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Ricci Solitons on Four-dimensional Neutral Lie Groups

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Abstract. After giving a full classification of four-dimensional pseudo-Riemannian Lie groups of signature $(2,2)$, we study Ricci solitons on these spaces.
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Key Words and Phrases: Lie group, neutral signature, Ricci soliton.

1. Introduction

A homogeneous manifold (M,g) is identified by the action of its group of isometries $I(M)$ on M. In fact, (M,g) is homogeneous if $I(M)$ acts transitively on M; that means, for each $p,q \in M$, there exists an isometry $\phi : M \to M$ such that $\phi(p) = q$.

In addition to their mathematical interest, homogeneous spaces are studied for their wide applications in mathematical physics and relativity. Four dimensional homogeneous Riemannian manifolds were classified in [3] by Bérard-Bérgery. Based on this classification, a simply connected four dimensional homogeneous Riemannian manifold is either symmetric, or isometric to a Lie group equipped with a left-invariant Riemannian metric. Simply connected Riemannian Lie groups of dimension four were also studied in [1].

In the pseudo-Riemannian setting, three-dimensional homogeneous manifolds were classified in [4], where the author proved the pseudo-Riemannian counterpart of the results in [19], i.e., a connected, simply connected and complete homogeneous Lorentzian manifold is either symmetric or isometric to a three dimensional Lorentzian Lie group (M,g), equipped with a left-invariant metric g. Lorentzian Lie groups of dimension three were classified in [18] and [13]. In dimension four, a full classification of homogeneous manifolds with non-trivial isotropy was given in [16]. This classification has been used for deep investigations of the homogeneous four-dimensional manifolds with non-trivial isotropy (see for example [7, 8, 12, 20]). The problem of classifying Lorentzian Lie groups of dimension four remained open.

Up to recent years, Riemannian Lie groups were the only field of study for left-invariant metrics (see for example [1]). Curvature conditions on four-

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dimensional Lorentzian and neutral Lie groups, with special emphasis on Einstein and Ricci parallel cases, were studied in [9] and [10] respectively.

In this paper we give a full classification of four-dimensional pseudo-Riemannian Lie groups of signature $(2,2)$ and we study the existence and some geometric properties of Ricci solitons, that is self-similar solutions of the Ricci flow, on these spaces.

The paper is organized in the following way. In Section 2 we recall some basic facts about four dimensional neutral Lie groups. Section 3 and 4 are devoted to present a full classification of four-dimensional neutral Lie groups where the restriction of the left invariant metric on a three dimensional subgroup is Lorentzian or degenerate respectively. Using this classification, in the last Section we study Ricci solitons on four-dimensional neutral Lie groups and some of their geometric properties.

Computer software has been used to check all the needed calculations.

2. Four-dimensional neutral Lie groups

Homogenous Riemannian manifolds of dimension four were classified by Bérard-Bérgery in [3]. This classification highlighted the important role of Riemannian Lie groups, since any homogeneous Riemannian manifold of dimension four is either symmetric or isometric to a Lie group which is equipped with a left-invariant Riemannian metric. The classification of four-dimensional Riemannian Lie groups was first obtained by Arias-Marco and Kowalski in [1]. We summarize this classification in the following proposition.

Proposition 2.1. [1] *A simply connected four-dimensional Riemannian Lie group is:*

(i) either one of the unsolvable direct products $SU(2)\times\mathbb{R}$ *and* $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$*; or*

(ii) one of the following solvable Lie groups:

- *(ii1) the non-trivial semi-direct products* $E(2)\rtimes\mathbb{R}$ *and* $E(1,1)\rtimes\mathbb{R}$*;*
- *(ii2) the non-nilpotent semi-direct products* $H\rtimes\mathbb{R}$*, where* H *denotes the Heisenberg group;*
- *(ii3) the semi-direct products* $\mathbb{R}^3\rtimes\mathbb{R}$*.*

Note that, semi-direct products obtained by a three-dimensional non-unimodular Lie algebra are absent in this classification. Indeed, one can check that a semi-direct product $\widetilde{\mathfrak{g}}_3\rtimes\mathfrak{r}$ with $\widetilde{g}_3$ non-unimodular, is also isomorphic to a semi-direct product $\mathfrak{g}_3\rtimes\mathfrak{r}$, with $\mathfrak{g}_3$ unimodular, i.e., is isomorphic to one of the cases listed in the above Proposition 2.1.

Pseudo-Riemannian Lie groups of dimension four were not studied until recently, when the classification of these spaces was studied in [10], with special emphasis on Einstein and Ricci parallel examples. This classification based on the coincidence of Riemannian and pseudo-Riemannian four dimensional Lie groups. We summarize this result in the following proposition.

Proposition 2.2. [10] *Every n-dimensional simply connected Lie group G admits left-invariant metrics of any prescribed signature $(p, n-p)$. In particular, if G is a four-dimensional simply connected Lie group equipped with a left-invariant metric of neutral signature, then G is one of Lie groups listed in the* Proposition 2.1.

By the above proposition, the class of simply connected four-dimensional Lie groups of neutral signature coincides with the class of simply connected four-dimensional Riemannian Lie groups, but this does not mean that the geometry of these two categories is also the same. In fact, our study shows more complexities in the neutral signature case. The difference between Riemannian and pseudo-Riemannian cases arises when we restrict the invariant metric to the subspaces of the corresponding Lie algebra. In other words, the restriction of a left invariant Riemannian metric to each subspace of the Lie algebra is again Riemannian, while in the neutral case, the restriction of a left invariant metric to each subspace (two or three dimensional) can be Riemannian, neutral, Lorentzian or even degenerate. In particular, since $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$, the restriction to $\mathfrak{g}_3$ is either:

(a) of signature $(2,1)$, (a$'$) of signature $(1,2)$, or (b) *degenerate.*

The first two cases are referred to as "(a)" and "(a$'$)", because they are equivalent to each other, up to reversing the metric. In fact, in case (a$'$), g is a left invariant metric of neutral signature over a four-dimensional Lie algebra $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$, and a space-like vector e_4 (spanning $\mathfrak{r}$) acts as a derivation over a three-dimensional Lie algebra $\mathfrak{g}_3$, such that the restriction of g on $\mathfrak{g}_3$ has signature $(1,2)$. It now suffices to *reverse the metric* [17] to find the same Lie algebra $\mathfrak{g}$ equipped with the neutral inner product $-g$, for which a time-like vector e_4 acts as a derivation over the three-dimensional Lorentzian Lie algebra $\mathfrak{g}_3$ of signature $(2,1)$ (case (a)).

We summarize the above discussion in the following lemma.

Lemma 2.3. [10] *Let $\mathfrak{g}$ denote any four-dimensional Lie algebra and g be an inner product on $\mathfrak{g}$, of neutral signature. Then, there exists a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$, such that*

- $\mathfrak{g}_3 = \mathrm{Span}(e_1, e_2, e_3)$ *is a three-dimensional Lie algebra and e_4 acts as a derivation on $\mathfrak{g}_3$ (that is, $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$, where $\mathfrak{r} = \mathrm{Span}(e_4)$), and*
- *with respect to $\{e_1, e_2, e_3, e_4\}$, the neutral inner product g takes one of the following forms:*

$$(a) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad (b) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Based on the Lemma 2.3, in order to study four dimensional Lie algebras $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ of neutral signature, where $\mathfrak{g}_3 = \mathrm{Span}(e_1, e_2, e_3)$ and $\mathfrak{r} = \mathrm{Span}(e_4)$, it would be sufficient to consider the following two cases:

a) $g|_{\mathfrak{g}_3}$ is Lorentzian and the time-like vector e_4 acts as a derivation on $\mathfrak{g}_3$.

b) $g|_{\mathfrak{g}_3}$ is degenerate and the light-like vector e_4 acts as a derivation on $\mathfrak{g}_3$.

3. Restricting the metric on $\mathfrak{g}_3$ is Lorentzian

In this section, we start by the case (a) of the Lemma 2.3. The restriction of the left-invariant metric g on $\mathfrak{g}_3$ is Lorentzian, so that the metric g is described by the neutral inner product as in the case (a) of the Lemma 2.3. A classification of the three dimensional homogeneous Lorentzian manifolds was given in [4]. This classification contains a full classification of three dimensional Lorentzian Lie groups through 7 classes, named $\mathfrak{g}1, \cdots, \mathfrak{g}7$, corresponding to the unimodular and non-unimodular Lie groups (Theorem 4.1 of [4]). So, we apply all these 7 cases to $\mathfrak{g}_3$, and considering that the time-like vector e_4 acts as a derivation on $\mathfrak{g}_3$, we define, in general, the Lie brackets $[e_i, e_4],\ 1 \leq i \leq 4$ in the following way

$$\begin{aligned}
[e_1, e_4] &= p_1e_1 + p_2e_2 + p_3e_3, \\
[e_2, e_4] &= q_1e_1 + q_2e_2 + q_3e_3, \\
[e_3, e_4] &= r_1e_1 + r_2e_2 + r_3e_3.
\end{aligned} \tag{3.1}$$

Now, we check the validity of the Jacobi identity for all possible cases of $\mathfrak{g}_3$. That is, we set

$$[[e_i, e_j], e_k] + [[e_j, e_k], e_i] + [[e_k, e_i], e_j] = 0, \quad i, j, k = 1, \cdots 4. \tag{3.2}$$

So, we have the following classification theorem for $\mathfrak{g}$.

Theorem 3.1. *Let $(G = G_3 \rtimes \mathbb{R}, g)$ be a four dimensional Lie group of neutral signature, where the restriction of g on $\mathfrak{g}_3$ is Lorentzian. In this case, there exists a pseudo-orthonormal basis, such that $\mathfrak{g}$ is isometric to one of the following cases:*

a- $\mathfrak{g}_3$ is of type $\mathfrak{g}1$)

1-
$$\begin{aligned}
&[e_1, e_2] = \alpha e_1, && [e_1, e_3] = -\alpha e_1, \qquad \alpha \neq 0, \\
&[e_1, e_4] = Ae_1, && [e_2, e_3] = \alpha e_2 + \alpha e_3, \\
&[e_2, e_4] = Be_1 + Ce_2 + Ce_3, && [e_3, e_4] = -Be_1 + De_2 + De_3,
\end{aligned}$$

2-
$$\begin{aligned}
&[e_1, e_2] = \alpha e_1 - \beta e_3, \qquad [e_1, e_3] = -\alpha e_1 - \beta e_2, \\
&[e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \quad [e_1, e_4] = -(A + B)e_1 - \tfrac{\beta B}{\alpha} e_2 + \tfrac{\beta A}{\alpha} e_3, \\
&[e_2, e_4] = \tfrac{\beta(A+B)-\alpha C}{\alpha} e_1 + Be_2 + \tfrac{\beta\alpha C-\beta^2 A+\alpha^2 B}{\alpha^2} e_3, \\
&[e_3, e_4] = Ce_1 + \tfrac{\beta\alpha C+(\alpha^2-\beta^2)A}{\alpha^2} e_2 + Ae_3, \quad \alpha \neq 0.
\end{aligned}$$

b- $\mathfrak{g}_3$ is of type $\mathfrak{g}2$)

1-
$$\begin{aligned}
&[e_1, e_2] = -\gamma e_2, \quad [e_1, e_3] = \gamma e_3, \quad [e_1, e_4] = Ae_2 + Be_3, \\
&[e_2, e_4] = Ce_2, \quad [e_3, e_4] = De_3, \quad \gamma \neq 0,
\end{aligned}$$

2- $[e_1,e_2]=-\gamma e_2-\beta e_3,\quad [e_1,e_3]=-\beta e_2+\gamma e_3,$
$[e_1,e_4]=Ae_2+Be_3,\quad [e_2,e_4]=Ce_2+De_3,$
$[e_3,e_4]=De_2-\frac{2\gamma D-\beta C}{\beta}e_3,\quad \beta\gamma\neq 0,$

3- $[e_1,e_2]=-\gamma e_2-\beta e_3,\quad [e_1,e_3]=-\beta e_2+\gamma e_3,$
$[e_2,e_3]=\alpha e_1,\quad [e_1,e_4]=\frac{(\gamma^2+\beta^2)A-\beta\alpha B}{\alpha\gamma}e_2+Be_3,$
$[e_2,e_4]=\frac{\alpha B-\beta A}{\gamma}e_1-Ce_2-\frac{\beta C}{\gamma}e_3,\quad [e_3,e_4]=Ae_1-\frac{\beta C}{\gamma}e_2+Ce_3,\ \alpha\gamma\neq 0.$

c- $\mathfrak{g}_3$ is of type $\mathfrak{g}3$)

1- $[e_1,e_2]=-\gamma e_3,\quad [e_1,e_3]=-\beta e_2,\quad [e_2,e_3]=\alpha e_1,\ \alpha\beta\neq 0,$
$[e_1,e_4]=-\frac{\beta A}{\alpha}e_2+\frac{\gamma B}{\alpha}e_3,\quad [e_2,e_4]=Ae_1+\frac{\gamma C}{\beta}e_3,\quad [e_3,e_4]=Be_1+Ce_2,$

2- $[e_1,e_2]=-\gamma e_3,\quad [e_2,e_3]=\alpha e_1,\quad \alpha\neq 0,$
$[e_1,e_4]=Ae_1+\frac{\gamma B}{\alpha}e_3,\quad [e_2,e_4]=Ce_1+De_3,\quad [e_3,e_4]=Be_1+Ae_3,$

3- $[e_1,e_2]=-\gamma e_3,\quad [e_1,e_3]=-\beta e_2,\quad \beta\neq 0,$
$[e_1,e_4]=Ae_2+Be_3,\quad [e_2,e_4]=De_2+\frac{\gamma C}{\beta}e_3,\quad [e_3,e_4]=Ce_2+De_3,$

4- $[e_1,e_3]=-\beta e_2,\quad [e_2,e_3]=\alpha e_1,\quad \alpha\neq 0,$
$[e_1,e_4]=Ae_1-\frac{\beta B}{\alpha}e_2,\quad [e_2,e_4]=Be_1+Ae_2,\quad [e_3,e_4]=Ce_1+De_2,$

5- $[e_1,e_2]=-\gamma e_3,\quad [e_1,e_4]=(A-B)e_1+Ce_2+De_3,$
$[e_2,e_4]=Ee_1+Be_2+Fe_3,\quad [e_3,e_4]=Ae_3,\quad \gamma\neq 0,$

6- $[e_1,e_3]=-\beta e_2,\quad [e_1,e_4]=(A-B)e_1+Ce_2+De_3,$
$[e_2,e_4]=Ae_2,\quad [e_3,e_4]=Ee_1+Fe_2+Be_3,\quad \beta\neq 0,$

7- $[e_2,e_3]=\alpha e_1,\quad [e_1,e_4]=(A+B)e_1,$
$[e_2,e_4]=Ce_1+Ae_2+De_3,\quad [e_3,e_4]=Ee_1+Fe_2+Be_3,\quad \alpha\neq 0,$

8- $[e_1,e_4]=Ae_1+Be_2+Ce_3,\quad [e_2,e_4]=De_1+Ee_2+Fe_3,$
$[e_3,e_4]=Ge_1+He_2+Ke_3.$

d- $\mathfrak{g}_3$ is of type $\mathfrak{g}4$)

1- $[e_1,e_2]=-e_2+(2\varepsilon-\beta)e_3,\quad [e_1,e_3]=-\beta e_2+e_3,$
$[e_1,e_4]=Ae_2+Be_3,\quad [e_2,e_4]=Ce_2+(D-C)(\varepsilon-\frac{\beta}{2})e_3,$
$[e_3,e_4]=\frac{\beta}{2}(C-D)e_2+De_3,\quad \varepsilon^2=1,$

2- $[e_1,e_2]=-e_2+\varepsilon e_3,\quad [e_1,e_3]=-\varepsilon e_2+e_3,$
$[e_1,e_4]=Ae_1+Be_2+Ce_3,$
$[e_2,e_4]=De_1+Ee_2-\frac{\varepsilon}{2}(A+E-F)e_3,$
$[e_3,e_4]=\varepsilon De_1-\frac{\varepsilon}{2}(A-E+F)e_2+Fe_3,\quad \varepsilon^2=1,$

3- $[e_1,e_2]=-e_2+(2\varepsilon-\beta)e_3,\quad [e_1,e_4]=Ae_2-\frac{(2\varepsilon-\beta)(\beta B+\alpha A)-B}{\alpha}e_3,$
$[e_1,e_3]=-\beta e_2+e_3,\quad [e_2,e_3]=\alpha e_1,$
$[e_2,e_4]=Be_1-Ce_2+(2\varepsilon-\beta)Ce_3,$
$[e_3,e_4]=(B\beta+A\alpha)e_1-C\beta e_2+Ce_3,\quad \alpha\neq 0,\quad \varepsilon^2=1,$

4- $[e_1,e_2]=-e_2+\varepsilon e_3,\quad [e_1,e_3]=-\varepsilon e_2+e_3,$
$[e_1,e_4]=(A+B)e_1-\varepsilon Ce_2+Ce_3,\quad [e_2,e_3]=\alpha e_1,$
$[e_2,e_4]=De_1+Ae_2-\varepsilon Ae_3,$
$[e_3,e_4]=\varepsilon(D-\alpha C)e_1-\varepsilon Be_2+Be_3,\quad \varepsilon^2=1.$

e- $\mathfrak{g}_3$ is of type $\mathfrak{g}5$)

1- $[e_1,e_3]=\alpha e_1+\beta e_2,\quad [e_2,e_3]=\gamma e_1+\delta e_2,$
$[e_1,e_4]=(B+\frac{A}{\gamma}(\alpha-\delta))e_1+\frac{\beta A}{\gamma}e_2,\quad [e_2,e_4]=Ae_1+Be_2,$
$[e_3,e_4]=Ce_1+De_2,\quad \alpha+\delta\neq 0,\ \alpha\gamma+\beta\delta=0,\ \gamma\neq 0,$

2- $[e_1,e_3]=\alpha e_1,\quad [e_2,e_3]=\delta e_2,$
$[e_1,e_4]=Ae_1,\quad [e_2,e_4]=Be_2,\quad [e_3,e_4]=Ce_1+De_2,\quad \alpha+\delta\neq 0,$

3- $[e_1,e_3]=\alpha e_1,\quad [e_2,e_3]=\alpha e_2,\quad \alpha\neq 0,$
$[e_1,e_4]=Ae_1+Be_2,\quad [e_2,e_4]=Ce_1+De_2,\quad [e_3,e_4]=Ee_1+Fe_2,$

4- $[e_1,e_3]=\alpha e_1+\beta e_2,\quad [e_1,e_4]=Ae_1+\frac{\beta(A-B)}{\alpha}e_2,$
$[e_2,e_4]=Be_2,\quad [e_3,e_4]=Ce_1+De_2,\quad \alpha\neq 0.$

f- $\mathfrak{g}_3$ is of type $\mathfrak{g}6$)

1- $[e_1,e_2]=\alpha e_2+\beta e_3,\quad [e_1,e_3]=\gamma e_2+\delta e_3,$
$[e_1,e_4]=Ae_2+Be_3,\quad [e_2,e_4]=(D+\frac{C}{\gamma}(\alpha-\delta))e_2+\frac{\beta C}{\gamma}e_3,$
$[e_3,e_4]=Ce_2+De_3,\quad \alpha+\delta\neq 0,\ \alpha\gamma-\beta\delta=0,\ \gamma\neq 0,$

2- $[e_1,e_2]=\alpha e_2,\quad [e_1,e_3]=\delta e_3,$
$[e_1,e_4]=Ae_2+Be_3,\quad [e_2,e_4]=Ce_2,\quad [e_3,e_4]=De_3,\quad \alpha+\delta\neq 0,$

3- $[e_1,e_2]=\alpha e_2,\quad [e_1,e_3]=\alpha e_3,\quad \alpha\neq 0,$
$[e_1,e_4]=Ae_2+Be_3,\quad [e_2,e_4]=Ce_2+De_3,\quad [e_3,e_4]=Ee_2+Fe_3,$

4- $[e_1,e_2]=\alpha e_2+\beta e_3,\quad [e_1,e_4]=Ae_2+Be_3,$
$[e_2,e_4]=Ce_2+\frac{\beta(C-D)}{\alpha}e_3,\quad [e_3,e_4]=De_3,\quad \alpha\neq 0.$

g- $\mathfrak{g}_3$ is of type $\mathfrak{g}7$)

1- $[e_1,e_2]=-\beta e_2-\beta e_3,\quad [e_1,e_3]=\beta e_2+\beta e_3,$
$[e_2,e_3]=\gamma e_1+\delta e_2+\delta e_3,$
$[e_1,e_4]=Ae_1-\frac{\beta(A-B-C)}{\delta}e_2-\frac{\beta(A-B-C)}{\delta}e_3,\quad \delta\neq 0,$
$[e_2,e_4]=-\frac{\gamma(A-B-C)+\delta D}{\delta}e_1+Be_2+Be_3,\quad [e_3,e_4]=De_1+Ce_2+Ce_3,$

2- $[e_1,e_2]=-\alpha e_1-\beta e_2-\beta e_3,\quad [e_1,e_3]=\alpha e_1+\beta e_2+\beta e_3,$
$[e_1,e_4]=Ae_1+\frac{\beta(A-B-C)}{\alpha}e_2+\frac{\beta(A-B-C)}{\alpha}e_3,\quad [e_2,e_4]=-De_1+Be_2+Be_3,$
$[e_3,e_4]=De_1+Ce_2+Ce_3,\quad \alpha\neq 0.$

Proof. Let $(G=G_3\rtimes\mathbb{R},g)$ be a four dimensional Lie group of neutral signature, where the restriction of g on $\mathfrak{g}_3$ is Lorentzian. Since the classification of three dimensional Lorentzian Lie groups is critical in this proof, we recall here that classification, as reported in [4, Theorem 4.1]. There exists a pseudo-orthonormal frame field $\{e_1,e_2,e_3\}$ with e_3 timelike, such that the Lie algebra of G_3 is one of the following:

- If G_3 is unimodular,

$$(\mathfrak{g}1):\begin{cases}[e_1,e_2]&=&\alpha e_1-\beta e_3,\\ [e_1,e_3]&=&-\alpha e_1-\beta e_2, & \alpha\neq 0,\\ [e_2,e_3]&=&\beta e_1+\alpha e_2+\alpha e_3,\end{cases}$$

$$(\mathfrak{g}2):\begin{cases}[e_1,e_2]&=&-\gamma e_2-\beta e_3,\\ [e_1,e_3]&=&-\beta e_2+\gamma e_3, & \gamma\neq 0,\\ [e_2,e_3]&=&\alpha e_1,\end{cases}$$

$$(\mathfrak{g}3):\ [e_1,e_2]=-\gamma e_3,\quad [e_1,e_3]=-\beta e_2,\quad [e_2,e_3]=\alpha e_1,$$

$$(\mathfrak{g}4):\begin{cases}[e_1,e_2]&=&-e_2+(2\varepsilon-\beta)e_3, & \varepsilon=\pm1,\\ [e_1,e_3]&=&-\beta e_2+e_3,\\ [e_2,e_3]&=&\alpha e_1.\end{cases}$$

- If G_3 is non-unimodular,

$$(\mathfrak{g}5):\begin{cases}[e_1,e_2]&=&0,\\ [e_1,e_3]&=&\alpha e_1+\beta e_2, & \text{where } \alpha+\delta\neq 0,\ \alpha\gamma+\beta\delta=0,\\ [e_2,e_3]&=&\gamma e_1+\delta e_2,\end{cases}$$

$$(\mathfrak{g}6):\ \begin{cases} [e_1,e_2] &= \alpha e_2+\beta e_3,\\ [e_1,e_3] &= \gamma e_2+\delta e_3, \quad \text{where } \alpha+\delta\neq 0,\ \alpha\gamma-\beta\delta=0,\\ [e_2,e_3] &= 0, \end{cases}$$

$$(\mathfrak{g}7):\ \begin{cases} [e_1,e_2] &= -\alpha e_1-\beta e_2-\beta e_3,\\ [e_1,e_3] &= \alpha e_1+\beta e_2+\beta e_3, \quad \text{where } \alpha+\delta\neq 0,\ \alpha\gamma=0.\\ [e_2,e_3] &= \gamma e_1+\delta e_2+\delta e_3, \end{cases}$$

We start with $\mathfrak{g}_3$ to be of type $\mathfrak{g}1$, and the Lie brackets $[e_i,e_4]$, $1\leq i\leq 4$ are as in the Equation (3.1). In this case, the Jacobi identity is established, if and only if

$$\begin{cases} \beta p_1-2\alpha p_2-\beta q_2+\beta r_3=0,\\ 2\alpha p_3+\beta p_1-\beta r_3+\beta q_2=0,\\ \alpha r_3-\beta p_2-\beta q_1-\alpha r_2=0,\\ \beta p_3-\beta r_1+\alpha q_3-\alpha q_2=0,\\ \beta q_3+\alpha p_3+\alpha p_2-\beta r_2=0,\\ \beta q_2-2\alpha r_1-2\alpha q_1-\beta p_1+\beta r_3=0. \end{cases} \tag{3.3}$$

Since $\alpha\neq 0$, we set $\beta=0$. Therefore

$$\begin{cases} -2\alpha p_2=0,\quad 2\alpha p_3=0,\\ \alpha(r_3-r_2)=0,\quad \alpha(q_3-q_2)=0,\\ \alpha(p_3+p_2)=0,\quad 2\alpha(r_1+q_1)=0, \end{cases}$$

so, immediately, we get $p_2=p_3=0$, $q_1=-r_1$, $r_3=r_2$ and $q_2=q_3$, which show the case $\mathbf{a}(1)$ of the statement in the Theorem, if we set $p_1=A$, $r_1=-B$, $q_2=C$ and $r_3=D$. Now if $\beta\neq 0$, we get an algebraic system of 6 equations with 11 unknowns. By analyzing this system, we obtain the case $\mathbf{a}(2)$ of the statement just after setting $r_3=A$, $q_2=B$, $r_1=C$ and $r_3=D$. For the other possible cases of $\mathfrak{g}_3$, we will complete the proof after checking the validity of the Jacobi identity case by case. ∎

Remark 3.2. All Lie algebras of the above Theorem 3.1, except the cases $\mathbf{a}(2)$, $\mathbf{b}(3)$, $\mathbf{c}(1)$, $\mathbf{d}(3)$, are solvable examples.

4. Restricting the metric on $\mathfrak{g}_3$ is degenerate

Throughout this section, the restriction of the left-invariant metric g on $\mathfrak{g}_3$ is degenerate, so that the metric g is described by the neutral inner product as in the case (b) of the Lemma 2.3 unless the degenerate metric is stated. Since $\mathfrak{g}=\mathfrak{g}_3\rtimes\mathfrak{r}$, we take, generally, the Lie brackets $[e_i,e_4]$, $1\leq i\leq 4$ as in the Equation (3.1). First, let $\mathfrak{g}'=[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}_3$. With regard to the subalgebra $\mathfrak{g}_3$, by setting $\mathfrak{g}_3'=[\mathfrak{g}_3,\mathfrak{g}_3]$, the following possibilities are noteworthy.

1) $\mathfrak{g}_3'$ is trivial: In this case, $\mathfrak{g}_3 = \mathfrak{r}^3$ is abelian and so, the Lie algebra $\mathfrak{g}$ will be as Equation (3.1), which is equipped with the left-invariant metric

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4.4}$$

where p_i, q_i and r_i are arbitrary real constants. Clearly, the Jacobi identity is valid without setting any restriction on the above coefficients.

2) $\mathfrak{g}_3'$ is one dimensional: In this case, $\mathfrak{g}_3 = \mathfrak{h}$ is the three dimensional Heisenberg Lie algebra and we set $\mathfrak{g}_3' = Span\{X\}$. We can write $X = V + \lambda e_3$, where V is space-like (resp. time-like), $e_3 \perp V$ is light-like, and λ is a real constant. One of the following cases may occur.

(i) $V \neq 0$: We set $e_1 = X/\|X\|$, which is space-like and complete the basis of $\mathfrak{g}_3$ with a time-like unit vector e_2, such that the Lie algebra $\mathfrak{g}$ will be

$$[e_1, e_2] = \alpha e_1, \quad [e_1, e_3] = \beta e_1, \quad [e_2, e_3] = \mu e_1,$$

which is equipped with the left-invariant metric (4.4).

(ii) $V \neq 0$: We set $e_2 = X/\|X\|$, which is time-like and complete the basis of $\mathfrak{g}_3$ with a space-like unit vector e_1, such that the Lie algebra $\mathfrak{g}$ will be

$$[e_1, e_2] = \alpha e_2, \quad [e_1, e_3] = \beta e_2, \quad [e_2, e_3] = \mu e_2,$$

which is equipped with the left-invariant metric (4.4).

(iii) $V = 0$: We choose an orthogonal basis $\{e_1, e_2, e_3\}$ for $\mathfrak{g}_3$, such that the Lie algebra $\mathfrak{g}$ will be

$$[e_1, e_2] = \alpha e_3, \quad [e_1, e_3] = \beta e_3, \quad [e_2, e_3] = \mu e_3,$$

which is equipped with the left-invariant metric (4.4).

In the above Lie algebras, α, β and μ are arbitrary real constants.

3) $\mathfrak{g}_3'$ is two dimensional: In this case, $\mathfrak{g}_3 = \mathfrak{e}(1,1)$ or $\mathfrak{g}_3 = \mathfrak{e}(2)$. Suppose that $\mathfrak{g}_3 = Span\{X_1, X_2\}$, where $X_i = V_i + \lambda_i e_3$, V_1 is space-like, V_2 is time-like, e_3 is a light-like vector orthogonal to V_1, and V_2 and λ_i are arbitrary real constants. One of the following cases may occur:

(i) **V_1 and V_2 are linearly independent**: In this case there exist a space-like vector e_1 and a time-like vector e_2, such that $\mathfrak{g}_3' = Span\{e_1, e_2\}$ and the Lie algebra $\mathfrak{g}$ will be

$$[e_1, e_2] = a_1 e_1 + a_2 e_2, \quad [e_1, e_3] = b_1 e_1 + b_2 e_2, \quad [e_2, e_3] = c_1 e_1 + c_2 e_2,$$

which is equipped with the left-invariant metric (4.4).

(ii) V_1 **and** V_2 **are linearly dependent**: In this case, we choose $\{V_1, e_3\}$ as a basis for $\mathfrak{g}_3'$, and we set $e_1 = V_1/\|V_1\|$, which is space-like. We choose a time-like vector e_2 orthogonal to e_1 and e_3, such that the Lie algebra $\mathfrak{g}$ will be

$$[e_1, e_2] = a_1 e_1 + a_3 e_3, \quad [e_1, e_3] = b_1 e_1 + b_3 e_3, \quad [e_2, e_3] = c_1 e_1 + c_3 e_3,$$

which is equipped with the left-invariant metric (4.4).

(iii) V_1 **and** V_2 **are linearly dependent**: In this case, we choose $\{V_2, e_3\}$ as a basis for $\mathfrak{g}_3'$, and we set $e_2 = V_2/\|V_2\|$, which is time-like. We choose a space-like vector e_1 orthogonal to e_2 and e_3 such that the Lie algebra $\mathfrak{g}$ will be

$$[e_1, e_2] = a_1 e_2 + a_3 e_3, \quad [e_1, e_3] = b_1 e_2 + b_3 e_3, \quad [e_2, e_3] = c_1 e_2 + c_3 e_3,$$

which is equipped with the left-invariant metric (4.4).

In all the above Lie algebras, a_i, b_i and c_i are arbitrary real constants.

4) $\mathfrak{g}_3'$ is three dimensional: In this case, $\mathfrak{g}_3 = \mathfrak{sl}(2)$ or $\mathfrak{g}_3 = \mathfrak{su}(2)$; and suppose that the light-like vector $e_3 \in \mathfrak{g}_3$ is orthogonal to $\mathfrak{g}_3$ itself. We consider the map $\mathrm{ad}_{e_3} : \mathfrak{g}_3 \to \mathfrak{g}_3$, which is of rank 2, since $\mathfrak{g}_3' = \mathfrak{g}_3$. Besides 0, ad_{e_3} has either two real eigenvalues or two conjugate complex eigenvalues. In addition, if we write $e_3 = [X_1, X_2]$, we have

$$\mathrm{ad}_{e_3} = \mathrm{ad}_{X_1} \circ \mathrm{ad}_{X_2} - \mathrm{ad}_{X_2} \circ \mathrm{ad}_{X_1},$$

and so, $\mathrm{tr}(\mathrm{ad}_{e_3}) = 0$. Now one of the following cases may occur:

(i) **Eigenvalues of** ad_{e_3} **are** $0, \lambda \neq 0$ **and** $-\lambda$: We choose e_1 and e_2 as unit space-like and time-like vectors, respectively; and the Lie algebra $\mathfrak{g}$ will be

$$[e_1, e_3] = \lambda e_1, \quad [e_2, e_3] = -\lambda e_3, \quad [e_1, e_3] = e_3,$$

which is equipped with the left-invariant metric

$$\begin{pmatrix} 1 & k & 0 & 0 \\ k & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4.5}$$

where k is an arbitrary real constant.

(ii) **Eigenvalues of** ad_{e_3} **are** $0, i\beta \neq 0$ **and** $-i\beta$: We chose e_1 and e_2 as unit space-like and time-like vectors, respectively; and the Lie algebra $\mathfrak{g}$ will be

$$[e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = \beta e_1, \quad [e_1, e_2] = \beta e_3,$$

which is equipped with the left-invariant metric (4.5).

Now, let $\dim \mathfrak{g}' = 2$; in this case at least two linear independent vectors, namely, e_4 and v, act as derivations on $\mathfrak{g}$, so according to the Lemma 2.3, the vector v is time-like or null. In the first case, we are, in fact, in one of the non-degenerate cases, where $g|_{\mathfrak{g}_3}$ is Lorentzian. But in the latter case, clearly, the null vector v belongs to $\mathfrak{g}_3$. Thus, the Lie algebra $\mathfrak{g}$ takes the following form

$$\begin{aligned}
&[e_1, e_2] = a_1e_1 + a_2e_2, \\
&[e_1, e_3] = b_1e_1 + b_2e_2, \quad [e_1, e_4] = p1e_1 + p2e_2, \\
&[e_2, e_3] = c_1e_1 + c_2e_2, \quad [e_2, e_4] = q1e_1 + q2e_2, \quad [e_3, e_4] = r_1e_1 + r_2e_2,
\end{aligned}$$

with the left-invariant metric g of the Equation (4.4). Here it should be noted that this case is also studied in case 3-(i) above.

Finally, if $\dim \mathfrak{g}' = 1$, since at most, two linear independent null vectors exist, surly, one time-like vector acts as a derivation on $\mathfrak{g}$; so, again one of the cases where $g|_{\mathfrak{g}_3}$ is Lorentzian may happen.

It is evident that the Jacobi identity must be valid in all the above arguments. Thus we may check, case by case, the Jacobi identity and state the following theorem for the case when the $g|_{\mathfrak{g}_3}$ is degenerate.

Theorem 4.1. *Let $(G = G_3 \rtimes \mathbb{R}, g)$ be a four dimensional Lie group of neutral signature, where the restriction of g on $\mathfrak{g}_3$ is degenerate. In this situation, one of the following cases may happen:*

a) *There exists a basis $\{e_1, \ldots, e_4\}$ for $\mathfrak{g}$, such that $\mathfrak{g}_3 = \mathrm{Span}(e_1, e_2, e_3)$, $\mathfrak{r} = \mathrm{Span}(e_4)$ and the inner product g is specified by the matrix (4.4), where the non-zero Lie brackets are:*

a1) $\mathfrak{g}_3'$ is trivial: *In this case, $\mathfrak{g} = \mathfrak{r}^3 \rtimes \mathfrak{r}$ is isometric to the following Lie algebra:*

$$\begin{aligned}
&[e_1, e_4] = Ae_1 + Be_2 + Ce_3, \quad [e_2, e_4] = De_1 + Ee_2 + Fe_3, \\
&[e_3, e_4] = Ge_1 + He_2 + Ke_3.
\end{aligned}$$

a2) $\mathfrak{g}_3'$ is one dimensional: *In this case $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{r}$ is isometric to one of the following Lie algebras, where $\mathfrak{h}$ is the three dimensional Heisenberg Lie algebra:*

1-
$$\begin{aligned}
&[e_1, e_2] = Ae_1, \quad [e_1, e_3] = Be_1, \quad [e_2, e_3] = Ce_1, \\
&[e_1, e_4] = De_1, \quad [e_2, e_4] = Ee_1 - \tfrac{BF}{A}e_2 + Fe_3, \\
&[e_3, e_4] = \tfrac{(ACG-BCF+ABE-ACD)}{A^2}e_1 - \tfrac{BG}{A}e_2 + Ge_3, \qquad A \neq 0,
\end{aligned}$$

2-
$$\begin{aligned}
&[e_1, e_3] = Ae_1, \quad [e_2, e_3] = Be_1, \quad [e_1, e_4] = Ce_1, \\
&[e_2, e_4] = \tfrac{B(C-D)}{A}e_1 + De_2, \quad [e_3, e_4] = Ee_1 + Fe_2, \qquad A \neq 0,
\end{aligned}$$

3-
$$\begin{aligned}
&[e_2, e_3] = Ae_1, \quad [e_1, e_4] = (B + C)e_1, \\
&[e_2, e_4] = De_1 + Ce_2 + Ee_3, \quad [e_3, e_4] = Fe_1 + Ge_2 + Be_3, \quad A \neq 0,
\end{aligned}$$

4-
$$\begin{aligned}
&[e_1, e_2] = Ae_2, \quad [e_1, e_3] = Be_2, \quad [e_2, e_3] = Ce_2, \\
&[e_1, e_4] = \tfrac{CE}{A}e_1 + De_2 + Ee_3, \quad [e_2, e_4] = Fe_2, \\
&[e_3, e_4] = \tfrac{CG}{A}e_1 + \tfrac{ABF-ACD-BCE-ABG}{A^2}e_2 + Ge_3, \quad A \neq 0,
\end{aligned}$$

5- $[e_1, e_3] = Ae_2, \quad [e_1, e_4] = (B - C)e_1 + De_2 + Ee_3,$
$[e_2, e_4] = Be_2, \quad [e_3, e_4] = Fe_1 + Ge_2 + Ce_3, \quad A \neq 0,$

6- $[e_2, e_3] = Ae_2, \quad [e_1, e_4] = Be_1,$
$[e_2, e_4] = Ce_2, \quad [e_3, e_4] = De_1 + Ee_2, \quad A \neq 0,$

7- $[e_1, e_3] = Ae_2, \quad [e_2, e_3] = Be_2, \quad [e_1, e_4] = \frac{AC-BD}{A}e_1 + De_2,$
$[e_2, e_4] = Ce_2, \quad [e_3, e_4] = Ee_1 + Fe_2, \quad A \neq 0,$

8- $[e_1, e_2] = Ae_3, \quad [e_1, e_3] = Be_3, \quad AB \neq 0,$
$[e_2, e_3] = Ce_3, \quad [e_1, e_4] = -\frac{CD}{B}e_1 + De_2 + Ee_3,$
$[e_2, e_4] = -\frac{CF}{B}e_1 + Fe_2 + Ge_3, \quad [e_3, e_4] = \frac{A(BF-CD)+B(BG-CE)}{AB}e_3,$

9- $[e_1, e_2] = Ae_3, \quad [e_1, e_4] = Be_1 + Ce_2 + De_3,$
$[e_2, e_4] = E_1 + Fe_2 + Ge_3, \quad [e_3, e_4] = (B + F)e_3, \quad A \neq 0,$

10- $[e_1, e_2] = Ae_3, \quad [e_2, e_3] = Be_3, \quad [e_1, e_4] = Ce_1 + De_3,$
$[e_2, e_4] = Ee_1 + Fe_3, \quad [e_3, e_4] = \frac{AC-BD}{A}e_3, \quad A \neq 0,$

11- $[e_1, e_3] = Ae_3, \quad [e_1, e_4] = Ce_1 + \frac{AC}{B}e_2 + \frac{AD}{B}e_3,$
$[e_2, e_3] = Be_3, \quad [e_2, e_4] = Ee_1 - \frac{AE}{B}e_2 + De_3,$
$[e_3, e_4] = Fe_3, \quad B \neq 0,$

12- $[e_1, e_3] = Ae_3, \quad [e_1, e_4] = Be_2 + Ce_3,$
$[e_2, e_4] = De_2, \quad [e_3, e_4] = Ee_3, \quad A \neq 0.$

a3) $\mathfrak{g}_3'$ is two dimensional: *In this case,* $\mathfrak{g} = \mathfrak{e}(2) \rtimes \mathfrak{r}$ *or* $\mathfrak{g} = \mathfrak{e}(1,1) \rtimes \mathfrak{r}$ *and* $\mathfrak{g}$ *is isometric to one of the following Lie algebras:*

1- $[e_1, e_3] = Ae_1 + Be_2, \quad [e_1, e_4] = \frac{E(A-D)+CF}{C}e_1 + \frac{BE}{C}e_2,$
$[e_2, e_3] = Ce_1 + De_2, \quad [e_2, e_4] = Ee_1 + Fe_2,$
$[e_3, e_4] = Ge_1 + He_2, \quad C \neq 0, \; AD - BC \neq 0,$

2- $[e_1, e_3] = Ae_1, \quad [e_1, e_4] = Ce_1, \quad [e_2, e_3] = Be_2,$
$[e_2, e_4] = De_2, \quad [e_3, e_4] = Ee_1 + Fe_2, \quad AB \neq 0,$

3- $[e_1, e_3] = Ae_1, \quad [e_1, e_4] = Be_1 + Ce_2, \quad [e_2, e_3] = Ae_2,$
$[e_2, e_4] = De_1 + Ee_2, \quad [e_3, e_4] = Fe_1 + Ge_2, \quad A \neq 0,$

4- $[e_1,e_3]=Ae_1+Be_2,\quad [e_1,e_4]=\frac{BD+AE-CE}{B}e_1+Ee_2,\quad [e_2,e_3]=Ce_2,$
$[e_2,e_4]=De_2,\quad [e_3,e_4]=Fe_1+Ge_2,\quad B\neq 0,\ AC\neq 0,$

5- $[e_1,e_2]=Ae_1+Be_3,\quad [e_1,e_4]=\frac{CH-AG-DG}{C}e_1-\frac{BG}{C}e_3,$
$[e_2,e_3]=Ce_1+De_3,\quad [e_2,e_4]=Ee_1+Fe_3,$
$[e_3,e_4]=Ge_1+He_3,\quad C\neq 0,\ AD-BC\neq 0,$

6- $[e_1,e_2]=Ae_1,\quad [e_2,e_3]=Be_3,\quad [e_1,e_4]=Ce_1,$
$[e_2,e_4]=De_1+Ee_3,\quad [e_3,e_4]=Fe_3,\quad AB\neq 0,$

7- $[e_1,e_2]=Ae_1,\quad [e_2,e_3]=-Ae_3,\quad [e_1,e_4]=Be_1+Ce_3,$
$[e_2,e_4]=De_1+Ee_3,\quad [e_3,e_4]=Fe_1+Ge_3,\quad A\neq 0,$

8- $[e_1,e_2]=Ae_1+Be_3,\quad [e_2,e_3]=Ce_3,\quad [e_1,e_4]=\frac{CD+AD+BE}{B}e_1+De_3,$
$[e_2,e_4]=Fe_1+Ge_3,\quad [e_3,e_4]=Ee_3,\quad B\neq 0,\ AC\neq 0,$

9- $[e_1,e_2]=Ae_2,\quad [e_1,e_3]=Be_3,\quad [e_1,e_4]=Ce_2+De_3,$
$[e_2,e_4]=Ee_2,\quad [e_3,e_4]=Fe_3,\quad AB\neq 0,$

10- $[e_1,e_2]=Ae_2,\quad [e_1,e_3]=Ae_3,\quad [e_1,e_4]=Be_2+Ce_3,$
$[e_2,e_4]=De_2+Ee_3,\quad [e_3,e_4]=Fe_2+Ge_3,\quad A\neq 0,$

11- $[e_1,e_2]=Ae_2+Be_3,\quad [e_1,e_3]=Ce_3,\quad [e_1,e_4]=Ce_2+Ee_3,$
$[e_2,e_4]=\frac{AF+BG-CF}{B}e_2+Fe_3,\quad [e_3,e_4]=Ge_3,\quad B\neq 0,\ AC\neq 0,$

12- $[e_1,e_2]=Ae_2+Be_3,\quad [e_1,e_3]=Ce_2+De_3,$
$[e_1,e_4]=Ee_2+Fe_3,\quad [e_2,e_4]=\frac{AG+CH-DG}{C}e_2+\frac{BG}{C}e_3,$
$[e_3,e_4]=Ge_2+He_3,\quad C\neq 0,\ AD-BC\neq 0.$

b) *There exist a basis* $\{e_1,\dots,e_4\}$ *for* $\mathfrak{g}$*, such that* $\mathfrak{g}_3=\mathrm{Span}(e_1,e_2,e_3)$ *and* $\mathfrak{r}=\mathrm{Span}(e_4)$*; also, the inner product* g *is specified by the Equation* (4.5) *and* $\mathfrak{g}$ *is isometric to one of the following Lie algebras:*

1- $[e_1,e_2]=-e_3,\quad [e_1,e_3]=-Ae_1,\quad [e_2,e_3]=Ae_2,\quad A\neq 0,$
$[e_1,e_4]=-Be_1+Ce_3,\quad [e_2,e_4]=Be_2+De_3,\quad [e_3,e_4]=ADe_1+ACe_2,$

2- $[e_1,e_2]=Ae_3,\quad [e_1,e_3]=-Ae_2,\quad [e_2,e_3]=Ae_1,\quad A\neq 0,$
$[e_1,e_4]=-Be_2-Ce_3,\quad [e_2,e_4]=Be_1-De_3,\quad [e_3,e_4]=Ce_1+De_2.$

Proof. The validity of the Jacobi identity can be checked in all cases. We give the details only for the case **(2-i)**, since the other cases are treated in a similar way. In this case, the Jacobi identity is established if and only if the following equations are valid

$$\begin{cases} \mu p_2 = \mu p_3 = 0, \quad \alpha p_2 = \alpha p_3 = 0, \quad \beta p_2 = \beta p_3 = 0, \\ \mu p_2 + \beta r_3 + \alpha r_2 = 0, \\ \mu p_3 - \beta q_3 - \alpha q_2 = 0, \\ \mu r_3 + \mu q_2 - \alpha r_1 + \beta q_1 - \mu p_1 = 0. \end{cases}$$

Since $\alpha\beta\mu \neq 0$, from the first equation we immediately get $p_2 = p_3 = 0$. If $\alpha \neq 0$, then, from the second and third equations, we have $q_2 = -\frac{\beta q_3}{\alpha}$ and $r_2 = -\frac{\beta r_3}{\alpha}$. Now, the last equation gives

$$r_1 = \frac{\alpha(\mu r_3 + \beta q_1 - \mu p_1) - \beta\mu q_3}{\alpha^2},$$

which shows the case **a2**(1) of the statement, after setting $\alpha = A,\ \beta = B,\ \mu = C,\ p_1 = D,\ q_1 = E,\ q_3 = F$ and $r_3 = G$. If $\alpha = 0$, we may also suppose $\beta = 0$; since $\mu \neq 0$, the last equation gives $p_1 = r_3 + q_2$, which shows the case **a2**(3) of the statement. Finally, if $\beta \neq 0$, from the second and third equations, we have $q_3 = r_3 = 0$; then the last equation gives $q_1 = \frac{\mu(p_1 - q_2)}{\beta}$, which proves the case **a2**(2) of the statement, and this ends the proof. ■

Remark 4.2. All Lie algebras of the above Theorem 4.1, except the two cases **b**(1) and **b**(2), are solvable.

5. Ricci solitons

In this section, using the complete classification of four dimensional neutral Lie groups given in Theorems 3.1 and 4.1 we study Ricci solitons. A pseudo-Riemannian manifold (M, g) is called a *Ricci soliton* if it admits a smooth vector field X, such that

$$\mathcal{L}_X g + \varrho = \lambda g, \tag{5.6}$$

where $\mathcal{L}_X$ and ϱ, denote the Lie derivative in the direction of X and the Ricci tensor, respectively, and λ is an arbitrary real number. Depending on the value of λ, a Ricci soliton is called *shrinking, steady*, or *expanding*, according to whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

Ricci solitons play an important role in understanding the singularities of the Ricci flow, of which they are the self-similar solutions. A survey and further references on the geometry of Ricci solitons may be found in [11].

Ricci solitons on homogeneous manifolds have been the subject of several studies. Ricci solitons on four dimensional pseudo-Riemannian homogeneous manifolds with non-trivial isotropy have been studied in [6]; especially, Ricci solitons on non-reductive cases have been considered in [5]. Here we complete the analysis of Ricci solitons on homogeneous manifolds by classifying the solitons on pseudo-Riemannian Lie groups.

Let us now consider, for example, the Lie algebra in the case (a-1) of Theorem 3.1; we apply the well known *Koszul formula* and describe the Levi-Civita connection ∇ of g. With respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$, with e_3 and e_4 time-like, we set $\nabla_{e_i} = \Lambda_i$ for all indices $i = 1, \cdots, 4$. Then, we get

$$\Lambda_1 = \begin{pmatrix} 0 & \alpha & -\alpha & A \\ -\alpha & 0 & 0 & \frac{B}{2} \\ -\alpha & 0 & 0 & \frac{B}{2} \\ A & \frac{B}{2} & -\frac{B}{2} & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 & 0 & \frac{B}{2} \\ 0 & 0 & \alpha & C \\ 0 & \alpha & 0 & -\frac{D+C}{2} \\ \frac{B}{2} & C & \frac{D-C}{2} & 0 \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} 0 & 0 & 0 & -\frac{B}{2} \\ 0 & 0 & -\alpha & -\frac{D-C}{2} \\ 0 & -\alpha & 0 & D \\ -\frac{B}{2} & -\frac{C+D}{2} & -D & 0 \end{pmatrix}, \quad \Lambda_4 = \begin{pmatrix} 0 & -\frac{B}{2} & \frac{B}{2} & 0 \\ -\frac{B}{2} & 0 & -\frac{D-C}{2} & 0 \\ \frac{B}{2} & -\frac{C+D}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The curvature tensor is calculated by $R(e_i, e_j) = \Lambda_i\Lambda_j - \Lambda_j\Lambda_i - \Lambda_{[e_i,e_j]}$, for all indices $1 \leq i, j \leq 4$. Then, by contraction over the first and third indices of the curvature tensor, we have the components of the Ricci tensor with respect to the basis $\{e_1, e_2, e_3, e_4\}$:

$$\begin{aligned} \varrho_{11} &= A(C+D+A), \quad \varrho_{12} = -\varrho_{13} = AB, \\ \varrho_{22} &= -2\alpha^2 + C(A+D) + \tfrac{B^2+C^2+D^2}{2}, \\ \varrho_{23} &= 2\alpha^2 + \tfrac{A(C-D)-B^2}{2}, \\ \varrho_{24} &= -\varrho_{34} = \alpha(\tfrac{C+D}{2} - A), \\ \varrho_{33} &= -2\alpha^2 - D(A+C) - \tfrac{C^2+D^2-B^2}{2}, \\ \varrho_{44} &= -(A^2 + \tfrac{C+D}{2} + CD). \end{aligned}$$

Let $X = X_1e_1 + X_2e_2 + X_3e_3 + X_4e_4$, for arbitrary real coefficients $X_1, \ldots, X_4$, be an arbitrary vector filed on G. By a direct calculation, one can easily obtain the Lie derivative of g; i.e., $\mathcal{L}_X g$ is

$$\begin{pmatrix} 2\alpha(X_2-X_3)+2X_4A & -X_1\alpha+X_4B & X_1\alpha-X_4B & -X_1A+B(X_3-X_2) \\ -X_1\alpha+X_4B & 2(X_3\alpha+X_4C) & -\alpha(X_2+X_3)+X_4(D-C) & -X_2C-X_3D \\ X_1\alpha-X_4B & -\alpha(X_2+X_3)+X_4(D-C) & 2(X_2\alpha-X_4D) & X_2C+X_3D \\ -X_1A+B(X_3-X_2) & -X_2C-X_3D & X_2C+X_3D & 0 \end{pmatrix}.$$

Now, Equation (5.6) gives the system of equations presented in case (a-1) of the following Theorem 5.1. We will have exactly four sets of solutions; for example, one non-trivial solution is

$$\begin{aligned} &A = \alpha = -D, \ B = 0, \ C + 3D \neq 0, \ \lambda = \tfrac{3D^2+C^2}{2} + CD, \\ &X_1 = 0, \ X_2 = -\tfrac{9D^2+6CD+C^2}{4C}, \ X_3 = \tfrac{3D^2+4CD+C^2}{4D}, \ X_4 = \tfrac{9D^2+6CD+C^2}{4C}. \end{aligned}$$

As we know, Einstein manifolds (where $\varrho = \eta g$ for a real constant η) are trivially Ricci soliton. In the above case (a-1), the Einstein condition is satisfied if and only if $C = -3D$, so we exclude this trivial Ricci soliton.

By the same method, we can find all Ricci solitons for all cases in Theorems 3.1 and 4.1. The results are summarized in the Theorems 5.1 and 5.4.

Theorem 5.1. *Let $(G = G_3 \rtimes \mathbb{R}, g)$ be one of the four dimensional Lie groups of neutral signature, which are listed in the Theorem 3.1. Then (G, g) is a Ricci soliton if and only if it is one of the following classes.*

a-1) *and the following equations are satisfied*

$$\begin{cases} 2\alpha(X_2 - X_3) + 2AX_4 + A(C + D) + A^2 - \lambda = 0, \\ AX_1 + B(X_2 - X_3) = 0, \quad \alpha X_1 - B(X_4 + A) = 0, \\ 2\alpha X_3 + 2CX_4 - 2\alpha^2 + AC + CD + \frac{B^2+C^2+D^2}{2} - \lambda = 0, \\ \alpha(X_2 + X_3) + X_4(C - D) - 2\alpha^2 + \frac{A(C-D)+B^2}{2} = 0 \\ CX_2 + DX_3 - \frac{\alpha(C+D)}{2} + \alpha A = 0, \quad A^2 + \frac{D^2+C^2}{2} + CD - \lambda = 0, \\ 2\alpha X_2 - 2DX_4 - 2\alpha^2 - D(A + C) - \frac{C^2+D^2-B^2}{2} + \lambda = 0. \end{cases}$$

In particular, one non-trivial solution is

$$A = \alpha = -D,\ B = 0,\ C + 3D \neq 0,\ \lambda = \frac{3D^2+C^2}{2} + CD,$$
$$X_1 = 0,\ X_2 = -\frac{9D^2+6CD+C^2}{4C},\ X_3 = \frac{3D^2+4CD+C^2}{4D},\ X_4 = \frac{9D^2+6CD+C^2}{4C}.$$

a-2) *and*

$$A = -B,\ \beta = 0,\ C \neq \pm 2\alpha,\ \lambda = 0,$$
$$X_1 = -\frac{CX_4}{\alpha},\ X_2 = -\frac{4BX_4-4\alpha^2+C^2}{4\alpha},\ X_3 = -\frac{4BX_4-4\alpha^2+C^2}{4\alpha}.$$

c-2) *and*

$$B = C = 0,\ \epsilon A = \alpha = \frac{\gamma}{3} = \varepsilon\frac{D}{4},\ D \neq 0,\ \lambda = -\frac{3D^2}{8},$$
$$X_1 = -\varepsilon X_4 = \frac{3\epsilon D}{2},\ X_2 = X_3 = 0,\ \epsilon^2 = \varepsilon^2 = 1.$$

c-3) *and*

$$A = C = 0,\ \epsilon D = \beta = \frac{\gamma}{3} = \varepsilon\frac{B}{4},\ D \neq 0,\ \lambda = -6D^2,$$
$$X_1 = X_3 = 0,\ \varepsilon X_2 = X_4 = -6D,\ \epsilon^2 = \varepsilon^2 = 1.$$

c-4) *and one of the following cases occurs*

1-

$$B = C = 0,\ \epsilon A = \alpha = \frac{\beta}{3} = \varepsilon\frac{D}{4},\ D \neq 0,\ \lambda = -\frac{3}{8}D^2,$$
$$X_1 = -\varepsilon X_4 = \frac{3\epsilon D}{2},\ X_2 = X_3 = 0, \epsilon^2 = \varepsilon^2 = 1,$$

2-

$$B = 0,\ \epsilon A = -\epsilon\alpha = \epsilon\beta = \frac{D}{\delta\sqrt{2}},\ C = \varepsilon D,\ D \neq 0,\ \lambda = 0,$$
$$X_1 = X_2 = 0,\ \varepsilon\epsilon X_3 = -\frac{X_4}{3} = \frac{D}{2\delta\sqrt{2}},\ \epsilon^2 = \varepsilon^2 = \delta^2 = 1,$$

3-

$$B = D = 0,\ \epsilon A = \beta = \varepsilon\frac{C}{4} = \frac{\alpha}{3},\ \alpha \neq 0,\ \lambda = -\frac{2}{3}\alpha^2,$$
$$X_1 = X_3 = 0,\ \varepsilon X_2 = X_4 = -2\epsilon\alpha,\ \epsilon^2 = \varepsilon^2 = 1.$$

c-8) *and the following set of equations are satisfied*

$$\begin{cases} AX_1 + DX_2 + GX_3 = 0,\ BX_1 + EX_2 + HX_3 = 0, \\ K(2X_4 + A) + \frac{C^2-G^2+F^2-H^2}{2} + EK + K^2 - \lambda = 0, \\ X_4(C - G) + \frac{EC-EG-BH-DF}{2} - AG + KC = 0, \\ X_4(F - H) + \frac{AF-AH-BC-DG}{2} - EH + KF = 0, \\ A(2X_4 + E) + \frac{B^2-D^2-C^2+G^2}{2} + AK + A^2 - \lambda = 0, \\ E(2X_4 + E) - \frac{B^2-D^2+F^2-H^2}{2} + EK + E^2 - \lambda = 0, \\ X_4(B + D) - \frac{CF-GH-KB-KD}{2} + AD + BE = 0, \quad CX_1 + FX_2 + KX_3 = 0, \\ \frac{D^2+B^2-G^2-C^2-H^2-F^2}{2} + A^2 + BD + CG + E^2 + FH + K^2 - \lambda = 0. \end{cases}$$

In particular, one solution is

$$A = \frac{G}{2} = -K,\ B = C = D = E = F = H = 0,\ K \neq 0,\ \lambda = 0,$$
$$X_1 = X_2 = X_3 = 0,\ X_4 = K.$$

d-1) *and one of the following cases occurs*

1-

$$A = -\varepsilon B,\ D = -C,\ \beta = \varepsilon\frac{B^2+2C^2-2}{2(C-1)(C+1)},\ B \neq 0,\ \lambda = 0,$$
$$X_1 = \frac{B(-2\varepsilon X_2 C+B)}{4(C-1)(C+1)},\ X_2 = -\varepsilon X_3,\ X_4 = -\frac{B(BC-2\varepsilon X_2)}{4(C-1)(C+1)}.$$

2-

$$A = B = 0,\ D = -C,\ (C \pm 1)(\beta - \varepsilon) \neq 0,\ \lambda = 0,$$
$$X_1 = -X_4 C + \varepsilon\beta - 1 + C^2 - \varepsilon\beta C^2,\ X_2 = X_3 = 0.$$

d-2) *and*

$$A = B = C = 0,\ E = -F,\ D \neq 0,\ \lambda = 0,$$
$$X_1 = -\frac{D^2}{4},\ X_2 = -\varepsilon X_3,\ X_4 = 0.$$

d-4) *and*

$$B = -A,\ C = \alpha = 0,\ D \neq 0,\ \lambda = 0,$$
$$X_1 = -\frac{D^2}{4},\ X_2 = -\varepsilon X_3,\ X_4 = 0.$$

e-2) *and one of the following cases occurs*

1-

$$A = \frac{D}{\varepsilon\sqrt{2}},\ B = C = \alpha = 0,\ \delta = \epsilon\frac{D}{\varepsilon\sqrt{2}},\ D \neq 0,\ \lambda = 0,$$
$$X_1 = X_2 = 0,\ \epsilon X_3 = 2X_4 = -\frac{D}{\varepsilon\sqrt{2}},\ \varepsilon^2 = \epsilon^2 = 1.$$

2-

$$B = \frac{C}{\varepsilon\sqrt{2}},\ A = D = \delta = 0,\ \alpha = \epsilon\frac{C}{\varepsilon\sqrt{2}},\ C \neq 0,\ \lambda = 0,$$
$$X_1 = X_2 = 0,\ \epsilon X_3 = 2X_4 = -\frac{C}{\varepsilon\sqrt{2}},\ \varepsilon^2 = \epsilon^2 = 1.$$

3-

$$C = \varepsilon\sqrt{2(A^2+B^2)},\ D = 0,\ \alpha = -\epsilon B,\ \delta = \epsilon A,\ \lambda = 0,$$
$$X_1 = X_2 = 0,\ X_3 = \epsilon(B - \frac{A}{2}),\ X_4 = -\frac{B}{2} - A,\ \varepsilon^2 = \epsilon^2 = 1.$$

4-

$$D = \varepsilon\sqrt{2(A^2+B^2)},\ C = 0,\ \alpha = -\epsilon B, \delta = \epsilon A,\ \lambda = 0,$$
$$X_1 = X_2 = 0,\ X_3 = \epsilon(\frac{B}{2} - A),\ X_4 = -\frac{A}{2} - B,\ \varepsilon^2 + \epsilon^2 = 1.$$

e-3) *and the following set of equations are satisfied*

$$\begin{cases} \alpha(A+D) = 0,\ \frac{E^2+F^2}{2} - 2\alpha^2 + \lambda = 0,\ AX_1 + CX_2 + EX_3 + \frac{3\alpha}{2}E = 0, \\ BX_1 + DX_2 + FX_3 + \frac{3\alpha}{2}F = 0,\ X_4(C+B) + \frac{FE}{2} + CA + BD = 0, \\ \alpha X_1 - EX_4 - \frac{ED+BF}{2} - AE = 0,\ \alpha X_2 - FX_4 - \frac{AF+CE}{2} - FD = 0, \\ \frac{C^2+B^2-E^2-F^2}{2} + A^2 + BC + D^2 - \lambda = 0, \\ 2\alpha X_3 + 2AX_4 + 2\alpha^2 + AD + \frac{B^2-C^2+E^2}{2} + A^2 - \lambda = 0, \\ 2\alpha X_3 + 2DX_4 + 2\alpha^2 + AD - \frac{B^2-C^2-F^2}{2} + D^2 - \lambda = 0. \end{cases}$$

In particular, one solution is

$$\begin{array}{l} A=D=F=0,\ B=-\alpha=-\varepsilon\frac{E}{4},\ C=\varepsilon\frac{3E}{4},\ E\neq 0,\ \lambda=-\frac{3E^2}{8},\\ X_1=X_4=0,\ X_2=-\varepsilon X_3=\frac{3E}{2},\ \varepsilon^2=1. \end{array}$$

e-4) *and one of the following cases occurs*

1-

$$\begin{array}{l} A=D=\beta=0,\ B=-\epsilon\alpha,\ C=\varepsilon\sqrt{2}\alpha,\ \alpha\neq 0,\ \lambda=0,\\ X_1=X_2=0,\ X_3=-2\epsilon X_4=-\alpha,\ \varepsilon^2+\epsilon^2=1. \end{array}$$

2-

$$\begin{array}{l} A=C=\beta=0,\ B=\epsilon\alpha,\ D=\varepsilon 2\alpha,\ \lambda=0,\\ X_1=X_2=0,\ 2X_3=\epsilon X_4=-\alpha,\ \varepsilon^2+\epsilon^2=1. \end{array}$$

f-1) *and the following set of equations are satisfied*

$$\left\{\begin{array}{l} (2DX_2+(2X_1+2\alpha+\delta)A-\beta B+2CX_3)\gamma+2CX_2(\alpha-\delta)=0,\\ (2BX_4+3BD+2\beta X_2+2\delta X_3+AC)\gamma+BC(\alpha-\delta)=0,\\ C\gamma^2-(2\beta C+2D(\alpha+\delta))\gamma+C(2\alpha\delta+\beta^2-2\alpha^2)=0,\\ A\gamma^2-(2BX_1+2B\delta+B\alpha+2DX_3)\gamma-2C\beta X_2=0,\\ 2X_3\gamma^2+(3AD+2AX_4+2X_2\alpha)\gamma+(2A\alpha-2A\delta-\beta B)C=0,\\ (2X_1+2\alpha)\gamma^2-(2CX_4+2CD+2\beta\delta+2\beta X_1+AB)\gamma-2((\alpha-\delta)C-\beta(D+X_4))C=0,\\ (A^2-B^2-C^2+4D^2-2\lambda)\gamma^2+2(\beta C+2D(\alpha-\delta))C\gamma-C^2(4\alpha\delta-2\delta^2+\beta^2-2\alpha^2)=0,\\ \gamma^4-(C^2-2\alpha^2+(-2\delta-4X_1)\alpha-A^2+4D^2+\beta^2-2\lambda+4X_4D)\gamma^2\\ \qquad -6C(\alpha-\delta)(\frac{2X_4}{3}+D)\gamma+C^2(4\alpha\delta-2\delta^2+\beta^2-2\alpha^2)=0,\\ \gamma^4-(2\delta^2+(2\alpha+4X_1)\delta+C^2-4X_4D-B^2+\beta^2+2\lambda-4D^2)\gamma^2+2DC(\alpha-\delta)\gamma+C^2\beta^2=0,\\ \frac{\gamma^2+\beta^2+A^2-B^2}{2}-\alpha^2-\gamma\beta-\delta^2-\lambda=0. \end{array}\right.$$

In particular, one solution is

$$\begin{array}{l} A=C=2\delta=\varepsilon\sqrt{3}\gamma,\ B=D=\gamma,\ \alpha=\beta=0,\ \lambda=\frac{3\gamma^2}{4},\\ X_1=-X_3=-\frac{3\varepsilon\sqrt{3}}{2}\gamma,\ X_2=-X_4=\frac{3}{2}\gamma. \end{array}$$

f-3) *and the following set of equations are satisfied*

$$\left\{\begin{array}{l} \alpha(C+F)=0,\ \frac{A^2-B^2}{2}-2\alpha^2-\lambda=0,\ AX_1+CX_2+EX_3+\frac{3}{2}A\alpha=0,\\ BX_1+DX_2+FX_3+\frac{3}{2}B\alpha=0,\ X_4(E-D)+\frac{1}{2}AB+EC-FD=0,\\ \alpha X_2+AX_4-\frac{BD+AF}{2}+AC=0,\\ \alpha X_3+BX_4+\frac{BC-AE}{2}+BF=0,\quad \frac{A^2-B^2-E^2-D^2}{2}+C^2+DE+F^2-\lambda=0,\\ 2\alpha X_1-2CX_4+\frac{A^2+D^2-E^2}{2}+2\alpha^2-CF-C^2+\lambda=0,\\ 2\alpha X_1-2FX_4-\frac{B^2+D^2-E^2}{2}+2\alpha^2-CF-F^2+\lambda=0. \end{array}\right.$$

In particular, one solution is

$$\begin{array}{l} A=\frac{4}{3}E,\ B=C=F=0,\ D=\alpha=\frac{1}{3}E,\ \lambda=\frac{2}{3}E^2,\\ X_1=-X_3=-2E,\ X_2=X_4=0. \end{array}$$

g-2) *and the following set of equations are satisfied*

$$\begin{cases} (2\alpha(X_2-X_3)-(2X_4+B+C)(A-C-B))\beta+2\alpha(-X_1\alpha+D(X_4+A))=0,\\ 2\alpha^4-(4\beta X_1+B^2+(2A+4X_4+2C)B+C^2+D^2-2\lambda)\alpha^2+\beta^2(A-C-B)^2=0,\\ 2\alpha^4-(4\beta X_1-C^2-(2A+4X_4+2B)C-B^2+D^2+2\lambda)\alpha^2+\beta^2(A-C-B)^2=0,\\ 2\alpha^4-(4\beta X_1+(B-C)A+D^2+2BX_4-2CX_4)\alpha^2+\beta^2(A-C-B)^2=0,\\ (B+C-2A)\alpha^2+(2BX_2+2CX_3)\alpha+2\beta X_1(A-C-B)=0,\\ (X_2-X_3)D-AX_1=0,\ A^2+(2X_4+B+C)A-2(X_2-X_3)\alpha-\lambda=0,\\ A^2+\frac{B^2+C^2}{2}+CB-\lambda=0. \end{cases}$$

In particular, one solution is

$$A=\beta=0,\ C=-B,\ D\neq\varepsilon\sqrt{2}\alpha,\ \lambda=0,$$
$$X_1=\frac{D(2\alpha^2-D^2)}{4B\alpha},\ X_2=X_3,\ X_4=\frac{2\alpha^2-D^2}{4B}.$$

Remark 5.2. All examples of Ricci solitons in the Theorem 5.1 are only shrinking, expanding or steady. This means, in each example, we just have $\lambda>0$, $\lambda<0$ or $\lambda=0$.

Remark 5.3. On three dimensional Riemannian Lie groups (Lie groups equipped with a left-invariant Riemannian metric), there are no non-trivial (invariant) Ricci solitons. This is true for any finite dimensional unimodular Riemannian Lie group [14]. With regard to the non-unimodular Riemannian Lie groups, similar results were proved in [15], for the dimension four. Based on results in the Theorem 5.1, non-trivial Ricci solitons exist on both unimodular and non-unimodular Lorentzian Lie groups in the dimension four. Thus the results in the pseudo-Riemannian are different from those in the Riemannian setting.

Theorem 5.4. *Let $(G=G_3\rtimes\mathbb{R},g)$ be one of the four dimensional Lie groups of neutral signature listed in Theorem 4.1. Then (G,g) is a Ricci soliton if and only if it is one of the following classes.*

a1) *and one of the following cases occurs*

1-

$$B=D,\ E=A,\ G=H=0,\ K=2A,$$
$$X_1=\frac{\lambda(AC+DF)}{2A(A-D)(A+D)},\ X_2=-\frac{\lambda(AF+DC)}{2A(A-D)(A+D)},\ X_3=\frac{2A^4-2A^2D^2+\lambda(F^2-C^2)}{4A(A+D)(A-D)},\ X_4=\frac{\lambda}{2A}.$$

2-

$$G=H=0,\ 2K(A+E)\neq2(A^2+E^2)-(B-D)^2,\ \lambda=0,$$
$$X_1=X_2=X_4=0,\ X_3=-\frac{2(A^2+E^2)-2K(E+A)-(B-D)^2}{4K}.$$

a2-1) *and*

$$B=C=E=0,\ D=\frac{AF}{G},\ \lambda=A^2,$$
$$X_1=0,\ X_2=-\frac{A^2F}{G^2},\ X_3=\frac{F^2A^2}{2G^3},\ X_4=\frac{A^2}{G}.$$

a2-3) *and*

$$A = F = G = \lambda = 0,\ D \neq 2\epsilon C,\ \epsilon^2 = 1,$$
$$X_1 = X_2 = X_4 = 0,\ X_3 = -\tfrac{4C^2-D^2}{4B}.$$

a2-4) *and*

$$B = C = D = E = F = 0,\ \lambda = -A^2,$$
$$X_1 = X_2 = X_3 = 0,\ X_4 = -\tfrac{A^2}{G}.$$

a2-8) *and the following equations are satisfied*

$$\begin{cases} 4CDX_4 + B^3 + 2\lambda B = 0, \\ 2(CF + DB)X_4 + B^2C = 0, \\ AB^2 + ((-2X_3 + F)\,C + (2A - 2D)\,X_1 - 2X_2F - 2X_4G)\,B - C^2D = 0, \\ (2X_3 - F)\,B^2 - ((A - D)\,C - 2AX_2 - 2EX_4)\,B + 2C\,(DX_1 + FX_2) = 0, \\ B^3A + (2AX_1 - 2GX_4)\,B^2 + \left(\left(-2FX_4 + 2\lambda + 2CX_2 - C^2\right)A + 2CEX_4\right)B + 2ACDX_4 = 0, \\ 2G\,(2X_3 - F)\,B^3 + \left(A^3 + \left(4X_3F + 4X_1E + 4X_2G - D^2\right)A + 2C\,(GD - E\,(2X_3 - F))\right)B^2 \\ \qquad -2CD\,(EC + A\,(2X_3 - F))\,B - AC^2F^2 = 0, \\ 2X_4F + \frac{C^2}{2} - \lambda = 0. \end{cases}$$

In particular, one non-trivial solution is

$$B = C,\ D = F,\ G = \tfrac{EB-A^2}{B},\ A^2 + B^2 \neq 0,\ \lambda = 0,$$
$$X_1 = \tfrac{AB-4DX_2}{4D},\ X_3 = \tfrac{ABD-4ADX_2+B^2E}{4BD},\ X_4 = -\tfrac{B^2}{4D}.$$

a2-9) *and the following equations are satisfied*

$$\begin{cases} (E-C)X_4 = 0,\ 2BX_4 - \lambda = 0,\ 2FX_4 - \lambda = 0,\ (B+F)X_4 - \lambda = 0, \\ (C - A)X_1 + FX_2 + GX_4 = 0, \quad X_1B + (E - A)X_2E - DX_4 = 0, \\ 2DX_1 + 2GX_2 + 2(B + F)X_3 - 2BF + \frac{A^2-E^2-C^2}{2} + EC = 0. \end{cases}$$

In particular, one non-trivial solution is

$$C = A = E \neq 0,\ B = F = 0,\ \lambda = 0,$$
$$X_1 = -\tfrac{4GX_2+E^2}{4D},\ X_4 = 0.$$

a2-10) *and*

$$B = 0,\ \lambda = 0,$$
$$X_1 = X_2 = X_4 = 0,\ X_3 = \tfrac{E^2-A^2}{4C}.$$

a2-11) *and the following equations are satisfied*

$$\begin{cases} \left(2CX_4 - B^2\right)A + 2EBX_4 = 0, \quad 4AEX_4 - B^3 + 2\lambda B = 0, \\ (C - 2X_3)\,B^2 - (2DX_4 + AE)\,B + 2A\,(CX_1 + EX_2) = 0, \\ A^2E + ((-C + 2X_3)\,B + 2DX_4)\,A - 2B\,(CX_1 + EX_2) = 0, \\ \left(E^2 - 4DX_2 - 4FX_3 + 2CF - 2C^2\right)B^2 \\ +2A\,(-2DX_1 + EC - EF)\,B + \left(C^2 - 2E^2\right)A^2 = 0, \\ 2CX_4 - \frac{A^2}{2} - \lambda = 0, \quad AX_1 + BX_2 - FX_4 + \frac{A^2-B^2}{2} + \lambda = 0. \end{cases}$$

In particular, one non-trivial solution is

$$A = -B,\ E = -C,\ F = 0,\ B \neq 0,\ \lambda = 0,$$
$$X_3 = -\tfrac{BD}{4C},\ X_4 = \tfrac{B^2}{4C}.$$

a3-1) *and the following equations are satisfied*

$$\begin{cases} (EG+(2X_3-3F)H+2FX_2)C-((A-D)H-2BX_1)E=0, \\ \left((C-B)^2-2(A-D)^2\right)E^2-4CF(A-D)E-4C^2F^2=0, \\ (2(A-D)G-2CX_2-2(A-D)X_1-HB)E-3C((2X_3-3F)G+2FX_1)=0, \\ EC^2-\left(2AF-G^2+2DF+2BE+H^2+2\lambda\right)C+\left(B^2-2A^2+2AD\right)E=0, \\ 2(F-X_3)C^2+(2(2A-X_4-D)E-2(F-X_3)B-GH)C+2EB(X_4-D)=0, \\ 2EC^2+\left((4X_3-6F)D-(2A-4X_4)F+H^2-2\lambda\right)C+2E\left(D^2-AD-B^2\right)=0, \\ 2EC^2+\left((4X_3+6F)A+2\lambda-4X_4F+2DF+G^2\right)C+2E\left(2A^2-(D+2X_4)A-B^2+2DX_4-D^2\right)=0, \\ (-2X_4+A+2D)H+2BX_1+2DX_2-CG=0, \quad (2X_4-D-2A)G-2AX_1-2CX_2+HB=0, \\ C^2+B^2-2A^2-2BC-2D^2=0. \end{cases}$$

In particular, a non-trivial solution is

$$\begin{array}{l} A=\varepsilon\sqrt{2}C,\ B=-C,\ D=F=G=H=0,\ E\neq 0,\ \lambda=0, \\ X_1=X_2=0,\ X_3=\frac{E(\varepsilon\sqrt{2}C-X_4)}{C}, \quad \varepsilon^2=1. \end{array}$$

a3-4) *and one of the following equations are satisfied*

$$\begin{cases} ((2X_3-3D)B-E(A-C))G+2B(EX_1+DX_2)=0, \\ \left(B^2-2(A-C)^2\right)E^2-4BD(A-C)E-4B^2D^2=0, \\ EB^2+\left(F^2-2AD-G^2-2DC-2a\right)B-2AE(A-C)=0, \\ (GB+2(X_1-F)(A-C))E+B((2X_3-3D)F+2DX_1)=0, \\ 2EB^2+\left((6D-4X_3)C+(2A-4X_4)D+2a-G^2\right)B+2EC(A-C)=0, \\ 2EB^2-\left((6D-4X_3)A-4DX_4+2DC+F^2+2a\right)B-2E(A-C)(2A-2X_4+C)=0, \\ B^2-2A^2-2C^2=0, \quad (-2X_3+2D)B+(2C-2X_4)E+FG=0, \\ (-2X_4+A+2C)G+2BX_1+2CX_2=0, \quad (2X_4-C-2A)F-2AX_1+GB=0. \end{cases}$$

In particular, a non-trivial solution is

$$\begin{array}{l} A=-C=\frac{B}{2},\ D=-\frac{E}{2},\ F=G=0,\ E\neq 0,\ \lambda=0, \\ X_1=X_2=0,\ X_3=-\frac{E(X_4+B)}{B}. \end{array}$$

a3-6) *and the following equations are satisfied*

$$\begin{cases} 2BX_2-2FX_4+AB-B^2+2\lambda=0, \\ 4X_2E+4X_3F+2AE-2CF+D^2+2C^2=0, \\ AX_1-DX_4=0, \quad 2A^2+B^2-2\lambda=0, \quad X_1C+X_2D+AD=0, \\ 2BX_3+2EX_4-CB-2AC=0, \quad 2AX_2+2CX_4+A^2-AB-\lambda=0. \end{cases}$$

In particular, one non-trivial solution is

$$\begin{array}{l} B=D=0,\ C=\frac{AE}{F},\ A\neq 0,\ \lambda=A^2 \\ X_1=0,\ X_2=-\frac{A^2E}{F^2},\ X_3=\frac{E^2A^2}{2F^3},\ X_4=\frac{A^2}{F}. \end{array}$$

a3-7) *and the following equations are satisfied*

$$\begin{cases} 4CX_1+4EX_2+4GX_3+2AE-2BG-D^2+2FC+2B^2=0, \quad FX_4=0, \\ 3A^2-2\lambda=0, \quad AX_1-DX_4=0, \quad 2AX_3-2EX_4+AB-DF=0, \\ 2AX_2+2GX_4+2A^2+F^2-2\lambda=0, \quad 4AX_2+4BX_4+4A^2-F^2-2\lambda=0, \\ BX_1+DX_2+FX_3-CX_4+AD-BF=0. \end{cases}$$

In particular, one non-trivial solution is

$$\begin{array}{l} A=F=0,\ 2GB\neq 2B^2-D^2,\ \lambda=0, \\ X_1=X_2=X_4=0,\ X_3=\frac{2BG+D^2-2B^2}{4G}. \end{array}$$

a3-8) *and the following equations are satisfied*

$$\begin{cases} \left(A^2+(2X_2-C)\,A-\lambda+2EX_4\right)B+2DX_4\,(C+A)=0,\\ 2B^2X_1+((E-2X_3)\,C-2GX_4+2AE)\,B+D\,(C+2A)\,(C+A)=0,\\ (2X_2-C)\,B^2-(2EX_1-2DX_4+2F\,(X_2+A))\,B-2DX_1\,(C+A)=0,\\ B^4+\left(2AG+4EX_3-F^2+4DX_1+4GX_2\right)B^2+2DE\,(C+A)\,B+2D^2\,(C+A)^2=0,\\ AX_1-FX_4=0,\quad 2A^2+C^2-2\lambda=0,\\ C^2-(A+2X_2)\,C-2\lambda+2EX_4=0. \end{cases}$$

In particular, a non-trivial solution is

$$A=C=0,\ B\neq\varepsilon F,\ \lambda=0,\\ X_1=X_2=X_4=0,\ X_3=\tfrac{F^2-B^2}{4E},\quad \varepsilon^2=1.$$

a3-9) *and one of the following cases occurs*

1-

$$C=0,\ F=\tfrac{2A^2D+E^2B}{2AE},\ B-2A\neq 0,\ \lambda=-\tfrac{2A^2+B^2}{2},\\ X_1=-\tfrac{(2A-B)(2(2E^2+BD)A^2+E^2B^2)}{4A(2A^2D-E^2B)},\ X_2=0,\\ X_3=\tfrac{E(E^2+BD)(2A-B)}{2(2A^2D-E^2B)},\ X_4=-\tfrac{E(B^2+2A^2)(2A-B)}{2(2A^2D-E^2B)}.$$

2-

$$A=B=0,\ 2EF\neq 2E^2-C^2,\ \lambda=0,\\ X_2=-\tfrac{CX_1}{E},\ X_3=\tfrac{-4DX_1+C^2+2EF-2E^2}{4F},\ X_4=0.$$

a3-10) *and*

$$A=\tfrac{B(G-D)}{E},\ C=\tfrac{E(D^2-2DG+B^2)}{2B(D-G)},\ F=0,\quad \lambda=-\tfrac{3B^2(D-G)^2}{2E^2},\\ X_1=-\tfrac{B(2D+G)}{4E},\ X_2=\tfrac{3B^2}{4E},\ X_3=\tfrac{-D^2-2DG+3B^2}{8(D-G)},\ X_4=\tfrac{3B^2(D-G)}{4E^2},\\ (B^2+D^2)(D-\varepsilon\sqrt{2}B)\neq 0,\qquad \varepsilon^2=1.$$

a3-11) *and the following equations are satisfied*

$$\begin{cases} (A^2+(2X_1+C)A-2X_4G+\lambda)B-2X_4F(A-C)=0,\\ 2X_2B^2-((G-2X_3)C-2EX_4-2AG)B+F(2A-C)(A-C)=0,\\ (2X_1+C)B^2-((2X_1+2A)C+2X_4F+2X_2G)B-2X_2F(A-C)=0,\\ B^4+(4X_1E+4X_2F-C^2+2AE+4X_3G)B^2+2GF(A-C)B+2F^2(A-C)^2=0,\\ X_2A+X_4C=0,\quad 2A^2+C^2+2\lambda=0,\quad 2X_1C-2X_4G+C^2+AC+2\lambda=0. \end{cases}$$

In particular, one non-trivial solution is

$$A=C=0,\ B\neq 0,\lambda=0,\\ X_1=X_2=X_4=0,\ X_3=-\tfrac{B^2}{4G}.$$

Remark 5.5. All Ricci soliton examples of Theorem 5.4, except the case **a1**(1), are only shrinking, expanding or steady.

Remark 5.6. According to the result of [2], all three dimensional Ricci solitons on Lorentzian Lie groups are solvable (solvsolitons). By using Remarks 3.2 and 4.2, and the Theorems 3.1 and 4.1, this is still true in the four-dimensional neutral metrics.

We can now study some curvature properties of our classification of Ricci solitons on four-dimensional Lie groups. So, for each of the cases listed in Theorems 5.1 and 5.4, we consider the tensor components with respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$, with e_3 and e_4 time-like being used to describe the Lie algebra $\mathfrak{g}$. The Ricci parallel condition means that the covariant derivative of the Ricci tensor will be zero. Clearly, each Ricci flat manifold (i.e., a manifold on which the Ricci tensor vanishes) is Einstein and every Einstein manifold is Ricci parallel.

If the covariant derivative of the curvature tensor vanishes, i.e., $\Lambda(R) = 0$, the manifold is called locally symmetric. Also, Conformal fatness is translated into the following system of algebraic equations:

$$\begin{aligned} W_{ijkh} &= R_{ijkh} - \frac{1}{2}(g_{ik}\varrho_{jh} + g_{jh}\varrho_{ik} - g_{ih}\varrho_{jk} - g_{jk}\varrho_{ih}) + \frac{r}{6}(g_{ik}g_{jh} - g_{ih}g_{jk}) \\ &= 0, \end{aligned}$$

for all indices $i, j, k, h = 1, \cdots 4$, where W denotes the Weyl tensor and r is the scalar curvature.

Now by a straightforward calculations, the results of the previous sections yield the following result.

Corollary 5.7. *Consider the four-dimensional neutral Ricci soliton examples of Lie groups in Theorem 5.1. The Ricci parallel, locally symmetric and conformally flat examples are listed in the following* Table I*, where the check mark "✓" (respectively, "✗") means that the corresponding condition holds for all Lie algebras of that given form (respectively, never holds for Lie algebras of that given form).*

case	Ricci parallel	locally symmetric	conformally flat
a−1	$A = B \pm 2\alpha = C + D = 0$, $AD + \alpha^2 = C + D = B = 0$	$AD + \alpha^2 = 0$ $C + D = B = 0$	✗
d−2	✓	✗	✗
d−4	✓	✗	✗
g−2	$A = B + C = 0$	$A = B + C = D = 0$	✗

Table I: Ricci Parallel, locally symmetric and conformal flatness examples of the Theorem 5.1.

Corollary 5.8. *Consider the four-dimensional neutral Ricci soliton examples of Lie groups in Theorem 5.4. The Ricci parallel, locally symmetric and conformally flat examples are listed in the following* Table II*, where the check mark "✓" (respectively, "✗") means that the corresponding condition holds for all Lie algebras of that form (respectively, never holds for Lie algebras of that form).*

case	Ricci parallel	locally symmetric	conformally flat
a1(1)	✗	✗	✓
a1(2)	✗	✗	$E+\varepsilon B-A-\varepsilon D$ $=K+2\varepsilon B-2A=0,$ $D+B=K-A-E=0,$ $D-B=E-A=0$
a2−1	✓	✓	✗
a2−4	✓	✓	✗
a2−8	✓	✓	✓
a2−9	✓	✓	✓
a2−11	✓	✓	✓
a3−6	✓	✓	✗
a3−7	$G=0$	$D=G=0$	$B-\varepsilon D=G-2\varepsilon D=0$
a3−9(1)	$B=0$	$B=0$	✗
a3−9(2)	$F=0$	$C=F=0$	$E-\varepsilon C=F-2C=0$

Table II: Ricci Parallel, locally symmetric and conformal flatness examples of the Theorem 5.4.

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