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Maximal Antipodal Subgroups of some Compact Classical Lie Groups

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Abstract. We classify maximal antipodal subgroups of the quotient groups of the compact classical Lie groups and explicitly describe them by using the dihedral group of order 8. The maximal antipodal subgroups are not unique up to conjugation in almost all cases.
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1. Introduction

A subset A of a compact Riemannian symmetric space M is called an *antipodal set* if $s_x(y) = y$ holds for any elements x, y in A. Here s_x denotes the geodesic symmetry at x in M. An antipodal set is a finite set. The 2*-number* of M, denoted by $\#_2 M$, is defined by $\#_2 M = \sup\{|A| \mid A \subset M \text{ antipodal}\}$, where $|A|$ denotes the cardinality of A. Chen-Nagano introduced these notions and determined the 2-numbers of most but not all compact Riemannian symmetric spaces in [2]. In [5] the second author studied maximal antipodal sets of oriented real Grassmann manifolds $\tilde{G}_k(\mathbb{R}^n)\,(2k \leq n)$ for $k = 3, 4$, where $\#_2\tilde{G}_k(\mathbb{R}^n)\,(k \neq 1, 2)$ were not concerned in [2].

A compact Lie group G has a bi-invariant Riemannian metric, with respect to which G is a Riemannian symmetric space. Hence we can consider antipodal sets of G. As we will show later, it is enough to consider maximal antipodal sets which are subgroups of G. Thus we mainly consider maximal antipodal subgroups of G. Although to determine the 2-numbers is a main purpose of [2], we determine all maximal antipodal subgroups in this article. As a corollary we also obtain the 2-numbers. We extend the claims relating with 2-numbers in [2] to the claims relating with maximal antipodal subgroups in the case of compact Lie groups.

The present paper is organized as follows. In Section 2 we introduce some notions like as polars and centrosomes and their properties which we need later. In Section 3 we review the definition of antipodal sets of compact Riemannian

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symmetric spaces and related notions. We mainly consider the case of compact Lie groups. In Section 4 we prepare certain finite subgroups of matrices for describing maximal antipodal subgroups of the quotient groups of the classical compact Lie groups in the following sections. We also calculate their cardinalities in order to determine the maximums of them. We classify and explicitly describe maximal antipodal subgroups of the quotient groups of $U(n)$ in Section 5, $SU(n)$ in Section 6 and $O(n), SO(n), Sp(n)$ in Section 7. The classification of those in the cases of $O(n), SO(n)$ and $Sp(n)$ is simultaneously treated.

Griess [3] and Yu [6] classified conjugate classes of elementary abelian p-subgroups of algebraic groups by means of algebraic methods. An elementary abelian 2-subgroup is just an antipodal subgroup. We classify conjugate classes of maximal antipodal subgroups of the quotient groups of the compact classical Lie groups and give explicit expressions of their representatives. Our method is geometric one in which we use polars and centrioles introduced by Chen-Nagano [2].

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2. Preliminaries

In this section we introduce some notions and their properties which we will need later.

Let M be a compact Riemannian symmetric space and $o \in M$. Each connected component of the fixed point set $F(s_o, M)$ of the geodesic symmetry s_o at o is called a *polar* of M with respect to o. If a polar which is not $\{o\}$ consists of a point $p \in M$, p is called a *pole* of o in M. Let p be a pole of o in M. We denote by $C(o,p)$ the set of the midpoints of geodesic segments joining o and p. We call $C(o,p)$ the *centrosome* for the pair (o,p). By [2, Proposition 2.9], p is a pole of o in M if and only if there exists a double covering map $\pi : M \to M'$ onto some compact Riemannian symmetric space M' with $\pi(p) = \pi(o)$ which satisfies $\pi \circ s_x = s_{\pi(x)} \circ \pi$ for every $x \in M$. In this case we denote by $\gamma : M \to M$ the covering transformation of $\pi : M \to M'$.

Proposition 2.1 ([2] Proposition 3.4). *Let M be a compact Riemannian symmetric space. The following five conditions are equivalent to each other for any two distinct points $o, q \in M$.*

(1) $s_o \circ s_q = s_q \circ s_o$.

(2) $Q(q)^2 = id$*, where* $Q(q) := s_q \circ s_o$.

(3) Either s_o fixes q or q is a point in the centrosome $C(o,p)$ for some pole p of o.

(4) Either s_o fixes q or $s_o(q) = \gamma(q)$ for the covering transformation γ for some pole $p = \gamma(o)$ of o.

(5) *Either s_o fixes q or there is a double covering map $\pi : M \to M'$ onto some compact Riemannian symmetric space M' satisfying $\pi \circ s_x = s_{\pi(x)} \circ \pi$ for any $x \in M$ such that $s_{\pi(o)}$ fixes $\pi(q)$.*

Let M be a compact Riemannian symmetric space. We denote by $I(M)$ the group of isometries of M and by $I_0(M)$ its identity component. Let A_1, A_2 be subsets of M. Then A_1 is *congruent* to A_2 if there is an element in $I_0(M)$ which maps A_1 to A_2.

3. Maximal antipodal subgroups of compact Lie groups

We define antipodal sets of compact Riemannian symmetric spaces and related notions in this section. We mainly discuss the case of compact Lie groups.

Let M be a compact Riemannian symmetric space. A subset $A \subset M$ is called an *antipodal set* if $s_x(y) = y$ holds for any $x, y \in A$. Since an antipodal set is a discrete subset in a compact Hausdorff space, it is finite. The maximum of the cardinalities of antipodal sets of M is called the *2-number* of M denoted by $\#_2 M$. If the cardinality of an antipodal set A attains $\#_2 M$, A is called a *great* antipodal set. A great antipodal set is a maximal antipodal set. But a maximal antipodal set is not necessarily a great antipodal set. In fact, we constructed a maximal antipodal set which is not a great antipodal set in the adjoint group of $SU(4)$ in [4] and many maximal antipodal sets which are not great in oriented real Grassmann manifolds in [5].

Let G be a compact Lie group. It is known that G is a Riemannian symmetric space with respect to a bi-invariant Riemannian metric. The geodesic symmetry s_x at $x \in G$ is given by $s_x(y) = xy^{-1}x$ for $y \in G$. In particular, for the identity element $e \in G$ we have $s_e(y) = y^{-1}$.

Lemma 3.1. *Let G be a compact Lie group. We have the following for any $x, y \in G$.*

(1) *$s_e(x) = x$ if and only if $x^2 = e$.*

(2) *If $x^2 = y^2 = e$, $s_x(y) = y$ if and only if $xy = yx$.*

We can see this lemma directly from the definition of the geodesic symmetry. We obtain the following lemma from Lemma 3.1 and the fundamental theorem of finite abelian groups.

Lemma 3.2. *If a maximal antipodal set $A \subset G$ satisfies $e \in A$, then A is an abelian subgroup of G which is isomorphic to a product $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ of some copies of $\mathbb{Z}_2$. Here $\mathbb{Z}_2$ denotes the cyclic group of order 2.*

By this lemma we have $\#_2 G = 2^l$ for some natural number l, where l is the so-called 2-*rank* of G ([1], [2]). We note that $l \geq \mathrm{rank}(G)$ and moreover $l > \mathrm{rank}(G)$ is possible.

Let Z be the center of G and let $Z' \subset Z$ be a discrete subgroup of Z.

Then the quotient group $G' := G/Z'$ is a compact Lie group locally isomorphic to G. Let $\pi : G \to G'$ be the natural projection which is a covering homomorphism whose kernel $\mathrm{Ker}(\pi) = Z'$. We have $\pi \circ s_x = s_{\pi(x)} \circ \pi$ for any $x \in G$. If A is an antipodal subgroup of G, $\pi(A)$ is an antipodal subgroup of G'. We note that if A is a maximal antipodal subgroup of G, $\pi(A)$ is not necessarily a maximal antipodal subgroup of G'.

Lemma 3.3. *Let G, G' be compact Lie groups and let $\pi : G \to G'$ be a covering homomorphism whose covering degree is odd. If A' is an antipodal subgroup of G', then there exists an antipodal subgroup B of G which satisfies the following conditions.*

(1) B is a 2-Sylow subgroup of $\pi^{-1}(A')$ such that $|B| = |A'|$.

(2) The restriction of π to B is an isomorphism from B onto A'.

We can obtain this lemma by Sylow's theorem.

Proposition 3.4. *Under the same assumption of Lemma 3.3 moreover we assume that G is connected. If A is a maximal antipodal subgroup of G, then $\pi(A)$ is a maximal antipodal subgroup of G' which is isomorphic to A under π. Conversely, if A' is a maximal antipodal subgroup of G', then there exists a maximal antipodal subgroup A of G such that A is isomorphic to A' under π.*

Proof. We set $Z' := \mathrm{Ker}(\pi)$. Since G is connected, the center Z of G includes Z'. Let A be a maximal antipodal subgroup of G. We can show that $\pi|_A : A \to \pi(A)$ is an isomorphism. In order to prove that $\pi(A)$ is a maximal antipodal subgroup of G', we assume that A' is an antipodal subgroup of G' satisfying $\pi(A) \subset A'$. By Lemma 3.3 there exists an antipodal subgroup B of G such that $\pi|_B : B \to A'$ is an isomorphism and B is a 2-Sylow subgroup of $\tilde{A} := \pi^{-1}(A')$. We can prove $\tilde{A} = BZ'$. If we write $a \in A \subset \tilde{A}$ as $a = bx\,(b \in B,\, x \in Z')$, then $x = e$. Therefore we obtain $A \subset B$ and by the maximality of A we have $A = B$. Hence $\pi(A) = \pi(B) = A'$ and so $\pi(A)$ is a maximal antipodal subgroup of G'.

Conversely, let A' be a maximal antipodal subgroup of G'. By Lemma 3.3 there exists an antipodal subgroup A of G such that A is isomorphic to A' under π. We can show that A is a maximal antipodal subgroup of G. ∎

4. Subgroups of the compact classical groups

In order to describe the classifications of maximal antipodal subgroups of the quotient groups of the compact classical groups we define certain finite subgroups of the compact classical groups. We need the cardinalities of each finite subgroups for determining great antipodal subgroups among maximal antipodal subgroups later.

For a set X consisting of square matrices we define

$$X^{\pm} := \{x \in X \mid \det x = \pm 1\}.$$

We define

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n).$$

Δ_n is a unique maximal antipodal subgroup up to conjugation in $U(n)$, $O(n)$ and $Sp(n)$. Δ_n^+ is a unique maximal antipodal subgroup up to conjugation in $SU(n)$ and $SO(n)$ (cf. Proposition 4.7).

We denote by 1_m the identity matrix of degree m. When $n = 2n'$, we define

$$I_{n'} := \begin{bmatrix} -1_{n'} & \\ & 1_{n'} \end{bmatrix}, \quad J_{n'} := \begin{bmatrix} & -1_{n'} \\ 1_{n'} & \end{bmatrix}, \quad K_{n'} := \begin{bmatrix} & 1_{n'} \\ 1_{n'} & \end{bmatrix} \in O(n).$$

Let

$$D[4] := \{\pm 1_2, \pm I_1, \pm J_1, \pm K_1\} \subset O(2)$$

be a dihedral group, which is the automorphism group of a square in the plane.

$$Q[8] := \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$$

is the quaternion group, where $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are elements of the standard basis of the quaternions $\mathbb{H}$.

The following lemma will be needed later in the proof of Theorem 7.1. For $x \in Q[8]$ we denote by L_x (resp. R_x) the action of x on $\mathbb{H}$ from the left (resp. right). We consider $Q[8] \cdot Q[8] = \{L_x \circ R_y \mid x, y \in Q[8]\}$ and $D[4] \otimes D[4] = \{A \otimes B \mid A, B \in D[4]\}$ as subgroups of $O(4)$, where

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} \in O(4) \quad (A = [a_{ij}], B \in D[4]).$$

Lemma 4.1. *$Q[8] \cdot Q[8]$ coincides with $D[4] \otimes D[4]$.*

Proof. The matrix representations of $L_{\mathbf{i}}, L_{\mathbf{j}}, L_{\mathbf{k}}, R_{\mathbf{i}}, R_{\mathbf{j}}, R_{\mathbf{k}}$ with respect to the standard basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ of $\mathbb{H}$ as a real vector space are given by

$$\begin{aligned} L_{\mathbf{i}} &= 1_2 \otimes J_1, \quad L_{\mathbf{j}} = J_1 \otimes I_1, \quad L_{\mathbf{k}} = J_1 \otimes K_1, \\ R_{\mathbf{i}} &= -I_1 \otimes J_1, \quad R_{\mathbf{j}} = J_1 \otimes 1_2, \quad R_{\mathbf{k}} = K_1 \otimes J_1, \end{aligned}$$

which implies $Q[8] \cdot Q[8] = D[4] \otimes D[4]$. ■

We decompose a natural number n as $n = 2^k \cdot l$ into the product of the k-th power 2^k of 2 and an odd number l.

For s with $0 \leq s \leq k$ we define

$$D(s, n) := \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s} \subset O(n)$$

as a tensor product of s copies of $D[4]$ and $\Delta_{n/2^s}$.

In general, for $x \in M_m(\mathbb{C})$ and $y \in M_n(\mathbb{C})$ we have $\det(x \otimes y) = (\det x)^n (\det y)^m$, which leads the following lemma.

Lemma 4.2. *Let $n = 2^k \cdot l$ where l is odd and let $D^{\pm}(s,n) = D(s,n) \cap O^{\pm}(n)$.*

(a) The case of $k = 1$.

$$D^+(1,n) = D^+[4] \otimes \Delta_l, \quad D^-(1,n) = D^-[4] \otimes \Delta_l.$$

(b) The case of $k \geq 2$.
If $s \geq 1$,

$$D^+(s,n) = D(s,n).$$

Here we refer to the cardinality of $D(s,n)$. We know

$$|D[4]| = 2^3, \quad |D[4] \otimes D[4]| = 2^{3+3}/2 = 2^5.$$

Moreover, we obtain inductively the cardinality of the tensor product of s copies of $D[4]$ as

$$|\underbrace{D[4] \otimes \cdots \otimes D[4]}_{s}| = 2^{2s+1}.$$

Hence we have

$$|D(s,n)| = |\underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s}| = 2^{2s+1} \cdot 2^{n/2^s}/2 = 2^{2s+2^{k-s} \cdot l}.$$

The following lemma will be used later.

Lemma 4.3. *Let $n = 2^k \cdot l$ where l is odd. We define a function f_n on $\{0, 1, \ldots, k\}$ by*

$$|D(s,n)| = 2^{f_n(s)}.$$

Then $f_n(s) = 2s + 2^{k-s}l$. The maximum of f_n is given as follows.

(1) When $n = 2$, f_2 takes its maximum at $s = 1$ only and the maximum is $f_2(1) = 3$.

(2) When $n = 4$, f_4 takes its maximum at $s = 2$ only and the maximum is $f_4(2) = 5$.

(3) Otherwise, f_n takes its maximum at $s = 0$ only and the maximum is $f_n(0) = n$.

Proof. We already proved $f_n(s) = 2s + 2^{k-s}l$. Since the definition of f_n is valid when s is a real number, we consider f_n as a function defined on the interval $[0,k]$, which is smooth with respect to s. Since the second derivative of f_n is positive, f_n is a convex function of s. Therefore f_n takes its maximum at $s = 0$ or $s = k$. When $k = 0$, f_n takes its maximum at $s = 0$ and the maximum is $f_n(0) = n$. We assume $k \geq 1$. When $n = 2$, that is, when $k = 1$ and $l = 1$, $f_2(1)$ is the maximum. When $n = 4$, that is, when $k = 2$ and $l = 1$, $f_4(1)$ is the maximum. We can show $f_n(0) = n = 2^k \cdot l > 2k + l = f_n(k)$ when $(k,l) \neq (1,1), (2,1)$. ■

We refer to the cardinality of $D^+(s,n) = D(s,n) \cap SO(n)$.

Lemma 4.4. *Let $n = 2^k \cdot l$ where l is odd. We define a function g_n on $\{0, 1, \ldots, k\}$ by*

$$|D^+(s,n)| = 2^{g_n(s)}.$$

The maximum of g_n is given as follows.

(1) When $n = 2$, g_2 takes its maximum at $s = 1$ only and the maximum is $g_2(1) = 2$.

(2) When $n = 4$, g_4 takes its maximum at $s = 2$ only and the maximum is $g_4(2) = 5$.

(3) When $n = 8$, g_8 takes its maximum at $s = 0, 3$ only and the maximum is $g_8(0) = g_8(3) = 7$.

(4) Otherwise, g_n takes its maximum at $s = 0$ only and the maximum is $g_n(0) = n - 1$.

Proof. $D^+(0,n) = \Delta_n^+$ for any n. By Lemma 4.2, when $k = 1$, we have $D^+(1,n) = D^+[4] \otimes \Delta_l$ and when $k \geq 2$, we have $D^+(s,n) = D(s,n)$ for $s \geq 1$. Therefore $g_n(s)$ is given as follows.

$$g_n(0) = n - 1$$
$$g_n(1) = \begin{cases} l + 1 & (k = 1) \\ 2 + 2^{k-1} \cdot l & (k \geq 2) \end{cases}$$
$$g_n(s) = 2s + 2^{k-s} \cdot l \quad (2 \leq s \leq k)$$

When $k = 0$, $g_n(0) = n - 1$ is the maximum trivially. When $k = 1$, $g_n(0) = n - 1 = 2l - 1$ and $g_n(1) = l + 1$. Therefore, when $l = 1$, i.e., $n = 2$, $g_2(1) = 2$ is the maximum and when $l \geq 3$, $g_n(0) = 2l-1 = n-1$ is the maximum. When $k \geq 2$, if $s \geq 1$, $g_n(s)$ is equal to $f_n(s)$ in Lemma 4.3. Therefore the maximum is $g_n(0), g_n(1)$ or $g_n(k)$. We have

$$g_n(0) = 2^k \cdot l - 1, \quad g_n(1) = 2 + 2^{k-1} \cdot l, \quad g_n(k) = 2k + l.$$

When $k = 2$, if $l = 1$, i.e., $n = 4$, $g_4(2) = 5$ is the maximum. If $l \geq 3$, we can see $g_n(0) > g_n(1)$ and $g_n(0) > g_n(k)$. Hence $g_n(0) = n - 1$ is the maximum. When $k \geq 3$, we can see $g_n(0) > g_n(1)$. In order to compare $g_n(0)$ and $g_n(k)$ we consider $\frac{2k+1}{2^k-1}$ similarly as in the proof of Lemma 4.3. We obtain

$$\frac{2k+1}{2^k-1} \begin{cases} = 1 & (k = 3) \\ < 1 & (k \geq 4) \end{cases}.$$

Therefore when $k = 3, l = 1$, i.e., $n = 8$, $g_8(0) = g_8(3) = 7$, which is the maximum. When $k = 3, l \geq 3$ or $k \geq 4$, we have $g_n(0) > g_n(k)$ and $g_n(0) = n - 1$ is the maximum. ∎

We refer to some fundamental properties of $D[4]$ and $Q[8]$. The following are hold.

$$\{x \in D[4] \mid x^2 = 1_2\} = \{\pm 1_2, \pm I_1, \pm K_1\},$$
$$\{x \in D[4] \mid x^2 = -1_2\} = \{\pm J_1\},$$
$$\{x \in Q[8] \mid x^2 = 1\} = \{\pm 1\},$$
$$\{x \in Q[8] \mid x^2 = -1\} = \{\pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}.$$

If we set $a = I_1, b = K_1 \in D[4]$, we have $a^2 = b^2 = 1_2$. Moreover,

$$aba^{-1}b^{-1} = abab = J_1 J_1 = -1_2.$$

If we set $a = \mathbf{i}$ and $b = \mathbf{j}$, we have $a^2 = b^2 = -1_2$. Moreover,

$$aba^{-1}b^{-1} = ab(-a)(-b) = abab = \mathbf{kk} = -1.$$

We denote by $M_n(\mathbb{K})$ the set of $n \times n$ matrices whose entries belong to $\mathbb{K}$. We denote by $GL(n, \mathbb{K})$ the group of invertible elements in $M_n(\mathbb{K})$. We can see easily the following two lemmas.

Lemma 4.5. *If $a, b \in M_n(\mathbb{K})$ satisfy*

$$a^2 = b^2 = 1_n, \quad aba^{-1}b^{-1} = -1_n, \tag{1}$$

the subgroup $\langle a, b\rangle$ of $GL(n, \mathbb{K})$ generated by a, b is isomorphic to $D[4]$.

Lemma 4.6. *If $a, b \in M_n(\mathbb{K})$ satisfy*

$$a^2 = b^2 = -1_n, \quad aba^{-1}b^{-1} = -1_n, \tag{2}$$

the subgroup $\langle a, b\rangle$ of $GL(n, \mathbb{K})$ generated by a, b is isomorphic to $Q[8]$.

First we refer to maximal antipodal subgroups of $U(n)$, $SU(n)$, $O(n)$, $SO(n)$ and $Sp(n)$. Using simultaneous diagonalizations or the conjugacy of maximal tori we can obtain the following proposition.

Proposition 4.7 (cf. [2]). *A maximal antipodal subgroup of $U(n), O(n)$ or $Sp(n)$ is conjugate to Δ_n. A maximal antipodal subgroup of $SU(n)$ or $SO(n)$ is conjugate to Δ_n^+. In particular, Δ_n is a unique great antipodal subgroup of $U(n), O(n)$ and $Sp(n)$ up to conjugation and $\#_2 U(n) = \#_2 O(n) = \#_2 Sp(n) = 2^n$. Δ_n^+ is a unique great antipodal subgroup of $SU(n)$ and $SO(n)$ up to conjugation and $\#_2 SU(n) = \#_2 SO(n) = 2^{n-1}$.*

5. Maximal antipodal subgroups of the quotient groups of $U(n)$

The center of $U(n)$ is $\{\alpha 1_n \mid \alpha \in U(1)\}$ and we identify it with $U(1)$. Let μ be a natural number, let $\mathbb{Z}_\mu$ be the cyclic group of degree μ which lies in the center of $U(n)$. Now we give the classification of maximal antipodal subgroups of $U(n)/\mathbb{Z}_\mu$.

Theorem 5.1. *Let $\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$ be the natural projection. Let θ be a primitive 2μ-th root of 1. We decompose n as $n = 2^k \cdot l$ into the product of the k-th power 2^k of 2 and an odd number l. Then a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ is conjugate to one of the following.*

(1) In the case where n or μ is odd,

$$\pi_n(\{1, \theta\}D(0, n)) = \pi_n(\{1, \theta\}\Delta_n).$$

(2) In the case where both n and μ are even,

$$\pi_n(\{1, \theta\}D(s, n)) \quad (0 \leq s \leq k),$$

where the case $(s, n) = (k - 1, 2^k)$ is excluded.

Remark 5.2. Since we have an inclusion $\Delta_2 \subsetneq D[4]$ which implies $D(k-1, 2^k) \subsetneq D(k, 2^k)$, the case $(s, n) = (k - 1, 2^k)$ is excluded.

Corollary 5.3. *Great antipodal subgroups, their cardinalities and the 2-number of $U(n)/\mathbb{Z}_\mu$ are as follows.*

(1) In the case where n or μ is odd, $\pi_n(\{1, \theta\}\Delta_n)$ is a unique great antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ up to conjugation. $|\pi_n(\{1, \theta\}\Delta_n)| = 2^n$, which is $\#_2(U(n)/\mathbb{Z}_\mu)$.

(2) In the case where both n and μ are even,

(2-1) when $n = 2$, $\pi_2(\{1, \theta\}D[4])$ is a unique great antipodal subgroup of $U(2)/\mathbb{Z}_\mu$ up to conjugation. $|\pi_2(\{1, \theta\}D[4])| = 2^3 = 2^{n+1}$, which is $\#_2(U(2)/\mathbb{Z}_\mu)$.

(2-2) When $n = 4$, $\pi_4(\{1, \theta\}D(2, 4))$ is a unique great antipodal subgroup of $U(4)/\mathbb{Z}_\mu$ up to conjugation. $|\pi_4(\{1, \theta\}D(2, 4))| = 2^5 = 2^{n+1}$, which is $\#_2(U(4)/\mathbb{Z}_\mu)$.

(2-3) Otherwise, $\pi_n(\{1, \theta\}\Delta_n)$ is a unique great antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ up to conjugation. $|\pi_n(\{1, \theta\}\Delta_n)| = 2^n$, which is $\#_2(U(n)/\mathbb{Z}_\mu)$ if $n \neq 2, 4$.

Before we prove Theorem 5.1, we prepare some lemmas. Let A be a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ and let $B = \pi_n^{-1}(A)$. B is a subgroup of $U(n)$. We denote by e the identity element of $U(n)/\mathbb{Z}_\mu$.

We can easily see the following lemma.

Lemma 5.4. *When μ is even, $\pi_n(1) = \pi_n(-1)$. When μ is odd, $\pi_n(\theta) = \pi_n(-1)$. In particular, we have $\pi_n(\Delta_n) = \pi_n(\theta\Delta_n)$ if μ is odd.*

The following Lemmas from 5.5 to 5.9 are proved by Chen-Nagano [2] in the case where A is great. We can similarly prove them in the case where A is maximal, so we omit their proofs.

Lemma 5.5 ([2] Lemma 5.3). *$\theta \in B$.*

Lemma 5.6 ([2] Lemma 5.4). *An element of A is conjugate to an element of $\pi_n(\Delta_n \cup \theta\Delta_n)$ in $U(n)/\mathbb{Z}_\mu$. Moreover, when B is commutative, A is conjugate to $\pi_n(\Delta_n \cup \theta\Delta_n)$ in $U(n)/\mathbb{Z}_\mu$. In particular, $|A| = |\Delta_n| = 2^n$ when B is commutative.*

Lemma 5.7 ([2] Lemma 5.5). *If $a, b \in B$, then $ab = ba$ or $ab = -ba$.*

Lemma 5.8 ([2] Lemma 5.6). *If $a, b, c \in B$ and $ab = -ba$, then at least one of c, ac, bc and abc commutes with a and b.*

Lemma 5.9 ([2] Lemma 5.7). *Let $a, b \in B$ satisfy $ab = -ba$. Then we have the following.*

(i) $\mathrm{Tr}(a) = \mathrm{Tr}(b) = 0$, *where* $\mathrm{Tr}(x)$ *denotes the trace of* x.

(ii) n *is even. (We set* $n = 2n'$.*)*

(iii) *Each of* a *and* b *is conjugate to an element of* $\{1, \theta, \theta^2, \ldots, \theta^{2\mu-1}\}I_{n'}$ *in* $U(n)$.

(iv) μ *is even.*

In Chen-Nagano [2] some details of the proof of the next lemma are omitted. Since the argument of the proof needs the knowledge of properties of centrioles, which is not familiar, we give its detailed proof.

Lemma 5.10 ([2] Lemma 5.8). *If B is not commutative, B is conjugate to a subgroup of $D[4] \otimes U(n')$ in $U(n)$ and A is conjugate to a subgroup of $\pi_n(D[4] \otimes U(n'))$ in $U(n)/\mathbb{Z}_\mu$.*

Before we prove Lemma 5.10, we need some preparations. The connected components of $F(s_{1_n}, U(n))$ are conjugate classes of involutive elements by Lemma 3.1 (1). We write

$$F(s_{1_n}, U(n)) = \bigcup_{j=0}^{n} M_j^+,$$

as a disjoint union of connected components

$$M_j^+ := \{gI_{j,n-j}g^{-1} \mid g \in U(n)\} \quad (0 \leq j \leq n).$$

For $j \in \{1, \ldots, n-1\}$, we have

$$M_j^+ \cong U(n)/U(j) \times U(n-j) = G_j(\mathbb{C}^n).$$

We define $\alpha_j : G_j(\mathbb{C}^n) \to U(n)$ by $\alpha_j(V) := 1_V - 1_{V^\perp}$ for $V \in G_j(\mathbb{C}^n)$, where 1_V and $1_{V^\perp}$ denote the orthogonal projections from $\mathbb{C}^n$ onto V and onto $V^\perp$, the orthogonal complement of V in $\mathbb{C}^n$, respectively. Then α_j gives an isometry between $G_j(\mathbb{C}^n)$ and M_j^+ with respect to their suitable invariant Riemannian metrics, therefore we can identify $G_j(\mathbb{C}^n)$ with M_j^+. We consider the case where $n = 2n'$ is even and $j = n'$. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{C}^n$. Let $o := \langle e_1, \dots, e_{n'} \rangle_{\mathbb{C}}$ be the n'-dimensional complex subspace of $\mathbb{C}^n$ spanned by $e_1, \dots, e_{n'}$. Then $o \in M_{n'}^+ = G_{n'}(\mathbb{C}^n)$. We have

$$F(s_o, M_{n'}^+) = \bigcup_{j=0}^{n'} G_{n'-j}(\langle e_1, \dots, e_{n'} \rangle_{\mathbb{C}}) \times G_j(\langle e_1, \dots, e_{n'} \rangle_{\mathbb{C}}^{\perp}).$$

In particular, $o^\perp = \langle e_{n'+1}, \dots, e_n \rangle_{\mathbb{C}}$ is the pole of o in $M_{n'}^+ = G_{n'}(\mathbb{C}^n)$. We define $\gamma : G_{n'}(\mathbb{C}^n) \to G_{n'}(\mathbb{C}^n)$ by $\gamma(V) := V^\perp$ for $V \in G_{n'}(\mathbb{C}^n)$. We have $\alpha_{n'}(\gamma(V)) = \alpha_{n'}(V^\perp) = -\alpha_{n'}(V)$ in $U(n)$. This means that the action of γ on $M_{n'}^+$ is -1 times the identity map under the identification of $G_{n'}(\mathbb{C}^n)$ and $M_{n'}^+$ given by $\alpha_{n'}$.

Now we prove Lemma 5.10.

Proof. Since B is not commutative, there exist $a, b \in B$ which satisfy $ab \neq ba$. By Lemma 5.7 we have $ab = -ba$. By Lemma 5.9 (iii) each of a and b is conjugate to an element of $\{1, \theta, \theta^2, \dots, \theta^{2\mu-1}\} I_{n'}$. Let a be conjugate to $\theta^m I_{n'}$ for some m. Then $a' := \theta^{2\mu-m} a$ is conjugate to $I_{n'}$. Since $\theta \in B$ by Lemma 5.5, $a' \in B$. We have $a'b = -ba'$ since $a'b = \theta^{2\mu-m} ab = -\theta^{2\mu-m} ba = -b\theta^{2\mu-m} a = -ba'$. Similarly we can take $b' \in B$ which is conjugate to $I_{n'}$ and satisfies $a'b' = -b'a'$. Therefore we may assume that a and b are conjugate to $I_{n'}$. There exit $u_a, u_b \in U(n)$ such that $a = u_a I_{n'} u_a^{-1}$ and $b = u_b I_{n'} u_b^{-1}$. Hence $a^2 = b^2 = 1_n$. Let $M^+ := \{u I_{n'} u^{-1} \mid u \in U(n)\}$. Then $a, b \in M^+$ and M^+ is a polar of 1_n in $U(n)$ which is isometric to $G_{n'}(\mathbb{C}^n) = U(n)/U(n') \times U(n')$. Therefore $s_a(b) = ab^{-1}a = aba = -ba^2 = -b = \gamma(b)$ by the previous argument. Thus we have (4) in Proposition 2.1 and so we have (3) in Proposition 2.1, that is, $b \in C(a, -a)$. Without loss of generality we may assume $a = I_{n'}$. For $\theta = (\theta_1, \dots, \theta_{n'}) \in \mathbb{R}^{n'}$ we set

$$c(\theta) := \begin{bmatrix} \cos\theta_1 & & \\ & \ddots & \\ & & \cos\theta_{n'} \end{bmatrix}, \quad s(\theta) := \begin{bmatrix} \sin\theta_1 & & \\ & \ddots & \\ & & \sin\theta_{n'} \end{bmatrix},$$

$$g(\theta) := \begin{bmatrix} c(\theta) & -s(\theta) \\ s(\theta) & c(\theta) \end{bmatrix}.$$

Then

$$T := \{g(\theta) I_{n'} g(\theta)^{-1} \mid \theta \in \mathbb{R}^{n'}\}$$

is a maximal torus of M^+ containing $I_{n'}$. Therefore we have

$$\begin{aligned} C(I_{n'}, -I_{n'}) &= C(I_{n'}, -I_{n'}) \cap M^+ \\ &= C(I_{n'}, -I_{n'}) \cap \left(\bigcup_{g \in U(n') \times U(n')} gTg^{-1} \right) \\ &= \bigcup_{g \in U(n') \times U(n')} g[C(I_{n'}, -I_{n'}) \cap T]g^{-1}. \end{aligned}$$

We can see easily $C(I_{n'}, -I_{n'}) \cap T = \{\pm K_{n'}\}$. Hence

$$\begin{aligned} C(I_{n'}, -I_{n'}) &= \bigcup_{g \in U(n') \times U(n')} g\{\pm K_{n'}\}g^{-1} \\ &= \left\{ \begin{bmatrix} 0 & x \\ x^{-1} & 0 \end{bmatrix} \middle| x \in U(n') \right\}, \end{aligned}$$

which is connected. Since $b \in C(I_{n'}, -I_{n'})$, b is conjugate to $K_{n'}$ under the action of $U(n') \times U(n')$ which fixes a. Hence we may assume $b = K_{n'}$. Then the subgroup $\langle a, b\rangle$ generated by a and b is $\langle a, b\rangle = D[4] \otimes 1_{n'}$. Let $c \in B$ be an arbitrary element. By Lemma 5.8, at least one of c, ac, bc and abc is commutative with a and b. Since $a = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \otimes 1_{n'}$ and $b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes 1_{n'}$, an element $g \in U(n)$ is commutative with a and b if and only if $g \in \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes U(n')$. There is $g \in D[4] \otimes 1_{n'}$ such that gc is commutative with a and b. Therefore $c \in g^{-1}(1_2 \otimes U(n')) \subset D[4] \otimes U(n')$. Hence $B \subset D[4] \otimes U(n')$ and $A \subset \pi_n(D[4] \otimes U(n'))$. ∎

Lemma 5.11. *If B is not commutative, there exists a maximal antipodal subgroup A' of $U(n')/\mathbb{Z}_\mu$ such that A is conjugate to $\pi_n(D[4] \otimes \pi_{n'}^{-1}(A'))$. Conversely, if C is a maximal antipodal subgroup of $U(n')/\mathbb{Z}_\mu$, then $\pi_n(D[4] \otimes \pi_{n'}^{-1}(C))$ is a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$.*

Proof. When B is not commutative, B is conjugate to a subgroup of $D[4] \otimes U(n')$, where $n = 2n'$, and A is conjugate to a subgroup of $\pi_n(D[4] \otimes U(n'))$ by Lemma 5.10. By changing B in its conjugacy class, if necessary, we may assume that B is a subgroup of $D[4] \otimes U(n')$ and A is a subgroup of $\pi_n(D[4] \otimes U(n'))$. Then $D[4] \otimes 1_{n'} \subset B$. If we set

$$B' := \{y \in U(n') \mid \text{there exists}\, x \in D[4]\, \text{such that}\, x \otimes y \in B\},$$

$1_{n'} \in B'$. In particular, B' is not empty. B' is a subgroup of $U(n')$ and we have $B = D[4] \otimes B'$. Moreover we caan see that $\pi_{n'}(B')$ is a maximal antipodal subgroup of $\pi_{n'}(U(n')) = U(n')/\mathbb{Z}_\mu$ by the maximality of A.

Conversely, if C is a maximal antipodal subgroup of $U(n')/\mathbb{Z}_\mu$, we can see that $\pi_n(D[4] \otimes \pi_{n'}^{-1}(C))$ is a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$. ∎

Now we turn to the proof of Theorem 5.1.

Proof. Let A be a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ and let $B = \pi_n^{-1}(A)$. We decompose n as $n = 2^k \cdot l$ into the product of the k-th power 2^k of 2 and an odd number l. In order to prove the theorem by induction on k, we rewrite the statement of the theorem.

(i) μ is odd. A is conjugate to $\pi_n(\{1, \theta\}D(0, n)) = \pi_n(\{1, \theta\}\Delta_n)$.

(ii) μ is even.

(ii-1) $k = 0$. A is conjugate to $\pi_n(\{1, \theta\}\Delta_n)$.

(ii-2) $k \geq 1$. A is conjugate to $\pi_n(\{1, \theta\}D(s, n))\,(0 \leq s \leq k)$, where the case $(s, n) = (k - 1, 2^k)$ is excluded.

(i) Let μ be odd. Then B is commutative by Lemma 5.9 and A is conjugate to $\pi_n(\Delta_n \cup \theta\Delta_n) = \pi_n(\{1, \theta\}\Delta_n)$ by Lemma 5.6.

(ii) Let μ be even. We prove the claim by induction on k. Let $k = 0$. Then B is commutative by Lemma 5.9 and A is conjugate to $\pi_n(\{1, \theta\}\Delta_n)$ by Lemma 5.6. Thus (ii-1) holds. Let $k \geq 1$. We assume that the claim is valid for k' satisfying $0 \leq k' < k$ and prove that the claim is valid for k. If B is commutative, A is conjugate to $\pi_n(\{1, \theta\}\Delta_n) = \pi_n(\{1, \theta\}D(0, n))$ by Lemma 5.6. If B is not commutative, there exits a maximal antipodal subgroup A' in $U(n')/\mathbb{Z}_\mu$ such that B is conjugate to $D[4] \otimes \pi_{n'}^{-1}(A')$ by Lemma 5.11 where $n' = n/2 = 2^{k-1} \cdot l$. By the assumption of induction, A' is conjugate to

$$\pi_n(\{1, \theta\}D(s', n')) \quad (0 \leq s' \leq k - 1)$$

in $U(n')/\mathbb{Z}_\mu$, where the case $(s', n') = (k - 2, 2^{k-1})$ is excluded. Since

$$D[4] \otimes \{1, \theta\}D(s', n') = \{1, \theta\}D[4] \otimes D(s', n') = \{1, \theta\}D(s' + 1, n),$$

B is conjugate to

$$\{1, \theta\}D(s, n) \quad (1 \leq s \leq k), \tag{3}$$

where the case $(s, n) = (k - 1, 2^k)$ is excluded. By the last half of Lemma 5.11 each of (3) except for the case $(s, n) = (k - 1, 2^k)$ can be occurred. Thus (ii-2) is valid for k. Therefore A is conjugate to

$$\pi_n(\{1, \theta\}D(s, n)) \quad (0 \leq s \leq k),$$

where the case $(s, n) = (k - 1, 2^k)$ is excluded. Thus (ii-2) is valid for k. ■

We prove Corollary 5.3.

Proof. When μ is odd, since $\pi_n(\theta) = \pi_n(-1)$ by Lemma 5.4, we have

$$|\pi_n(\{1, \theta\}D(s, n))| = |\pi_n(D(s, n))| = |D(s, n)| = 2^{2s+2^{k-s}\cdot l}.$$

Here we remark that $|D(s, n)| = 2^{2s+2^{k-s}\cdot l}$ by Lemma 4.3. When μ is even, since $\pi_n(1) = \pi_n(-1)$ by Lemma 5.4, we have

$$\begin{aligned} |\pi_n(\{1, \theta\}D(s, n))| &= |\pi_n(D(s, n))| + |\pi_n(\theta D(s, n))| \\ &= 2^{2s+2^{k-s}\cdot l}/2 + 2^{2s+2^{k-s}\cdot l}/2 = 2^{2s+2^{k-s}\cdot l}. \end{aligned}$$

Therefore, in both cases we have

$$|\pi_n(\{1, \theta\}D(s, n))| = 2^{2s+2^{k-s}\cdot l} = 2^{f_n(s)},$$

where $f_n(s) = 2s + 2^{k-s} \cdot l$ is the function defined in Lemma 4.3.

(1) The case where n or μ is odd. By Theorem 5.1, $\pi_n(\{1, \theta\}\Delta_n)$ is a unique maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ up to conjugation. Hence $\pi_n(\{1, \theta\}\Delta_n)$ is a unique great antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ up to conjugation. When μ is odd, we have $|\pi_n(\{1, \theta\}\Delta_n)| = |\pi_n(\Delta_n)| = |\Delta_n| = 2^n$ by Lemma 5.4. When μ is

even, we have $|\pi_n(\{1,\theta\}\Delta_n)| = 2|\pi_n(\Delta_n)| = 2|\Delta_n|/2 = 2^n$ by Lemma 5.4. Hence $\#_2(U(n)/\mathbb{Z}_\mu) = 2^n$.

(2) The case where n and μ are even. Every maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ is conjugate to $\pi_n(\{1,\theta\}D(s,n))$ $(0 \le s \le k)$ except for the case of $(s,n) = (k-1, 2^k)$ by Theorem 5.1. Since $|\pi_n(\{1,\theta\}D(s,n))| = 2^{f_n(s)}$, if s gives the maximum of $\{f_n(s') \mid 0 \le s' \le k\}$, $\pi_n(\{1,\theta\}D(s,n))$ is a great antipodal subgroup of $U(n)/\mathbb{Z}_\mu$. By Lemma 4.3, when $n = 2$, $f_2(s)$ takes the maximum at $s = 1$ only and $f_2(1) = 3$. When $n = 4$, $f_4(s)$ takes the maximum at $s = 2$ only and $f_4(2) = 5$. Otherwise, $f_n(s)$ takes the maximum at $s = 0$ only and $f_n(0) = n$. Therefore, when $n = 2$, $\pi_2(\{1,\theta\}D(1,2)) = \pi_n(\{1,\theta\}D[4])$ is a unique great antipodal subgroup of $U(2)/\mathbb{Z}_\mu$ up to conjugation and $\#_2(U(2)/\mathbb{Z}_\mu)) = 2^3$. When $n = 4$, $\pi_4(\{1,\theta\}D(2,4))$ is a unique great antipodal subgroup of $U(4)/\mathbb{Z}_\mu$ up to conjugation and $\#_2(U(4)/\mathbb{Z}_\mu) = 2^5$. Otherwise, $\pi_n(\{1,\theta\}D(0,n)) = \pi_n(\{1,\theta\}\Delta_n)$ is a unique great antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ up to conjugation and $\#_2(U(n)/\mathbb{Z}_\mu) = 2^n$. ■

6. Maximal antipodal subgroups of the quotient groups of $SU(n)$

We give the classification of maximal antipodal subgroups of $SU(n)/\mathbb{Z}_\mu$ by using the results of the previous section.

Theorem 6.1. *Let n and μ be natural numbers with the condition that n is divided by μ. We decompose n as $n = 2^k \cdot l$ into the product of the k-th power 2^k of 2 and an odd number l. Let $\mathbb{Z}_\mu$ be the cyclic group of degree μ which lies in the center of $SU(n)$ and let θ be a primitive 2μ-th root of 1. Then a maximal antipodal subgroup of $SU(n)/\mathbb{Z}_\mu$ is conjugate to one of the following.*

(1) In the case where n or μ is odd,

$$\pi_n(\Delta_n^+).$$

(2) In the case where both n and μ are even,

(a) when $k = 1$,

$$\pi_n(\Delta_n^+ \cup \theta\Delta_n^-), \quad \pi_n((D^+[4] \cup \theta D^-[4]) \otimes \Delta_l),$$

where $\pi_2(\Delta_2^+ \cup \theta\Delta_2^-)$ is excluded when $n = \mu = 2$.

(b) When $k \ge 2$, under the expression $\mu = 2^{k'} \cdot l'$ where $1 \le k' \le k$ and l is divided by l',

(b1) if $k' = k$,

$$\pi_n(\Delta_n^+ \cup \theta\Delta_n^-), \quad \pi_n(D(s,n)) \quad (1 \le s \le k),$$

where the case $(s,n) = (k-1, 2^k)$ is excluded.

(b2) If $1 \le k' < k$,

$$\pi_n(\{1,\theta\}\Delta_n^+), \quad \pi_n(\{1,\theta\}D(s,n)) \quad (1 \le s \le k),$$

where the case $(s,n) = (k-1, 2^k)$ is excluded and when $n = 4$, moreover, $\pi_4(\{1,\theta\}\Delta_4^+)$ is excluded.

Remark 6.2. Since we have an inclusion $\Delta_4^+ = \Delta_2 \otimes \Delta_2 \subsetneq D[4] \otimes D[4] = D(2,4)$, $\pi_4(\{1,\theta\}\Delta_4^+)$ is excluded.

Corollary 6.3. *Great antipodal subgroups, their cardinalities and the 2-number of $SU(n)/\mathbb{Z}_\mu$ are as follows.*

(1) In the case where n or μ is odd, $\pi_n(\Delta_n^+)$ is a unique great antipodal subgroup up to conjugation. $|\pi_n(\Delta_n^+)| = 2^{n-1}$, which is $\#_2(SU(n)/\mathbb{Z}_\mu)$.

(2) In the case where both n and μ are even,

(a) $k = 1$. When $n = \mu = 2\,(l = 1)$, $\pi_2(D^+[4] \cup \theta D^-[4])$ is a unique great antipodal subgroup up to conjugation. $|\pi_2(D^+[4] \cup \theta D^-[4])| = 2^2 = 2^n$, which is $\#_2(SU(2)/\mathbb{Z}_2)$. Otherwise $(l \geq 3)$, $\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)$ is a unique great antipodal subgroup up to conjugation. $|\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)| = 2^{n-1}$, which is $\#_2(SU(n)/\mathbb{Z}_\mu)$.

(b) $k \geq 2$. Under the expression $\mu = 2^{k'} \cdot l'$ where $1 \leq k' \leq k$ and l is divided by l',

(b1) $k' = k$. When $n = 4$, $\pi_4(D(2,4))$ is a unique great antipodal subgroup up to conjugation. $|\pi_4(D(2,4))| = 2^4 = 2^n$, which is $\#_2 SU(4)/\mathbb{Z}_4$. When $n \geq 8$, $\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)$ is a unique great antipodal subgroup up to conjugation. $|\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)| = 2^{n-1}$, which is $\#_2(SU(n)/\mathbb{Z}_\mu)$.

(b2) $1 \leq k' < k$. When $n = 4$, $\pi_4(\{1,\theta\}D(2,4))$ is a unique great antipodal subgroup up to conjugation. $|\pi_4(\{1,\theta\}D(2,4))| = 2^5 = 2^{n+1}$, which is $\#_2(SU(4)/\mathbb{Z}_2)$. When $n = 8$, $\pi_8(\{1,\theta\}\Delta_8^+)$ and $\pi_8(\{1,\theta\}D(3,8))$ are the great antipodal subgroups up to conjugation. Their cardinalities are $2^7 = 2^{n-1}$, which is $\#_2(SU(8)/\mathbb{Z}_\mu)$. Otherwise, $\pi_n(\{1,\theta\}\Delta_n^+)$ is a unique great antipodal subgroup of $SU(n)/\mathbb{Z}_\mu$ up to conjugation. $|\pi_n(\{1,\theta\}\Delta_n^+)| = 2^{n-1}$, which is $\#_2(SU(n)/\mathbb{Z}_\mu)$.

First we prove Theorem 6.1.

Proof. Let A be a maximal antipodal subgroup of $SU(n)/\mathbb{Z}_\mu$. Since $SU(n)/\mathbb{Z}_\mu$ is a subgroup of $U(n)/\mathbb{Z}_\mu$, A is an antipodal subgroup of $U(n)/\mathbb{Z}_\mu$. Hence there is a maximal antipodal subgroup $\tilde{A}$ such that $\tilde{A} \cap SU(n)/\mathbb{Z}_\mu = A$. By Theorem 5.1, $\tilde{A}$ is conjugate to $\pi_n(\{1,\theta\}D(s,n))$ for some $s\,(0 \leq s \leq k)$ by an element of $U(n)/\mathbb{Z}_\mu$. Hence there is $g \in U(n)$ such that

$$\tilde{A} = \pi_n(g)\pi_n(\{1,\theta\}D(s,n))\pi_n(g)^{-1}.$$

Since $SU(n)/\mathbb{Z}_\mu$ is a normal subgroup of $U(n)/\mathbb{Z}_\mu$, we have

$$A = \tilde{A} \cap SU(n)/\mathbb{Z}_\mu = \pi_n(g)\left(\pi_n(\{1,\theta\}D(s,n)) \cap SU(n)/\mathbb{Z}_\mu\right)\pi_n(g)^{-1}.$$

We can write $g = g_1 z$ where $g_1 \in SU(n)$ and $z \in U(1)$. Since z is an element of the center of $U(n)$, $\pi_n(z)$ is an element of the center of $\pi_n(U(n))$. Hence we have

$$\begin{aligned}&\pi_n(g)\,(\pi_n(\{1,\theta\}D(s,n)) \cap SU(n)/\mathbb{Z}_\mu)\,\pi_n(g)^{-1}\\&= \pi_n(g_1)\,(\pi_n(\{1,\theta\}D(s,n)) \cap SU(n)/\mathbb{Z}_\mu)\,\pi_n(g_1)^{-1}.\end{aligned}$$

Therefore A is conjugate to $\pi_n(\{1,\theta\}D(s,n)) \cap SU(n)/\mathbb{Z}_\mu$ by an element of $SU(n)/\mathbb{Z}_\mu$. We have

$$\pi_n(\{1,\theta\}D(s,n)) \cap SU(n)/\mathbb{Z}_\mu = \pi_n(\{1,\theta\}D(s,n) \cap SU(n)).$$

In fact, it is clear that

$$\begin{aligned}\pi_n(\{1,\theta\}D(s,n) \cap SU(n)) &\subset \pi_n(\{1,\theta\}D(s,n)) \cap \pi_n(SU(n))\\&= \pi_n(\{1,\theta\}D(s,n)) \cap SU(n)/\mathbb{Z}_\mu.\end{aligned}$$

Conversely, if $d \in D(s,n)$ satisfies $\pi_n(d) \in \pi_n(SU(n))$, then $\theta^{2m}d \in SU(n)$ for some m. Since $\det(\theta^{2m}d) = \theta^{2mn}\det(d) = (\theta^{2\mu})^{mn/\mu}\det(d) = \det(d)$, $\theta^{2m}d \in SU(n)$ is equivalent to $d \in SU(n)$. Hence we have $\pi_n(d) \in \pi_n(D(s,n) \cap SU(n))$. If $d \in D(s,n)$ satisfies $\pi_n(\theta d) \in \pi_n(SU(n))$, we have $\pi_n(\theta d) \in \pi_n(\theta D(s,n) \cap SU(n))$ by a similar argument above. Therefore A is conjugate to
$\pi_n(\{1,\theta\}D(s,n) \cap SU(n))$ by an element of $SU(n)/\mathbb{Z}_\mu$.
It is sufficient to determine $(\{1,\theta\}D(s,n))^+$ for each case in Theorem 5.1.

(1) The case where n or μ is odd. In this case μ is odd. Hence $\pi_n(\{1,\theta\}\Delta_n) = \pi_n(\Delta_n)$ by Lemma 5.4 and so A is conjugate to $\pi_n(\Delta_n^+)$.

(2) The case where both n and μ are even.

(a) When $k = 1$, $\mu = 2 \cdot l'$ where l' divides l. Since $\det(\theta 1_n) = \theta^n = (\theta^\mu)^{\frac{n}{\mu}} = -1$, we have

$$(\{1,\theta\}D(0,n))^+ = \Delta_n^+ \cup \theta\Delta_n^-.$$

On the other hand, by Lemma 4.2

$$(\{1,\theta\}D(1,n))^+ = D^+[4] \otimes \Delta_l \cup \theta D^-[4] \otimes \Delta_l = (D^+[4] \cup \theta D^-[4]) \otimes \Delta_l.$$

Therefore A is conjugate to one of

$$\pi_n(\Delta_n^+ \cup \theta\Delta_n^-), \quad \pi_n((D^+[4] \cup \theta D^-[4]) \otimes \Delta_l).$$

However $\pi_2(\Delta_2^+ \cup \theta\Delta_2^-)$ is excluded when $n = \mu = 2$ because $\Delta_2 \subsetneq D[4]$ mentioned in Remark 5.2.

(b) When $k \geq 2$, $\mu = 2^{k'} \cdot l'$ where $1 \leq k' \leq k$ and l' divides l. We have

$$\det(\theta 1_n) = (\theta^\mu)^{\frac{n}{\mu}} = (-1)^{2^{(k-k')} \cdot \frac{l}{l'}} = \begin{cases} -1 & (k' = k) \\ 1 & (1 \leq k' < k). \end{cases}$$

(b1) When $k' = k$, we have $(\{1,\theta\}\Delta_n)^+ = \Delta_n^+ \cup \theta\Delta_n^-$ and $(\{1,\theta\}D(s,n))^+ = D(s,n)$ by Lemma 4.2. Therefore A is conjugate to one of

$$\pi_n(\Delta_n^+ \cup \theta\Delta_n^-), \quad \pi_n(D(s,n)) \ (1 \leq s \leq k)$$

but the case $(s,n)=(k-1,2^k)$ is excluded.

(b2) When $1 \le k' < k$, we have $(\{1,\theta\}\Delta_n)^+ = \{1,\theta\}\Delta_n^+$ and
$$(\{1,\theta\}D(s,n))^+ = \{1,\theta\}D(s,n)$$
by Lemma 4.2. Therefore A is conjugate to one of

$$\pi_n(\{1,\theta\}\Delta_n^+), \quad \pi_n(\{1,\theta\}D(s,n))\ (1 \le s \le k)$$

but the case $(s,n)=(k-1,2^k)$ is excluded. ■

We prove Corollary 6.3.

Proof. (1) The case where n or μ is odd. By Theorem 6.1, $\pi_n(\Delta_n^+)$ is the unique maximal antipodal subgroup of $SU(n)/\mathbb{Z}_\mu$ up to conjugation. Hence $\pi_n(\Delta_n^+)$ is a unique great antipodal subgroup of $SU(n)/\mathbb{Z}_\mu$ up to conjugation and $\#_2(SU(n)/\mathbb{Z}_\mu) = |\pi_n(\Delta_n^+)| = 2^{n-1}$.

(2) The case where n and μ are even.

(a) When $k=1$, every maximal antipodal subgroup is conjugate to
$$\pi_n(\Delta_n^+ \cup \theta\Delta_n^-) \text{ or } \pi_n((D^+[4] \cup \theta D^-[4]) \otimes \Delta_l),$$
where $\pi_2(\Delta_2^+ \cup \theta\Delta_2^-)$ is excluded when $n=\mu=2$, by Theorem 6.1. We have

$$|\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)| = |\pi_n(\Delta_n^+)| + |\pi_n(\theta\Delta_n^-)| = 2 \cdot 2^{n-2} = 2^{n-1}$$

and

$$|\pi_n((D^+[4] \cup \theta D^-[4]) \otimes \Delta_l)| = |\pi_n(D^+[4] \otimes \Delta_l)| + |\pi_n(\theta D^-[4] \otimes \Delta_l)| = 2 \cdot 2^l = 2^{l+1}.$$

Since $2^{n-1} = 2^{g_n(0)}$ and $2^{l+1} = 2^{g_n(s)}$ where $g_n(s)$ is the function defined in Lemma 4.4, we can apply Lemma 4.4. Therefore, when $l=1$, i.e., $n=\mu=2$, $\pi_2(D^+[4] \cup \theta D^-[4])$ is a unique great antipodal subgroup of $SU(2)/\mathbb{Z}_2$ up to conjugation and $\#_2(SU(2)/\mathbb{Z}_2) = 4$. When $l \ge 3$, $\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)$ is a unique great antipodal subgroup of $SU(n)/\mathbb{Z}_\mu$ up to conjugation and $\#_2(SU(n)/\mathbb{Z}_\mu) = 2^{n-1}$.

(b) The case of $k \ge 2$.

(b1) When $k'=k$, every maximal antipodal subgroup is conjugate to $\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)$ or $\pi_n(D(s,n))$ $(1 \le s \le k)$ except for the case of $(s,n) = (k-1,2^k)$ by Theorem 6.1. We have $|\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)| = 2^{n-1}$ and $|\pi_n(D(s,n))| = 2^{2s+2^{k-s}\cdot l-1}$ $(1 \le s \le k)$. Since $2^{n-1} = 2^{f_n(0)-1}$ and $2^{2s+2^{k-s}\cdot l} - 1 = 2^{f_n(s)-1}$ where $f_n(s)$ is the function defined in Lemma 4.3, we can apply Lemma 4.3. Therefore, when $n=4$, i.e., $n=\mu=4$, $\pi_4(D(2,4))$ is a unique great antipodal subgroup of $SU(4)/\mathbb{Z}_4$ up to conjugation and $\#_2(SU(4)/\mathbb{Z}_4) = 2^4$. When $n \ge 8$, $\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)$ is a unique great antipodal subgroup of $SU(n)/\mathbb{Z}_\mu$ up to conjugation and $\#_2(SU(n)/\mathbb{Z}_\mu) = 2^{n-1}$.

(b2) When $1 \le k' < k$, every maximal antipodal subgroup is conjugate to $\pi_n(\{1,\theta\}\Delta_n^+)$ or $\pi_n(\{1,\theta\}D(s,n))$ $(1 \le s \le k)$ except for the cases of $(s,n) = (k-1,2^k)$ and $\pi_4(\{1,\theta\}\Delta_4^+)$ by Theorem 6.1. We have

$$|\pi_n(\{1,\theta\}\Delta_n^+)| = 2 \cdot 2^{n-1}/2 = 2^{n-1}$$

and
$$|\pi_n(\{1,\theta\}D(s,n))| = 2 \cdot 2^{2s+2^{k-s}\cdot l}/2 = 2^{2s+2^{k-s}\cdot l} \quad (1 \le s \le k).$$

Since $2^{n-1} = 2^{g_n(0)}$ and $2^{2s+2^{k-s}\cdot l} = 2^{g_n(s)}$ where $g_n(s)$ is the function defined in Lemma 4.4, we can apply Lemma 4.4. Therefore, when $n = 4$ and $\mu = 2$, $\pi_4(\{1,\theta\}D(2,4))$ is a unique great antipodal subgroup of $SU(4)/\mathbb{Z}_2$ up to conjugation and $\#_2(SU(4)/\mathbb{Z}_2) = 2^5$. When $n = 8$, $\pi_8(\{1,\theta\}\Delta_8^+)$ and $\pi_8(\{1,\theta\}D(3,8))$ are the great antipodal subgroups of $SU(8)/\mathbb{Z}_\mu$ up to conjugation and $\#_2(SU(8)/\mathbb{Z}_\mu) = 2^7$. Otherwise, $\pi_n(\{1,\theta\}\Delta_n^+)$ is a great antipodal subgroup of $SU(n)/\mathbb{Z}_\mu$ up to conjugation and $\#_2(SU(n)/\mathbb{Z}_\mu) = 2^{n-1}$. ∎

7. Maximal antipodal subgroups of the quotient groups of $O(n)$, $SO(n)$ and $Sp(n)$

We give the classification of maximal antipodal subgroups of $O(n)/\{\pm 1_n\}$, S
We decompose n as $n = 2^k \cdot l$ and simultaneously show the classification of all of them by induction on k.

Theorem 7.1. *Let $\tilde{G}$=$O(n)$, $SO(n)$, $Sp(n)$ and G=$O(n)/\{\pm 1_n\}$, $SO(n)/\{\pm 1_n\}$, $Sp(n)/\{\pm 1_n\}$ respectively, where n is even when $\tilde{G} = SO(n)$. Let $\pi_n : \tilde{G} \to G$ denote the natural projection. We decompose n as $n = 2^k \cdot l$ into the product of the k-th power 2^k of 2 and an odd number l.*

(I) *A maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$ is conjugate to one of the following.*

$$\pi_n(D(s,n)) \quad (0 \le s \le k),$$

where the case $(s,n) = (k-1, 2^k)$ is excluded.

(II) *When n is even, a maximal antipodal subgroup of $SO(n)/\{\pm 1_n\}$ is conjugate to one of the following.*

(1) *In the case where $k = 1$,*

$$\pi_n(\Delta_n^+), \quad \pi_n(D^+[4] \otimes \Delta_l),$$

where $\pi_2(\Delta_2^+)$ is excluded when $n = 2$.

(2) *In the case where $k \ge 2$,*

$$\pi_n(\Delta_n^+), \quad \pi_n(D(s,n)) \quad (1 \le s \le k),$$

where the case $(s,n) = (k-1, 2^k)$ is excluded and moreover $\pi_4(\Delta_4^+)$ is excluded when $n = 4$.

(III) *A maximal antipodal subgroup of $Sp(n)/\{\pm 1_n\}$ is conjugate to one of the following.*

$$\pi_n(Q[8] \cdot D(s,n)) \quad (0 \le s \le k),$$

where the case $(s,n) = (k-1, 2^k)$ is excluded.

Corollary 7.2. *Great antipodal subgroups, their cardinalities and the 2-numbers of $O(n)/\{\pm 1_n\}, SO(n)/\{\pm 1_n\}$ and $Sp(n)/\{\pm 1_n\}$ are as follows.*

(I) $O(n)/\{\pm 1_n\}$.

(1) In the cases of $n=2$, $\pi_2(D[4])$ *is a unique great antipodal subgroup up to conjugation.* $|\pi_2(D[4])| = 2^2 = 2^n$, *which is* $\#_2(O(2)/\{\pm 1\})$.

(2) In the case of $n=4$, $\pi_4(D(2,4))$ *is a unique great antipodal subgroup up to conjugation.* $|\pi_4(D(2,4))| = 2^4 = 2^n$, *which is* $\#_2(O(4)/\{\pm 1\})$.

(3) Otherwise, $\pi_n(\Delta_n)$ *is a unique great antipodal subgroup up to conjugation.* $|\pi_n(\Delta_n)| = 2^{n-1}$, *which is* $\#_2(O(n)/\{\pm 1\})$.

(II) $SO(n)/\{\pm 1_n\}$.

(1) In the case of $n=2$, $\pi_2(D^+[4])$ *is a unique great antipodal subgroup up to conjugation.* $|\pi_2(D^+[4])| = 2^1 = 2^{n-1}$, *which is* $\#_2(SO(2)/\{\pm 1\})$.

(2) In the case of $n=4$, $\pi_4(D(2,4))$ *is a unique great antipodal subgroup up to conjugation.* $|\pi_4(D(2,4))| = 2^4 = 2^n$, *which is* $\#_2(SO(4)/\{\pm 1\})$.

(3) In the case of $n=8$, $\pi_8(\Delta_8^+)$ *and* $\pi_8(D(3,8))$ *are the great antipodal subgroups up to conjugation.* $|\pi_8(\Delta_8^+)| = |\pi_8(D(3,8))| = 2^6 = 2^{n-2}$, *which is* $\#_2(SO(8)/\{\pm 1\})$.

(4) Otherwise, $\pi_n(\Delta_n^+)$ *is a unique great antipodal subgroup up to conjugation.* $|\pi_n(\Delta_n^+)| = 2^{n-2}$, *which is* $\#_2(SO(n)/\{\pm 1\})$.

(III) $Sp(n)/\{\pm 1_n\}$.

(1) In the case of $n=2$, $\pi_2(Q[8]\cdot D[4])$ *is a unique great antipodal subgroup up to conjugation.* $|\pi_2(Q[8]\cdot D[4])| = 2^4 = 2^{n+2}$, *which is* $\#_2(Sp(2)/\{\pm 1\})$.

(2) In the case of $n=4$, $\pi_4(Q[8]\cdot D(2,4))$ *is a unique great antipodal subgroup up to conjugation.* $|\pi_4(Q[8]\cdot D(2,4))| = 2^6 = 2^{n+2}$, *which is* $\#_2(Sp(4)/\{\pm 1\})$.

(3) Otherwise, $\pi_n(Q[8]\cdot \Delta_n)$ *is a unique great antipodal subgroup up to conjugation.* $|\pi_n(Q[8]\cdot \Delta_n)| = 2^{n+1}$, *which is* $\#_2(Sp(n)/\{\pm 1\})$.

Before we prove Theorem 7.1, we need some preparations. We can show the following lemma by straightforward calculation.

Lemma 7.3. *If* $x \in M_n(\mathbb{H})$ *is conjugate to a real diagonal matrix by some element in* $Sp(n)$, *then* $\mathrm{Tr}(x)$ *is a real number which is invariant under the conjugation by* $Sp(n)$.

Remark 7.4. In general for $x \in M_n(\mathbb{H})$ the trace $\mathrm{Tr}(x)$ is not invariant under the conjugation by $Sp(n)$.

When we prove Theorem 7.1, we divide the case into four cases, which are explained below.

Let $A \subset G$ be a maximal antipodal subgroup. If we set $B := \pi_n^{-1}(A)$, B is a subgroup of $\tilde{G}$. There are two cases, that is, (1) B is commutative and

(2) B is not commutative. If $x \in B$, we have $\pi_n(x)^2 = e$, where e denotes the identity element of G. Hence $x^2 = \pm 1_n$. Therefore we divide the case (1) into the following two cases.

(1-1) The case where $x^2 = 1_n$ for every $x \in B$.

(1-2) The case where $c^2 = -1_n$ for some $c \in B$.

In the case (2), there exist $a, b \in B$ such that $ab \neq ba$. Since $\pi_n(a), \pi_n(b) \in A$ are commutative, we have $\pi_n(ab) = \pi_n(ba)$. Hence $ab = -ba$.

We divide the case (2) into the following two cases.

(2-1) The case where $x^2 = 1_n$ for some $x \in \{a, b, ab\}$.

(2-2) The case where $x^2 = -1_n$ for every $x \in \{a, b, ab\}$.

In the case (2-1) we may assume $a^2 = 1_n$ if we retake a and b. When $a^2 = 1_n$, there is nothing to do. When $b^2 = 1_n$, it is enough to change b to a. When $(ab)^2 = 1_n$, $\bar{a} := ab$ satisfies $\bar{a}b = abb = -bab = -b\bar{a}$. Hence it is enough to change $\bar{a}$ to a.

Since $a^2 = 1_n$, we have $b^2 = a^2b^2 = -(ab)^2$. Therefore if $b^2 = 1_n$ (resp. $(ab)^2 = 1_n$), then $(ab)^2 = -1_n$ (resp. $b^2 = -1_n$).

Lemma 7.5. *If $a^2 = 1_n$ or $b^2 = 1_n$, $\langle a, b\rangle$ is isomorphic to $D[4]$. If $a^2 = b^2 = -1_n$, $\langle a, b\rangle$ is isomorphic to $Q[8]$.*

Proof. If $a^2 = 1_n$ or $b^2 = 1_n$, it is enough to consider the case where $a^2 = 1_n$. In this case $b^2 = 1_n$ or $(ab)^2 = 1_n$ by the argument above. Therefore $\langle a, b\rangle$ is isomorphic to $D[4]$ by Lemma 4.5.

If $a^2 = b^2 = -1_n$, we have $aba^{-1}b^{-1} = -baa^{-1}b^{-1} = bb^{-1} = -1_n$. Hnece $\langle a, b\rangle$ is isomorphic to $Q[8]$ by Lemma 4.6. ∎

By Lemma 7.5 it turns out that the case (2-1) is the case where $\langle a, b\rangle \cong D[4]$ and the case (2-2) is the case where $\langle a, b\rangle \cong Q[8]$.

According to the four cases (1-1), (1-2), (2-1), (2-2) explained above, we obtain the following.

Proposition 7.6. *$SO(n)/\{\pm 1_n\}$ can be considered when n is even ($n = 2n'$) and any maximal antipodal subgroup of $SO(n)/\{\pm 1_n\}$ is conjugate to one of the following.*

(1-1) $\pi_n(\Delta_n^+)$.

(1-2) $\pi_n(D^+[4] \otimes \Delta_{n'})$.

(2-1) $\pi_n(D[4] \otimes B')$ *where* $\pi_{n'}(B')$ *is a maximal antipodal subgroup of* $O(n')/\{\pm 1_{n'}\}$.

(2-2) $n = 4n''$ *and* $\pi_n(Q[8] \cdot B')$ *where* $\pi_{n''}(B')$ *is a maximal antipodal subgroup of* $Sp(n'')/\{\pm 1_{n''}\}$.

Any maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$ *is conjugate to one of the following.*

(1-1) $\pi_n(\Delta_n)$.

(1-2) None.

(2-1) $n = 2n'$ *and* $\pi_n(D[4] \otimes B')$ *where* $\pi_{n'}(B')$ *is a maximal antipodal subgroup of* $O(n')/\{\pm 1_{n'}\}$.

(2-2) $n = 4n''$ *and* $\pi_n(Q[8] \cdot B')$ *where* $\pi_{n''}(B')$ *is a maximal antipodal subgroup of* $Sp(n'')/\{\pm 1_{n''}\}$.

Any maximal antipodal subgroup of $Sp(n)/\{\pm 1_n\}$ *is conjugate to one of the following.*

(1-1) None.

(1-2) None.

(2-1) $n = 2n'$ *and* $\pi_n(D[4] \otimes B')$ *where* $\pi_{n'}(B')$ *is a maximal antipodal subgroup of* $Sp(n')/\{\pm 1_{n'}\}$.

(2-2) $\pi_n(Q[8] \cdot B')$ *where* $\pi_n(B')$ *is a maximal antipodal subgroup of* $O(n)/\{\pm 1_n\}$.

Proof. Let $A \subset G$ be a maximal antipodal subgroup and we set $B := \pi_n^{-1}(A)$.

(1) The case where B is commutative.

(1-1) The case where $x^2 = 1_n$ for every $x \in B$. In this case B is an antipodal subgroup of $\tilde{G}$. The maximality of A implies that B is a maximal antipodal subgroup of $\tilde{G}$.

In the cases of $\tilde{G} = O(n), Sp(n)$, B is conjugate to Δ_n by Proposition 4.7 and A is conjugate to $\pi_n(\Delta_n)$. However, when $\tilde{G} = Sp(n)$, we obtain $\pi_n(\Delta_n) \subsetneq \pi_n(Q[8] \cdot \Delta_n)$ and we will show that $\pi_n(Q[8] \cdot \Delta_n)$ is an antipodal subgroup of $Sp(n)/\{\pm 1_n\}$ in (2-2), hence $\pi_n(\Delta_n)$ is not maximal.

In the case of $\tilde{G} = SO(n)$, B is conjugate to Δ_n^+ by Proposition 4.7 and A is conjugate to $\pi_n(\Delta_n^+)$.

(1-2) The case where $c^2 = -1_n$ for some $c \in B$. We consider the cases of $\tilde{G} = O(n), SO(n)$. There is $g \in \tilde{G}$ such that

$$gcg^{-1} = \begin{bmatrix} R(\theta_1) & & & \\ & \ddots & & \\ & & R(\theta_{n'}) & \\ & & & 1_{n-2n'} \end{bmatrix}, \quad R(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix} \ (1 \le i \le n').$$

Since $c^2 = -1_n$,

$$\begin{bmatrix} R(2\theta_1) & & & \\ & \ddots & & \\ & & R(2\theta_r) & \\ & & & 1_{n-2n'} \end{bmatrix} = (gcg^{-1})^2 = gc^2g^{-1} = -1_n.$$

Therefore $n = 2n'$ and $\cos 2\theta_i = -1$, $\sin 2\theta_i = 0\,(1 \leq i \leq n')$. Hence c is conjugate to

$$\begin{bmatrix} \pm J_1 & & \\ & \ddots & \\ & & \pm J_1 \end{bmatrix}, \quad J_1 := \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}.$$

Moreover, c is also conjugate to

$$J_{n'} := \begin{bmatrix} & -1_{n'} \\ 1_{n'} & \end{bmatrix} = J_1 \otimes 1_{n'}.$$

By retaking A in its conjugate class, we may assume that $c = J_1 \otimes 1_{n'}$. Since B is commutative,

$$B \subset C := \{x \in \tilde{G} \mid xc = cx\}.$$

C is isomorphic to $U(n')$. Hence $A = \pi_n(B) \subset \pi_n(C) \cong U(n')/\{\pm 1_{n'}\}$. By Theorem 5.1, A is conjugate to

$$\pi_n(\{1_2 \otimes 1_{n'}, J_1 \otimes 1_{n'}\} \cdot \Delta_{n'}) = \pi_n(\{1_2, J_1\} \otimes \Delta_{n'}) = \pi_n(D^+[4] \otimes \Delta_{n'})$$

since B is commutative. However, we obtain $\pi_n(D^+[4] \otimes \Delta_{n'}) \subsetneq \pi_n(D[4] \otimes \Delta_{n'})$ and we will show that $\pi_n(D[4] \otimes \Delta_{n'})$ is an antipodal subgroup of $O(n)/\{\pm 1_n\}$ in (2-2), hence $\pi_n(D^+[4] \otimes \Delta_{n'})$ is not maximal.

We consider the case of $\tilde{G} = Sp(n)$. There is $g \in \tilde{G}$ such that

$$gcg^{-1} = \begin{bmatrix} e^{\mathbf{i}\theta_1} & & \\ & \ddots & \\ & & e^{\mathbf{i}\theta_n} \end{bmatrix}.$$

Since $c^2 = -1_n$, c is conjugate to

$$\begin{bmatrix} \pm \mathbf{i} & & \\ & \ddots & \\ & & \pm \mathbf{i} \end{bmatrix}.$$

Moreover, since $\mathbf{j}\mathbf{i}\mathbf{j}^{-1} = -\mathbf{i}$, c is also conjugate to $\mathbf{i}1_n$. By retaking A in its conjugate class, we may assume that $c = \mathbf{i}1_n$. Then we have

$$B \subset C := \{x \in \tilde{G} \mid xc = cx\}$$

and C is isomorphic to $U(n)$. Hence $A = \pi_n(B) \subset \pi_n(C) \cong U(n)/\{\pm 1_n\}$. By Theorem 5.1, A is conjugate to $\pi_n(\{1, \mathbf{i}\}\Delta_n)$ since B is commutative. But we obtain $\pi_n(\{1, \mathbf{i}\}\Delta_n) \subsetneq \pi_n(Q[8] \cdot \Delta_n)$ and we will show that $\pi_n(Q[8] \cdot \Delta_n)$ is an antipodal subgroup of $Sp(n)/\{\pm 1_n\}$ in (2-2), hence $\pi_n(\{1, \mathbf{i}\}\Delta_n)$ is not maximal.

(2) The case where B is not commutative. We set

$$Z := \{z \in \tilde{G} \mid \ xz = zx \text{ for all } x \in \langle a, b \rangle\}.$$

We find that Lemma 5.8 can be applied to this case. Hence for any $c \in B$, at least one of c, ac, bc and abc commutes with a and b. Therefore at least one of c, ac, bc and abc belongs to Z. This implies $c \in \langle a, b \rangle \cdot Z$. Hence we have $B \subset \langle a, b \rangle \cdot Z$.

(2-1) The case of $\langle a,b\rangle \cong D[4]$. We consider the case of $\tilde{G} = SO(n)$. We may assume $a^2 = 1_n$. In this case a is conjugate to an element of Δ_n^+. Since $ab = -ba$, $\mathrm{Tr}(a) = -\mathrm{Tr}(bab^{-1}) = -\mathrm{Tr}(a)$, which implies $\mathrm{Tr}(a) = 0$. Hence a is conjugate to $I_{n'} = I_1 \otimes 1_{n'}$. Since $a \in SO(n)$, n' is even and so n is a multiple of 4. If $b^2 = 1_n$, b is conjugate to $I_{n'}$ similarly. If $b^2 = -1_n$, $b' := ab$ satisfies $(b')^2 = 1_n$, $ab' = -b'a$ and $\langle a,b\rangle = \langle a,b'\rangle$. Therefore by exchanging b for b' we obtain $b^2 = 1_n$. Without loss of generality, we may assume $a = I_{n'}$. If we set

$$M^+ := \{gI_{n'}g^{-1} \mid g \in SO(n)\},$$

M^+ is a polar of $SO(n)$ with respect to 1_n. Since a and b are conjugate to $I_{n'}$, we have $a, b \in M^+$. The assignment

$$G_{n'}(\mathbb{R}^n) \ni V \mapsto 1_V - 1_{V^\perp} \in M^+$$

gives an isometry between $G_{n'}(\mathbb{R}^n)$ and M^+ for suitable invariant Riemannian metrics. Since $V^\perp$ is the pole of V in $G_{n'}(\mathbb{R}^n)$, we find that $-x$ is the pole of x in M^+. Therefore the transformation on M^+ induced by

$$\gamma : G_{n'}(\mathbb{R}^n) \to G_{n'}(\mathbb{R}^n);\ V \mapsto V^\perp,$$

also denoted by γ, is -1 times the identity map. We have $s_a(b) = ab^{-1}a = -b = \gamma(b)$. By Proposition 2.1 we obtain $b \in C(a,-a) = C(I_{n'}, -I_{n'})$. In a similar way as the case of $\tilde{G} = U(n)$ we obtain

$$\begin{aligned} C(I_{n'}, -I_{n'}) &= \bigcup_{g \in S(O(n') \times O(n'))} g\{\pm K_{n'}\}g^{-1} \\ &= \left\{ \begin{bmatrix} 0 & x \\ x^{-1} & 0 \end{bmatrix} \,\middle|\, x \in SO(n') \right\}, \end{aligned}$$

which is connected. Since $b \in C(I_{n'}, -I_{n'})$, b is conjugate to $K_{n'}$ under the action of $S(O(n') \times O(n'))$ which fixes a. Hence we may assume $b = K_{n'} = K_1 \otimes 1_{n'}$. Then we obtain $\langle a,b\rangle = D[4] \otimes 1_{n'}$. Therefore we have

$$D[4] \otimes 1_{n'} \subset B \subset (D[4] \otimes 1_{n'}) \cdot Z \subset D[4] \otimes O(n').$$

If we set

$$B' := \{y \in O(n') \mid {}^\exists x \in D[4] \text{ s.t. } x \otimes y \in B\},$$

we obtain $B = D[4] \otimes B'$ and $\pi_{n'}(B')$ is a maximal antipodal subgroup of $O(n')/\{\pm 1_{n'}\}$ in the same way as the case of $\tilde{G} = U(n)$.

We consider the case of $\tilde{G} = O(n)$. When n is odd, $O(n)/\{\pm 1_n\}$ is isomorphic to $SO(n)$. Therefore a maximal antipodal subgroup is conjugate to Δ_n^+. Since $B = \pi_n^{-1}(\Delta_n^+) = \Delta_n$ is abelian, it contradicts to the assumption. Hence the case cannot occur. When n is even, in a similar way as the case of $\tilde{G} = SO(n)$ there exists $B' \subset O(n')$ such that $B = D[4] \otimes B'$ and $\pi_{n'}(B')$ is a maximal antipodal subgroup of $O(n')/\{\pm 1_{n'}\}$.

We consider the case of $\tilde{G} = Sp(n)$. Since $a^2 = 1_n$, a is conjugate to

$$I_{d,n-d} = \begin{bmatrix} -1_d & \\ & 1_{n-d} \end{bmatrix}.$$

Without loss of generality, we may assume $a = I_{d,n-d}$. Since b satisfies also $b^2 = 1_n$, b is conjugate to $I_{d',n-d'}$. Since $-a = bab^{-1}$, we have

$$-\mathrm{Tr}(a) = \mathrm{Tr}(-a) = \mathrm{Tr}(bab^{-1}) = \mathrm{Tr}(a)$$

by Lemma 7.3. Thus $\mathrm{Tr}(a) = 0$. Therefore $n = 2n'$ and $d = n'$. That is,

$$a = I_{n'} = \begin{bmatrix} -1_{n'} & \\ & 1_{n'} \end{bmatrix}.$$

Similarly we obtain $\mathrm{Tr}(b) = 0$. Hence b is conjugate to $I_{n'}$. If we set

$$M^+ = \{gI_{n'}g^{-1} \mid g \in Sp(n)\},$$

M^+ is a polar of $Sp(n)$ with respect to 1_n which is isomorphic to $G_{n'}(\mathbb{H}^n)$ as a Riemannian symmetric space. We have $a, b \in M^+$. In a similar way as the case of $\tilde{G} = U(n)$, we obtain $b \in C(a, -a) = C(I_{n'}, -I_{n'})$ and

$$\begin{aligned} C(I_{n'}, -I_{n'}) &= \bigcup_{g \in Sp(n') \times Sp(n')} g\{\pm K_{n'}\}g^{-1} \\ &= \left\{ \begin{bmatrix} 0 & x \\ x^{-1} & 0 \end{bmatrix} \middle| \, x \in Sp(n') \right\}, \end{aligned}$$

which is connected. Since $b \in C(I_{n'}, -I_{n'})$, b is conjugate to $K_{n'}$ under the action of $Sp(n') \times Sp(n')$ which fixes a. Hence we may assume $b = K_{n'} = K_1 \otimes 1_{n'}$. Then we obtain $\langle a, b\rangle = D[4] \otimes 1_{n'}$. Therefore we have

$$D[4] \otimes 1_{n'} \subset B \subset (D[4] \otimes 1_{n'}) \cdot Z \subset D[4] \otimes Sp(n').$$

If we set

$$B' := \{y \in Sp(n') \mid {}^{\exists}x \in D[4] \text{ s.t. } x \otimes y \in B\},$$

we obtain $B = D[4] \otimes B'$ and $\pi_{n'}(B')$ is a maximal antipodal subgroup of $Sp(n')/\{\pm 1_{n'}\}$ in the same way as the case of $\tilde{G} = U(n)$.

(2-2) The case of $\langle a, b\rangle \cong Q[8]$. We consider the case of $\tilde{G} = SO(n)$. Since $\langle a, b\rangle \cong Q[8]$ defines a quaternion structure on $\mathbb{R}^n$, we have $n = 4n''$. If we set

$$Z := \{z \in \tilde{G} \mid {}^{\forall}x \in \langle a, b\rangle,\ xz = zx\},$$

Z is a subgroup of $SO(4n'')$ which is isomorphic to $Sp(n'')$. We obtain $B \subset \langle a, b\rangle \cdot Z$ by Lemma 5.8. Since $\langle a, b\rangle \cdot Z \cong Q[8] \cdot Sp(n'')$, we obtain $A = \pi_n(B) \subset \pi_n(Q[8] \cdot Sp(n''))$ if we identify $\langle a, b\rangle \cdot Z$ with $Q[8] \cdot Sp(n'')$. Moreover, there exists $B' \subset Sp(n'')$ such that $A = \pi_n(Q[8] \cdot B')$ and $\pi_{n''}(B')$ is a maximal antipodal subgroup of $Sp(n'')/\{\pm 1_{n''}\}$.

We consider the case of $\tilde{G} = O(n)$. When n is odd, $O(n)/\{\pm 1_n\}$ is isomorphic to $SO(n)$. Therefore a maximal antipodal subgroup is conjugate to Δ_n^+. Since $B = \pi_n^{-1}(\Delta_n^+) = \Delta_n$ is abelian, it contradicts to the assumption. Hence the case cannot occur. When n is even, we have $B \subset Q[8] \cdot Sp(n'') \subset SO(n)$. Therefore a maximal antipodal subgroup of $SO(n)/\{\pm 1_n\}$ is also maximal in $O(n)/\{\pm 1_n\}$.

We consider the case of $\tilde{G} = Sp(n)$. Since $a^2 = -1_n$, a is conjugate to

$$\begin{bmatrix} \pm\mathbf{i} & & \\ & \ddots & \\ & & \pm\mathbf{i} \end{bmatrix}.$$

Moreover a is conjugate to $\mathbf{i}1_n$. Thus we may assume $a = \mathbf{i}1_n$. We have $a, b \in C(1_n, -1_n) = \{g \in Sp(n) \mid g^2 = -1_n\}$, the centrosome for the pair $\{1_n, -1_n\}$ in $Sp(n)$, and

$$C(1_n, -1_n) = \{g\mathbf{i}1_n g^{-1} \mid g \in Sp(n)\} \cong Sp(n)/U_{\mathbf{i}}(n),$$

where

$$U_{\mathbf{i}}(n) := \{g \in Sp(n) \mid g\mathbf{i} = \mathbf{i}g\} \cong U(n).$$

Since $s_a(b) = -b$, we have $b \in C(a, -a)$, the centrosome for the pair $\{a, -a\}$ in $C(1_n, -1_n)$. Since

$$C(a, -a) \cong U_{\mathbf{i}}(n)/O(n)$$

is connected and $\mathbf{j}1_n \in C(a, -a)$, we can transform b to $\mathbf{j}1_n$ with keeping a fixed. Therefore we may assume $a = \mathbf{i}1_n$ and $b = \mathbf{j}1_n$. Then we have

$$\langle a, b\rangle = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\} \cdot 1_n.$$

If we set

$$Z := \{z \in \tilde{G} \mid {}^{\forall}x \in \langle a, b\rangle,\ xz = zx\},$$

we have $B \subset \langle a, b\rangle \cdot Z$ by Lemma 5.8 and

$$Z = U_{\mathbf{i}}(n) \cap U_{\mathbf{j}}(n) \cap U_{\mathbf{k}}(n) = O(n).$$

Hence $\langle a, b\rangle \cdot 1_n \subset B \subset \langle a, b\rangle \cdot O(n)$. Therefore there exists $B' \subset O(n)$ such that $B = \langle a, b\rangle \cdot B'$ where $\pi_n(B')$ is a maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$. ■

Now we prove Theorem 7.1 by induction on k where $n = 2^k \cdot l\,(l : \text{odd})$.

Proof. First we prove the theorem for $\tilde{G} = O(n), Sp(n)$ in the case where n is odd.

When n is odd, $G = O(n)/\{\pm 1_n\}$ is isomorphic to $SO(n)$. Since a maximal antipodal subgroup of $SO(n)$ is conjugate to Δ_n^+, $B = \pi_n^{-1}(A)$ is conjugate to Δ_n for a maximal antipodal subgroup $A \subset O(n)/\{\pm 1_n\}$. Therefore a maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(\Delta_n) = \pi_n(D(0, n))$.

When $\tilde{G} = Sp(n)$, by Proposition 7.6 neither the case (1-1) or (1-2) occurs. Moreover,since n is odd, the case (2-1) does not occur too. In the case (2-2) by Proposition 7.6 a maximal antipodal subgroup of $G = Sp(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(Q[8] \cdot B')$ where $\pi_n(B')$ is a maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$. By what we proved for $G = O(n)/\{\pm 1_n\}\,(n : \text{odd})$, $\pi_n(B')$ is conjugate to $\pi_n(\Delta_n)$. Therefore a maximal antipodal subgroup of $Sp(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(Q[8] \cdot \Delta_n) = \pi_n(Q[8] \cdot D(0, n))$.
Next we prove the theorem for $\tilde{G} = SO(n)$ in the case where $n = 2l\,(l$ 3odd$)$.

(1-1) A maximal antipodal subgroup of $G = SO(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(\Delta_n^+)$ by Proposition 7.6. However, when $n = 2$, since $\Delta_2^+ \subsetneq D^+[4]$ and $\pi_n(D^+[4])$ is a maximal antipodal subgroup which appears in (1-2), $\pi_2(\Delta_2^+)$ is not maximal.

(1-2) A maximal antipodal subgroup of $G = SO(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(D^+[4] \otimes \Delta_l)$ by Proposition 7.6.

(2-1) A maximal antipodal subgroup of $G = SO(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(D[4] \otimes B')$ where $\pi_l(B')$ is a maximal antipodal subgroup of $O(l)/\{\pm 1_l\}$ by Proposition 7.6. By what we proved for $G = O(l)/\{\pm 1_l\}$, $\pi_l(B')$ is conjugate to $\pi_l(\Delta_l)$. Therefore a maximal antipodal subgroup of $SO(n)/\{\pm 1_n\}$ should be conjugate to $\pi_n(D[4] \otimes \Delta_l)$. However, since $D[4] \otimes \Delta_l \not\subset SO(n)$, we have $\pi_n(D[4] \otimes \Delta_l) \not\subset SO(n)/\{\pm 1_n\}$. Hence this case does not occur.

As a result, when $n = 2l\,(l : \text{odd})$, a maximal antipodal subgroup of $SO(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(\Delta_n^+)$ or $\pi_n(D^+[4] \otimes \Delta_l)$ except for $\pi_2(\Delta_2^+)$. Hence we proved Theorem 7.1 (II) (1).

Let $k \geq 1$ when $\tilde{G} = O(n), Sp(n)$ and let $k \geq 2$ when $\tilde{G} = SO(n)$. We assume that Theorem 7.1 holds for k' satisfying $k' < k$ and we will prove the theorem for k.

We prove the theorem for $k = 1$ when $\tilde{G} = O(n)$.

(1-1) $B = \pi_n^{-1}(A)$ is conjugate to Δ_n for a maximal antipodal subgroup $A \subset O(n)/\{\pm 1_n\}$ and A is conjugate to $\pi_n(\Delta_n)$ by Proposition 7.6. However, when $n = 2$, since $\Delta_2 \subsetneq D[4]$ and $\pi_n(D[4])$ is a maximal antipodal subgroup which appears in (2-1), $\pi_2(\Delta_2)$ is not maximal.

(1-2) This case does not occur by Proposition 7.6.

(2-1) A maximal antipodal subgroup is conjugate to $\pi_n(D[4] \otimes B')$ where $\pi_l(B')$ is a maximal antipodal subgroup of $O(l)/\{\pm 1_l\}$ by Proposition 7.6. Since l is odd, a maximal antipodal subgroup is conjugate to $\pi_n(D[4] \otimes \Delta_l) = \pi_n(D(1, n))$ by the previous result.

(2-2) Since n should be divided by 4, this case does not occur.

As a result, when $n = 2l\,(l : \text{odd})$, a maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(D(0, n))$ or $\pi_n(D(1, n))$ except for $\pi_2(\Delta_2) = \pi_2(D(0, 2))$.

We consider the case where $\tilde{G} = O(n)$ with $k \geq 2$.

(1-1) $B = \pi_n^{-1}(A)$ is conjugate to Δ_n for a maximal antipodal subgroup $A \subset O(n)/\{\pm 1_n\}$ and A is conjugate to $\pi_n(\Delta_n)$ by Proposition 7.6.

(1-2) This case does not occur by Proposition 7.6.

(2-1) A maximal antipodal subgroup is conjugate to $\pi_n(D[4] \otimes B')$ where $\pi_{n'}(B')$ is a maximal antipodal subgroup of $O(n')/\{\pm 1_{n'}\}$ by Proposition 7.6. By the induction assumption $\pi_{n'}(B')$ is conjugate to

$$\pi_{n'}(D(s, n')) \quad (0 \leq s \leq k - 1).$$

Hence a maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$ is conjugate to

$$\pi_n(D[4] \otimes D(s, n')) = \pi_n(D(s + 1, n)) \quad (0 \leq s \leq k - 1).$$

(2-2) Since $k \geq 2$, n is divided by 4. Set $n = 4n''$. A maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(Q[8] \cdot B')$ where $\pi_{n''}(B')$ is a maximal antipodal subgroup of $Sp(n'')/\{\pm 1_{n''}\}$ by Proposition 7.6. By the induction assumption $\pi_{n''}(B')$ is conjugate to

$$\pi_{n''}(Q[8] \cdot D(s, n'')) \quad (0 \le s \le k-2).$$

Hence a maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$ is conjugate to

$$\pi_n(Q[8] \cdot Q[8] \cdot D(s, n'')) \quad (0 \le s \le k-2),$$

which is equal to

$$\pi_n(D[4] \otimes D[4] \otimes D(s, n'')) = \pi_n(D(s+2, n)) \quad (0 \le s \le k-2)$$

by Lemma 4.1.

As a result, a maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$ with $k \ge 2$ is conjugate to

$$\pi_n(D(s, n)) \quad (0 \le s \le k).$$

We consider the case where $\tilde{G} = SO(n)$ with $k \ge 2$.

(1-1) $B = \pi_n^{-1}(A)$ is conjugate to Δ_n^+ for a maximal antipodal subgroup $A \subset SO(n)/\{\pm 1_n\}$ and A is conjugate to $\pi_n(\Delta_n^+)$ by Proposition 7.6.

(1-2) A maximal antipodal subgroup is conjugate to $\pi_n(D^+[4] \otimes \Delta_{n'})$ where $n' = n/2$ by Proposition 7.6. However, since $D^+[4] \subsetneq D[4]$ and n is even, $D^+[4] \otimes \Delta_{n'} \subsetneq D[4] \otimes \Delta_{n'} \subset SO(n)$. Therefore $\pi_n(D^+[4] \otimes \Delta_{n'})$ is not maximal.

(2-1) A maximal antipodal subgroup is conjugate to $\pi_n(D[4] \otimes B')$ where $\pi_{n'}(B')$ is a maximal antipodal subgroup of $O(n')/\{\pm 1_{n'}\}$ by Proposition 7.6. By the induction assumption $\pi_{n'}(B')$ is conjugate to

$$\pi_{n'}(D(s, n')) \quad (0 \le s \le k-1).$$

Hence a maximal antipodal subgroup of $SO(n)/\{\pm 1_n\}$ is conjugate to

$$\pi_n(D[4] \otimes D(s, n')) = \pi_n(D(s+1, n)) \quad (0 \le s \le k-1).$$

(2-2) Since $k \ge 2$, n is divided by 4. Set $n = 4n''$. A maximal antipodal subgroup of $SO(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(Q[8] \cdot B')$ where $\pi_{n''}(B')$ is a maximal antipodal subgroup of $Sp(n'')/\{\pm 1_{n''}\}$ by Proposition 7.6. By the induction assumption $\pi_{n''}(B')$ is conjugate to

$$\pi_{n''}(Q[8] \cdot D(s, n'')) \quad (0 \le s \le k-2).$$

Hence a maximal antipodal subgroup of $SO(n)/\{\pm 1_n\}$ is conjugate to

$$\begin{aligned}\pi_n(Q[8] \cdot Q[8] \cdot D(s, n'')) &= \pi_n(D[4] \otimes D[4] \otimes D(s, n'')) \\ &= \pi_n(D(s+2, n)) \quad (0 \le s \le k-2).\end{aligned}$$

As a result, a maximal antipodal subgroup of $SO(n)/\{\pm 1_n\}$ with $k \ge 2$ is conjugate to

$$\pi_n(\Delta_n^+) \quad \text{or} \quad \pi_n(D(s, n)) \quad (0 \le s \le k).$$

We consider the case where $\tilde{G} = Sp(n)$ with $k \ge 1$.

(1-1) and (1-2) do not occur by Proposition 7.6.

(2-1) A maximal antipodal subgroup is conjugate to $\pi_n(D[4] \otimes B')$ where $\pi_{n'}(B')$ is a maximal antipodal subgroup of $Sp(n')/\{\pm 1_{n'}\}$ by Proposition 7.6. By the induction assumption $\pi_{n'}(B')$ is conjugate to

$$\pi_{n'}(Q[8] \cdot D(s, n')) \quad (0 \le s \le k-1).$$

Hence a maximal antipodal subgroup of $Sp(n)/\{\pm 1_n\}$ is conjugate to

$$\begin{aligned} \pi_n(D[4] \otimes Q[8] \cdot D(s, n')) &= \pi_n(Q[8] \cdot D[4] \otimes D(s, n')) \\ &= \pi_n(Q[8] \cdot D(s+1, n)) \quad (0 \le s \le k-1). \end{aligned}$$

(2-2) A maximal antipodal subgroup of $Sp(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(Q[8] \cdot B')$ where $\pi_n(B')$ is a maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$ by Proposition 7.6. By what we proved for $O(n)/\{\pm 1_l\}$, $\pi_n(B')$ is conjugate to

$$\pi_n(D(s, n)) \quad (0 \le s \le k).$$

Hence a maximal antipodal subgroup of $Sp(n)/\{\pm 1_n\}$ is conjugate to

$$\pi_n(Q[8] \cdot D(s, n)) \quad (0 \le s \le k).$$

As a result, a maximal antipodal subgroup of $Sp(n)/\{\pm 1_n\}$ with $k \ge 1$ is conjugate to

$$\pi_n(Q[8] \cdot D(s, n)) \quad (0 \le s \le k).$$

We complete the proof of Theorem 7.1. ■

We prove Corollary 7.2.

Proof.

(I) Every maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(D(s, n))$ $(0 \le s \le k)$ except for the case where $(s, n) = (k-1, 2^k)$ by Theorem 7.1. We obtain the following by Lemma 4.3. When $n = 2$, $\pi_2(D(1, 2)) = \pi_2(D[4])$ is a unique great antipodal subgroup of $O(2)/\{\pm 1_2\}$ up to conjugation and $\#_2(O(2)/\{\pm 1_2\}) = 2^2$. When $n = 4$, $\pi_4(D(2, 4))$ is a unique great antipodal subgroup of $O(4)/\{\pm 1_4\}$ up to conjugation and $\#_2(O(4)/\{\pm 1_4\}) = 2^4$. Otherwise, $\pi_n(D(0, n)) = \pi_n(\Delta_n)$ is a unique great antipodal subgroup of $O(n)/\{\pm 1_n\}$ up to conjugation and $\#_2(O(n)/\{\pm 1_n\}) = 2^{n-1}$.

(II) By Theorem 7.1 and Lemma 4.2, every maximal antipodal subgroup is conjugate to $\pi_n(D^+(s, n))$ $(0 \le s \le k)$ except for the case where $(s, n) = (k-1, 2^k)$, $\pi_2(D^+(0, 2)) = \pi_2(\Delta_2^+)$ and $\pi_4(D^+(0, 4)) = \pi_4(\Delta_4^+)$. We have $|\pi_n(D^+(s, n))| = |D^+(s, n)|/2 = 2^{g_n(s)-1}$ by Lemma 4.4. We obtain the following by Lemma 4.4. When $n = 2$, $\pi_2(D^+(1, 2)) = \pi_2(D^+[4])$ is a unique great antipodal subgroup of $SO(2)/\{\pm 1_n\}$ up to conjugation and $\#_2(SO(2)/\{\pm 1_n\}) = 2$. When $n = 4$, $\pi_4(D^+(2, 4)) = \pi_4(D(2, 4))$ is a unique great antipodal subgroup of $SO(4)/\{\pm 1_n\}$ up to conjugation and $\#_2(SO(4)/\{\pm 1_n\}) = 2^4$. When $n = 8$, $\pi_8(D^+(0, 8)) = \pi_8(\Delta_8^+)$ and $\pi_8(D^+(3, 8)) = \pi_8(D(3, 8))$ are the great antipodal subgroups of $SO(8)/\{\pm 1_n\}$ up to conjugation and $\#_2(SO(8)/\{\pm 1_n\}) = 2^6$.

Otherwise, $\pi_n(D^+(0,n)) = \pi_n(\Delta_n^+)$ is a unique great antipodal subgroup of $SO(n)/\{\pm 1_n\}$ up to conjugation and $\#_2(SO(n)/\{\pm 1_n\}) = 2^{n-2}$.

(III) Every maximal antipodal subgroup of $Sp(n)/\{\pm 1_n\}$ is conjugate to $\pi_n(Q[8]\cdot D(s,n))$ $(0 \le s \le k)$ except for the case where $(s,n) = (k-1, 2^k)$ by Theorem 7.1. We have $|\pi_n(Q[8]\cdot D(s,n))| = |Q[8]|\cdot|D(s,n)|/2^2 = 2^{f_n(s)+1}$ by Lemma 4.3. We obtain the following by Lemma 4.3. When $n = 2$, $\pi_2(Q[8]\cdot D(1,2)) = \pi_n(Q[8]\cdot D[4])$ is a unique great antipodal subgroup of $Sp(2)/\{\pm 1_2\}$ up to conjugation and $\#_2(Sp(2)/\{\pm 1_2\})) = 2^4$. When $n = 4$, $\pi_4(Q[8]\cdot D(2,4))$ is a unique great antipodal subgroup of $Sp(4)/\{\pm 1_4\}$ up to conjugation and $\#_2(Sp(4)/\{\pm 1_4\}) = 2^6$. Otherwise, $\pi_n(Q[8]\cdot D(0,n)) = \pi_n(Q[8]\cdot \Delta_n)$ is a unique great antipodal subgroup of $Sp(n)/\{\pm 1_n\}$ up to conjugation and $\#_2(Sp(n)/\{\pm 1_n\}) = 2^{n+1}$. ■

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