^{d-1}$ and $G/N = \mathbb{Z}$, so that $G = N \rtimes A$, where A is an infinite cyclic group. Since (RFP) holds for N , N is abelian by the inductive hypothesis and hence $N = V \times \mathbb{Z}^{d-1}$. Since G is compact-free, so is $G/Z(G)$. So G satisfies the hypotheses of Lemma 2.1. Thus, assuming that G is not abelian, $\text{ind}_A^G \alpha \succ \widehat{A}$ for every $\alpha \in \widehat{A}$. This contradicts property (RFP) and therefore G has to be abelian. ■

Theorem 2.3. *For a compactly generated nilpotent locally compact group G the following conditions are equivalent.*

- (i) (FP) holds for G .
- (ii) (RFP) holds for G .
- (iii) The quotient group G/G^c is abelian.

Proof. (i) \Rightarrow (ii) is trivial, and (iii) \Rightarrow (i) is true since G^c is compact and every group with relatively compact conjugacy classes satisfies (FP). (ii) \Rightarrow (iii) follows from Lemma 2.2 since (RFP) holds for G/G^c and G/G^c is a compact-free and compactly generated nilpotent locally compact group. ■

Corollary 2.4. *Let G be a nilpotent locally compact group. If G satisfies (RFP), then G/G^c is abelian.*

Proof. Let \mathcal{H} denote the collection of all compactly generated open subgroups of G . Then (RFP) holds for each $H \in \mathcal{H}$ and hence H/H^c is abelian by Theorem 2.3. Since $G^c = \bigcup_{H \in \mathcal{H}} H^c$, it follows that $[G, G] = \bigcup_{H \in \mathcal{H}} [H, H] \subseteq \bigcup_{H \in \mathcal{H}} H^c = G^c$. ■

The converse of Corollary 2.4 does not hold since every discrete group with property (RFP) has to be an FC-group [20, Theorem 2.7].

3. (FP1) for nilpotent SIN-groups

The Frobenius property (FP1) does not hold for general nilpotent locally compact groups. Indeed, as shown in [8], the 3-step nilpotent group of all upper triangular real 4×4 -matrices with 1 on the main diagonal does not satisfy (FP1) (see [30, Example 6.39]). On the other hand, every 2-step nilpotent locally compact group, which is a projective limit of Lie groups, satisfies (FP1) [5, Theorem 3.4]. The same is true for nilpotent discrete groups [25, Theorem 1.2]. The purpose of this section is to extend this latter result to the much larger class of nilpotent SIN-groups. Recall that a locally compact group G is called an SIN-group (a group with *small invariant neighbourhoods*) if there exists a neighbourhood basis \mathcal{V} of the identity such that $x^{-1}Vx = V$ for all $V \in \mathcal{V}$ and $x \in G$.

In the sequel, for any locally compact group G , we denote by G_d the group G equipped with the discrete topology. Moreover, a representation π of G , viewed as a representation of G_d , is denoted π_d . Clearly, if $\pi \prec \rho$ then $\pi_d \prec \rho_d$, and if π is irreducible then so is π_d .

Lemma 3.1. *Let G be a nilpotent locally compact group containing an open subgroup H of the form $H = \mathbb{R}^d \times K$, $d \in \mathbb{N}_0$, where K is a compact group. Suppose that σ and ρ are representations of G such that $\sigma|_K \sim \chi \sim \rho|_K$ for some character χ of K and $1_G \prec \bar{\sigma} \otimes \rho$. If σ is irreducible, then $\sigma \prec \rho$.*

Proof. To show that $\sigma \prec \rho$, let $\xi, \eta \in \mathcal{H}(\sigma)$, a compact subset C of G and $\epsilon > 0$ be given. Enlarging C , we can assume that C is of the form $C = \bigcup_{x \in F} xD$, where F is a finite subset of G and $D = [-R, R]^d \times K \subseteq H$ for some $R > 0$. Thus every element of C is of the form $x \left(\sum_{i=1}^d \lambda_i e_i \right) z$, where $x \in F$, $z \in K$, $|\lambda_i| \leq R$ and e_i denotes the standard i^{th} unit vector in \mathbb{R}^d . Since $1_G \prec \bar{\sigma}_d \otimes \rho_d$, it follows from [25, Theorem 1.2] that $\text{supp } \sigma_d \cap \text{supp } \rho_d \neq \emptyset$. Since discrete nilpotent groups have a T_1 primitive ideal space [37] and σ_d is irreducible, we get $\sigma_d \prec \rho_d$.

Thus there exist $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_m \in \mathcal{H}(\rho)$ such that

$$\left| \langle \sigma(xe_i)\xi, \eta \rangle - \sum_{j=1}^m \langle \rho(xe_i)\xi_j, \eta_j \rangle \right| \leq \epsilon/dR$$

for all $x \in F$ and $1 \leq i \leq d$. Since $\sigma|_K \sim \chi \sim \rho|_K$ and χ is a character, we have $\sigma(az) = \chi(z)\sigma(a)$ and $\rho(az) = \chi(z)\rho(a)$ for every $a \in G$ and $z \in K$.

It follows that, for $x \in F$, $z \in K$ and $y = \sum_{i=1}^d \lambda_i \cdot e_i$, where $\lambda_i \in [-R, R]$, $1 \leq i \leq d$,

$$\begin{aligned} \left| \langle \sigma(xyz)\xi, \eta \rangle - \sum_{j=1}^m \langle \rho(xyz)\xi_j, \eta_j \rangle \right| &= \left| \chi(z) \langle \sigma(xy)\xi, \eta \rangle - \chi(z) \sum_{j=1}^m \langle \rho(xy)\xi_j, \eta_j \rangle \right| \\ &\leq \sum_{i=1}^d |\lambda_i| \cdot \left| \langle \sigma(xe_i)\xi, \eta \rangle - \sum_{j=1}^m \langle \rho(xe_i)\xi_j, \eta_j \rangle \right| \leq \epsilon. \end{aligned}$$

Thus every coordinate function of σ can be approximated uniformly on compact subsets of G by sums of coordinate functions associated with ρ . This shows that $\sigma \prec \rho$. ■

Lemma 3.2. *Let K be an abelian, compact, normal subgroup of the locally compact group G , $\chi \in \widehat{K}$ and $\sigma \in \widehat{G}_\chi$ such that $\sigma|_K \sim \chi$. Suppose that G_χ has finite index in G . Let M and N be closed subgroups of G containing G_χ such that $M \subseteq N$, M is normal in N and N/M is abelian. Let $\pi_M = \text{ind}_{G_\chi}^M \sigma \in \widehat{M}$ and $\pi_N = \text{ind}_{G_\chi}^N \sigma \in \widehat{N}$. Then the following two conditions are equivalent.*

- (i) *If $1_M \prec \pi_M \otimes \bar{\rho}$ for some representation ρ of M such that $\rho|_K \sim M(\chi)$, then $\pi_M \prec a \cdot \rho$ for some $a \in N$.*
- (ii) *If $1_N \prec \pi_N \otimes \bar{\tau}$ for some representation τ of N such that $\tau|_K \sim N(\chi)$, then $\pi_N \prec \tau$.*

Proof. (i) \Rightarrow (ii) Observe that since M is normal in N ,

$$\pi_N|_M = \left(\text{ind}_{G_\chi}^N \sigma \right)|_M \sim N \left(\text{ind}_{G_\chi}^M \sigma \right) = N(\pi_M).$$

Let τ be as in (ii) and let F be a coset representative system for M in N . Then

$$1_M \prec \pi_N|_M \otimes \bar{\tau}|_M \sim N(\pi_M) \otimes \bar{\tau}|_M \sim \{x \cdot (\pi_M \otimes \bar{\tau}|_M) : x \in F\}.$$

Since F is finite, it follows that $1_M \prec x \cdot (\pi_M \otimes \bar{\tau}|_M)$ for some $x \in F$ and so $1_M \prec \pi_M \otimes \bar{\tau}|_M$. On the other hand, since

$$\tau|_K \sim N(\chi) = \bigcup_{x \in F} x \cdot M(\chi),$$

we have $\tau|_M \sim \bigcup_{x \in F} x \cdot \rho$, where ρ is a representation of M such that $\rho|_K \sim M(\chi)$. Indeed, with $R = \text{supp}(\tau|_M) \cap \text{supp}(\text{ind}_K^M \chi)$, we have $\tau|_M \sim \bigcup_{x \in F} x \cdot R$ since $\text{supp}(\pi|_M)$ is N -invariant, and hence for ρ we can choose any representation of

M with support equal to R . Again using that F is finite, we conclude that $1_M \prec \pi_M \otimes x \cdot \bar{\rho}$ for some $x \in F$. This in turn implies that

$$1_K \prec \pi_M|_K \otimes x \cdot (\bar{\rho}|_K) \sim M(\chi) \otimes x \cdot M(\bar{\chi}).$$

Since K is compact and abelian, this means that there exist $m_1, m_2 \in M$ such that $1_K = (m_1 \cdot \chi)((xm_2) \cdot \bar{\chi})$ and so $m_1 \cdot \chi = (xm_2) \cdot \chi$, which gives $m_1^{-1}xm_2 \in M$ and therefore $x \in M$. Thus we have seen that $1_M \prec \pi_M \otimes \bar{\rho}$, and since $\rho|_K \sim M(\chi)$, condition (i) shows that $\pi_M \prec a \cdot \rho$ for some $a \in N$. Now, since $a \in N$,

$$\pi_N = \text{ind}_M^N \pi_M \prec \text{ind}_M^N (a \cdot \rho) = \text{ind}_M^N \rho \prec \text{ind}_M^N (\tau|_M) \sim \tau \otimes \widehat{N/M}.$$

Since $\widehat{N/M}$ is finite and π_N is irreducible, we get $\pi_N \prec \tau \otimes \bar{\alpha}$ for some $\alpha \in \widehat{N/M}$. Since π_N is induced from M , it follows that $\pi_N = \pi_N \otimes \alpha \prec \tau$, as was to be shown.

The proof of (ii) \Rightarrow (i) is much simpler. Let ρ be a representation of M such that $\rho|_K \sim M(\chi)$ and $1_M \prec \pi_M \otimes \bar{\rho}$. Choose any representation τ of N such that $\tau|_M \sim N(\rho)$. Then

$$\begin{aligned} 1_N &\prec \text{ind}_M^N 1_M \prec \text{ind}_M^N (\pi_M \otimes \bar{\rho}) \prec \text{ind}_M^N (\pi_M \otimes \bar{\tau}|_M) \\ &= \bar{\tau} \otimes \text{ind}_M^N \pi_M = \bar{\tau} \otimes \pi_N. \end{aligned}$$

Since $\tau|_K \sim N(\rho)|_K = N(\rho|_K) \sim N(M(\chi)) = N(\chi)$, (ii) implies that $\pi_N \prec \tau$. Restricting to M gives

$$\pi_M \prec (\text{ind}_M^N \pi_M)|_M = \pi_N|_M \prec \tau|_M \sim N(\rho).$$

Since π_M is irreducible and N/M is finite, it follows that $\pi_M \prec a \cdot \rho$ for some $a \in N$. This shows that (i) holds. ■

Before proceeding to establish the result announced at the outset of this section, we have to introduce some more notation, which will also be used in Section 4. For any group G , we denote by $Z(G)$ the centre of G and by

$$\{e\} = Z_0(G) \subseteq Z(G) = Z_1(G) \subseteq \dots \subseteq Z_m(G) \subseteq \dots$$

the ascending central series of G , that is, $Z_{j+1}(G)/Z_j(G) = Z(G/Z_j(G))$, $j = 0, 1, \dots$. Then G is said to be m -step nilpotent if $Z_m(G) = G$, but $Z_{m-1}(G) \neq G$.

Let N be a closed normal subgroup of the locally compact group G . We write $N \in [\text{FC}]_G^-$, if every element of N has a relatively compact conjugacy class in G , and $N \in [\text{SIN}]_G$, if the identity element has a neighbourhood basis \mathcal{V} in N such that $x^{-1}Vx = V$ for every $V \in \mathcal{V}$ and $x \in G$. Thus G is a SIN-group if $G \in [\text{SIN}]_G$. Moreover, let $\text{Aut}(N)$ denote the group of all topological automorphisms of N , endowed with the usual group topology [22, Theorem (26.5)], and let $I(N, G)$ be the subgroup of $\text{Aut}(N)$ consisting of all inner automorphisms $y \rightarrow x^{-1}yx$, $x \in G$, of N . Then the closure $\overline{I(N, G)}$ in $\text{Aut}(N)$ is compact if and only if $N \in [\text{FC}]_G^- \cap [\text{SIN}]_G$ [17, Theorem 0.1]. For a SIN-group G , let G_F denote the set of all $x \in G$ such that the conjugacy class of x is relatively compact. Then G_F is an open normal subgroup of G ; in particular, $G_F \in [\text{FC}]_G^- \cap [\text{SIN}]_G$.

Theorem 3.3. *Let G be a nilpotent SIN-group and let π be an irreducible representation of G . Then, for any representation ρ of G , $1_G \prec \bar{\pi} \otimes \rho$ implies $\pi \prec \rho$. In particular, (FP1) holds for G .*

Proof. By Lemma 1.1, it suffices to prove the first statement. Note first that any SIN-group G is a projective limit of Lie groups G/C_α [32, Lemma 4.3] and that then $\widehat{G} = \bigcup_\alpha \widehat{G/C_\alpha}$ [32, Proposition 3.2]. Thus there exists a compact normal subgroup C of G such that $\pi \in \widehat{G/C}$ and G/C is a Lie group. Now ρ can be written as a direct sum $\rho = \rho_1 \oplus \rho_2$, where $\rho_1(C) = \{I\}$ and $\text{supp}(\rho_2) \subseteq \widehat{G} \setminus \{1_G\}$. Then $1_G \prec \bar{\pi} \otimes \rho$ implies that $1_G \prec \bar{\pi} \otimes \rho_1$. Therefore, passing to G/C , we can henceforth assume that G is a Lie group.

Then G_0 , the connected component of the identity of G , is open in G and of the form $G_0 = \mathbb{R}^d \times K$, where K is compact and connected and both K and \mathbb{R}^d are normal in G [17, Theorem 2.13]. Being a compact, connected, solvable Lie group, K is abelian. Indeed, this appears to be folklore and can be verified by exploiting the fact that the action of a connected group on the discrete dual of any abelian, compact, normal subgroup is trivial and arguing by induction on the length of the commutator series of G . Since $\overline{I(K, G)}$ is compact, the group consisting of all automorphisms $\alpha_x : \chi \rightarrow x \cdot \chi$, $x \in G$, of \widehat{K} , is also relatively compact [22, Theorem (26.9)]. Hence all G -orbits in \widehat{K} are finite since \widehat{K} is discrete. Choose $\chi \in \widehat{K}$ such that $\pi|_K \sim G(\chi)$. Then the stability group G_χ has finite index in G , and there exists $\sigma \in \widehat{G_\chi}$ such that $\pi = \text{ind}_{G_\chi}^G \sigma$ (see [30, Theorem 4.22]).

Let G be m -step nilpotent and let $H_j = G_\chi Z_j(G)$ and $\pi_j = \text{ind}_{G_\chi}^{H_j} \sigma \in \widehat{H_j}$, $0 \leq j \leq m$. Then it is easily verified that H_j is normal in H_{j+1} , and H_{j+1}/H_j is abelian. We prove by induction on j that if σ is a representation of H_j such that $\sigma|_K \sim H_j(\chi)$ and $1_{H_j} \prec \bar{\pi}_j \otimes \sigma$, then $\pi_j \prec \sigma$. The case $j = 0$ is covered by Lemma 3.1. For the inductive step $j \rightarrow j + 1$, let σ be a representation of H_{j+1} such that $\sigma|_K \sim H_{j+1}(\chi)$ and $1_{H_{j+1}} \prec \bar{\pi}_{j+1} \otimes \sigma$. Let, as in the proof of Lemma 3.2, F be a representative system for the cosets of H_j in H_{j+1} . Then there exists a representation ρ of H_j such that $\rho|_K \sim H_j(\chi)$ and $\sigma|_{H_j} \sim F(\rho)$. It follows that

$$1_{H_j} \prec \bar{\pi}_{j+1}|_{H_j} \otimes \sigma|_{H_j} \sim F(\bar{\pi}_j) \otimes F(\rho),$$

and this in turn implies that $1_{H_j} \prec \bar{\pi}_j \otimes a \cdot \rho$ for some $a \in F^{-1}F \subseteq N$. Now, the implication (i) \Rightarrow (ii) of Lemma 3.2 applies and yields $\pi_{j+1} \prec \sigma$.

Finally, if ρ is an arbitrary representation of G such that $1_G \prec \bar{\pi} \otimes \rho$, then as in the first paragraph of the proof, $\rho = \rho_1 \oplus \rho_2$, where $\rho_1|_K \sim G(\chi)$ and $\rho_2|_K \prec \widehat{K} \setminus G(\chi)$. Then 1_G cannot be weakly contained in $\bar{\pi} \otimes \rho_2$, and hence $1_G \prec \bar{\pi} \otimes \rho_1$ and this implies $\pi \prec \rho_1 \prec \rho$, completing the proof. ■

From Theorem 3.3 and Lemma 1.2(iii) we in particular conclude the following

Corollary 3.4. *Let G be a nilpotent SIN-group. Then $\text{Prim}(C^*(G))$ is a T_1 -space.*

Remark 3.5. Let G be a countable, metabelian, discrete group and let $\pi \in \widehat{G}$. Then (FP1) holds for π if (and, by Lemma 1.2(iii), only if) $\{\ker \pi\}$ is closed in $\text{Prim}(C^*(G))$. To see this, let H be a subgroup of G and $\tau \in \widehat{H}$ such that $\tau \prec \pi|_H$. Then, since G is amenable,

$$1_G \prec \text{ind}_H^G(\bar{\tau} \otimes \tau) \prec \text{ind}_H^G(\bar{\pi}|_H \otimes \tau) = \bar{\pi} \otimes \text{ind}_H^G \tau,$$

and Proposition 1 of [24] shows that $\text{supp } \pi \cap \text{supp}(\text{ind}_H^G \tau) \neq \emptyset$. So $\pi \prec \text{ind}_H^G \tau$ since $\text{supp } \pi \sim \{\pi\}$ by assumption.

4. (FP2) for some nilpotent groups

The problem of when (FP2) holds for a given irreducible representation of a locally compact group G is much more intricate than the analogous (FP1) question. There are only a few isolated results for simply connected nilpotent Lie groups on the one hand (see [5], [9] and [27]) and for some classes of nilpotent discrete groups on the other hand (see [3] and [28]). The purpose of this section is to provide a criterion for irreducible representations of compact-free nilpotent SIN-groups (Theorem 4.7).

Proposition 4.1. *Let G be an amenable SIN-group and let $\pi \in \widehat{G}$ such that $\pi \sim \text{ind}_{G_F}^G(\pi|_{G_F})$. Then (FP2) holds for π .*

Proof. Let H be a closed subgroup of G and $\tau \in \widehat{H}$ such that $\pi \prec \text{ind}_H^G \tau$. Then, since H is amenable,

$$\text{ind}_H^G \tau \prec \text{ind}_H^G (\text{ind}_{H \cap G_F}^H(\tau|_{H \cap G_F})) = \text{ind}_{H \cap G_F}^G(\tau|_{H \cap G_F}).$$

Let $\Delta = \overline{I(H \cap G_F, H)} \subseteq \text{Aut}(H \cap G_F)$. Then Δ is compact since $H \cap G_F \in [\text{SIN}]_H \cap [\text{FC}]_H^-$ and Δ acts on the primitive ideal space $\text{Prim}(C^*(H \cap G_F))$ in the usual manner. By [33, Section 4; in particular Proposition 4.8], the map $\gamma \rightarrow \ker \pi_\gamma$ is a homeomorphism between $E(H \cap G_F, H)$ and the set $H\text{-Max}(C^*(H \cap G_F))$ of maximal modular H -invariant ideals of $C^*(H \cap G_F)$. On the other hand, $P \cap C^*(H \cap G_F) \in H\text{-Max}(C^*(H \cap G_F))$ for each $P \in \text{Prim}(C^*(G))$ (see [23, Theorem 1]). Therefore, there exists $\sigma \in \widehat{H \cap G_F}$ such that $\tau|_{H \cap G_F} \sim \Delta(\sigma)$. Since $\text{ind}_{H \cap G_F}^G \delta(\sigma) = \text{ind}_{H \cap G_F}^G \sigma$ for every $\delta \in \Delta$, we get $\pi \prec \text{ind}_{H \cap G_F}^G \sigma$. Now [21, Theorem 3.7] implies that $\pi|_{H \cap G_F} \succ \sigma$. Since, by hypothesis, $\pi \sim \text{ind}_{G_F}^G(\pi|_{G_F})$, it follows that

$$\pi|_H \sim \text{ind}_{G_F}^G(\pi|_{G_F})|_H \succ \text{ind}_{H \cap G_F}^H(\pi|_{H \cap G_F}) \succ \text{ind}_{H \cap G_F}^H \sigma \succ \tau,$$

as required. ■

Proposition 4.3 below, a special case of which is the following lemma, generalizes [30, Proposition 5.65]. At the same time, the proof simplifies the one given in [30].

Lemma 4.2. *Let G be an arbitrary locally compact group and Z a closed subgroup of the centre $Z(G)$ of G , and let $N = \{y \in G : [y, G] \subseteq Z\}$, i.e. $N/Z = Z(G/Z)$. Let $\pi \in \widehat{G}$ and $\lambda \in \widehat{Z}$ such that $\pi|_Z \sim \lambda$, and assume that λ is faithful. Then there exists a character χ of $Z(G)$ such that χ extends λ and $\pi|_N \sim \text{ind}_{Z(G)}^N \chi$.*

Proof. Let χ be the unique character of $Z(G)$ such that $\pi|_{Z(G)} \sim \chi$. Take any $\xi \in \mathcal{H}(\pi)$ with $\|\xi\| = 1$, and let φ denote the associated coefficient function of π . Then, since $\pi|_Z \sim \lambda$, for $y \in N$ and $x \in G$,

$$x^{-1} \cdot \varphi(y) = \varphi([x, y]y) = \langle \pi([x, y]y)\xi, \xi \rangle = \lambda([x, y])\varphi(y).$$

Moreover, since $[N, G] \subseteq Z \subseteq Z(G)$, for $y, y_1, y_2 \in N$ and $x, x_1, x_2 \in G$, we have

$$[x, y_1y_2] = [x, y_1][x, y_2] \quad \text{and} \quad [x_1x_2, y] = [x_1, y][x_2, y].$$

Defining $\gamma_x : N \rightarrow \mathbb{T}$ by $\gamma_x(y) = \lambda([x, y])$, the preceding two equations show that each γ_x is a character of $N/Z(G)$, and the map $\gamma : x \rightarrow \gamma_x$ is a homomorphism from G into the dual group $\Gamma = \widehat{N/Z(G)}$. Since λ is faithful, the subgroup $\gamma(G)$ of Γ separates the points of $N/Z(G)$ and hence is dense in Γ .

Now, by the above equation $x^{-1} \cdot \varphi|_N = \gamma_x \varphi|_N$ for every $x \in G$. Since $x^{-1} \cdot \pi = \pi$ and every coefficient function of π is a finite linear combination of such functions φ , it follows that $\pi|_N \sim \pi|_N \otimes \gamma_x$ for every $x \in G$. The density of $\gamma(G)$ in Γ implies that $\pi|_N \otimes \gamma \prec \pi|_N$ for every $\gamma \in \Gamma$. It follows that

$$\text{ind}_{Z(G)}^N \chi \sim \text{ind}_{Z(G)}^N (\pi|_{Z(G)}) \sim \pi|_N \otimes \widehat{G/Z(G)} \prec \pi|_N$$

and hence $\pi|_N \sim \text{ind}_{Z(G)}^N \chi$, as claimed.

It is clear that χ extends λ since $\lambda \sim \pi|_Z \sim \chi|_Z$ and λ and $\chi|_Z$ are both characters of Z . ■

Proposition 4.3. *Let G be an arbitrary locally compact group, $\pi \in \widehat{G}$ and $\lambda \in \widehat{Z(G)}$ such that $\pi|_{Z(G)} \sim \lambda$. Let*

$$K_\lambda = \{z \in Z(G) : \lambda(z) = 1\} \quad \text{and} \quad L_\lambda = \{y \in G : [y, G] \subseteq K_\lambda\}.$$

Then there exists a unique G -invariant character χ of L_λ such that χ extends λ and $\pi|_{Z_2(G)} \sim \text{ind}_{L_\lambda}^{Z_2(G)} \chi$.

Proof. Let $q : G \rightarrow G/K_\lambda$ denote the quotient homomorphism and let $\dot{\pi} \in \widehat{G/K_\lambda}$ and $\dot{\lambda} \in \widehat{Z(G)/K_\lambda}$ be defined by $\dot{\pi} \circ q = \pi$ and $\dot{\lambda} \circ q = \lambda$. Then $\dot{\lambda}$ is faithful and therefore Lemma 4.2 applies and yields that $\dot{\pi} \sim \text{ind}_{Z(G/K_\lambda)}^N \dot{\chi}$, where $\dot{\chi} \in \widehat{Z(G/K_\lambda)}$ and

$$N = \{yK_\lambda : [yK_\lambda, G/K_\lambda] \subseteq Z(G)/K_\lambda\} = \{yK_\lambda : [y, G] \subseteq Z(G)\}.$$

Then $\chi := \dot{\chi} \circ q \in \widehat{L_\lambda}$, $q^{-1}(Z(G/K_\lambda)) = L_\lambda$ and $q^{-1}(N) = Z_2(G)$. Thus

$$\pi = \dot{\pi} \circ q \sim (\text{ind}_{Z(G/K_\lambda)}^N \dot{\chi}) \circ q = \text{ind}_{L_\lambda}^{Z_2(G)} (\dot{\chi} \circ q) = \text{ind}_{L_\lambda}^{Z_2(G)} \chi,$$

as was to be shown. The G -invariance of χ is easy to check, and the uniqueness follows from the fact that $(\text{ind}_{L_\lambda}^{Z_2(G)} \chi)|_{L_\lambda} \sim \chi$. ■

Let G be any locally compact group and N a closed normal subgroup of G , and suppose that π is an irreducible representation of G and χ is a G -invariant character of N . We shall frequently use in the sequel that $\pi \prec \text{ind}_N^G \chi$ implies $\pi|_N \sim \chi$. To see this, let $K = \{x \in N : \pi(x) = I\}$ and $q : G \rightarrow G/K$ the quotient homomorphism. Then $\chi = 1$ on K and, as in the proof of the preceding proposition, let $\dot{\pi}$ denote the irreducible representation of G/K and $\dot{\chi}$ the G/K -invariant character of N/K such that $\pi = \dot{\pi} \circ q$ and $\chi = \dot{\chi} \circ q$. Then $\dot{\pi} \prec \text{ind}_{N/K}^{G/K} \dot{\chi}$ and, because $\dot{\chi}$ is faithful and G/K -invariant, $N/K \subseteq Z(G/K)$. Moreover, since $\dot{\pi}|_{N/K}$ is a multiple of a character and $(\text{ind}_{N/K}^{G/K} \dot{\chi})|_{N/K} \sim \dot{\chi}$, it follows that $\dot{\pi}|_{N/K} \sim \dot{\chi}$ and hence $\pi|_N \sim \chi$.

Theorem 4.4. *Let G be a SIN-group such that $Z_2(G)/(Z_2(G) \cap G_F)$ is torsion-free, and let π be an irreducible representation of G . If (FP2) holds for π , then there exist a closed normal subgroup L of G such that $Z(G) \subseteq L \subseteq Z_2(G) \cap G_F$ and a G -invariant character χ of L such that $\pi|_{Z_2(G)} \sim \text{ind}_L^{Z_2(G)} \chi$. In particular, if $G_F = Z(G)$, then $\pi|_{Z_2(G)} \sim \text{ind}_{Z(G)}^{Z_2(G)} \chi$.*

Proof. By Proposition 4.3, there exist a closed normal subgroup L of G such that $Z(G) \subseteq L \subseteq Z_2(G)$ and a G -invariant character χ of L such that $\pi|_{Z_2(G)} \sim \text{ind}_L^{Z_2(G)} \chi$. We have to show that $L \subseteq G_F$. Towards a contradiction, assume that there exists $x \in L \setminus G_F$ and let $H = \langle x \rangle$. Then $H \cap G_F = \{e\}$ since $Z_2(G)/(Z_2(G) \cap G_F)$ is torsion-free. It follows that $\text{ind}_H^G \tau \succ \widehat{G}$ for every $\tau \in \widehat{H}$ [21, Proposition 2.3]. Since (FP2) holds for π , we get $\pi|_H \succ \widehat{H}$. On the other hand, since L is normal and χ is G -invariant, $(\text{ind}_L^{Z_2(G)} \chi)|_L \sim \chi$ and hence $\pi|_H \sim \chi|_H$. This forces $L = G_F$. ■

From Theorem 4.4 and Proposition 4.1 we can now deduce the following

Corollary 4.5. *Let G be a 2-step nilpotent SIN-group and let $\pi \in \widehat{G}$. Then (FP2) holds for π if and only if there exist a closed normal subgroup L of G such that $L \subseteq G_F$ and a G -invariant character χ of L such that $\pi \sim \text{ind}_L^G \chi$.*

Proof. In view of Theorem 4.4, we only have to show the 'if' part. Since $\pi \sim \text{ind}_L^G \chi$ and χ is G -invariant, we have $\pi|_L \sim \chi$ and $\pi \sim \pi \otimes \text{ind}_L^G 1_L$. This implies

$$\pi|_{G_F} \sim \pi|_{G_F} \otimes (\text{ind}_L^G 1_L)|_{G_F} \sim \pi|_{G_F} \otimes \text{ind}_L^{G_F} 1_L = \text{ind}_L^{G_F}(\pi|_L) \sim \text{ind}_L^{G_F} \chi$$

and hence, by induction in stages,

$$\pi \sim \text{ind}_L^G \chi = \text{ind}_{G_F}^G (\text{ind}_L^{G_F} \chi) \sim \text{ind}_{G_F}^G (\pi|_{G_F}).$$

Thus, by Proposition 4.1, (FP2) holds for π . ■

The statement of the next lemma is known to be true for nilpotent discrete groups [38].

Lemma 4.6. *Let G be a nilpotent SIN-group such that G_F is compact-free. Then $G_F = Z(G)$.*

Proof. We suppose that G is non-discrete. Then, since G_F is compact-free, G contains an open normal subgroup of the form \mathbb{R}^d , $d \geq 1$, such that $\mathbb{R}^d \subseteq G_F$ [17, Theorem 2.13]. We are going to prove by induction on d that $G_F \subseteq Z(G)$.

To that end, we first observe that $Z(G)_0 \neq \{e\}$. For a contradiction, assume that $Z(G)_0 = \{e\}$. Then, since $[Z_2(G)_0, G]$ is a connected subgroup of $Z(G)$, it follows that $Z_2(G)_0 = \{e\}$. Continuing in this manner, we get $Z_m(G)_0 = \{e\}$ for all $m \in \mathbb{N}$, and so $G_0 = \{e\}$. This contradiction shows that $Z(G)_0 \neq \{e\}$. We therefore have $(G/Z(G)_0)_0 = \mathbb{R}^k$ for some $k < d$, and hence $(G/Z(G)_0)_F \subseteq Z(G/Z(G)_0)$ by the induction hypothesis. In particular, $G_F \subseteq Z_2(G)$.

Finally, let $x \in G_F$ and let $\Gamma = \overline{I(G_F, G)}$. Then $\Gamma(x) = \overline{\{[y, x] : y \in G\}} \cdot x$ and, since $x \in Z_2(G)$, $[y_1, x][y_2, x] = [y_1 y_2, x]$ for all $y_1, y_2 \in G$. Since $\Gamma(x)$ is compact, this shows that the closed subgroup of $Z(G)$ generated by all commutators $[y, x]$, $y \in G$, is compact. Because $Z(G)$ is compact-free, it follows that $x \in Z(G)$. This proves that $G_F = Z(G)$. ■

For any locally compact group G and a closed normal subgroup N of G , let $K(N, G)$ denote the convex set of all continuous positive definite functions φ on N such that $\varphi(e) = 1$ and $\varphi(xyx^{-1}) = \varphi(y)$ for all $y \in N$ and $x \in G$. Moreover, let $E(N, G)$ be the set of all extreme points of $K(N, G)$ (that is, those $\varphi \in K(N, G)$ which cannot be written as a non-trivial convex combination of two different elements of $K(N, G)$). When $N = G$, we simply write $K(G)$ and $E(G)$ instead of $K(N, G)$ and $E(N, G)$.

Suppose that G is a SIN-group and N is an open normal subgroup of G such that $N \subseteq G_F$. Let $\gamma \in K(N, G)$ and denote by $\tilde{\gamma}$ the trivial extension of γ to G , that is, $\tilde{\gamma}(x) = 0$ for $x \in G \setminus N$. Then, as is known and easily verified, the representation $\pi_{\tilde{\gamma}}$ is unitarily equivalent to the induced representation $\text{ind}_N^G \pi_\gamma$.

Let G be an amenable group with open centre and suppose that $\text{Prim}(C^*(G))$ is a T_1 -space. Let $\pi \in \widehat{G}$ and $\chi \in \widehat{Z(G)}$ such that $\pi|_{Z(G)} \sim \chi$. Then there exists $\varphi \in K(G)$ such that $\pi \sim \pi_\varphi$ and $\varphi|_{Z(G)} = \chi$. To verify this, let $P_\chi(G) = \{\psi \in P(G) : \psi|_{Z(G)} = \chi\}$ and view $P_\chi(G)$ as a subset of the Banach space dual of $C^*(G)$. Then, equipped with the w^* -topology, $P_\chi(G)$ is a compact convex set because $Z(G)$ is open in G and hence $C^*(Z(G)) \subseteq C^*(G)$. Now, G acts on $P_\chi(G)$ by $(x, \psi) \rightarrow x \cdot \psi$, and this mapping is affine and continuous. Since G is amenable, by Day's fixed point theorem (see [15, Section 3.3]) there exists $\varphi \in P_\chi(G)$ such that $x \cdot \varphi = \varphi$ for all $x \in G$, that is, $\varphi \in K(G)$. Since $\pi_{x \cdot \varphi} \sim \pi_\varphi$ for all $x \in G$, it follows that $\pi_\varphi \prec \pi$ and so $\pi_\varphi \sim \pi$ since $\text{Prim}(C^*(G))$ is a T_1 -space.

Theorem 4.7. *Let G be a compact-free nilpotent SIN-group. For an irreducible representation π of G , the following are equivalent.*

- (i) (FP2) holds for π .
- (ii) $\pi \sim \text{ind}_{Z(G)}^G(\pi|_{Z(G)})$.

Proof. Since G is compact-free, $G_F = Z(G)$ by Lemma 4.6 and $Z_2(G)/Z(G)$ is torsion-free. By Proposition 4.1, we only have to show that (i) implies (ii). Assuming (i), by Theorem 4.4 there exists a character λ of $Z(G)$ such that $\pi|_{Z_2(G)} \sim \text{ind}_{Z(G)}^{Z_2(G)} \lambda$. As pointed out above, since $\text{Prim}(C^*(G))$ is a T_1 -space (Corollary 3.4) and G is amenable and $Z(G)$ is open, there exists $\varphi \in K(G)$ such that $\varphi|_{Z(G)} = \lambda$ and $\pi \sim \pi_\varphi$.

Since $\pi_{\tilde{\lambda}} = \text{ind}_{Z(G)}^G \lambda$, it suffices to show that $\varphi = \tilde{\lambda}$. To that end, we first prove that $\varphi(y) = 0$ for every $y \in Z_2(G) \setminus Z(G)$. Using the notation of Proposition 4.3, we have $\pi|_{Z_2(G)} \sim \text{ind}_{L_\lambda}^{Z_2(G)} \chi$, where χ is G -invariant character of L_λ extending λ . Since, on the other hand, $\pi|_{Z_2(G)} \sim \text{ind}_{Z(G)}^{Z_2(G)} \lambda$, it follows that $\text{ind}_{Z(G)}^{L_\lambda} \lambda \sim \pi|_{L_\lambda} \sim \chi$. This implies that $\chi = \chi\alpha$ for every character α of $L_\lambda/Z(G)$ and hence $L_\lambda = Z(G)$.

Now, let $y \in Z_2(G)$ be such that $\varphi(y) \neq 0$. We recall that if ψ is any positive definite function with $\psi(e) = 1$ and $a \in G$ is such that $|\psi(a)| = 1$, then $\psi(ab) = \psi(a)\psi(b)$ for every $b \in G$ [22, Theorem (32.4), (v)]. Therefore, for all $x \in G$, we have

$$\varphi(y) = \varphi(xyx^{-1}) = \varphi([x, y]y) = \varphi([x, y])\varphi(y) = \lambda([x, y])\varphi(y).$$

This implies that $\lambda([x, y]) = 1$ for all $x \in G$ and hence $y \in L_\lambda = Z(G)$.

To show that φ even vanishes on $G \setminus Z(G)$, we consider G with the discrete topology and let $i : G_d \rightarrow G$ denote the identity mapping. Then $\varphi \circ i \in K(G_d)$ and [3, Lemma 1.2] yields that $\varphi \circ i = 0$ on $G_d \setminus Z(G)$ since for a torsion-free nilpotent discrete group the ascending FC-series coincides with the ascending central series. ■

Remark 4.8. It is worth mentioning the following topological aspects in connection with the results obtained in this section.

(1) Let A be an arbitrary C^* -algebra. Then a primitive ideal I of A is a separated point of $\text{Prim}(A)$ if and only if I is a minimal primal ideal of A [1, Proposition 4.5]. On the other hand, for an amenable SIN-group G and $\pi \in \widehat{G}$, $\ker \pi$ is a minimal primal ideal in $C^*(G)$ precisely when $\pi \sim \text{ind}_{G_F}^G(\pi|_{G_F})$ [29, Theorem 3.6].

(2) Let G be a second countable SIN-group. Then the set Γ of all $\gamma \in E(G_F, G)$ such that $\tilde{\gamma}$, the trivial extension of γ to all of G , belongs to $E(G)$, is dense in $E(G_F, G)$ for the topology of uniform convergence on compact subsets of G_F [18, Satz 1]. This implies

$$\bigcap \{ \ker \pi_\gamma : \gamma \in \Gamma \} = \{0\}; \text{ equivalently, } \{ \pi_\gamma : \gamma \in \Gamma \} \sim \widehat{G}_F.$$

Thus the set of all induced representations $\text{ind}_{G_F}^G \pi_\gamma, \gamma \in \Gamma$, is weakly equivalent to the left regular representation of G , and hence to \widehat{G} provided that G is amenable. Now, for every $\varphi \in E(G)$, π_φ is a factorial representation of $C^*(G)$ [7, Proposition 17.3.4], and the kernel of a factorial representation of any separable C^* -algebra is a primitive ideal [7, (3.9.1) and (5.7.6)]. Then, for each $\gamma \in \Gamma$, we can choose some $\rho_\gamma \in \widehat{G}$ such that $\rho_\gamma \sim \pi_{\tilde{\gamma}} = \text{ind}_{G_F}^G \pi_\gamma$. It now follows from Proposition 4.1 that if G is a second countable amenable SIN-group, then the set of all $\pi \in \widehat{G}$ for which (FP2) holds is dense in \widehat{G} .

(3) There are also examples of torsion-free nilpotent discrete groups of arbitrary nilpotence length with the property that the set of all $\pi \in \widehat{G}$, for which (FP2) holds, is open (and dense) in \widehat{G} . One such example was given in [2], namely the multiplicative group of upper triangular $n \times n$ -matrices, $n \geq 3$, with 1 on the main diagonal and entries from an infinite countable field of characteristic zero.

(4) In [19] an example was given of an uncountable, 2-step nilpotent, torsion-free discrete group G with the property that $E(G) \cap E(\widehat{G_f}, G) = \emptyset$. Thus, by either Theorem 4.4 or Theorem 4.7, (FP2) fails to hold for every irreducible representation of G .

5. Motion groups

By a general motion group we mean a semidirect product $G = N \rtimes K$, where N is a locally compact abelian group and K is a compact group. The most prominent examples are of course the groups G_N of rigid motions of \mathbb{R}^N , $N \geq 2$, that is, the semidirect products $\mathbb{R}^N \rtimes SO(N)$, where $SO(N)$ acts on \mathbb{R}^N by rotation. It is known that none of the groups G_N has property (FP2) [30, Theorem 6.20] and that (FP1) holds only for $N = 2$ [30, Theorem 6.23]. Our main aim in this section is to identify the irreducible representations of G_N which satisfy either of the properties (FP1) and (FP2).

We briefly recall the description of the dual space $\widehat{G_N}$. We embed $SO(N-1)$ into $SO(N)$ by $SO(N-1) \rightarrow \text{diag}(1, SO(N-1))$. Then $SO(N-1)$ is the stability group of characters χ_t of \mathbb{R}^N corresponding to vectors $(t, 0, \dots, 0) \in \mathbb{R}^N$ with $t \neq 0$. Then

$$\widehat{G_N} = \widehat{SO(N)} \cup \left\{ \text{ind}_{\mathbb{R}^N \rtimes SO(N-1)}^{G_N} (\chi_t \times \sigma) : t > 0, \sigma \in \widehat{SO(N-1)} \right\},$$

where $\widehat{SO(N)}$ is considered as a subset of $\widehat{G_N}$ since $G_N/\mathbb{R}^N = SO(N)$.

Lemma 5.1. *Let π be an infinite dimensional irreducible representation of $G = \mathbb{R}^N \rtimes SO(N)$ and let H be a closed subgroup of G containing \mathbb{R}^N . Then, for $\tau \in \widehat{H}$, $\pi \prec \text{ind}_H^G \tau$ implies that $\pi|_H \succ \tau$.*

Proof. For $\chi \in \widehat{\mathbb{R}^N}$, let K_χ denote the stabilizer of χ in $SO(N)$ and $G_\chi = \mathbb{R}^N \rtimes K_\chi$. Then $H = \mathbb{R}^N \rtimes (H \cap SO(N))$ and $H_\chi = \mathbb{R}^N \rtimes (H \cap K_\chi)$. Since $\tau \in \widehat{H}$, there exist $\chi \in \widehat{\mathbb{R}^N}$ and $\omega \in \widehat{H \cap K_\chi}$ such that $\tau = \text{ind}_{H_\chi}^H (\chi \times \omega)$. Since

$$\pi|_{\mathbb{R}^N} \prec (\text{ind}_H^G \tau)|_{\mathbb{R}^N} \sim G(\tau|_{\mathbb{R}^N}) \sim K(\chi),$$

π is of the form $\pi = \text{ind}_{G_\chi}^G (\chi \times \sigma)$ for some $\sigma \in \widehat{K_\chi}$. By hypothesis,

$$\pi \prec \text{ind}_H^G \tau = \text{ind}_H^G \left(\text{ind}_{H_\chi}^H (\chi \times \omega) \right) = \text{ind}_{G_\chi}^G \left(\text{ind}_{H_\chi}^{G_\chi} (\chi \times \omega) \right).$$

The support of $\text{ind}_{G_\chi}^G \left(\text{ind}_{H_\chi}^{G_\chi} (\chi \times \omega) \right)$ consists precisely of all representations $\text{ind}_{G_\chi}^G (\chi \times \rho)$, where $\chi \times \rho \in \text{supp} \left(\text{ind}_{H_\chi}^{G_\chi} (\chi \times \omega) \right)$.

Now $\chi \times \sigma$ and $\text{ind}_{H_\chi}^{G_\chi}(\chi \times \omega)$ can both be viewed as representations of the compact quotient group $G_\chi/\{x \in \mathbb{R}^N : \chi(x) = 1\}$. Therefore the classical Frobenius reciprocity theorem implies that $(\chi \times \sigma)|_{H_\chi} \geq \chi \times \omega$. Since G is second countable and H and G_χ are clearly regularly related, because both contain \mathbb{R}^N and G_N/\mathbb{R}^N is compact, it follows that

$$\begin{aligned} \pi|_H &= \left(\text{ind}_{G_\chi}^G(\chi \times \sigma)\right)|_H \succ \text{ind}_{H_\chi}^H((\chi \times \sigma)|_{H_\chi}) \\ &\geq \text{ind}_{H_\chi}^H(\chi \times \omega) = \tau \end{aligned}$$

[11, Theorem 5.2]. ■

Lemma 5.2. *Let $G = \mathbb{R}^2 \rtimes SO(2)$ and let π be an infinite dimensional irreducible representation of G . Then (FP) holds for π .*

Proof. Since (FP1) holds for G , we only have to show that π satisfies (FP2).

Recall that $\pi = \text{ind}_{\mathbb{R}^2}^G \chi$ for some character χ of \mathbb{R}^2 . Let H be a closed subgroup of G and let $\tau \in \widehat{H}$ be such that $\pi \prec \text{ind}_H^G \tau$. We now distinguish the following three cases which exhaust all possibilities for H : (1) $H \supseteq \mathbb{R}^2$, (2) $H \cap \mathbb{R}^2 = \{0\}$ and (3) $\{0\} \neq H \cap \mathbb{R}^2 \neq \mathbb{R}^2$. In case (1), we already know from Lemma 5.1 that $\pi|_H \succ \tau$.

(2) Since $H \cap \mathbb{R}^2 = \{0\}$, we have

$$\pi|_H = \left(\text{ind}_{\mathbb{R}^2}^G \chi\right)|_H \succ \text{ind}_{\mathbb{R}^2 \cap H}^H(\chi|_{\mathbb{R}^2 \cap H}) = \lambda_H \succ \widehat{H}.$$

Thus $\pi|_H \succ \tau$ for every $\tau \in \widehat{H}$.

(3) We observe first that in this case τ has to be finite dimensional. Indeed, since $\{0\} \neq H \cap \mathbb{R}^2 \neq \mathbb{R}^2$, it is easy to see that the normalizer of $H \cap \mathbb{R}^2$ in G is a finite extension of \mathbb{R}^2 . Thus this normalizer is of the form $\mathbb{R}^2 \rtimes E$ for some finite subgroup E of $SO(2)$. In particular, H has an abelian subgroup of finite index, namely $H \cap \mathbb{R}^2$, and therefore every irreducible representation of H is finite dimensional. Let N denote the subgroup of G generated by H and \mathbb{R}^2 . Then $N = \mathbb{R}^2 \rtimes F$ for some subgroup F of E .

Notice next that it suffices to show $\pi|_H \otimes \bar{\tau} \succ 1_H$, because then [10, Lemma 4] implies that $\tau \prec \pi|_H$ since τ is finite dimensional. Now

$$\pi = \text{ind}_{\mathbb{R}^2}^G \chi \sim \text{ind}_{\mathbb{R}^2}^G(\pi|_{\mathbb{R}^2}) = \pi \otimes \text{ind}_{\mathbb{R}^2}^G 1_{\mathbb{R}^2}$$

and, analogously, $\tau \sim \tau \otimes \text{ind}_{\mathbb{R}^2}^H 1_{\mathbb{R}^2}$. Since G is amenable, we get $1_G \prec \bar{\pi} \otimes \pi$, and hence, using that $\pi \sim \pi \otimes \text{ind}_{\mathbb{R}^2}^G 1_{\mathbb{R}^2}$ and $\bar{\pi} \prec \text{ind}_H^G \bar{\tau}$,

$$\begin{aligned} 1_G &\prec \pi \otimes \text{ind}_H^G \bar{\tau} = \text{ind}_H^G(\pi|_H \otimes \bar{\tau}) \\ &\sim \text{ind}_H^G(\pi|_H \otimes \text{ind}_{H \cap \mathbb{R}^2}^H 1_{H \cap \mathbb{R}^2} \otimes \bar{\tau}) \\ &\sim \text{ind}_H^G(\text{ind}_{H \cap \mathbb{R}^2}^H(\pi|_{H \cap \mathbb{R}^2} \otimes \bar{\tau}|_{H \cap \mathbb{R}^2})). \end{aligned}$$

Thus, with $\rho = \pi|_{H \cap \mathbb{R}^2} \otimes \bar{\tau}|_{H \cap \mathbb{R}^2}$,

$$1_G \prec \left\{ \text{ind}_{H \cap \mathbb{R}^2}^G \omega : \omega \in \text{supp } \rho \right\}.$$

Since N is normal in G and $H \cap \mathbb{R}^2$ is normal in N ,

$$(\text{ind}_{H \cap \mathbb{R}^2}^G \omega)|_N \sim \text{ind}_N^G(\text{ind}_{H \cap \mathbb{R}^2}^N \omega)|_N \sim SO(2) (\text{ind}_{H \cap \mathbb{R}^2}^N \omega)$$

for each $\omega \in \widehat{H \cap \mathbb{R}^2}$. Let

$$S = \overline{\bigcup_{\omega \in \text{supp } \rho} \text{supp}(\text{ind}_{H \cap \mathbb{R}^2}^N \omega)} \subseteq \widehat{N}.$$

Then $SO(2)(S)$ is also closed in \widehat{N} . Indeed, let $(a_\alpha, \sigma_\alpha)_\alpha$ be a net in $SO(2) \times S$ such that $a_\alpha \cdot \sigma_\alpha \rightarrow \sigma$ for some $\sigma \in \widehat{N}$. As $SO(2)$ is compact, after passing to a subnet if necessary, we may assume that $a_\alpha \rightarrow a$ for some $a \in SO(2)$. Then $\sigma_\alpha = a_\alpha^{-1} \cdot (a_\alpha \cdot \sigma_\alpha) \rightarrow a^{-1} \cdot \sigma$, so that $a^{-1} \cdot \sigma \in S$ and $\sigma \in SO(2)(S)$.

Now, since $1_N = 1_G|_N$ is weakly contained in

$$\bigcup_{\omega \in \text{supp } \rho} (\text{ind}_{H \cap \mathbb{R}^2}^G \omega)|_N \sim \bigcup_{\omega \in \text{supp } \rho} SO(2) (\text{ind}_{H \cap \mathbb{R}^2}^N \omega),$$

we obtain that $1_N \in SO(2)(S)$. This in turn implies that $1_N \in S$. Since N/\mathbb{R}^2 is finite and $N = H\mathbb{R}^2$, there exists a finite subset A of H such that $N(\omega) = A(\omega)$ for every $\omega \in \widehat{H \cap \mathbb{R}^2}$. Notice next that since $H \cap \mathbb{R}^2$ is normal in N and $N(\omega) = A(\omega)$,

$$(\text{ind}_{H \cap \mathbb{R}^2}^N \omega)|_H \sim \{\text{ind}_{H \cap \mathbb{R}^2}^H(a \cdot \omega) : a \in N\} \sim \{\text{ind}_{H \cap \mathbb{R}^2}^H(a \cdot \omega) : a \in A\}.$$

From $\pi \sim \pi \otimes \widehat{G/\mathbb{R}^2}$ we conclude

$$\pi|_H \sim \pi|_H \otimes \widehat{G/\mathbb{R}^2}|_H \sim \pi|_H \otimes \widehat{H/H \cap \mathbb{R}^2}.$$

Since A is finite and $1_H = 1_N|_H \in S|_H$, there exists $a \in A$ such that

$$\begin{aligned} 1_H &\prec \{\text{ind}_{H \cap \mathbb{R}^2}^H(a \cdot \omega) : \omega \in \text{supp } \rho\} \prec \text{ind}_{H \cap \mathbb{R}^2}^H(a \cdot \rho) \\ &= \text{ind}_{H \cap \mathbb{R}^2}^H \rho = \text{ind}_{H \cap \mathbb{R}^2}^H (\pi|_{H \cap \mathbb{R}^2} \otimes \bar{\tau}|_{H \cap \mathbb{R}^2}) \\ &= \bar{\tau} \otimes \text{ind}_{H \cap \mathbb{R}^2}^H (\pi|_{H \cap \mathbb{R}^2}) \sim \bar{\tau} \otimes \pi|_H. \end{aligned}$$

This completes the proof of the lemma. ■

Lemma 5.3. *Let $N \geq 2$ and let π be a finite dimensional irreducible representation of G_N . Then (FP2) fails for π .*

Proof. We embed G_2 into G_N by

$$((x_1, x_2), A) \rightarrow ((0, \dots, 0, x_1, x_2), \text{diag}(1, \dots, 1, A)),$$

$(x_1, x_2) \in \mathbb{R}^2$, $A \in SO(2)$. Now consider the subgroup $SO(2)$ of G_2 and let $\Gamma \subseteq \widehat{SO(2)}$ denote the finite support of $\pi|_{SO(2)}$ and choose $\chi \in \widehat{SO(2)}$ such that $\chi\Gamma \cap \Gamma = \emptyset$. By [30, Proposition 6.19] or [9, Lemma 4], $1_{G_2} \prec \text{ind}_{SO(2)}^{G_2} \chi$ and hence

$$\begin{aligned} \pi &= \pi \otimes 1_{G_N} \prec \pi \otimes \text{ind}_{G_2}^{G_N} 1_{G_2} \prec \pi \otimes \text{ind}_{G_2}^{G_N} (\text{ind}_{SO(2)}^{G_2} \chi) \\ &= \pi \otimes \text{ind}_{SO(2)}^{G_N} \chi = \text{ind}_{SO(2)}^{G_N} (\pi|_{SO(2)} \otimes \chi) \\ &\sim \{\text{ind}_{SO(2)}^{G_N}(\chi\gamma) : \gamma \in \Gamma\}. \end{aligned}$$

Since Γ is finite, it follows that $\pi \prec \text{ind}_{SO(2)}^{G_N}(\chi\gamma)$ for some $\gamma \in \Gamma$. However, $\pi|_{SO(2)}$ does not (weakly) contain $\chi\gamma$ since $\chi\gamma \notin \Gamma$. ■

Lemma 5.4. *Let $G = N \rtimes K$, where N is a second countable abelian group and K is a compact Lie group. Let $\chi \in \widehat{N}$ be such that K_χ is normal in K . Then (FP1) holds for every irreducible representation π of G with $\pi|_N \sim K(\chi)$.*

Proof. It was shown in [26, p. 218] that if χ and π are as in the lemma, then for any representation ρ of G , $1_G \prec \bar{\pi} \otimes \rho$ implies $\pi \prec \rho$. Thus (FP1) holds for π by Lemma 1.1. ■

Corollary 5.5. *Property (FP1) holds for every semidirect product $N \rtimes K$, where N is a second countable locally compact abelian group and K is a compact abelian Lie group. In particular, (FP1) holds for the motion group of the plane.*

It is not known whether the hypothesis that K be a Lie group in Corollary 5.5 can be dropped.

Lemma 5.6. *Let $N \geq 3$ and let π be an infinite dimensional irreducible representation of G_N . Then (FP1) does not hold for π .*

Proof. Embed $SO(N - 1)$ into $SO(N)$ by $SO(N - 1) \rightarrow \text{diag}(1, SO(N - 1))$. It is easily verified (for instance, by induction on $N = 3, 4, \dots$) that $SO(N - 1)$ does not contain any nontrivial normal subgroup of $SO(N)$. Since π is infinite dimensional, $\pi = \text{ind}_{\mathbb{R}^N \rtimes K_\chi}^G (\chi \times \sigma)$ for some $\chi \in \mathbb{R}^N$ and $\sigma \in \widehat{K_\chi}$, and K_χ is conjugate to $SO(N - 1)$. The statement of the lemma now follows from [26, Theorem 1, (iii) \Rightarrow (iv)]. ■

Combining Lemmas 1.2, 5.1, 5.2 and 5.6 and Corollary 5.5, we obtain the following theorem.

Theorem 5.7. *Let $G_N = \mathbb{R}^N \rtimes SO(N)$, $N \geq 2$.*

- (i) *(FP1) holds for G_2 , and (FP2) holds for $\pi \in \widehat{G_2}$ if and only if π is infinite dimensional.*
- (ii) *If $N \geq 3$, then (FP1) holds for $\pi \in \widehat{G_N}$ if and only if π is finite dimensional.*
- (iii) *If $N \geq 3$, then (FP2) fails to hold for every finite dimensional $\pi \in \widehat{G_N}$.*

Unfortunately, for $N \geq 3$, we have not been able to decide whether (FP2) may hold for an infinite dimensional irreducible representation of G_N . This is mainly due to the vast variety of closed subgroups of G_N and their dual spaces. We next establish property (FP1) for a class of motion groups which includes G_2 . This result will subsequently be applied to a p -adic motion group.

Proposition 5.8. *Let G be a semidirect product $G = N \rtimes K$, where N is abelian and K is compact. Suppose that for each $\chi \in \widehat{N}$, the stability group $K_\chi = \{a \in K : a \cdot \chi = \chi\}$ is either trivial or equals K . Then, for $\pi \in \widehat{G}$ and an arbitrary representation ρ of G , $1_G \prec \bar{\pi} \otimes \rho$ implies that $\pi \prec \rho$. In particular, (FP1) holds for G .*

Proof. For every $\tau \in \widehat{G}$, choose $\chi_\tau \in \widehat{N}$ such that $\tau|_N \sim K(\chi_\tau)$, and let

$$T_1 = \{\tau \in \text{supp } \rho : K_{\chi_\tau} = \{e\}\} \quad \text{and} \quad T_2 = \{\tau \in \text{supp } \rho : K_{\chi_\tau} = K\}.$$

Then $\tau = \text{ind}_N^G \chi_\tau$ for every $\tau \in T_1$ and, for each $\tau \in T_2$, χ_τ is a K -invariant character of N and hence $\tau = \chi_\tau \times \omega$ for some $\omega \in \widehat{K}$. Thus every $\tau \in T_2$ is finite dimensional.

Since $1_G \prec \bar{\pi} \otimes \rho$, we have $1_G \prec \bar{\pi} \otimes T_1$ or $1_G \prec \bar{\pi} \otimes T_2$. Assuming first that $1_G \prec \bar{\pi} \otimes T_2$, let

$$M = \{x \in N : \chi_\tau(x) = 1 \text{ for all } \tau \in T_2\}.$$

Then M is a closed normal subgroup of G and $T_2 \subseteq \widehat{G/M}$ and $1_G \prec \bar{\pi} \otimes T_2 \prec \bar{\pi} \otimes \widehat{G/M}$ and hence $1_M \prec \bar{\pi}|_M$. We claim that this implies $\pi \in \widehat{G/M}$. To see this, observe that since N is abelian, G/N is compact and M is normal in G , the G -orbits in the dual group \widehat{M} are compact (and hence closed). On the other hand, because π is irreducible, $\pi|_M$ is weakly equivalent to some such orbit $G(\omega)$, $\omega \in \widehat{M}$. But then $1_M \prec \bar{\pi}|_M$ forces $\omega = 1_M$. Now

$$\chi_\tau([a, x]) = a \cdot \chi_\tau(x)\chi_\tau(x^{-1}) = 1$$

for all $\tau \in T_2$, $x \in N$ and $a \in K$. Thus $[G, N] \subseteq M$, and this implies that N/M is contained in the centre of G/M . Thus G/M is the direct product of N/M and K . In particular, all the irreducible representations of G/M are finite dimensional. Since $1_G \prec \bar{\pi} \otimes T_2$, [10, Lemma 4] implies that $\pi \prec T_2 \prec \rho$.

Finally, suppose that $1_G \prec \bar{\pi} \otimes T_1$. Then

$$1_N \prec \bar{\pi}|_N \otimes T_1|_N \sim K(\bar{\chi}_\pi) \otimes \bigcup_{\tau \in T_1} K(\chi_\tau),$$

and hence there exist nets $(\tau_\alpha)_\alpha$ in T_1 and $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ in K such that $a_\alpha \cdot \bar{\chi}_\pi \otimes b_\alpha \cdot \chi_{\tau_\alpha} \rightarrow 1_N$. Since K is compact, we can assume that $a_\alpha \rightarrow a$ for some $a \in K$. It follows that $a \cdot \bar{\chi}_\pi \otimes b_\alpha \cdot \chi_{\tau_\alpha} \rightarrow 1_N$, which in turn implies that $\chi_\pi \prec \bigcup_\alpha K(\chi_{\tau_\alpha}) \prec T_1|_N$. Because $\tau = \text{ind}_N^G \chi_\tau$ for every $\tau \in T_1$, we conclude

$$\pi \prec \text{ind}_N^G \chi_\pi \prec \{\text{ind}_N^G(\tau|_N) : \tau \in T_1\} \sim T_1 \prec \rho.$$

In view of Lemma 1.1, this completes the proof. ■

We conclude this section with studying a totally disconnected group, which may either be viewed as a p -adic motion group or a p -adic Heisenberg group. This example also shows that, even for motion groups with abelian acting compact group K , identifying the irreducible representations, which satisfy either of the properties (FP1) or (FP2), can be technically complicated.

Example 5.9. Let p be a prime number, Ω_p the p -adic number field equipped with the usual topology defined by the p -adic metric and Δ_p the compact open subring of p -adic integers. Let G be the group of upper triangular matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in \Delta_p, y, z \in \Omega_p.$$

In what follows we denote such a matrix by the triple (x, y, z) . Then G is 2-step nilpotent with centre $Z(G) = \{(0, 0, z) : z \in \Omega_p\}$, but also the semidirect product $N \rtimes K$, where $K = \{(x, 0, 0) : x \in \Delta_p\} = \Delta_p$ and $N = \{(0, y, z) : y, z \in \Omega_p\} = \Omega_p^2$. We are going to show that (FP1) holds for G and (FP2) holds for $\pi \in \widehat{G}$ if and only if π is infinite dimensional.

We briefly recall the facts from representation theory of G we need. There exists a character λ of Ω_p such that the group kernel of λ equals Δ_p . For each $y \in \Omega_p$, define $\lambda_y \in \widehat{\Omega_p}$ by $\lambda_y(x) = \lambda(yx)$, $x \in \Omega_p$. Then the map $y \rightarrow \lambda_y$ is a topological isomorphism between Ω_p and $\widehat{\Omega_p}$. Using this description of $\widehat{\Omega_p}$ and the action of Δ_p on $N = \Omega_p^2$, it is not difficult to determine \widehat{G} . For $k \in \mathbb{Z}$, let

$$\Lambda_k = \{\lambda_y : y \in p^k \Delta_p \setminus p^{k+1} \Delta_p\} \quad \text{and} \quad G_k = \{0\} \times p^{-k} \Delta_p \times \Omega_p.$$

Then $\widehat{\Omega_p} \setminus \{1_{\Omega_p}\}$ is the disjoint union of the open and closed sets Λ_k , $k \in \mathbb{Z}$, and every $\pi \in \widehat{G}$ such that $\pi|_{Z(G)} \sim \lambda$ for some $\lambda \in \Lambda_k$, is weakly equivalent to $\text{ind}_{G_k}^G \sigma$ for some (G -invariant) character σ of G_k . For all this, compare [12, Section 4.1], [22, Section 10] and [30, Example 5.70].

Next we show that if H is any closed subgroup of G , then H and G_k are regularly related in the sense of Mackey [31, p. 127]. This means that there exist measurable subsets E_0, E_1, \dots of G , each of which is a union of double cosets of H and G_k , such that E_0 has measure zero and each double coset not in E_0 is the intersection of the E_j containing it. To start with, note that since $G_k \supseteq Z(G)$ and $G/Z(G)$ is abelian, HG_k is a normal subgroup of G and every double coset of H and G_k is actually a coset of HG_k . If $(x, y, z) \in G$, then

$$(x, y, z)HG_k = (x, 0, 0)(0, y, z - xy)HG_k = (x, 0, 0)(0, y, 0)HG_k.$$

Let Y be a coset representative system for the subgroup $p^{-k} \Delta_p$ in Ω_p . Then Y is countable since $p^{-k} \Delta_p$ is open in Ω_p and Ω_p is σ -compact. Moreover, for any $y \in p^{-k} \Delta_p$, we have $(0, y, 0)HG_k = HG_k$ because $G/Z(G)$ is abelian and $G_k/Z(G) = p^{-k} \Delta_p$. Let \mathcal{V} be a open neighbourhood basis for $\Delta_p \setminus \{0\}$, and for each $V \in \mathcal{V}$ and $y \in Y$, set

$$E_{V,y} = (V \times \{y\} \times \{0\})HG_k.$$

The sets $E_{V,y}$ form a countable family of measurable subsets of G , each of which is a union of cosets of HG_k .

Let $q : G \rightarrow G/N = \Delta_p$ denote the quotient homomorphism and assume first that $\overline{q(H)} = \overline{q(HG_k)}$ has measure zero in Δ_p . Then $\overline{q(H)} = \{0\}$ since every nontrivial closed subgroup of Δ_p is open in Δ_p . Given a coset

$$C = (x, 0, 0)(0, y, 0)HG_k, \quad y \in Y, x \neq 0,$$

it is clear that C is the intersection of all sets $E_{V,y}$, where $x \in V$. Since $q(H) = \{0\}$, the set $E_0 = \overline{HG_k} \subseteq N$ has measure zero in G and is a union of cosets of HG_k . Consequently, using the collection of sets

$$\{E_0\} \cup \{E_{V,y} : V \in \mathcal{V}, y \in Y\},$$

shows that H and G_k are regularly related.

Therefore, we are left with the case that $\overline{q(H)}$ is an open subgroup of Δ_p . If the measurable subgroup $q(H)$ of Δ_p also does have positive Haar measure, then it must be open too and hence equal to $\overline{q(H)}$. But then

$$HG_k = q^{-1}(q(H)) \cap (\Delta_p \times G_k)$$

is open (hence closed) in G , and H and G_k are regularly related. Thus we can assume that $q(H) = \{0\}$. Then let $E_0 = N$, which is of measure zero and a union of HG_k -cosets. Arguing as before, it follows that every coset of HG_k which is not contained in E_0 , is the intersection of the sets $E_{V,y}$ containing it. This completes the proof that H and G_k are regularly related.

Since $G_k \subseteq G_F = N$ and since H and G_k are regularly related for each closed subgroup H of G , it follows from [27, Lemma 1.1] that (FP2) holds for every infinite dimensional $\pi \in \widehat{G}$. However, (FP2) fails to hold for every finite dimensional irreducible representation of G , that is, every character γ of $G/Z(G) = \Delta_p \times \Omega_p$. To see this, observe first that for any $\alpha \in \Delta_p$, we have

$$\left(\text{ind}_{\Delta_p}^G \alpha\right)|_N \succ \text{ind}_{\Delta_p \cap N}^N(\alpha|_{\Delta_p \cap N}) = \lambda_N \sim \widehat{N}.$$

Since the sets $\Lambda_k, k \in \mathbb{Z}$, are open in $\widehat{\Omega}_p$, it follows that for each $k \in \mathbb{Z}$, there exists a G -invariant character χ_k of G_k such that $\text{ind}_{\Delta_p}^G \alpha \succ \text{ind}_{G_k}^G \chi_k$. Consequently, since $1_G \prec \text{ind}_{Z(G)}^G 1_{Z(G)}$ and $(G_\chi, \chi_k) \rightarrow (Z(G), 1_{Z(G)})$ as $k \rightarrow \infty$ in Fell's subgroup representation topology, using continuity of induction with respect to this topology, we get

$$1_G \in \overline{\bigcup_{k \in \mathbb{Z}} \text{supp}(\text{ind}_{G_k}^G \chi_k)} \subseteq \text{supp}(\text{ind}_{\Delta_p}^G \alpha).$$

It now follows as in the proof of Lemma 5.3 that $\gamma \prec \text{ind}_{\Delta_p}^G(\gamma|_{\Delta_p} \alpha)$. Taking any $\alpha \neq 1_{\Delta_p}$, we see that (FP2) cannot hold for γ .

Finally, since for every $\chi \in \widehat{N}$, the stabilizer K_χ of χ either equals K or is trivial, Proposition 5.8 shows that (FP1) holds for G .

6. The $ax + b$ -group

The following 2-dimensional simply connected solvable Lie group G has the property that every irreducible representation of G satisfies exactly one of (FP1) and (FP2).

Example 6.1. Let G be the $ax + b$ -group, that is, the semidirect product $G = \mathbb{R} \rtimes \mathbb{R}_+^\times$, where the multiplicative group of positive real numbers \mathbb{R}_+^\times acts on \mathbb{R} by multiplication. Identify the normal subgroup $\mathbb{R} \times \{1\}$ with \mathbb{R} . For $t \in \mathbb{R}$, let χ_t denote the character of \mathbb{R} defined by $\chi_t(x) = \exp(2\pi itx)$, $x \in \mathbb{R}$. Moreover, let Γ denote the lift of $\widehat{\mathbb{R}_+^\times}$ to G . Then $\widehat{G} = \Gamma \cup \{\pi_+, \pi_-\}$, where $\pi_+ = \text{ind}_{\mathbb{R}}^G \chi_1$ and $\pi_- = \text{ind}_{\mathbb{R}}^G \chi_{-1}$. The singletons $\{\pi_+\}$ and $\{\pi_-\}$ are open in \widehat{G} and the closure of each of them contains Γ . Consequently, (FP2) cannot hold for any $\gamma \in \Gamma$.

We claim that π_+ satisfies (FP2). Thus let H be a closed subgroup of G and $\tau \in \widehat{H}$ such that $\pi_+ \prec \text{ind}_H^G \tau$. Recall that, by [10, Theorem 5.3],

$$\pi_+|_H \sim \{\text{ind}_{H \cap \mathbb{R}}^H(\chi_t|_{H \cap \mathbb{R}}) : t > 0\}.$$

In particular, $\pi_+|_H \succ \widehat{H}$ if $H \cap \mathbb{R} = \{0\}$ since H is amenable. Suppose next that $H \cap \mathbb{R}$ is a proper nontrivial subgroup of \mathbb{R} , so that $H \cap \mathbb{R} = \alpha\mathbb{Z}$ for some $\alpha > 0$. It is easily verified that the normalizer of $\alpha\mathbb{Z}$ in G equals \mathbb{R} . Thus $H = H \cap \mathbb{R} = \alpha\mathbb{Z}$. Then $\pi_+|_H \sim \{\chi_s|_{\alpha\mathbb{Z}} : s > 0\}$, and since $\chi_s(\alpha n) = (\exp(2\pi i s \alpha))^n$ for $n \in \mathbb{Z}$ and each character of $\alpha\mathbb{Z}$ is of the form $\alpha n \rightarrow z^n$ for some $z \in \mathbb{T}$, it follows that $\pi_+ \succ \widehat{H}$.

The remaining case is where $H \supseteq \mathbb{R}$, and only here will we use the hypothesis that $\pi_+ \prec \text{ind}_H^G \tau$. Then $\tau|_{\mathbb{R}} \not\sim 1_{\mathbb{R}}$ and hence $\tau = \text{ind}_{\mathbb{R}}^H \chi_s$ for some $s \neq 0$. Hence $\text{ind}_H^G \tau = \text{ind}_{\mathbb{R}}^G \chi_s$ which equals either π_+ or π_- . Since π_+ is not weakly contained in π_- , we must have $\text{ind}_H^G \tau = \pi_+$. This implies that $s > 0$ and therefore

$$\pi_+|_H \sim \{\text{ind}_{\mathbb{R}}^H \chi_t : t > 0\} \succ \text{ind}_{\mathbb{R}}^H \chi_s = \tau.$$

Similarly, it is shown that (FP2) holds for π_- .

It is easy to identify those $\pi \in \widehat{G}$ which satisfy (FP1). In fact, if $\gamma \in \Gamma$, H and τ are such that $\gamma|_H \succ \tau$, then $\gamma|_H = \tau$ and hence $\gamma \prec \text{ind}_H^G(\gamma|_H) = \text{ind}_H^G \tau$. Alternatively, this follows from Lemma 1.2. On the other hand, property (FP1) does not hold for π_+ and π_- because neither of the singletons $\{\pi_+\}$ and $\{\pi_-\}$ is closed in \widehat{G} (compare Lemma 1.2(iii)).

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Robert J. Archbold
Institute of Mathematics
University of Aberdeen
King’s College
Aberdeen AB24 3UE, Scotland, UK
r.archbold@abdn.ac.uk

Eberhard Kaniuth
Institut für Mathematik
Universität Paderborn
D-33095 Paderborn, Germany
kaniuth@math.uni-paderborn.de

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