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Gröbner–Shirshov Basis of the Universal Enveloping Rota–Baxter Algebra of a Lie Algebra

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Abstract. We consider Lie algebras equipped with a Rota–Baxter operator. The forgetful functor from this category to the category of Lie algebras has a left adjoint one denoted by U_{RB}. We prove an operator analogue of the Poincaré–Birkhoff–Witt theorem for U_{RB} by means of Gröbner–Shirshov bases theory for Lie algebras with an additional operator.
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1. Introduction

A Rota–Baxter algebra is a linear space A over a field $\Bbbk$ equipped with a bilinear product $(a,b)\mapsto ab$, $a,b\in A$, and with a linear map $R: A\to A$ such that

$$R(a)R(b) = R(R(a)b) + R(aR(b)) + \lambda R(ab), \tag{1}$$

where λ is a constant from $\Bbbk$. A linear operator R satisfying (1) is called a Rota–Baxter operator of weight λ.

This notion initially appeared in analysis [1], and then in combinatorics [14] and quantum field theory [6]. We refer the reader to the book [11] and references therein for more details. There is a number of studies on associative and commutative Rota–Baxter algebras. Let us briefly review some results closely related with the topic of this paper.

A linear basis of the free associative Rota–Baxter algebra was found in [8], where it was also shown that the universal enveloping Rota–Baxter algebra of a free dendriform (or tridendriform, for nonzero weight) algebra is free. A simpler proof of the same fact follows from [9]. Another method for finding a basis of free associative Rota–Baxter algebra was proposed in [3]; in a more modern form the same approach was exposed in [10].

Rota–Baxter Lie algebras are of special interest since they are closely related with pre-Lie (left/right-symmetric) algebras. Namely, if L is a Lie algebra with

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a product $[\cdot,\cdot]$ equipped with a Rota–Baxter operator R then the same space L with new operation $ab=[R(a),b]$, $a,b\in L$, is a pre-Lie algebra.

Moreover, there is a natural relation between Rota–Baxter operators and solutions of the classical Yang–Baxter equation (CYBE) [15]. Namely, if L is a Lie algebra equipped with a symmetric invariant bilinear form $\langle\cdot,\cdot\rangle$ (not necessarily non-degenerate) then there is a natural map $L\to L^*$, $a\mapsto\langle a,\cdot\rangle$, and thus we have a map $\Phi: L\otimes L\to L^*\otimes L\hookrightarrow \mathrm{End}\,(L)$. If $X\in L\otimes L$ is a skew-symmetric solution of CYBE

$$[X^{12},X^{13}]+[X^{12},X^{23}]+[X^{13},X^{23}]=0.$$

then $R=\Phi(X)$ is a Rota–Baxter operator on L.

This paper is devoted to the study of combinatorial structure of Lie algebras with a Rota–Baxter operator. The main problem we solve is an analogue of the Poincaré–Birkhoff–Witt (PBW) Theorem for the universal enveloping Rota–Baxter Lie algebra $U_{RB}(L)$ of an arbitrary Lie algebra L. We prove that $U_{RB}(L)$ carries a natural filtration such that the corresponding associated graded algebra $\mathrm{gr}\,U_{RB}(L)$ is isomorphic (as a Lie algebra) to the universal enveloping system $U_{RA}(L)$ in the class RALie of Lie algebras equipped with linear operator R satisfying the identity $[R(x),R(y)]=0$. The same statement holds for the varieties of associative and commutative algebras.

The main tool of the proof is a version of the Composition-Diamond Lemma (CD-Lemma) for Lie algebras with an additional linear operator. A more general approach to this Lemma (for Lie algebras with an arbitrary set of linear operators) was developed in [13]. In the proof of CD-Lemma we use terminology of [10] and some combinatorial statements of [16] (for more detailed exposition of the latter results see [2]).

2. Algebras with additional operator

Suppose Var is a variety of linear algebras (associative, commutative, Lie, Poisson, etc.). Denote by RVar the variety of Var-algebras equipped with an additional linear operator R. Denote the forgetful functor from RVar to Var by Θ_R, and let U_R stand for its left adjoint functor from Var to RVar.

Obviously, if $\mathrm{Var}\langle V\rangle$ is a free Var-algebra generated by a linear space V then $U_R(\mathrm{Var}\langle V\rangle)$ is isomorphic to the free RVar-algebra $\mathrm{RVar}\langle V\rangle$.

Let us state explicit construction of $U_R(A)$. Given $A\in\mathrm{Var}$, denote by $\bar{A}$ a copy of the linear space A. Let $\rho: A\to\bar{A}$ stand for the isomorphism $a\to\bar{a}$.

Define a series of algebras $\{A_n\}_{n\geq 0}$ by the following rule:

$$A_0=A,$$
$$A_1=A*\mathrm{Var}\langle\bar{A}\rangle,$$
$$\dots$$
$$A_n=A*\mathrm{Var}\langle\bar{A}_{n-1}\rangle,$$
$$\dots$$

where $*=*_{\mathrm{Var}}$ is the free product in the variety Var.

As above, let $\bar{A}_n$ be a copy of the space A_n. Denote the linear isomorphism $a \to \bar{a}$, $a \in A_n$, by ρ_n.

Construct a series of Var-homomorphisms $\tau_n : A_n \to A_{n+1}$, $n \geq 0$, as follows. Start with the canonical embedding $\tau_0 : A \to A_1$, and proceed by induction:

$$
\begin{array}{ccc}
A_{n-1} & \xrightarrow{\tau_{n-1}} & A_n \\
\downarrow \rho_{n-1} & & \downarrow \rho_n \\
\bar{A}_{n-1} & \xrightarrow{\bar{\tau}_{n-1}} & \bar{A}_n \\
\downarrow \subset & & \downarrow \subset \\
\mathrm{Var}\langle \bar{A}_{n-1} \rangle & \xrightarrow{\tau_{n-1}^0} & \mathrm{Var}\langle \bar{A}_n \rangle \\
\downarrow \subseteq & & \downarrow \subseteq \\
A * \mathrm{Var}\langle \bar{A}_{n-1} \rangle & \xrightarrow{\tau_n} & A * \mathrm{Var}\langle \bar{A}_n \rangle,
\end{array}
$$

where $\bar{\tau}_{n-1} = \rho_{n-1}^{-1} \circ \tau_{n-1} \circ \rho_n$ is a linear homomorphism, τ_{n-1}^0 is the induced Var-homomorphism of free algebras, and τ_n is a Var-homomorphism coming from the universal property of the free product.

Lemma 2.1. *Homomorphisms τ_n are injective for all $n \geq 0$.*

Proof. Assume τ_{n-1} is injective. Consider

$$
\begin{array}{ccccc}
\bar{A}_{n-1} & \underset{\subseteq}{\xrightarrow{\bar{\tau}_{n-1}}} & \bar{A}_n & \xrightarrow{\varphi} & \bar{A}_{n-1} \\
\subseteq \downarrow & & \subseteq \downarrow & & \downarrow \\
\mathrm{Var}\langle \bar{A}_{n-1} \rangle & \longrightarrow & \mathrm{Var}\langle \bar{A}_n \rangle & \xrightarrow{\varphi^0} & \mathrm{Var}\langle \bar{A}_{n-1} \rangle
\end{array}
$$

where φ is a projection of the linear space $\bar{A}_n$ onto $\bar{A}_{n-1}$: $\varphi\tau_{n-1}^0 = id_{\bar{A}_{n-1}}$, φ^0 is the extension of φ to a homomorphism of free algebras. Then the universal property of free product implies the existence of ψ such that

$$A_n = A * \mathrm{Var}\langle \bar{A}_{n-1} \rangle \xrightarrow{\tau_n} A * \mathrm{Var}\langle \bar{A}_n \rangle \xrightarrow{\psi} A * \mathrm{Var}\langle \bar{A}_{n-1} \rangle = A_n.$$

Hence, $\psi\tau_n = id_{A_n}$ and τ_n is injective. ■

Lemma 2.2. *For every* RVar*-algebra B and for every homomorphism of* Var*-algebras $\psi : A \to B$ there exists unique family $\{\psi_n\}_{n\geq 0}$ of* Var*-homomorphisms $\psi_n : A_n \to B$ such that*

$$\rho_n \circ \psi_{n+1} = \psi_n \circ R$$

and

$$\psi_0 = \psi, \quad \psi_n = \tau_n \circ \psi_{n+1}.$$

Proof. Let us show existence and uniqueness by induction. Given $\psi_n : A_n \to B$, construct

$$\begin{array}{ccccccc} A_n & \xrightarrow{\rho_n} & \bar{A}_n & \xrightarrow{\subseteq} & \mathrm{Var}\langle \bar{A}_n\rangle & \xrightarrow{\subseteq} & A_{n+1} = A * \mathrm{Var}\langle \bar{A}_n\rangle \\ {\scriptstyle\psi_n}\downarrow & & {\scriptstyle\bar{\psi}_n}\downarrow & & {\scriptstyle\psi_n^0}\downarrow & & \uparrow{\scriptstyle\subseteq} \\ B & \xrightarrow{R} & B & \xrightarrow{\mathrm{id}} & B & \xleftarrow{\psi} & A \end{array}$$

Here the rightmost vertical arrow is the canonical embedding of A into the free product which coincides with $\tau_0 \circ \cdots \circ \tau_n$, $\bar{\psi}_n = \rho_n^{-1} \circ \psi_n \circ R$ is a linear map, ψ_n^0 is a homomorphism of Var-algebras induced by $\bar{\psi}_n$. The right-hand square on the diagram above induces Var-homomorphism $\psi_{n+1} : A_{n+1} = A * \mathrm{Var}\langle \bar{A}_n\rangle \to B$.

Let us prove $\psi_n = \psi_{n+1}\tau_n$. For $n = 0$, it follows from the definition of ψ_1. Assume $n > 0$ and $\psi_{n-1} = \psi_n\tau_{n-1}$. Then for all $y \in \bar{A}_{n-1}$

$$\bar{\tau}_{n-1}(y) = \rho_n\tau_{n-1}\rho_{n-1}^{-1}(y)$$

Since $\bar{\psi}_n\rho_n(z) = R\psi_n(z)$ for all $z \in A_n$, we have

$$\bar{\psi}_n\bar{\tau}_{n-1}(y) = \bar{\psi}_n\rho_n\tau_{n-1}\rho_{n-1}^{-1}(y) = R\psi_n\tau_{n-1}\rho_{n-1}^{-1}(y) = R\psi_{n-1}\rho_{n-1}^{-1}(y) = \bar{\psi}_{n-1}(y).$$

Therefore, the induced Var-homomorphisms are related in the same way:

$$\psi_{n-1}^0 = \psi_n^0\tau_{n-1}^0.$$

Now, for all $x \in \mathrm{Var}\langle \bar{A}_{n-1}\rangle \subseteq A_n$ we have

$$\tau_n(x) = \tau_{n-1}^0(x).$$

Hence,

$$\psi_{n+1}\tau_n(x) = \psi_n^0\tau_{n-1}^0(x) = \psi_{n-1}^0(x) = \psi_n(x)$$

by the definition of ψ_{n+1}. Since ψ_n is uniquely determined by its action on $\bar{A}_{n-1}$ (uniqueness property of the universal map on free product), we have the required equality $\psi_{n+1}\tau_n = \psi_n$ on the entire A_n. ■

The chain

$$A \xrightarrow{\tau_0} A_1 \xrightarrow{\tau_1} A_2 \longrightarrow \dots \longrightarrow A_n \xrightarrow{\tau_n} A_{n+1} \longrightarrow \dots$$

naturally defines direct system of Var-algebras. Let

$$A_\infty = \lim_{\to} A_n,$$

$$\rho : A_\infty \to A_\infty, \quad \rho = \lim_{\to} \rho_n.$$

Theorem 2.3. *The* Var*-algebra* A_∞ *with linear map* ρ *is isomorphic to the universal* RVar *enveloping* $U_R(A)$.

Proof. The universal property of (A_∞, ρ) follows from Lemma 2.2. ■

Let us consider a particular case Var = Lie. Recall that if Y is a well-ordered set of generators then a linear basis of $\mathrm{Lie}\langle Y\rangle$ may be constructed in the following way [16]. A word $u \in Y^*$ is called an (associative) *Lyndon–Shirshov word* (LS-word) if either $u \in Y$ or for every presentation $u = vw$, $v, w \in Y^*$, we have $u > wv$ lexicographically. Denote the set of all such words by $\mathrm{LS}(Y)$. For every $u \in \mathrm{LS}(Y)$ there exists *standard* bracketing $[u]$ such that $[u] = ([v][w])$, where w is the longest proper LS-suffix of u (then v is also an LS-word, $[v]$ and $[w]$ are standard bracketings on these shorter words). The set $\{[u] \mid u \in \mathrm{LS}(Y)\}$ is a linear basis of $\mathrm{Lie}\langle Y\rangle$.

It is not hard to construct a linear basis of free RLie-algebra $\mathrm{RLie}\langle X\rangle$ for a given well-ordered set X of generators. Let $\mathrm{RLS}_0(X) = \{[u] \mid u \in \mathrm{LS}(X)\}$ be the basis of $\mathrm{Lie}\langle X\rangle$ equipped with deg-lex ordering:

$$[u] < [v] \iff u <_{deglex} v$$

Assume the set $\mathrm{RLS}_n(X)$ is already constructed and equipped with a well order. Consider the alphabet $U_n = X \cup \{R([u]) \mid [u] \in \mathrm{RLS}_n(X)\}$ with the following order:

$$x < R([u]),$$
$$R([u]) < R([v]) \iff [u] < [v],$$

for $x \in X$, $[u], [v] \in \mathrm{RLS}_n(X)$. Then

$$\mathrm{RLS}_{n+1}(X) := \{[w] \mid w \in \mathrm{LS}(U_n)\}$$

equipped with the deg-lex order. Obviously, $\mathrm{RLS}_n(X) \subset \mathrm{RLS}_{n+1}(X)$ for all $n \geq 0$.

Corollary 2.4. *The set*

$$\mathrm{RLS}(X) = \bigcup_{n\geq 0} \mathrm{RLS}_n(X)$$

is a linear basis of $\mathrm{RLie}\langle X\rangle$.

Proof. Let $L = \mathrm{Lie}\langle X\rangle$, $U_R(L) \simeq \mathrm{RLie}\langle X\rangle \xrightarrow{\theta} L_\infty$. Consider the images of RLS-words as elements of $\mathrm{RLie}\langle X\rangle$ under the isomorphism θ iduced by $x \mapsto x$, $x \in X$. By definition, $\theta(\mathrm{RLS}_0(X))$ is a basis of $L_0 = L$. Assume $\theta(\mathrm{RLS}_k(X))$ is a basis of L_k for all $k \leq n$, and the embedding $\mathrm{RLS}_{k-1}(X) \subset \mathrm{RLS}_k(X)$ is compatible with $\tau_{k-1} : L_{k-1} \to L_k$, $k = 1, \ldots, n$. Then $\theta(R(\mathrm{RLS}_n(X))) = \rho_n(\theta(\mathrm{RLS}_n(X)))$ is the set of free generators for $\mathrm{Lie}\langle \bar{L}_n\rangle$. Moreover, θ is compatible with τ_n.

Recall that the free product of two free Lie algebras is the free Lie algebra generated by disjoint union of the generating sets of factors. In our case, these generating sets are X and $\theta(R(\mathrm{RLS}_n(X)))$. Therefore, $\theta(\mathrm{RLS}_{n+1}(X))$ is the linear basis of L_{n+1}. ∎

In particular, $\mathrm{RLie}\langle X\rangle$ as a Lie algebra is isomorphic to $\mathrm{Lie}\langle U\rangle$, where $U = \bigcup_{n\geq 0} U_n$. Therefore, $\mathrm{RLie}\langle X\rangle$ has a natural ascending filtration

$$\mathrm{RLie}^{(n)}\langle X\rangle = \{f \in \mathrm{RLie}\langle X\rangle \mid \deg f \leq n\}, \tag{2}$$

where $\deg f$ is the degree of $f \in \mathrm{Lie}\langle U\rangle$ relative to the alphabet U.
Note that U may not be a well-ordered set, e.g., $xy > R(xy) > R^2(xy) > \dots$ for $x > y$. However, for every $n \geq 0$ the subset U_n is obviously well-ordered.

For $[u] \in \mathrm{RLS}(X)$, denote by $\deg_R u$ (R-degree) the total number of operators R appearing in u. For $f \in \mathrm{RLie}\langle X\rangle$, let $\deg_R f$ be the maximal R-degree among all its monomials. Note that for every $n \geq 0$ there exists N such that $\{[u] \in \mathrm{RLS}(X) \mid \deg_R u \leq n\} \subseteq \mathrm{LS}(U_N)$.

Remark 2.5. Denote by RAVar the subvariety of RVar defined by identity $R(x)R(y) = 0$ (image of R is abelian). The following construction provides the universal enveloping RAVar-algebra $U_{RA}(A)$ of $A \in \mathrm{Var}$, the proof is similar to the one stated above.

For $A \in \mathrm{Var}$, let $A_0 = A$ and $A_{n+1} = A * A_n^0$, $n \geq 0$, where A_n^0 is the same space as A_n considered as an algebra with zero multiplications. Then $U_{RA}(A)$ is isomorphic to $\varinjlim A_n$.

Let us construct a series of $\mathrm{RALS}_n(X) \subset \mathrm{RLS}_n(X)$ by induction. Start with $\mathrm{RALS}_0(X) = \mathrm{RLS}_0(X)$ and assume $\mathrm{RALS}_n(X)$ is already defined. Consider the alphabet

$$U_n = X \cup \{R([u]) \mid [u] \in \mathrm{RALS}_n(X)\}$$

with the same order as in the definition of RLS, and set $\mathrm{RALS}_{n+1}(X)$ to be the set of all $[w] \in \mathrm{LS}(U_n)$ such that w does not contain subwords of the form $R([u])R([v])$, $u, v \in \mathrm{RALS}_n(X)$, $[u] > [v]$. It is easy to see ([17]) that $\mathrm{RALS}_n(X)$ is the linear basis of the Lie algebra L_n constructed from $L_0 = \mathrm{Lie}\langle X\rangle$ as above. Therefore,

$$\mathrm{RALS}(X) = \bigcup_{n\geq 0} \mathrm{RALS}_n(X)$$

is the linear basis of $U_{RA}(\mathrm{Lie}\langle X\rangle) \simeq \mathrm{RALie}\langle X\rangle$. As a Lie algebra, $\mathrm{RALie}\langle X\rangle$ is generated by $U = \bigcup\limits_{n\geq 0} U_n$.

Corollary 2.6. *The free* RALie*-algebra* $\mathrm{RALie}\langle X\rangle$ *is isomorphic as a Lie algebra to the partially commutative (see [7]) Lie algebra* $\mathrm{Lie}\langle U \mid uv = 0,\ u, v \in U \setminus X\rangle$. ■

3. CD-lemma for RLie algebras

Let us call elements of $\mathrm{RLie}\langle X\rangle$ by RLie-polynomials, and let $\bar{f} \in \mathrm{RLS}(X)$ stand for the leading word (principle monomial) of an RLie-polynomial f.

Recall an important statement which plays a key role in the combinatorial theory of Lie algebras.

Lemma 3.1 (Shirshov bracketing, [16, Lemma 4])**.** *Let U be an ordered set, and $w, u \in \mathrm{LS}(U)$. Suppose u is a subword of w, i.e., $w = aub$, where a and b are some words in U (either of them may be empty). Denote by $w_{u\leftarrow *}$ a word in the alphabet $U \dot\cup \{*\}$ obtained from aub by replacing of this occurence of u with* *a new symbol* $*$: $w_{u\leftarrow *} = a{*}b$. *Then there exists a unique bracketing on* $w_{u\leftarrow *}$, *denoted by* $\{w_{u\leftarrow *}\}$, *such that*

$$\{a[u]b\} = [w] + \sum_i \alpha_i [w_i], \quad \alpha_i \in \Bbbk,\ w_i \in \mathrm{LS}(U),\ [w_i] < [w].$$

Uniqueness of the Shirshov bracketing implies the following property: let $w, u, z \in \mathrm{LS}(U)$, u is a subword of z, and z is a subword of w. Consider the words $w_{z\leftarrow *} = a{*}b$, $w_{u\leftarrow *}$, and $z_{u\leftarrow *}$ with the corresponding Shirshov bracketings $\{\dots\}$. Then

$$\{a\{z_{u\leftarrow *}\}b\} = \{w_{u\leftarrow *}\}.$$

Suppose S is a set of monic RLie polynomials. Construct $\hat S$ as follows. For every $f \in S$, $\bar f = [u]$, consider the associative word $u \in \mathrm{LS}(U)$ and consider all $[w] \in \mathrm{RLS}(X)$ such that the corresponding $w \in \mathrm{LS}(U)$ contain u as a subword: $w = aub$. Let $\{w_{u\leftarrow *}\} = \{a{*}b\}$ be the Shirshov bracketing. Denote by $\hat S_0$ the collection of all RLie-polynomials $\{w_{u\leftarrow f}\} = \{afb\}$ corresponding to all possible occurences of u, $[u] = \bar f$, $f \in S$, in all RLS-words $[w]$. Then $\overline{\{w_{u\leftarrow f}\}} = [w]$ and $\{w_{u\leftarrow f}\}$ belongs to the ideal of Lie algebra $\mathrm{Lie}\langle U\rangle$ generated by S. All these polynomials are monic, and $S \subset \hat S_0$.

Proceed by induction: given $\hat S_n = \Sigma$, define $\hat S_{n+1} = \Sigma \cup \widehat{R(\Sigma)}_0 \supset \hat S_n$, and

$$\hat S = \bigcup_{n\ge 0} \hat S_n$$

Lemma 3.2. *Denote by* $I_R(S)$ *the ideal generated by* S *in* $\mathrm{RLie}\langle X\rangle$. *Then* $f \in I_R(S)$ *if and only if* $f = \sum_i \alpha_i h_i$, $h_i \in \hat S$, $\alpha_i \in \Bbbk$.

Proof. The ideal $I_R(S)$ in $\mathrm{RLie}\langle X\rangle$ is the minimal R-invariant ideal in the Lie algebra $\mathrm{Lie}\langle U\rangle$ which contains S. By the construction, $I_R(S) \supseteq \hat S$.

Conversely, it follows from [18, Lemma 3] that the ideal $I(\Sigma)$ generated by a set Σ in $\mathrm{Lie}\langle U\rangle$ coincides with the linear span of $\hat\Sigma_0$. Hence, the linear span of $\hat S$ is an ideal in $\mathrm{Lie}\langle U\rangle$. Obviously, this ideal is R-invariant, so $I_R(S) \subseteq \Bbbk\hat S$. ■

Recall that a *rewriting system* is an oriented graph $\mathcal{G} = (V, E)$ which has no infinite oriented paths. A vertex $v \in V$ is called *terminal* if there are no edges of the form $v \to w$ in E.

Define an oriented graph $\mathcal{G}_R(X, S)$ on the set of vertices $\mathrm{RLie}\langle X\rangle$ based on a set of monic RLie-polynomials S, assuming that two RLie-polynomials f and g are connected by an edge $f \to g$ if and only if $f = f_0 + \alpha[u] + f_1$ (all monomials of f_0 are larger than $[u]$ and $[u] > \bar f_1$, $\alpha \in \Bbbk$, $\alpha \ne 0$) such that $[u] = \bar h$ for some $h \in \hat S$, and $g = f - \alpha h$. Every edge obviously corresponds to unique $h \in \hat S$, and therefore has a well-defined *level* which is the minimal n such that $h \in \hat S_n$.

From now on, assume the following additional condition on S: $\deg_R \bar s \ge \deg_R s$ for every $s \in S$, i.e., the number of operators R in the leading word $\bar s$ is greater or equal than R-degrees of all other monomials in s. Obviously, the same condition holds for $h \in \hat S$. In this case, $\mathcal{G}_R(X, S)$ is a rewriting system since for every vertex $f \in \mathrm{RLie}\langle X\rangle$ its cone (set of all vertices g such that there exists an oriented path $f \to \cdots \to g$) belongs to $\Bbbk\,\mathrm{RLS}_n(X)$ for some n and $\mathrm{RLS}_n(X)$ is well ordered. Terminal vertices of this rewriting system are also called *S-reduced* RLie-polynomials.

Let notation $f \sim_d g$ express that $f, g \in \mathrm{RLie}\langle X\rangle$ are connected by a non-oriented path of length $d \geq 1$, and $f \sim g$ says $f \sim_d g$ for some $d \geq 1$.

The following two lemmas are almost obvious but we still state their proofs for readers' convenience.

Lemma 3.3. *Let V be a subspace of* $\mathrm{RLie}\langle X\rangle$*, and let $\mathcal{G}(V)$ stand for the subgraph of $\mathcal{G}_R(X,S)$ with vertices V. Then for every $f, g, h \in V$*

$$f \sim g \text{ in } \mathcal{G}(V) \iff f + h \sim g + h \text{ in } \mathcal{G}(V).$$

Proof. It is enough to show ($\Rightarrow$). Suppose $f \sim_d g$ and proceed by induction on d. In fact, we only need $d = 1$ since the induction step is obvious. Assume $f \to g$, $f = f_0 + \alpha[u] + f_1$, $[u] = \bar{s}$, $s \in \hat{S}$, $g = f - \alpha s$, $\alpha \neq 0$, as in the definition of $\mathcal{G}_R(X,S)$. In particular, $s \in V$. Apply the same principle to write down the decompositions of $h = h_0 + \beta[u] + h_1$ (for some $\beta \in \Bbbk$) and $g = f - \alpha s = g_0 + g_1$. Then $f + h = f_0 + h_0 + (\alpha + \beta)[u] + f_1 + h_1$, $g + h = g_0 + h_0 + \beta[u] + h_1 + g_1$. If $\alpha + \beta \neq 0$ and $\beta = 0$ then $f + h \to g + h$. If $\alpha + \beta \neq 0$ and $\beta \neq 0$ then

$$f + h \to f + h - (\alpha + \beta)s = g + h - \beta s \leftarrow g + h,$$

so $f + h \sim_2 g + h$. Finally, if $\alpha + \beta = 0$ then

$$f + h = f_0 + h_0 + f_1 + h_1 = g + \alpha s + h = g + h - \beta s \leftarrow g + h. \qquad \blacksquare$$

Lemma 3.4. *Under the conditions of Lemma 3.3, the following statement holds: for every $f, g \in V$*

$$f - g = \sum_i \alpha_i s_i, \quad \alpha_i \in \Bbbk,\ s_i \in \hat{S} \cap V,$$

if and only if $f \sim g$ in $\mathcal{G}(V)$.

Proof. ($\Leftarrow$) It follows from the definition of edges in $\mathcal{G}_R(X,S)$.

($\Rightarrow$) Assume $f - g = \alpha_1 s_1 + \cdots + \alpha_n s_n$, $s_i \in \hat{S} \cap V$. If $n = 1$ then we simply have $f - g \to 0$, so $f \sim g$ by Lemma 3.3. If $n > 1$ then $f - (g + \alpha_1 s_1) \sim 0$ by induction, so $f - g \sim \alpha_1 s_1 \to 0$ by Lemma 3.3. $\blacksquare$

By Lemma 3.2, the ideal $I_R(S)$ coincides with the linear span of $\hat{S}$. Therefore, connected components (in the non-oriented sense) of $\mathcal{G}_R(X,S)$ are exactly the elements of the quotient algebra $\mathrm{RLie}\langle X\rangle / I_R(S)$.

We will mainly use the following subspaces of $\mathrm{RLie}\langle X\rangle$:

$$V_n = \Bbbk\{[u] \in \mathrm{RLS}(X) \mid \deg_R u \leq n\}, \quad n \geq 0,$$
$$V^{[w]} = \Bbbk\{[u] \in \mathrm{RLS}(X) \mid [u] \leq [w]\}, \quad [w] \in \mathrm{RLS}(X),$$
$$V_n^{[w]} = V_n \cap V^{[w]}.$$

Note that

$$V_n = \bigcup_{[w]\in \mathrm{RLS}(X)\cap V_n} V_n^{[w]},$$

and $\mathrm{RLS}(X)\cap V_n$ is a well-ordered subset of $\mathrm{RLS}(X)$.

Recall that a rewriting system is said to be *confluent* if for every vertex v there exists unique terminal vertex t such that v is connected with t by an oriented path (i.e., $v \to \dots \to t$, or $v \rightsquigarrow t$). In particular, every non-oriented connected component of a confluent rewriting system contains unique terminal vertex. Therefore, if the rewriting system $\mathcal{G}_R(X,S)$ is confluent then there exists unique normal form of an element in $\mathrm{RLie}\langle X\rangle/I_R(S)$ which may be found by straightforward walk on the graph. The following statement is a well-known criterion of confluence.

Theorem 3.5 (Diamond Lemma, [19]). *A rewriting system $\mathcal{G}=(V,E)$ is confluent if and only if for every $v\in V$ and for every two edges $v\to w_1$, $v\to w_2$ there exists a vertex $u\in V$ such that $w_1\rightsquigarrow u$ and $w_2\rightsquigarrow u$.* ∎

It is easy to see that a rewriting system $\mathcal{G}_R(X,S)$ is confluent if and only if so is each subsystem $\mathcal{G}(V_n)$, $n\ge 0$. The latter is confluent if and only if so is $\mathcal{G}(V_n^{[w]})$ for all $[w]\in \mathrm{RLS}(X)\cap V_n$.

Proposition 3.6. *Let $S\subset \mathrm{RLie}\langle X\rangle$ be a set of monic RLie-polynomials, $n\ge 0$. Suppose the rewriting system $\mathcal{G}(V_n)\subset \mathcal{G}_R(X,S)$ has the following property: for every RLS-word $[w]\in V_n$ and for every pair of edges $[w]\to g_1$, $[w]\to g_2$ in $\mathcal{G}(V_n)$ we have*

$$g_1-g_2=\sum_i \alpha_i h_i,\quad h_i\in \hat S\cap V_n,\ \bar h_i<[w]. \tag{3}$$

Then the system $\mathcal{G}(V_n)$ is confluent.

Proof. Let us check the Diamond Condition from Theorem 3.5 for the rewriting system $\mathcal{G}_n^{[v]}=\mathcal{G}(V_n^{[v]})\subset \mathcal{G}_R(X,S)$, $[v]\in \mathrm{RLS}(X)\cap V_n$.

Proceed by induction on $[v]$. Assume the rewriting system $\mathcal{G}_n^{[u]}$ is confluent for all $[u]\in V_n$, $[u]<[v]$, and consider an ambiguity in the graph $\mathcal{G}_n^{[v]}$, i.e., a pair of edges $f\to g_1$, $f\to g_2$. Here $g_1=f-\alpha h_1$, $g_2=f-\beta h_2$, $h_i\in \hat S\cap V_n^{[v]}$. There are three possible cases:

Case 1: $\bar h_1\ne \bar h_2$.

Then f may be written (with ordered monomials) as $f=f_0+\alpha[u_1]+f_{01}+\beta[u_2]+f_1$, $[u_i]=\bar h_i$. Suppose $[u_1]-h_1=\gamma[u_2]+h$, where h does not contain monomial $[u_2]$, $\gamma\in\Bbbk$. It is now easy to see that if $\gamma\alpha+\beta\ne 0$ then there exist edges $g_1\to g$, $g_2\to g_2'\to g$ in $\mathcal{G}_n^{[v]}$, where $g_2'=g_2-\alpha h_1$, $g=f_0+f_{01}+f_1+\alpha h+(\beta+\gamma\alpha)([u_2]-h_2)$. If $\gamma\alpha+\beta=0$ then there exist edges $g_2\to g_2'\to g_1$ for the same g_2'. Therefore, in this case the Diamond Condition holds.

Case 2: $\bar h_1=\bar h_2<\bar f$.

Then $f=f_0+\alpha[u]+f_1$, $[u]=\bar h_i<\bar f\le [v]$. Hence, there in an ambiguity

$f' \to g'_1$, $f' \to g'_2$ in $\mathcal{G}_n^{[u]}$, where $f' = \alpha[u] + f_1$, $g_i = f_0 + g'_i$. By the inductive hypothesis, there exist two paths in $\mathcal{G}_n^{[u]}$: $g'_i \to \dots \to g'$, $i = 1,2$. Therefore, $g_i \to \dots \to f_0 + g'$ in $\mathcal{G}_n^{[v]}$ since all monomials in f_0 are greater than $[u]$.

Case 3: $\bar h_1 = \bar h_2 = \bar f$.
Without loss of generality, assume $\bar f = [v]$. Then $g_i = \alpha([v] - h_i) + f_1$ and the difference $g_1 - g_2 = \alpha(h_2 - h_1)$ coincides (up to a scalar) with one that appears in the pair of edges $[v] \to [v] - h_i$, $i = 1,2$. Therefore, the condition of the statement implies g_1 and g_2 are connected by a non-oriented path in $\mathcal{G}_R^{[u]}(X,S)$ for some $[u] < [v]$. The last rewriting system is confluent by induction hypothesis, so there exist oriented paths $g_1 \rightsquigarrow g$ and $g_2 \rightsquigarrow g$ in $\mathcal{G}_R^{[u]}(X,S)$, so the Diamond Condition holds for $\mathcal{G}_n^{[v]}$. ■

Recall the Shirshov's definition of a composition [18] in the free Lie algebra $\mathrm{Lie}\langle U\rangle$.

Let $f, g \in \mathrm{Lie}\langle U\rangle$ be monic Lie-polynomials, $\bar f = [u]$, $\bar g = [v]$. We say that f and g form a composition relative to a word w if $u = u_1u_2$, $v = v_1v_2$, $u_2 = v_1$ ($u_i, v_i \in U^*$). Here $w = u_1u_2v_2 = u_1v_1v_2$ is a LS-word, and there are two Shirshov braketings:

$$\{w_{u\leftarrow *}\} = \{*v_2\}_1, \quad \{w_{v\leftarrow *}\} = \{u_1*\}_2.$$

The Lie polynomial

$$(f,g)_w = \{fv_2\}_1 - \{u_1g\}_2$$

is called a *composition of f and g relative to w*. It is important that

$$\overline{(f,g)_w} < [w]. \tag{4}$$

It follows from the definition of $\mathcal{G}_R(X,S)$ that if $f, g \in S$ then

$$[w] \to g_1 = [w] - \{fv_2\}_1$$

is an edge in $\mathcal{G}_R(X,S)$, and so is

$$[w] \to g_2 = [w] - \{u_1g\}_2.$$

Therefore, $g_1 - g_2 = (f,g)_w$.

Suppose S is a set of monic RLie-polynomials such that the rewriting system $\mathcal{G}_R(X,S)$ is *reduced*, i.e., it has the following property: every vertex $s \in S$ is a terminal vertex of the graph $\mathcal{G}_R(X, S \setminus \{s\})$. Then S itself is said to be reduced.

Proposition 3.7. *Let S be a reduced set of monic RLie-polynomials. Suppose all compositions of type $(s_1,s_2)_w$, $s_1, s_2 \in S$, $[w] \in V_n \cap \mathrm{RLS}(X)$, have the following presentation:*

$$(s_1,s_2)_w = \sum_i \alpha_i h_i, \quad h_i \in \hat S \cap V_n,\ \bar h_i < [w]. \tag{5}$$

Then the rewriting system $\mathcal{G}(V_n) \subset \mathcal{G}_R(X,S)$ is confluent.

Proof. Check the conditions of Proposition 3.6 for a word $[w] \in \mathrm{RLS}(X) \cap V_n$. Assume there is a pair of edges in $\mathcal{G}(V_n)$: $[w] \to g_1$, $[w] \to g_2$.

There are several possible cases.

Case 1. Both edges are of level 0. (This case is actually covered by the classical Composition-Diamond Lemma [18], but we prefer to consider it in our terminology to make the exposition complete.) Then

$$g_1 = [w] - h_1, \quad g_2 = [w] - h_2,$$

$h_i \in \widehat{\{s_i\}}_0 \cap V_n$, $s_i \in S$. Let $[u] = \bar{s}_1$ and $[v] = \bar{s}_2$.

Recall the following

Lemma 3.8 ([5]). *Suppose $u, v, w \in \mathrm{LS}(U)$, $w = aubvc$, where a, b, and c are some words in U (either of them may be empty). Then there exists a bracketing $\{a * b \star c\}$ such that $\{a[u]b[v]c\} = [w] + \sum\limits_i \alpha_i [w_i]$, $[w_i] < [w]$.*

This statement also implicitely appears in [18].

Case 1.1. Let the corresponding occurences of subwords u and v in w do not intersect. Then $w = aubvc$, $h_1 = \{as_1bvc\}_1$, $h_2 = \{aubs_2c\}_2$, where $\{\dots\}_1$ and $\{\dots\}_2$ are the Shirshov bracketings on $w_{u\leftarrow *}$ and $w_{v\leftarrow *}$, respectively. Therefore,

$$\begin{aligned} g_1 - g_2 &= \{aubs_2c\}_2 - \{as_1bvc\}_1 \\ &= \{aubs_2c\}_2 - \{a[u]bs_2c\}_{12} + \{as_1b[v]c\}_{12} - \{as_1bvc\}_1 \\ &+ \{a[u]bs_2c\}_{12} - \{as_1bs_2c\}_{12} + \{as_1bs_2c\}_{12} - \{as_1b[v]c\}_{12} \\ &= \big(\{aubs_2c\}_2 - \{a[u]bs_2c\}_{12}\big) + \big(\{as_1b[v]c\}_{12} - \{as_1bvc\}_1\big) \\ &+ \{a([u] - s_1)bs_2c\}_{12} + \{as_1b(s_2 - [v])c\}_{12}, \end{aligned}$$

where $\{a * b \star c\}_{12}$ is the bracketing from Lemma 3.8. In the last expression, all summands belong to linear span of $h_i \in \hat{S} \cap V_n$ with $\bar{h}_i < [w]$, so $g_1 - g_2$ has the required presentation (3).

Case 1.2. Let the corresponding occurences of u and v in w intersect: $u = u_1u_2$, $v = v_1v_2$, $u_2 = v_1$. Then $z = uv_2 = u_1v$ is a LS-word, $w = azb = auv_2b = au_1vb$, $h_1 = \{as_1v_2b\}_1$, $h_2 = \{au_1s_2b\}_2$, where $\{a * v_2b\}_1$ and $\{au_1 * b\}$ are the Shirshov bracketings on $w_{u\leftarrow *}$ and $w_{v\leftarrow *}$, respectively. Consider also the Shirshov bracketings $\{a * b\}_0$ on $w_{z\leftarrow *}$ and $\{*v_2\}_{01}$, $\{u_1*\}_{02}$ on $z_{u\leftarrow *}$ and $z_{v\leftarrow *}$, respectively. Then

$$\begin{aligned} g_1 - g_2 &= \{au_1s_2b\}_2 - \{as_1v_2b\}_1 \\ &= \{a\{u_1s_2\}_{02}b\}_0 - \{a\{s_1v_2\}_{01}b\}_0 \\ &= -\{afb\}_0, \end{aligned}$$

where $f = (s_1, s_2)_z$. Since $f = \sum\limits_i \alpha_i h_i$, $\bar{h}_i < [z]$, $h_i \in \hat{S} \cap V_n$, RLie polynomial $g_1 - g_2$ may be presented in the form (3).

Case 2. The edge $[w] \to g_1$ is of positive level d, $[w] \to g_2$ is of level 0.

In this case, $w = a_1 \dots a_m$, $a_i \in U$, $a_k = R([v])$ for some k, where $[v] = \bar{h}$, $h \in \widehat{\{s_1\}}_{d-1}$, $s_1 \in S$. Therefore, $h_1 = [a_1 \dots a_{k-1} R(h) a_{k+1} \dots a_m]$. As above,

$$w = aub, \quad [u] = \bar{s}_2,$$

and $h_2 = \{as_2b\}$. Since S is reduced, the occurence of letter $a_k = R([v]) \in U$ considered above may appear in either of the subwords a or b. Suppose $a = ca_kc'$ (the second case is analogous). Then

$$w = cR([v])c'ub, \quad c, c', b \in U^* \cup \{\epsilon\},$$

where ϵ is the empty word. Therefore,

$$\begin{aligned} g_1 - g_2 &= \{cR([v])c's_2b\} - [cR(h)c'ub] \\ &= \{cR([v])c's_2b\} - \{cR(h)c's_2b\} + \big(\{cR(h)c's_2b\} - [cR(h)c'ub]\big), \end{aligned}$$

and the same reasonings as in Case 1.1 show the required relation (3) holds.

Case 3. Both edges $[w] \to g_1$, $[w] \to g_2$ have positive level. In this case, $w = a_1 \dots a_k \dots a_l \dots a_m$, $a_i \in U$, where $a_k = R([u])$, $a_l = R([v])$, where $[u] \to g_1'$ and $[v] \to g_2'$ are edges of smaller level, and

$$h_1 = [a_1 \dots R(g_1') \dots a_l \dots a_m], \quad h_2 = [a_1 \dots a_k \dots R([v]) \dots a_m].$$

Case 3.1. If $k \neq l$ then one may proceed as in Case 2.

Case 3.2. If $k = l$, proceed by induction on the level of edges. Consider $a_k = a_l = R([u])$ with edges $[u] \to g_1'$, $[u] \to g_2'$ in $\mathcal{G}(V_{n-1}) \subset \mathcal{G}(V_n)$. Inductive hypothesis claims $g_1' - g_2' = \sum_i \alpha_i h_i'$, $\bar{h}_i' < [u]$. Therefore,

$$g_1 - g_2 = [a_1 \dots R(g_1' - g_2') \dots a_m]$$

also has a required presentation (3). ■

The entire system S is *closed with respect to composition* if for every $s_1, s_2 \in S$ every their composition $(s_1, s_2)_w$ may be presented as

$$(s_1, s_2)_w = \sum_i \alpha_i h_i, \quad h_i \in \hat{S}, \ \bar{h}_i < [w], \ \deg_R h_i \leq \deg_R w.$$

A reduced set of monic RLie-polynomials which is closed with respect to composition is called a *Gröbner–Shirshov basis* (GSB) in $\mathrm{RLie}\langle X\rangle$.

Propositions 3.6 and 3.7 immediately imply

Theorem 3.9. *If S is a GSB in* $\mathrm{RLie}\langle X\rangle$. *Then the rewriting system $\mathcal{G}_R(X, S)$ is confluent.* ■

Corollary 3.10. *If S is a GSB in* $\mathrm{RLie}\langle X\rangle$ *then the set of S-reduced words forms a linear basis of the algebra* $\mathrm{RLie}\langle X \mid S\rangle = \mathrm{RLie}\langle X\rangle / I_R(S)$.

Proof. Terminal vertices of $\mathcal{G}(X, S)$ are exactly linear combinations of S-reduced words. ■

Example 3.11. Let S consist of all $R([u])R([v])$, $[u],[v] \in \mathrm{RALS}(X)$, $[u] > [v]$. Then S is a reduced system closed with respect to compositions, and the set of S-reduced words coincides with $\mathrm{RALS}(X)$.

Obviously, $\mathrm{RLie}\langle X \mid S\rangle \simeq \mathrm{RALie}\langle X\rangle$, so $\mathrm{RALS}(X)$ is indeed the linear basis of $\mathrm{RALie}\langle X\rangle$.

More generally, let L be a Lie algebra, and let $(X,\leq)$ be an ordered linear basis of L.

Example 3.12. The set W of all words $[w] \in \mathrm{RALS}(X)$ such that w do not contain subwords of type xy, $x,y \in X$, $x > y$, form a linear basis of $U_{RA}(L)$.

It is easy to see that
$S = \{R([u])R([w]) \mid [u],[w] \in W, u > w\} \cup \{xy - [x,y] \mid x,y \in, x > y\}$
is a GSB, and $\mathrm{RLie}\langle X \mid S\rangle \simeq U_{RA}(L)$.

4. Rota–Baxter Lie algebras

Let RBLie denotes the variety of Lie algebras equipped with a Rota–Baxter operator R of weight $\lambda \in \Bbbk$, i.e., a linear map satisfying (1).

Consider the forgetful functor $\mathrm{RBLie} \to \mathrm{Lie}$. For every $L \in \mathrm{Lie}$ there exists universal enveloping algebra $U_{RB}(L) \in \mathrm{RBLie}$: $L \subset U_{RB}(L)$ is a Lie subalgebra such that for every $B \in \mathrm{RBLie}$ and for every homomorphism $\varphi : L \to B$ of Lie algebras there exists unique homomorphism of RBLie algebras $\bar\varphi : U_{RB}(L) \to B$ such that $\bar\varphi|_L = \varphi$. In this Section, we clarify the structure of $U_{RB}(L)$ and prove an analogue of the Poincaré–Birkhoff–Witt Theorem.

Suppose L is a Lie algebra with a linear basis X. Assume X is well ordered. Consider

$$S^{(0)} = \{xy - [x,y] \mid \ x,y \in X,\ x > y\} \subset \mathrm{Lie}\langle X\rangle \subset \mathrm{RLie}\langle X\rangle.$$

Here $[x,y]$ is a linear form in X equal to the product of x and y in L. Then $S^{(0)}$ is a GSB in $\mathrm{Lie}\langle X\rangle$ and, therefore, in $\mathrm{RLie}\langle X\rangle$. Moreover, $L \simeq \mathrm{Lie}\langle X \mid S^{(0)}\rangle$.

Now, consider

$$\rho(x,y) = R(x)R(y) - R(R(x)y) + R(R(y)x) - \lambda R([x,y]), \quad x,y \in X,\ x > y,$$

and let $S^{(2)} \subset \mathrm{RLie}\langle X\rangle$ be the union of $S^{(0)}$ and the set of all $\rho(x,y)$. Denote by $\mathcal{G}^{(2)}$ the subgraph $\mathcal{G}(V_2)$ of $\mathcal{G}_R(X,S^{(2)})$. Obviously, $S^{(2)}$ is a GSB: it is reduced, and the graph $\mathcal{G}_R(X,S^{(2)})$ has no ambiguities.

Proceed by induction on R-degree. Assume a reduced system $S^{(n)}$, $n \geq 2$, is already constructed in such a way that the subgraph $\mathcal{G}^{(n)} = \mathcal{G}(V_n) \subset \mathcal{G}_R(X,S^{(n)})$ is a confluent rewriting system. Denote by T_n the set of terminal vertices of $\mathcal{G}^{(n)}$, and let $t_n : V_n \to T_n$ be the linear map that turns an RLie-polynomial f, $\deg_R f \leq n$, into the terminal vertex $t_n(f)$ connected with f.

For every two terminal words $a,b \in T_n \cap \mathrm{RLS}(X)$, $\deg_R a + \deg_R b = n-1$, $a > b$, consider

$$\rho(a,b) = R(a)R(b) - R(t_n([R(a),b])) + R(t_n([R(b),a])) - \lambda R(t_n([a,b])),$$

where $[\cdot,\cdot]$ stands for the product in $\mathrm{RLie}\langle X\rangle$. Construct

$$S^{(n+1)} = S^{(n)} \cup \{\rho(a,b) \mid a,b\in T_n\cap \mathrm{RLS}(X),\ \deg_R a+\deg_R b = n-1,\ a>b\}.$$

It is easy to see from the construction that the subgraph $\mathcal{G}(V_n)\subset \mathcal{G}_R(X,S^{(n+1)})$ coincides with $\mathcal{G}^{(n)}$.

We have to resolve the following problems:

- Prove the confluence of $S^{(n+1)}$ (assuming $S^{(n)}$ is confluent);
- Show the isomorphism of RLie-algebras $\mathrm{RLie}\langle X\mid S\rangle \simeq U_{RB}(L)$, where S is the union of all $S^{(n)}$, $n\ge 0$;
- Describe the set of all S-reduced words in $\mathrm{RLS}(X)$.

Lemma 4.1. *Let $f,g\in \mathrm{RLie}\langle X\rangle$, $\deg_R f+\deg_R g = n$. Then $t_n([f,t_n(g)]) = t_n([f,g])$.*

Proof. It follows from Lemma 3.4 that $[f,g]$ and $[f,t_n(g)]$ belong to the same connected component of $\mathcal{G}^{(n)}$. Since the latter is confluent, $t_n([f,g]) = t_n([f,t_n(g)])$. ■

Lemma 4.2. *The rewriting system $\mathcal{G}^{(n+1)} = \mathcal{G}(V_{n+1})\subset \mathcal{G}_R(X,S^{(n+1)})$ is confluent.*

Proof. Here we assume by induction that $\mathcal{G}^{(n)} = \mathcal{G}(V_n)\subseteq \mathcal{G}_R(X,S^{(n+1)})$ is confluent. It is enough to check the conditions of Proposition 3.7 for compositions $(s_1,s_2)_w$, $s_1,s_2\in S^{(n+1)}$, $[w]\in \mathrm{RLS}(X)$, $\deg_R w = n+1$.
Suppose $s_1{=}\rho(a,b)$, $s_2{=}\rho(b,c)$, $w{=}R(a)R(b)R(c)$, $a,b,c{\in} T_n\cap \mathrm{RLS}(X)$, $a{>}b{>}c$.
Denote

$$\rho(a,b) = R(a)R(b) - \sum_i \gamma_i R(c_i),$$
$$\rho(b,c) = R(b)R(c) - \sum_j \alpha_j R(a_j),$$
$$\rho(a,c) = R(a)R(c) - \sum_l \beta_l R(b_l),$$

where $\deg_R c_i, \deg_R a_j, \deg_R b_l < n-1$. Then

$$\begin{aligned}(s_1,s_2)_w &= [\rho(a,b),R(c)] - [R(a),\rho(b,c)] = [R(a)R(b),R(c)] - [R(a),R(b)R(c)]\\ &\quad - \sum_i \gamma_i[R(c_i),R(c)] + \sum_j \alpha_j[R(a),R(a_j)]\\ &= -[R(b),\rho(a,c)] - \sum_l \beta_l[R(b),R(b_l)]\\ &\quad - \sum_i \gamma_i[R(c_i),R(c)] + \sum_j \alpha_j[R(a),R(a_j)]\\ &= -[R(b),\rho(a,c)] + K(a,b,c).\end{aligned}$$

Here $h = [R(b), \rho(a,c)] \in \hat{S}^{(n)}$, $\bar{h} = [R(b)R(a)R(c)] < [w]$, and all monomials in

$$K(a,b,c) = \sum_j \alpha_j [R(a), R(a_j)] - \sum_l \beta_l [R(b), R(b_l)] - \sum_i \gamma_i [R(c_i), R(c)]$$

are smaller than $[w]$ since they are of degree two in U. Straightforward computations show

$$K(a,b,c) = \sum_k \xi_k h_k + R(J(a,b,c)),$$

where $h_k \in \hat{S}^{(n)}$,

$$\begin{aligned} J(a,b,c) = t_n\big(&[R(a), t_n([R(b),c] + [b,R(c)] + \lambda[b,c])] + [a, t_n([R(b),R(c)])] \\ &+ \lambda[a, t_n([R(b),c] + [b,R(c)] + \lambda[b,c])] \\ &- [R(b), t_n([R(a),c] + [a,R(c)] + \lambda[a,c])] - [b, t_n([R(a),R(c)])] \\ &- \lambda[b, t_n([R(a),c] + [a,R(c)] + \lambda[a,c])] \\ &- [t_n([R(a),R(b)]), c] - [t_n([R(a),b] + [a,R(b)] + \lambda[a,b]), R(c)] \\ &- \lambda[t_n([R(a),b] + [a,R(b)] + \lambda[a,b]), c]\big). \end{aligned}$$

Indeed, $[R(b), R(c)] \to R(t_n([R(b),c] + [b,R(c)] + \lambda[b,c]))$ is an edge in $\mathcal{G}^{(n)}$, so $t_n([R(b),R(c)]) = \sum_j \alpha_j R(a_j)$. Moreover, $t_n(R(x)) = R(t_n(x))$ for all $x \in V_{n-1}$.

It remains to apply Lemma 4.1 to conclude

$$\begin{aligned} J(a,b,c) = t_n\big(&\mathrm{Jac}\,(R(a),R(b),c) + \mathrm{Jac}\,(R(a),b,R(c)) + \mathrm{Jac}\,(a,R(b),R(c)) \\ &+ \lambda \mathrm{Jac}\,(R(a),b,c) + \lambda \mathrm{Jac}\,(a,R(b),c) + \lambda \mathrm{Jac}\,(a,b,R(c)) \\ &+ \lambda^2 \mathrm{Jac}\,(a,b,c)\big) = 0, \end{aligned}$$

where $\mathrm{Jac}\,(x,y,z) = [x,[y,z]] - [y,[x,z]] - [[x,y],z]$ is the Jacobian. Hence, $(s_1,s_2)_w$ has a required presentation (5). ■

Denote $S = \bigcup\limits_{n\geq 1} S^{(n)}$. Obviously, S is a GSB. Denote by T the set of terminal vertices in $\mathcal{G}_R(X,S)$, $T = \bigcup\limits_{n\geq 1} T_n$.

Lemma 4.3. $\mathrm{RLie}\langle X \mid S\rangle$ *is a Rota–Baxter Lie algebra.*

Proof. We have to prove

$$[R(f), R(g)] - R([R(f),g]) - R([f,R(g)]) - \lambda R([f,g]) \in I_R(S) \tag{6}$$

for all $f, g \in \mathrm{RLie}\langle X\rangle$. Since for every $f \in \mathrm{RLie}\langle X\rangle$ there exists $t \in T$ such that $f - t \in I_R(S)$, it is enough to check (6) for $f = a$, $g = b$, where $a, b \in T \cap \mathrm{RLS}(X)$. Assume $a \in T_n$, $b \in T_m$. Then $[R(a), R(b)]$ and $R([R(a),b] + [a,R(b)] + \lambda[a,b])$ have the same terminal form in $\mathcal{G}^{(n+m+2)}$, so they are connected by a non-oriented path in $\mathcal{G}_R(X,S)$. Hence, (6) holds. ■

Corollary 4.4. $\mathrm{RLie}\langle X \mid S\rangle \simeq U_{RB}(L)$.

Proof. Let $B \in \mathrm{RBLie}$, and let $\varphi : L \to B$ be a homomorphism of Lie algebras. Identify L with the Lie subalgebra in $\mathrm{RLie}\langle X\rangle$ spanned by X. Then there exists unique homomorphism of RLie-algebras $\psi : \mathrm{RLie}\langle X\rangle \to B$ such that $\psi(x) = \varphi(x)$ for $x \in X$. Denote by τ the natural homomorphism $\mathrm{RLie}\langle X\rangle \to \mathrm{RLie}\langle X \mid S\rangle$, $\mathrm{Ker}\,\tau = I_R(S)$. Since for every $f \in V_n$ we have $f - t_n(f) \in \mathrm{Ker}\,\tau$, Lemma 4.3 implies $S^{(n)} \subset \mathrm{Ker}\,\tau$. Therefore, there exists a homomorphism of RLie-algebras $\bar{\varphi} : \mathrm{RLie}\langle X \mid S\rangle \to B$, $\bar{\varphi}(x) = \psi(x) = \varphi(x)$ for $x \in X$. ■

The universal enveloping Rota–Baxter Lie algebra $U_{RB}(L)$ of a Lie algebra L has a natural ascending filtration induced by (2):

$$U_{RB}^{(n)}(L) = \tau(\mathrm{RLie}^{(n)}\langle X\rangle).$$

Denote by $\mathrm{gr}\,U_{RB}(L)$ the associated graded RLie algebra. The following statement is ideologically similar to the classical Poincaré–Birkhoff–Witt Theorem.

Theorem 4.5. $\mathrm{gr}\,U_{RB}(L) \simeq U_{RA}(L)$ *as Lie algebras.*

Proof. A GSB of $U_{RA}(L)$ is given by Example 3.12. It is enough to compare GSBs of $U_{RA}(L)$ and $U_{RB}(L)$. The principal parts of these relations coincide, they are of degree 2. For the latter algebra, the right-hand sides of relations are of degree 1. Therefore, $U_{RA}(L)$ and $U_{RB}(L)$ have similar monomial bases of terminal words from RLS (more precisely, from RALS), and the principal part of the product of two such words in $U_{RB}(L)$ coincides with the product of these words in $U_{RA}(L)$. ■

Corollary 4.6 (c.f. [13]). *For* $L = \mathrm{Lie}\langle X\rangle$, *we have* $U_{RB}(L) \simeq \mathrm{RBLie}\langle X\rangle$. *The set of words* $\mathrm{RALS}(X)$ *described in Corollary 2.6 is a linear basis of* $\mathrm{RBLie}\langle X\rangle$.

Remark 4.7. For associative algebras the statement of Theorem 4.5 is easy to show by means of Gröbner–Shirshov bases technique for associative Rota–Baxter algebras [3]: multiplication table of an associative algebra A is closed under all compositions in the free associative Rota–Baxter algebra.

Remark 4.8. For commutative algebras, an analogue of Theorem 4.5 also holds. Moreover, there is an explicit construction of the universal enveloping commutative Rota–Baxter algebra $U_{RB}(A)$ for a given commutative algebra A (mixed shuffle algebra [11]).

Let us briefly state the construction from [11] (in the case of zero weight) in more natural terms. Consider

$$Ш(A) = A^\# \otimes B^\#, \quad B = \mathrm{preCom}\langle A^\#\rangle^{(+)},$$

where $\mathrm{preCom}\langle V\rangle$ stands for the free pre-commutative algebra (also known as Zinbiel algebra, see [20]) generated by a space V, $Z^{(+)}$ denotes the anti-commutator algebra of an algebra Z, and $A^\# = A \oplus \Bbbk 1_A$, $B^\# = B \oplus \Bbbk 1_B$ are the algebras obtained by joining external units. Note that B is an associative and commutative algebra.

Define a linear operator R on $Ш(A)$ by the following rule:

$$R(a\otimes 1_B) = 1_A\otimes a, \quad a\in A^{\#},$$

$$R(a\otimes b) = 1_A\otimes ab, \quad a\in A^{\#},\ b\in B.$$

Here ab is the product in pre-commutative algebra $\mathrm{preCom}\langle A^{\#}\rangle$. The Zinbiel identity $(xy)z = x(yz)+x(zy)$ on $\mathrm{preCom}\langle A^{\#}\rangle$ implies R to be a Rota–Baxter operator on $Ш(A)$. For example,

$$\begin{aligned} R(a_1\otimes 1_B)R(a_2\otimes b) &= (1_A\otimes a_1)(1_A\otimes a_2b)\\ &= 1_A\otimes(a_1(a_2b)+(a_2b)a_1) = 1_A\otimes a_1(a_2b)+1_A\otimes a_2(ba_1)+1_A\otimes a_2(a_1b).\end{aligned}$$

On the other hand,

$$\begin{aligned} &R((a_1\otimes 1_B)R(a_2\otimes b)+R(a_1\otimes 1_B)(a_2\otimes b)) = R((a_1\otimes 1_B)(1_A\otimes a_2b)+(1_A\otimes a_1)(a_2\otimes b))\\ &= R(a_1\otimes a_2b + a_2\otimes(a_1b+ba_1)) = 1_A\otimes a_1(a_2b)+1_A\otimes a_2(a_1b)+1_A\otimes a_2(ba_1).\end{aligned}$$

It is easy to check that the embedding

$$A\to Ш(A), \quad a\mapsto a\otimes 1_B,\ a\in A,$$

may be extended to a homomorphism of Rota–Baxter algebras $U_{RB}(A)\to Ш(A)$. Suppose X is a linear basis of A and consider the following elements of $U_{RB}(A)$:

$$u = R^{s_1}(x_1R^{s_2}(x_2R^{s_3}(\dots R^{s_{n-1}}(x_{n-1}R^{s_n}(x_n))\dots)), \tag{7}$$
$$x_i\in X,\ s_1\ge 0,\ s_2,\dots s_n>0, n\ge 1.$$

Images of these elements $Ш(A)$ are linearly independent since the linear base of $\mathrm{preCom}\langle A^{\#}\rangle$ is given by $x_1(x_2(\dots(x_{n-1}x_n)\dots))$, $x_i\in X\cup\{1_A\}$ [12]. On the other hand, the set of (7) obviously span $U_{RB}(A)$. Therefore, (7) is a linear basis of $U_{RB}(A)$ as well as of $U_{RA}(A)$ from Remark 2.5.

For nonzero weight, it is enough to replace $\mathrm{preCom}\langle A^{\#}\rangle$ with $\mathrm{postCom}\,\langle A^{\#}\rangle$ (commutative tridendriform algebra, or CTD-algebra, [20]), and set B to be the associated commutative algebra.

Remark 4.9. The statement of Theorem 4.5 also holds for algebras with a Nijenhuis operator (see [4]), i.e., a linear map N such that

$$[N(x),N(y)] = N([N(x),y]) + N([x,N(y)]) - N^2([x,y]).$$

The route of proof is completely similar to what is stated above. The key computation of a composition is based on the following relation which is easy to check by straightforward computation:

$$\begin{aligned}\mathrm{Jac}\,(N(a),N(b),N(c)) &= N\big(\mathrm{Jac}\,(a,N(b),N(c))+\mathrm{Jac}\,(N(a),b,N(c))\\ &+\mathrm{Jac}\,(N(a),N(b),c)\big) - N^2\big(\mathrm{Jac}\,(a,b,N(c))+\mathrm{Jac}\,(a,N(b),c)\\ &+\mathrm{Jac}\,(N(a),b,c)\big)+N^3\big(\mathrm{Jac}\,(a,b,c)\big).\end{aligned}$$

References

[1] Baxter, G., *An analytic problem whose solution follows from a simple algebraic identity*, Pacific J. Math. **10** (1960), 731–742.

[2] Bokut, L. A., and Y.-Q Chen, *Gröbner-Shirshov bases and their calculation*, Bull. Math. Sci. **4** (2014), 325–395.

[3] Bokut, L. A., Y.-Q. Chen, and J.-J. Qiu, *Gröbner–Shirshov bases for associative algebras with multiple operators and free Rota–Baxter algebras*, J. Pure Appl. Algebra **214** (2010), 89–100.

[4] Cariñena, J., J. Grabowski, and G. Marmo, *Quantum bi-Hamiltonian systems*, Internat. J. Modern Phys. A **15** (2000), 4797–4810.

[5] Chibrikov, E. S., *On free Lie conformal algebras*, Vestnik Novosibirsk State University **4** (2004), 65–83.

[6] Connes A., and D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem*, Comm. Math. Phys. **210** (2000), 249–273.

[7] Duchamp, G., and D. Krob, *The free partially commutative Lie algebra: Bases and ranks*, Adv. Math. **95** (1992), 92–126.

[8] Ebrahimi-Fard, K., and L. Guo, *Rota–Baxter algebras and dendriform algebras*, J. Pure Appl. Algebra **212** (2008), 320–339.

[9] Gubarev, V. Yu., and P. S. Kolesnikov, *Embedding of dendriform algebras into Rota–Baxter algebras*, Cent. Eur. J. Math. **11** (2013), 226–245.

[10] Gao, X., and L. Guo, *Rota's Classification Problem, rewriting systems and Gröbner-Shirshov bases*, J. Algebra **470** (2017), 219–253.

[11] Guo, L., "An Introduction to Rota-Baxter Algebra," International Press (US) and Higher Education Press (China), 2012.

[12] Loday, J.-L., *Dialgebras*, in: Loday J.-L., A. Frabetti, F. Chapoton, and F. Goichot, Eds., "Dialgebras and related operads," Lectures Notes in Mathematics **1763**, Springer, Berlin etc., 2001, 7–66,

[13] Qiu, J., and Y. Chen, *Gröbner-Shirshov bases for Lie Ω-algebras and free Rota–Baxter Lie algebras*, J. Algebra and its Appl., to appear, `arXiv:1604.06675`.

[14] Rota, G.-C., *Baxter algebras and combinatorial identities I, II*, Bull. Amer. Math. Soc. **75** (1969), 325–329 and ibid. **75** (1969), 330–334.

[15] Semenov-Tian-Shansky, M. A., *What is a classical r-matrix?*, Funct. Anal. Appl. **17** (1983), 259–272.

[16] Shirshov, A. I., *On free Lie rings*, Mat. Sb. **45** (1958), 113–122 (Russian).

[17] —, *On a hypothesis of the theory of Lie algebras*, Sibirsk. Mat. Zhurn. **3** (1962), 297–301 (Russian).

[18] —, *Some algorithmic problem for Lie algebras*, Sibirsk. Mat. Zhurn. **3** (1962), 292–296 (Russian).

[19] Newman, M. H. A., *On theories with a combinatorial definition of "equivalence"*, Ann. of Math. **43** (1942), 223–243.

[20] Zinbiel G. W., *Encyclopedia of types of algebras 2010*, `arXiv:1101.0267 [math.RA]`.

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