

Zero Sets of Abelian Lie Algebras of Vector Fields

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Communicated by K.-H. Neeb

Abstract. Assume M is a 3-dimensional real manifold without boundary, \mathcal{A} is an abelian Lie algebra of analytic vector fields on M , and $X \in \mathcal{A}$.

Theorem If K is a locally maximal compact set of zeroes of $X \in \mathcal{A}$ and the Poincaré-Hopf index of X at K is nonzero, there is a point in K at which all the elements of \mathcal{A} vanish.

Mathematics Subject Classification 2010: 37C10, 37C35.

Key Words and Phrases: keywords Analytic vector field, real manifold, abelian Lie algebra.

1. Introduction

Throughout this paper M denotes a real analytic, metrizable manifold that is connected and has finite dimension n , fixed at $n = 3$ in the main results.

The space of (continuous) vector fields on M endowed with the compact open topology is $\mathcal{V}(M)$, and $\mathcal{V}^r M$ is the subspace of C^r vector fields. Here r denotes a positive integer, ∞ , or ω (meaning analytic); this convention is abbreviated by $1 \leq r \leq \omega$.

The *zero set* of $X \in \mathcal{V}(M)$ is $Z(X) := \{p \in M : X_p = 0\}$. If $Z(X) = \emptyset$ (the empty set), X is *nonsingular*. The zero set of a subset $\mathcal{S} \subset \mathcal{V}(M)$ is $Z(\mathcal{S}) := \bigcap_{X \in \mathcal{S}} Z(X)$.

A compact set $K \subset Z(X)$ is a *block of zeros* for X — called an *X -block* for short— if it lies in a precompact open set $U \subset M$ whose closure \bar{U} contains no other zeros of X ; such an open set is *isolating* for X , and for (X, K) .

When U is isolating for X there is a unique maximal open neighborhood $\mathcal{N}_U \subset \mathcal{V}(M)$ of X with the following property (HIRSCH [10]):

If $Y \in \mathcal{N}_U$ has only finitely many zeros in U , the Poincaré-Hopf index of $Y|_U$ depends only on X and K .

This index is an integer denoted by $i_K(X)$, and also by $i(X, U)$, with the latter notation implying that U is isolating for X . The index can be equivalently defined as the intersection number of $X(U)$ with the zero section of the tangent bundle

of U ([2]); and as the the fixed-point index of the time- t map of the local flow of $X|U$ for sufficiently small $t > 0$. ([5, 9, 12].)

The celebrated POINCARÉ-HOPF Theorem [14, 23] connects the index to the Euler characteristic $\chi(M)$. A modern formulation (see MILNOR [19]) runs as follows:

Theorem 1.1 (POINCARÉ-HOPF). *Assume M is a compact n -manifold, $X \in \mathcal{V}(M)$, and $Z(X) \cap \partial M = \emptyset$. If X is tangent to ∂M at all boundary points, or points outward at all boundary points then $i(X, M) = \chi(M)$. If X points inward at all boundary points, $i(X, M) = (-1)^{n-1}\chi(M)$.*

For calculations of the index in more general settings see GOTTLIEB [6], JUBIN [15], MORSE [21], PUGH [24].

The X -block K is *essential* if $i_K(X) \neq 0$. When this holds every $Y \in \mathcal{N}_U(X)$ has an essential block of zeros in U (Theorem 2.3). If M is a closed manifold (compact, no boundary) and $\chi(M) \neq 0$, the Poincaré-Hopf Theorem implies $Z(X)$ is an essential X -block.

C. Bonatti's proved a remarkable extension of the Poincaré-Hopf Theorem to certain pairs of commuting analytic vector fields on manifolds that need not be compact:

Theorem 1.2 (BONATTI [2]). *Assume $\dim M \leq 4$ and $\partial M = \emptyset$. If $X, Y \in \mathcal{V}^\omega(M)$ and $[X, Y] = 0$, then $Z(Y)$ meets every essential X -block.¹*

Related results are in the articles [3, 10, 11, 13, 16, 17, 22, 27].

Our main result is an extension of Bonatti's Theorem:

Theorem 1.3. *Let M be a connected 3-manifold and $\mathcal{A} \subset \mathcal{V}^\omega(M)$ an abelian Lie algebra of analytic vector fields on M . Assume $X \in \mathcal{A}$ is nontrivial and $Z(X) \cap \partial M = \emptyset$. If K is an essential X -block, then $Z(\mathcal{A}) \cap K \neq \emptyset$.*

The proof, in Section 3, relies heavily on Bonatti's Theorem. An analog for surfaces is proved in HIRSCH [10, Thm. 1.3].

1.1. Application to attractors. The *interior* $\text{Int}(L)$ of a subset $L \subset M$ is the union of all open subsets of M contained in L .

Fix a metric on M . If $Q \subset M$ is closed, the minimum distance from $z \in M$ to points of Q is denoted by $\text{dist}(z, Q)$.

Let $X \in \mathcal{V}^1(M)$ have local flow Φ . An *attractor* for X (see [1, 4, 7, 25]) is a nonempty compact set $P \subset M$ that is invariant under Φ and has a compact neighborhood $N \subset M$ such that

$$\Phi_t(N) \subset (N)$$

and

¹ "The demonstration of this result involves a beautiful and quite difficult local study of the set of zeros of X , as an analytic Y -invariant set." —P. MOLINO [20]

$$\lim_{t \rightarrow \infty} \text{dist}(\Phi_t(x, P), P) = 0 \text{ uniformly in } x \in N. \quad (1)$$

Such an N can be chosen so that

$$t > s \geq 0 \implies \Phi_t(N) \subset \text{Int}(\Phi_s(N)). \quad (2)$$

Henceforth (2) is assumed. By F. W. WILSON [28, Thm. 2.2], we choose N so that:

$$N \text{ is a compact } C^1 \text{ submanifold and } X \text{ is inwardly transverse to } \partial N.^2 \quad (3)$$

Theorem 1.4. *Let M , \mathcal{A} and X be as in Theorem 1.3. If $P \subset M$ is a compact attractor for X and $\chi(P) \neq 0$, then $Z(\mathcal{A}) \cap P \neq \emptyset$.*

Proof. P is a proper subset of M , as otherwise M is a closed 3-manifold having nonzero Euler characteristic, contradicting the classical result that odd-dimensional closed manifolds have zero Euler characteristic (e.g., HIRSCH [8, Thm. 5.2.5]); Spanier[26, Thm. 6.2.18]. Fix N as above and note that $\chi(N) \neq 0$.

By (3) and the Poincaré-Hopf Theorem 1.1 there is an essential X -block $K \subset N \setminus \partial N$, and $K \subset P$ by (1). Standard homology theory and (2) imply that the inclusion map $P \hookrightarrow N$ induces an isomorphisms on singular homology, hence $\chi(N) = \chi(P) \neq 0$.

The conclusion follows from Theorem 1.3 applied to the data M', \mathcal{A}', X' :

$$M' := N, \quad \mathcal{A}' := \{Y|N : Y \in \mathcal{A}\}, \quad X' := X|N. \quad \blacksquare$$

Example 1.5. Denote the inner product of $x, y \in \mathbb{R}^3$ by $\langle x, y \rangle$ and the norm of x by $\|x\|$. Let $B_r \subset \mathbb{R}^3$ denote the open ball about the origin of radius $r > 0$.

- Assume \mathcal{A} is an abelian Lie algebra of analytic vector fields on an open set $M \subset \mathbb{R}^3$ that contains \overline{B}_r . Let $X \in \mathcal{A}$ and $r > 0$ be such that

$$\|x\| = r \implies \langle X_p, p \rangle < 0.$$

Then $Z(\mathcal{A}) \cap B_r \neq \emptyset$.

Proof. This is a consequence of Theorem 1.4: \overline{B}_r contains an attractor for X because X inwardly transverse to $\partial \overline{B}_r$ and $\chi(\overline{B}_r) = 1$.

2. Background material

Lemma 2.1 (INVARIANCE). *If $T, S \in \mathcal{A}$ then $Z(S)$ is invariant under T .*

Proof. Let $\Phi := \{\Phi_t\}_{t \in \mathbb{R}}$ and $\Psi := \{\Psi_s\}$ denote the local flows of T and S , respectively. If $t, s \in \mathbb{R}$ are sufficiently close to 0, because $[T, S] = 0$ we have

$$\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t,$$

² This means X_p is not tangent to ∂N if $p \in \partial N$.

and

$$Z(S) = \text{Fix}(\Psi) := \bigcap_s \text{Fix}(\Psi_s),$$

where Fix denotes the fixed point set. Suppose $p \in Z(S)$. Then $p \in \text{Fix}(\Psi)$, and

$$\Psi_s \circ \Phi_t(p) = \Phi_t \circ \Psi_s(p) = \Phi_t(p).$$

Consequently $\Phi_t(p) \in \text{Fix}(\Psi_s)$ for sufficiently small $|t|, |s|$, implying the conclusion. ■

A closed set $Q \subset M$ is an *analytic subspace* of M , or *analytic in* M , provided Q has a locally finite covering by zero sets of analytic maps defined on open subsets of M . This is abbreviated to *analytic space* when the ambient manifold M is clear from the context. The connected components of analytic spaces are also analytic spaces.

Analytic spaces have very simple local topology, owing to the theorem of ŁOJASIEWICZ [18]:

Theorem 2.2 (TRIANGULATION). *If T is a locally finite collection of analytic spaces in M , there is a triangulation of M such that each element of T is covered by a subcomplex.*

The proof of Theorem 1.3 uses the following folk theorem:

Theorem 2.3 (STABILITY). *Assume $X \in \mathcal{V}(M)$ and $U \subset M$ is isolating for X .*

- (a) *If $i(X, U) \neq 0$ then $Z(X) \cap U \neq \emptyset$.*
- (b) *If $Y \in \mathcal{V}(M)$ is sufficiently close to X , then U is isolating for Y and $i(Y, U) = i(X, U)$.*

Proof. See HIRSCH [10, Thm. 3.9]. ■

Let $Z(\mathcal{S})$ denote the set of common zeros of a subset $\mathcal{S} \subset \mathcal{V}^\omega(M)$.

Proposition 2.4. *The following conditions hold for every $\mathcal{S} \subset \mathcal{A}$:*

- (a) *$Z(\mathcal{S})$ is analytic in M .*
- (b) *Every zero dimensional \mathcal{A} -invariant set lies in $Z(\mathcal{A})$.*

Proof. Left to the reader. ■

3. Proof of Theorem 1.3

Recall the hypotheses of the Main Theorem:

- *M is a 3-dimensional manifold,*

- $\mathcal{A} \subset \mathcal{V}^\omega(M)$ is an abelian Lie algebra,
- $X \in \mathcal{A}$ is nontrivial, $Z(X) \cap \partial M = \emptyset$, and K is an essential block of zeroes for X .

The conclusion to be proved is: $Z(\mathcal{A}) \cap K \neq \emptyset$. It suffices to show that $Z(\mathcal{A})$ meets every neighborhood of K , because $Z(\mathcal{A})$ is closed and K is compact.

Case I: $\dim \mathcal{A} = d < \infty$. The special case $d \leq 2$ is covered by Bonatti's Theorem. We proceed by induction on d :

Induction Hypothesis

- $\dim \mathcal{A} = d + 1$, $d \geq 2$.
- The zero set of every d -dimensional subalgebra of \mathcal{A} meets K .

Arguing by contradiction, we assume *per contra*:

(PC) $Z(\mathcal{A}) \cap K = \emptyset$.

An important consequence is:

(A) $\dim K \leq 2$.

For otherwise $\dim K = 3$, which entails the contradiction that X is trivial: X is analytic and vanishes on a 3-simplex in the connected 3-manifold M .

The Stability Theorem (2.3) implies X has a neighborhood $\mathcal{N}_U \subset \mathcal{V}^\omega(X)$ with the following property:

(B) $Y \in \mathcal{N}_U \implies U$ is isolating for Y and $i(Y, U) = i(X, U) \neq 0$.

Let $G_d(\mathcal{A})$ denote d -dimensional Grassmann manifold of d -dimensional linear subspaces \mathcal{B} of \mathcal{A} ; these are abelian subalgebras.

The nonempty set

$$G_d(\mathbf{N}_U) := \{\mathcal{B} \in G_d(\mathcal{A}) : \mathcal{B} \cap \mathbf{N}_U \neq \emptyset\}$$

is open in $G_d(\mathcal{A})$, hence it is a d -dimensional analytic manifold.

Bonatti's Theorem and (B) imply:

(C) $Z(\mathcal{B}) \cap K \neq \emptyset$ for all $\mathcal{B} \in G_d(\mathbf{N}_U)$.

A key topological consequence of (C) is:

(D) If \mathcal{B} and \mathcal{B}' are distinct elements of $G_d(\mathbf{N}_U)$, then $Z(\mathcal{B}) \cap K$ and $Z(\mathcal{B}') \cap K$ are disjoint.

This holds because $\mathcal{B} \cup \mathcal{B}'$ spans \mathcal{A} , hence (PC) implies

$$(Z(\mathcal{B}) \cap K) \cap (Z(\mathcal{B}') \cap K) = Z(\mathcal{A}) \cap K = \emptyset.$$

Each set $Z(\mathcal{B}) \cap K$ is invariant (Lemma 2.1) and therefore has positive dimension by (PC). Moreover:

(E) The set $\Gamma_U := \{\mathcal{B} \in G_d(\mathbf{N}_U) : \dim Z(\mathcal{B}) \cap K = 2\}$ is finite.

For otherwise (D) implies K contains an infinite sequence of pairwise disjoint compact subsets that are 2-dimensional and analytic in M . But this is impossible by (A) and the Triangulation Theorem 2.2.

(E) shows that $\Gamma_U = \emptyset$ provided U is small enough. Therefore we can assume:

(F) $\dim Z(\mathcal{B}) \cap K = 1$ for all $\mathcal{B} \in G_d(\mathbf{N}_U)$.

The set $Q := \{(\mathcal{B}, p) \in G_d(\mathbf{N}_U) \times M : p \in Z(\mathcal{B}) \cap K\}$ is analytic in $G_d(\mathbf{N}_U) \times M$ (Proposition 2.4). The natural projections

$$\pi_1: Q \rightarrow G_d(\mathbf{N}_U), \quad \pi_2: Q \rightarrow K$$

are analytic, π_1 is surjective, π_2 is injective by (D).

The sets $Z(\mathcal{B}) \cap K$ are therefore pairwise disjoint, and each is a 1-dimensional analytic subspaces of Q by (F). Therefore $\dim Q = \dim G_d(\mathcal{A}) + \dim (Z(\mathcal{B}) \cap K) \leq \dim K$, whence $\dim Q = d + 1 \leq 2$. But this is impossible because $d \geq 2$ by the Induction Hypothesis. This completes the inductive proof of the Main Theorem in Case I.

Case II: $\dim \mathcal{A}$ is infinite. Consider the family \mathfrak{F} of compact subsets of K :

$$\mathfrak{F} := \{Z(\mathcal{A}') \cap K : \mathcal{A}' \subset \mathcal{A} \text{ is a finite-dimensional subalgebra}\}.$$

Evidently $\bigcap_{S \in \mathfrak{F}} S = Z(\mathcal{A}) \cap K$. Case I shows every finite subset of \mathfrak{F} has nonempty intersection. As K is compact, all the elements of \mathfrak{F} have nonempty intersection, proving $Z(\mathcal{A}) \cap K \neq \emptyset$. ■

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Received February 7, 2016
and in final form February 19, 2017