Restrictions from $\mathfrak{gl}_n$ to $\mathfrak{sl}_n$

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Communicated by M. Schlichenmaier

Abstract. Let K be an algebraically closed field, let n be a positive integer. Consider the general linear Lie algebra of all $(n \times n)$–matrices over K and its subalgebra of all matrices with trace equal to 0, the special linear Lie algebra. If the characteristic of K does not divide n, then the larger Lie algebra is the direct product of the smaller Lie algebra with a one dimensional Lie algebra; in this case each finite dimensional simple module for the general linear Lie algebra restricts to a simple module for the special linear Lie algebra. This is no longer the case when the characteristic of K divides n; the purpose of this paper is to describe what happens in this situation.
Mathematics Subject Classification 2010: 17B10, 17B50.
Key Words and Phrases: Lie algebras. representations.

Introduction

Let K be an algebraically closed field of characteristic $p > 0$, let n be a positive integer. We denote by $\mathfrak{gl}_n(K)$ the general linear Lie algebra of all $(n \times n)$–matrices over K and by $\mathfrak{sl}_n(K)$ its subalgebra of all matrices with trace equal to 0, the special linear Lie algebra. One of our results is:

Proposition 1. *The restriction of a simple $\mathfrak{gl}_n(K)$–module to $\mathfrak{sl}_n(K)$ is either simple or has length p with all composition factors isomorphic to each other.*

In order to get a more precise statement, we need some extra notation. Since K is algebraically closed, each simple $\mathfrak{gl}_n(K)$–module E has a p–character. This means that there exists a linear form χ on $\mathfrak{gl}_n(K)$ such that each $X^p - X^{[p]} - \chi(X)^p$ with $X \in \mathfrak{gl}_n(K)$ annihilates E. Here X^p is the p–th power of X in the enveloping algebra of $\mathfrak{gl}_n(K)$ while $X^{[p]}$ is the p–th power of X as a matrix.

The map $(X, Y) \mapsto \operatorname{tr}(XY)$ is a non-degenerate symmetric bilinear form on $\mathfrak{gl}_n(K)$. So for each $\chi \in \mathfrak{gl}_n(K)^*$ there exists a unique matrix $C_\chi \in \mathfrak{gl}_n(K)$ such that $\chi(X) = \operatorname{tr}(C_\chi X)$ for all $X \in \mathfrak{gl}_n(K)$.

Proposition 2. *If all blocks in the Jordan normal form of C_χ have size divisible by p, then there is a unique (up to isomorphism) simple $\mathfrak{gl}_n(K)$–module E with*

p–character χ such that the restriction of E to $\mathfrak{sl}_n(K)$ is not simple. Otherwise all simple $\mathfrak{gl}_n(K)$–modules with *p*–character χ restrict to simple $\mathfrak{sl}_n(K)$–modules.

A simple $\mathfrak{gl}_n(K)$–module is called restricted if it has *p*–character 0. Since $C_0 = 0$ has all Jordan blocks of size 1, the proposition says in particular that simple restricted $\mathfrak{gl}_n(K)$–modules restrict to simple $\mathfrak{sl}_n(K)$–modules. This is well-known as the simple restricted modules for $\mathfrak{gl}_n(K)$ and $\mathfrak{sl}_n(K)$ come from simple modules for the algebraic groups $\mathrm{GL}_n(K)$ and $\mathrm{SL}_n(K)$ respectively and since simple modules for $\mathrm{GL}_n(K)$ restrict to simple modules for $\mathrm{SL}_n(K)$.

1.   Induction and Restriction

1.1.   Let K be a field of characteristic $p > 0$, let $\mathfrak{g}$ be a finite dimensional restricted Lie algebra over K. See [5] for basic properties of restricted Lie algebras and their representations.

Let $\mathfrak{g}' \subset \mathfrak{g}$ be a restricted subalgebra of $\mathfrak{g}$. Any $\mathfrak{g}$–module M restricts to a $\mathfrak{g}'$–module; we denote this module by $\operatorname{res} M$ if we want to emphasize that we regard M as a $\mathfrak{g}'$–module.

Fix a linear form $\chi \in \mathfrak{g}^*$ and consider the reduced enveloping algebras $U_\chi(\mathfrak{g})$ and $U_\chi(\mathfrak{g}')$; here we write χ also for the restriction of χ to $\mathfrak{g}'$. Then $U_\chi(\mathfrak{g}')$ is a subalgebra of $U_\chi(\mathfrak{g})$. Each simple $U_\chi(\mathfrak{g}')$–module is a composition factor of $U_\chi(\mathfrak{g}')$, which is a $\mathfrak{g}'$–submodule of $U_\chi(\mathfrak{g})$. Therefore each simple $U_\chi(\mathfrak{g}')$–module is a composition factor of $\operatorname{res} E$ for some simple $U_\chi(\mathfrak{g})$–module E.

1.2.   Let $x_1, x_2, \ldots, x_n$ be a basis for a complement to $\mathfrak{g}'$ in $\mathfrak{g}$. Set $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ and $|\alpha| = \sum_{i=1}^n \alpha_i$ for each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathbf{N}^n$. Let I denote the set of all $\alpha \in \mathbf{N}^n$ with $\alpha_i < p$ for all i. Then all x^α with $\alpha \in I$ are a basis for $U_\chi(\mathfrak{g})$ considered as a $U_\chi(\mathfrak{g}')$–module under left as well as under right multiplication.

We see in particular that $U_\chi(\mathfrak{g}')$ is a direct summand of $\operatorname{res} U_\chi(\mathfrak{g})$. Therefore each indecomposable direct summand of $U_\chi(\mathfrak{g}')$ (i.e., each principal indecomposable $U_\chi(\mathfrak{g}')$–module) is isomorphic to a direct summand of $\operatorname{res} Q$ for some principal indecomposable $U_\chi(\mathfrak{g})$–module Q.

1.3.   We have an induction functor ind_χ and a coinduction functor cnd_χ, both from the category of all $U_\chi(\mathfrak{g}')$–modules to the category of all $U_\chi(\mathfrak{g})$–modules, defined by

$$\operatorname{ind}_\chi M = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}')} M \quad \text{and} \quad \operatorname{cnd}_\chi M = \operatorname{Hom}_{U_\chi(\mathfrak{g}')}(U_\chi(\mathfrak{g}), M). \qquad (1)$$

In the definition of $\operatorname{cnd}_\chi M$ one regards $U_\chi(\mathfrak{g})$ as a $U_\chi(\mathfrak{g}')$–module under left multiplication and one lets any $x \in \mathfrak{g}$ act on the Hom–space via $(x\varphi)(u) = \varphi(ux)$. These functors are left or right adjoint to the restriction functor, i.e., we have for each $U_\chi(\mathfrak{g}')$–module M and each $U_\chi(\mathfrak{g})$–module V canonical isomorphisms

$$\operatorname{Hom}_\mathfrak{g}(\operatorname{ind}_\chi M, V) \simeq \operatorname{Hom}_{\mathfrak{g}'}(M, V) \text{ and } \operatorname{Hom}_\mathfrak{g}(V, \operatorname{cnd}_\chi M) \simeq \operatorname{Hom}_{\mathfrak{g}'}(V, M). \qquad (2)$$

The basis property of all x^α, $\alpha \in I$, yields for each $U_\chi(\mathfrak{g}')$–module M direct sum decompositions as vecor spaces that

$$\operatorname{ind}_\chi M = \bigoplus_{\alpha \in I} x^\alpha \, U_\chi(\mathfrak{g}') \otimes_{U_\chi(\mathfrak{g}')} M \text{ and } \operatorname{cnd}_\chi M = \bigoplus_{\alpha \in I} \operatorname{Hom}_{U_\chi(\mathfrak{g}')}(U_\chi(\mathfrak{g}') \, x^\alpha, M). \quad (3)$$

Each summand is isomorphic to M as a vector space: Map any $m \in M$ to $x^\alpha \otimes m$ in the first case; in the second case map $\varphi \colon U_\chi(\mathfrak{g}') \, x^\alpha \to M$ to $\varphi(x^\alpha)$. We get in particular

$$\dim \operatorname{ind}_\chi M = p^{\dim \mathfrak{g}/\mathfrak{g}'} \dim M = \dim \operatorname{cnd}_\chi M. \quad (4)$$

1.4.   Lemma. *If $\mathfrak{g}'$ is a restricted ideal in $\mathfrak{g}$, then we have for any $U_\chi(\mathfrak{g}')$– module M an isomorphism* $\operatorname{ind}_\chi M \simeq \operatorname{cnd}_\chi M$

This follows from the fact that for each $x \in \mathfrak{g}'$ the trace of $\operatorname{ad}_{\mathfrak{g}'}(x)$ is equal to the trace of $\operatorname{ad}_\mathfrak{g}(x)$. See, e.g., [6], 1.14(6) combined with [6], 1.12(1) or [6], 1.15(1).

1.5. The following result is a special case of Lemma 2.14 in [1]. We include a proof for the convenience of the reader.

Proposition. *Supopose that $\mathfrak{g}'$ a restricted ideal in $\mathfrak{g}$. Let M be a $U_\chi(\mathfrak{g}')$–module. Then the $U_\chi(\mathfrak{g}')$–module $\operatorname{res} \operatorname{ind}_\chi M$ has a filtration with all quotients of subsequent terms isomorphic to M.*

Proof.   We have for all $u \in U_\chi(\mathfrak{g})$ and all $\alpha \in I$

$$x^\alpha u = \sum_\beta \binom{\alpha}{\beta} \operatorname{ad}(x_1)^{\alpha_1 - \beta_1} \circ \operatorname{ad}(x_2)^{\alpha_2 - \beta_2} \circ \cdots \circ \operatorname{ad}(x_n)^{\alpha_n - \beta_n}(u) \, x^\beta$$

and

$$u x^\alpha = \sum_\beta \binom{\alpha}{\beta} x^\beta \operatorname{ad}'(x_n)^{\alpha_n - \beta_n} \circ \cdots \circ \operatorname{ad}'(x_2)^{\alpha_2 - \beta_2} \circ \operatorname{ad}'(x_1)^{\alpha_1 - \beta_1}(u)$$

where we sum over all $\beta \in I$ with $\beta_i \leq \alpha_i$ for all i, where the binomial coefficient is the product of all $\binom{\alpha_i}{\beta_i}$, and where $\operatorname{ad}'(x)(y) = [y, x] = -\operatorname{ad}(x)(y)$.

If $u \in U_\chi(\mathfrak{g}')$, then $\operatorname{ad}(x_i)(u) \in U_\chi(\mathfrak{g}')$ for all i. It then follows that

$$x^\alpha u \in u x^\alpha + \sum_{|\beta| < |\alpha|} U_\chi(\mathfrak{g}') x^\beta \qquad \text{and} \qquad u x^\alpha \in x^\alpha u + \sum_{|\beta| < |\alpha|} x^\beta U_\chi(\mathfrak{g}'). \quad (*)$$

Choose a numbering $\alpha(1), \alpha(2), \ldots, \alpha(p^n)$ for the elements of I such that $|\alpha(i)| < |\alpha(j)|$ implies $i < j$. Now $(*)$ implies for each i, $0 \leq i \leq p^n$, that

$$Z_i := \sum_{j=1}^i x^{\alpha(j)} U_\chi(\mathfrak{g}') \otimes M$$

is a $\mathfrak{g}'$–submodule of $\operatorname{ind}_\chi M$. We have clearly $Z_i \subset Z_{i+1}$ for $i < p^n$. Using $(*)$ once more, we get that Z_{i+1}/Z_i is isomorphic to M. This yields the claim since $Z_0 = 0$ and $Z_{p^n} = \operatorname{ind}_\chi M$. ∎

Remark: The proposition implies that the exact functors $\operatorname{res} \circ \operatorname{ind}_\chi$ and $\operatorname{res} \circ \operatorname{cnd}_\chi$ induce multiplication by $p^{\dim \mathfrak{g}/\mathfrak{g}'}$ on the Grothendieck group of all finite dimensional $U_\chi(\mathfrak{g}')$–modules.

1.6. Corollary. *Supopose that $\mathfrak{g}'$ is a restricted ideal in $\mathfrak{g}$. Let L be a simple $U_\chi(\mathfrak{g}')$–module and E be a simple $U_\chi(\mathfrak{g})$–module.*

(a) *If E is a composition factor of $\mathrm{ind}_\chi L$, then each composition factor of $\mathrm{res}\, E$ is isomorphic to L.*

(b) *If L is a composition factor of $\mathrm{res}\, E$, then E is a composition factor of $\mathrm{ind}_\chi L$; each composition factor of $\mathrm{res}\, E$ is isomorphic to L.*

Proof.   (a) If E is a composition factor of $\mathrm{ind}_\chi L$, then $\mathrm{res}\, E$ is a subquotient of $\mathrm{res}\,\mathrm{ind}_\chi L$. The proposition implies that each composition factor of $\mathrm{res}\,\mathrm{ind}_\chi L$ is isomorphic to L. The claim follows.

(b) There exists a simple $U_\chi(\mathfrak{g}')$–module L' with $\mathrm{Hom}_{\mathfrak{g}'}(L', E) \neq 0$. Then 1.3(2) implies $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{ind}_\chi L', E) \neq 0$, so E is a composition factor of $\mathrm{ind}_\chi L'$. The claim in (a) now implies that L is isomorphic to L'. This yields the first part of (b). The second part follows by applying (a) once more.   ∎

1.7.   We assume again that $\mathfrak{g}'$ a restricted ideal in $\mathfrak{g}$. If E and E' are simple $U_\chi(\mathfrak{g})$–modules, let us write $E \approx E'$ if there exists a simple $U_\chi(\mathfrak{g}')$–module L that is a composition factor both of $\mathrm{res}\, E$ and of $\mathrm{res}\, E'$. Then $\approx$ defines an equivalence relation on the isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules. Corollary 1.6 shows that there is a bijection from the set of equivalence classes for this relation and the set of isomorphism classes of simple $U_\chi(\mathfrak{g}')$–modules: One maps the class of some E to the class of all composition factors of $\mathrm{res}\, E$.

The size of the equivalence classes for $\approx$ can vary much. For example, when $\mathfrak{g}$ is a direct product $\mathfrak{g} = \mathfrak{g}' \times \mathfrak{g}''$ and K is algebraically closed, then each of these classes is in bijection with the isomorphism classes of simple $U_\chi(\mathfrak{g}'')$–modules.

On the other hand, if we take $\mathfrak{g}' = 0$, then all simple $U_\chi(\mathfrak{g})$–modules form just one equivalence class.

2.   Codimension 1

2.1.   *Assumptions.* We assume in this section that K is algebraically closed, that $\mathfrak{g}'$ is a restricted ideal in $\mathfrak{g}$ and that the restricted Lie algebra $\mathfrak{g}/\mathfrak{g}'$ is a one dimensional toral algebra.

So our assumptions are a bit stronger than the "Codimension 1" of the header. Note that $\mathfrak{gl}_n(K)$ and $\mathfrak{sl}_n(K)$ satisfy our assumptions on $\mathfrak{g}$ and $\mathfrak{g}'$.

2.2.   The assumption says that $\mathfrak{g}/\mathfrak{g}'$ has a basis consisting of one element z such that $z^{[p]} = z$. Any finite dimensional $\mathfrak{g}/\mathfrak{g}'$–module is the direct sum of the eigenspaces for z; if the module is restricted, then the eigenvalues belong to $\mathbf{F}_p$. We have for each $a \in K$ a one dimensional $\mathfrak{g}/\mathfrak{g}'$–module K_a on which z acts as multiplication by a. These modules satisfy $K_a \otimes K_b \simeq K_{a+b}$ for all $a, b \in K$. The $\mathfrak{g}/\mathfrak{g}'$–module K_a is restricted if and only if $a \in \mathbf{F}_p$.

We usually regard any K_a as a $\mathfrak{g}$–module annihilated by $\mathfrak{g}'$. (In our $\mathfrak{gl}_n$– $\mathfrak{sl}_n$ example we can choose z such that K_a is the one dimensional $\mathfrak{gl}_n(K)$–module where each matrix $X \in \mathfrak{gl}_n(K)$ acts as multiplication by a times the trace of X.)

2.3. Fix a p–character χ for $\mathfrak{g}$. If M and N are finite dimensional $\mathfrak{g}$–modules, then $\mathrm{Hom}_{\mathfrak{g}'}(M, N)$ becomes a $\mathfrak{g}$–module under $(x\varphi)(m) = x\,\varphi(m) - \varphi(xm)$ for all $x \in \mathfrak{g}$, $\varphi \in \mathrm{Hom}_{\mathfrak{g}'}(M, N)$, and $m \in M$. Suppose that both M and N are $U_\chi(\mathfrak{g})$–modules; then $\mathrm{Hom}_{\mathfrak{g}'}(M, N)$ is a restricted $\mathfrak{g}$–module. Obviously $\mathfrak{g}'$ annihilates $\mathrm{Hom}_{\mathfrak{g}'}(M, N)$, so $\mathrm{Hom}_{\mathfrak{g}'}(M, N)$ is a restricted $\mathfrak{g}/\mathfrak{g}'$–module, hence a direct sum of eigenspaces for z with all eigenvalues in $\mathbf{F}_p$. A simple calculation shows that the eigenspace for the eigenvalue r identifies with

$$\mathrm{Hom}_{\mathfrak{g}}(M \otimes K_r, N) \simeq \mathrm{Hom}_{\mathfrak{g}}(M, N \otimes K_{-r}).$$

We get thus isomorphisms

$$\mathrm{Hom}_{\mathfrak{g}'}(M, N) \simeq \bigoplus_{r \in \mathbf{F}_p} \mathrm{Hom}_{\mathfrak{g}}(M \otimes K_r, N) \simeq \bigoplus_{r \in \mathbf{F}_p} \mathrm{Hom}_{\mathfrak{g}}(M, N \otimes K_r). \qquad (1)$$

2.4. If E is a simple $U_\chi(\mathfrak{g})$–module, then so is any $E \otimes K_r$ with $r \in \mathbf{F}_p$. As $(E \otimes K_r) \otimes K_s \simeq E \otimes K_{r+s}$, we see: If E is isomorphic to $E \otimes K_r$ for some $r \in \mathbf{F}_p$, then E is isomorphic to $E \otimes K_{mr}$ for all $m \in \mathbf{N}$; in case $r \neq 0$ we get then that E is isomorphic to $E \otimes K_s$ for all $s \in \mathbf{F}_p$. We say that

- E has type A if E is not isomorphic to any $E \otimes K_r$ with $r \in \mathbf{F}_p$, $r \neq 0$,

- E has type B if E is isomorphic to each $E \otimes K_r$ with $r \in \mathbf{F}_p$.

Each simple $U_\chi(\mathfrak{g})$–module E is either of type A or B. If E is of type A, then so are all $E \otimes K_r$ with $r \in \mathbf{F}_p$; these p modules belong then to p distinct isomorphism classes.

2.5. Lemma. *If E is a simple $U_\chi(\mathfrak{g})$–module of type B, then $\mathrm{res}\,E$ is not simple.*

Proof. We apply 2.3(1) and get that $\mathrm{Hom}_{\mathfrak{g}'}(E, E) \simeq \bigoplus_{r \in \mathbf{F}_p} \mathrm{Hom}_{\mathfrak{g}}(E, E \otimes K_r) \simeq \mathrm{Hom}_{\mathfrak{g}}(E, E)^p$ has dimension p. ∎

2.6. For any simple $U_\chi(\mathfrak{g})$–module E denote by Q_E its projective cover in the category of all $U_\chi(\mathfrak{g})$–modules. For any simple $U_\chi(\mathfrak{g}')$–module L denote by P_L its projective cover in the category of all $U_\chi(\mathfrak{g}')$–modules.

Lemma. *Let E be a simple $U_\chi(\mathfrak{g})$–module and let L be simple $U_\chi(\mathfrak{g}')$–module that is a composition factor of $\mathrm{res}\,E$. Then there exists a positive integer $m(E)$ such that*

$$\mathrm{res}\,Q_E \simeq P_L^{m(E)}.$$

Proof. Since $\mathrm{res}\,Q_E$ is a projective finite dimensional $U_\chi(\mathfrak{g}')$–module, cf. 1.2, it is a finite direct sum of projective covers of simple $U_\chi(\mathfrak{g}')$–modules. Suppose that

M is a simple $U_\chi(\mathfrak{g}')$–module such that P_M is a direct summand of $\operatorname{res} Q_E$. Then $\operatorname{Hom}_{\mathfrak{g}'}(P_M, M) \neq 0$ implies

$$0 \neq \operatorname{Hom}_{\mathfrak{g}'}(Q_E, M) \simeq \operatorname{Hom}_{\mathfrak{g}}(Q_E, \operatorname{ind}_\chi M).$$

This means that E is a composition factor of $\operatorname{ind}_\chi M$. Now Corollary 1.6(a) yields that each composition factor of $\operatorname{res} E$ is isomorphic to M, hence that $L \simeq M$. So each direct summand of $\operatorname{res} Q_E$ is isomorphic to P_L. ∎

Remark: We shall see below that $m(E) = 1$ for all E, i.e., that $\operatorname{res} Q_E$ is indecomposable. This indecomposability is for $\chi = 0$ a special case of Thm. 2.1.2 in [2]. The proof given there extends obviously to arbitrary p–characters and shows that $\operatorname{res} V$ is indecomposable for any indecomposable $U_\chi(\mathfrak{g})$–module V. However, we shall see that the claim in the present case has a very simple direct proof.

2.7. Let E be a simple $U_\chi(\mathfrak{g})$–module. Denote by $\ell(E)$ the length of $\operatorname{res} E$.

Lemma. *We have*

$$m(E) \cdot \ell(E) = \begin{cases} 1 & \text{if } E \text{ has type } A, \\ p & \text{if } E \text{ has type } B. \end{cases}$$

Proof. Let L be simple $U_\chi(\mathfrak{g}')$–module that is a composition factor of $\operatorname{res} E$. Lemma 2.6 and equation 2.3(1) show that

$$\operatorname{Hom}_{\mathfrak{g}'}(Q_E, E) \simeq \operatorname{Hom}_{\mathfrak{g}'}(P_L, E)^{m(E)}$$
$$\simeq \bigoplus_{r \in \mathbf{F}_p} \operatorname{Hom}_{\mathfrak{g}}(Q_E, E_E \otimes K_r).$$

The right hand side of the first line has dimension $m(E)\,\ell(E)$ since each composition factor of $\operatorname{res} E$ is isomorphic to L.

For E of type A the right hand side of the second line is isomorphic to $\operatorname{Hom}_{\mathfrak{g}}(Q_E, E)$ and has dimension 1. For E of type B the right hand side of the second line is isomorphic to $\operatorname{Hom}_{\mathfrak{g}}(Q_E, E)^p$ and has dimension p. The claim follows. ∎

2.8. Proposition. *Let E be a simple $U_\chi(\mathfrak{g})$–module and let L be a simple $U_\chi(\mathfrak{g}')$–module that is a composition factor of $\operatorname{res} E$.*

(a) *We have $\operatorname{res} Q_E \simeq P_L$.*

(b) *If E has type A, then $\operatorname{res} E$ is simple, isomorphic to L.*

(c) *If E_i has type B, then $\operatorname{res} E$ has length p and we have $\dim L = (\dim E)/p$.*

Proof. If E has type A, then Lemma 2.7 implies $m(E) = \ell(E) = 1$, hence (a) in this case and (b).

If E has type B, then $\ell(E) > 1$ by Lemma 2.5. Now Lemma 2.6 implies $m(E) = 1$ and $\ell(E) = p$ as p is a prime number. This yields (a) in this case as well as (c) where at the end we use the uniqueness of L. ∎

Remark: Note that this result yields Proposition 1 of the introduction.

2.9. Corollary. *Let E be a simple $U_\chi(\mathfrak{g})$–module and let L be a simple $U_\chi(\mathfrak{g}')$–module that is a composition factor of $\operatorname{res} E$.*

(a) *If E' is a simple $U_\chi(\mathfrak{g})$–module, then $\operatorname{res} E'$ has a composition factor isomorphic to L if and only if $E' \simeq E \otimes K_r$ for some $r \in \mathbf{F}_p$.*

(b) *If E has type A, then $\operatorname{ind}_\chi L \simeq \bigoplus_{r \in \mathbf{F}_p} E \otimes K_r$.*

(c) *If E has type B, then $\operatorname{ind}_\chi L \simeq E$.*

Proof. We know from Corollary 1.6(b) that E is a composition factor of $\operatorname{ind}_\chi L$. The claim in (c) for E of type B follows now from $\dim E = p \dim L = \dim \operatorname{ind}_\chi L$.

If E has type A, then we get for all $r \in \mathbf{F}_p$ that

$$\operatorname{Hom}_\mathfrak{g}(\operatorname{ind}_\chi L, E \otimes K_r) \simeq \operatorname{Hom}_{\mathfrak{g}'}(L, E) \neq 0.$$

So the head of $\operatorname{ind}_\chi L$ contains a direct summand isomorphic to $\bigoplus_{r \in \mathbf{F}_p} E \otimes K_r$. The claim in (b) follows as this summand has the same dimension as $\operatorname{ind}_\chi L$.

If E' is a simple $U_\chi(\mathfrak{g})$–module, then $\operatorname{res} E'$ has a composition factor isomorphic to L if and only if E' is a composition factor of $\operatorname{ind}_\chi L$, see Corollary 1.6(b). Therefore (a) follows from (b) and (c). ∎

Remark: Part (a) can be restated in the notation from 1.7 as follows: We have $E \approx E'$ if and only if there exists $r \in \mathbf{F}_p$ with $E' \simeq E \otimes K_r$.

3. Reduction to Jordan Normal Form

3.1. Assume from now on that $\mathfrak{g} = \mathfrak{gl}_n(K)$ and $\mathfrak{g}' = \mathfrak{sl}_n(K)$ where K is algebraically closed. In order to prove Proposition 2 in the introduction, we now have to ask for each $\chi \in \mathfrak{g}^*$: Are there simple $U_\chi(\mathfrak{g})$–modules of type B? If so, how many? Note that $\mathfrak{g}'$ does not play a role in these questions.

3.2. The group $G = \operatorname{GL}_n(K)$ acts by conjugation on $\mathfrak{g}$. This is an action by automorphisms of $\mathfrak{g}$ as restricted Lie algebra; it is called the adjoint representation of G. The contragredient representation of G on $\mathfrak{g}^*$ is then called the coadjoint representation of G.

Recall from the introduction that we associate to any $\chi \in \mathfrak{g}^*$ the unique matrix $C_\chi \in \mathfrak{g}$ such that $\chi(X) = \operatorname{tr}(C_\chi X)$ for all $X \in \mathfrak{g}$. We have then for any $g \in G$

$$(g\,\chi)\,(X) = \chi\,(g^{-1}Xg) = \operatorname{tr}\,(C_\chi g^{-1}Xg) = \operatorname{tr}\,(g^{-1}(gC_\chi g^{-1})Xg) = \operatorname{tr}\,((gC_\chi g^{-1})X)$$

for all X, hence $C_{g\chi} = gC_\chi g^{-1}$.

3.3. Let $g \in G$. For any $\mathfrak{g}$–module M we denote by ${}^g M$ the $\mathfrak{g}$–module that is equal to M as a vector space and where any $X \in \mathfrak{g}$ acts on ${}^g M$ as $g^{-1}Xg$ does on M. We call ${}^g M$ the twist of M by g.

Twisting by g is an equivalence of categories; twisting by g^{-1} is an inverse functor. We have for any $\mathfrak{g}$–modules M and N a natural isomorphism

$$^g(M \otimes N) \simeq {}^gM \otimes {}^gN. \tag{1}$$

Any one dimensional $\mathfrak{g}$–module K_a with $a \in K$ as in 2.2 — where any $X \in \mathfrak{g}$ acts as multiplication by $a \operatorname{tr}(X)$ — satisfies $^gK_a = K_a$ for each $g \in G$.

3.4. Let $\chi \in \mathfrak{g}^*$. If we twist a $U_\chi(\mathfrak{g})$–module by some $g \in G$, then a simple calculation shows that we get a $U_{g\chi}(\mathfrak{g})$–module.

Lemma. *Let $\chi \in \mathfrak{g}^*$ and $g \in G$. Then twisting by g induces a bijection between the isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules of type B and the isomorphism classes of simple $U_{g\chi}(\mathfrak{g})$–modules of type B.*

Proof. By 3.3 and the observation above, twisting by g induces a bijection between the isomorphism classes of all simple $U_\chi(\mathfrak{g})$–modules and the isomorphism classes of all simple $U_{g\chi}(\mathfrak{g})$–modules.

It is left show: A simple $U_\chi(\mathfrak{g})$–module E has type B if and only if ^gE has type B. This follows from

$$^g(E \otimes K_r) \simeq {}^gE \otimes {}^gK_r = {}^gE \otimes K_r$$

for all $r \in \mathbf{F}_p$, see 3.3. ∎

3.5. This lemma and the equation $C_{g\chi} = gC_\chi g^{-1}$ from 3.2 show that it suffices to consider χ with C_χ in Jordan normal form.

4. The Nilpotent Case

4.1. We keep the assumptions from 3.1. We first look at the case where C_χ is nilpotent. We assume that C_χ has Jordan normal form, so there exists a subset $I \subset \{1, 2, \ldots, n-1\}$ such that

$$C_\chi = \sum_{i \in I} E_{i,i+1}. \tag{1}$$

Here we denote more generally by E_{ij} for $1 \leq i, j \leq n$ the matrix in $\mathfrak{g}$ with (i, j)–entry equal to 1 and all other entries equal to 0.

The definition of C_χ then implies for all i and j that

$$\chi(E_{ij}) = \begin{cases} 1, & \text{if } i = j+1 \text{ and } j \in I, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

This means that χ has standard Levi form with respect to the standard triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{n}^+$ ($\mathfrak{n}^-$) is the space of all strict upper (lower) triangular $(n \times n)$–matrices over K, and $\mathfrak{h}$ the space of all diagonal matrices in $\mathfrak{g}$.

The notion of standard Levi form was introduced by Friedlander and Parshall in [4] for Lie algebras of more general reductive groups. There they also proved the results on simple $U_\chi(\mathfrak{g})$–modules that we shall state in 4.2 under some restriction on p. See also the survey in Section 10 of [5] for any p.

4.2.   For any i, $1 \leq i \leq n$, denote by $\varepsilon_i \in \mathfrak{h}^*$ the linear form that maps any diagonal matrix to its i–th diagonal entry. Set $\Lambda = \sum_{i=1}^{n} \mathbf{F}_p \varepsilon_i \subset \mathfrak{h}^*$ and $\varpi_n = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$. Then ϖ_n maps any diagonal matrix to its trace.

Each $\lambda \in \Lambda$ defines a one dimensional $(\mathfrak{h} \oplus \mathfrak{n})$–module where each $X + Y$ with $X \in \mathfrak{h}$ and $Y \in \mathfrak{n}$ acts as multiplication by $\lambda(X)$. This module turns out to be a $U_\chi(\mathfrak{h} \oplus \mathfrak{n})$–module. We apply to it the induction functor ind_χ as in 1.3 (with $\mathfrak{g}'$ replaced by $\mathfrak{h} \oplus \mathfrak{n}$) and get a $U_\chi(\mathfrak{g})$–module that we denote by $Z_\chi(\lambda)$ and usually call a baby Verma module. The "standard Levi" condition implies that any $Z_\chi(\lambda)$ has a unique simple homomorphic image; we denote it by $L_\chi(\lambda)$. It turns out that each simple $U_\chi(\mathfrak{g})$–module is isomorphic to some $L_\chi(\lambda)$.

The symmetric group S_n acts on $\mathfrak{h}$ by permuting the diagonal entries, hence on $\Lambda \subset \mathfrak{h}^*$ by permuting the ε_i. We write now W instead of S_n for this permutation group as it is really the Weyl group of G and $\mathfrak{g}$ that is acting. We shall consider its "dot" action $w \bullet \lambda = w(\lambda + \rho) - \rho$ for all $w \in W$ and $\lambda \in \mathfrak{h}^*$ where $\rho = \sum_{j=1}^{n} (n - j)\varepsilon_j$.

Set W_I the subgroup of W generated by all transpositions $(i\ i + 1)$ with $i \in I$. We can now complete the classification of the simple $U_\chi(\mathfrak{g})$–modules by stating that $L_\chi(\lambda) \simeq L_\chi(\mu)$ if and only if there exists $w \in W_I$ with $\mu = w \bullet \lambda$.

4.3. Lemma. *Let $\lambda \in \Lambda$. Then $L_\chi(\lambda)$ has type B if and only if there exists for each $r \in \mathbf{F}_p$ an element $w_r \in W_I$ with $w_r \bullet \lambda - \lambda = r\varpi_n$ if and only if there exists $w \in W_I$ such that $w \bullet \lambda - \lambda$ is a non-zero $\mathbf{F}_p$–multiple of ϖ_n.*

Proof.   The restriction of K_r to $\mathfrak{h} \oplus \mathfrak{n}$ is the one dimensional $(\mathfrak{h} \oplus \mathfrak{n})$–module where each $X + Y$ with $X \in \mathfrak{h}$ and $Y \in \mathfrak{n}$ acts as multiplication by $r\varpi_n(X)$. Therefore the tensor identity (e.g., 1.12(1) in [6]) implies that $Z_\chi(\lambda) \otimes K_r \simeq Z_\chi(\lambda + r\varpi_n)$. It follows that $L_\chi(\lambda) \otimes K_r \simeq L_\chi(\lambda + r\varpi_n)$. So $L_\chi(\lambda)$ has type B if and only if $\lambda + r\varpi_n \in W_I \bullet \lambda$ for all $r \in \mathbf{F}_p$, if and only if there exists $r \in \mathbf{F}_p$, $r \neq 0$, for which $\lambda + r\varpi_n \in W_I \bullet \lambda$. ∎

4.4.   The set I is given by integers

$$1 = k_1 < k_2 < \cdots < k_s < k_{s+1} = n + 1 \tag{1}$$

such that I is the union of all

$$I_j := \{\, k_j, k_j + 1, \ldots, k_{j+1} - 2 \,\} \tag{2}$$

with $1 \leq j \leq s$. Note that $I_j = \emptyset$ if $k_{j+1} = k_j + 1$. Note also that the blocks of $C_\chi = \sum_{i \in I} E_{i,i+1}$ have size $k_{j+1} - k_j$ with $1 \leq j \leq s$.

Lemma. *Let $\lambda \in \Lambda$ and write $\lambda + \rho = \sum_{i=1}^{n} a_i \varepsilon_i$ with all $a_i \in \mathbf{F}_p$. For all $r \in \mathbf{F}_p$ and for all j, $1 \leq j \leq s$, set $N_j(\lambda, r) := \{\, i \mid k_j \leq i < k_{j+1},\ a_i = r \,\}$. Then $L_\chi(\lambda)$ has type B if and only if $|N_j(\lambda, r)| = |N_j(\lambda, 0)|$ for all $r \in \mathbf{F}_p$ and for all j, $1 \leq j \leq s$.*

Proof.   By definition W_I is the subgroup of $W = S_n$ generated by all $(i\ i + 1)$ with $i \in I$. It follows that W_I consists of all permutations that preserve all subsets

$$N_j := \{\, k_j, k_j + 1, \ldots, k_{j+1} - 1 \,\}, \qquad 1 \leq j \leq s. \tag{3}$$

We have for any $\sigma \in W_I$ that

$$\sigma^{-1}(\lambda + \rho) = \sum_{i=1}^{n} a_{\sigma(i)}\varepsilon_i. \tag{4}$$

So we have $r\varpi_n = \sigma^{-1}\bullet\lambda - \lambda = \sigma^{-1}(\lambda + \rho) - (\lambda + \rho)$ if and only if

$$a_{\sigma(i)} = a_i + r \qquad \text{for all } i,\ 1 \le i \le n. \tag{5}$$

If so, then (5) says that σ maps the set of all i with $a_i = 0$ onto the set of all k with $a_k = r$; and since $\sigma \in W_I$ preserves all N_j, we get $\sigma(N_j(\lambda, 0)) = N_j(\lambda, r)$. If now $L_\chi(\lambda)$ has type B, then we get from Lemma 4.3 for each $r \in \mathbf{F}_p$ an element $\sigma \in W_I$ with $r\varpi_n = \sigma^{-1}\bullet\lambda - \lambda$; It follows that $|N_j(\lambda, r)| = |N_j(\lambda, 0)|$ for all r and j.

Suppose conversely that $|N_j(\lambda, r)| = |N_j(\lambda, 0)|$ for all r and i. Since $\{1, 2, \ldots, n\}$ is the disjoint union of all $N_j(\lambda, r)$ with $r \in \mathbf{F}_p$ and $1 \le j \le s$, there exists a permutation $\tau \in W$ with $\tau(N_j(\lambda, r)) = N_j(\lambda, r + 1)$ for all r and j. Each N_j is the disjoint union of all $N_j(\lambda, r)$ with $r \in \mathbf{F}_p$. It follows that $\tau(N_j) = N_j$ for all j, hence that $\tau \in W_I$. Finally a look at (4) and (5) shows that $\varpi_n = \sigma^{-1}\bullet\lambda - \lambda$, hence that $L_\chi(\lambda)$ has type B according to Lemma 4.3. ∎

4.5. Proposition. *There exists $\lambda \in \Lambda$ for which $L_\chi(\lambda)$ has type B if and only if p divides all $k_{j+1} - k_j$ with $1 \le j \le s$. If so, then $L_\chi(0)$ is up to isomorphism the only simple $U_\chi(\mathfrak{g})$–module of type B.*

Proof.   If $\lambda \in \Lambda$ satisfies the condition in Lemma 4.4, then we get for all i, in the notation from above, $p \cdot |N_j(\lambda, 0)| = \sum_{r \in \mathbf{F}_p} |N_j(\lambda, r)| = |N_j| = k_{j+1} - k_j$, hence $p \mid k_{j+1} - k_j$.

Suppose conversely that p divides all $k_{j+1} - k_j$. Consider $\lambda = 0$, hence $\lambda + \rho = \sum_{i=1}^{n}(n - i)\varepsilon_i = \sum_{i=1}^{n} a_i\varepsilon_i$. For each j the set of all a_i with $i \in N_j$ is the set of the reductions modulo p of all integers $n - k$ with $k_j \le k < k_{j+1}$. In this sequence of consecutive numbers each residue class modulo p has equally many representatives since the number of terms, equal to $k_{j+1} - k_j$, is a multiple of p. It follows that $|N_j(0, r)| = |N_j(0, 0)|$ for all r and i. So $L_\chi(0)$ has type B.

Finally, let $\lambda, \mu \in \Lambda$ both satisfy the condition in Lemma 4.4. We get then $|N_j(\lambda, r)| = (k_{j+1} - k_j)/p = |N_j(\mu, r)|$ for all r and all j. It follows that there exists $\tau \in W_I$ with $\tau(N_j(\lambda, r)) = N_j(\mu, r)$ for all r and j, hence $\tau^{-1}\bullet\lambda = \mu$ and $L_\chi(\lambda) \simeq L_\chi(\mu)$. ∎

Remark: Note that Proposition 4.5 proves Proposition 2 of the Introduction in the case where C_χ is nilpotent.

4.6.   Consider in particular the case where C_χ is regular nilpotent; this means in the notation above that $s = 1$, i.e., that C_χ has only one Jordan block (of size n). If p does not divide n, then work of Friedlander and Parshall (in [3] and [4]) shows that both $Z_\chi(\lambda)$ and its restriction to $\mathfrak{g}'$ are simple for each $\lambda \in \Lambda$. If p divides n, the each $Z_\chi(\lambda)$ is still simple, cf. [5], Prop. 10.5. Proposition 4.5 now says that in this case the restriction of $Z_\chi(0)$ to $\mathfrak{g}'$ is not simple. This was observed in the case $p = n = 3$ by Xin Wen in [8]; this paper grew out of my wish to understand this behaviour.

5.   The General Case

5.1.   We keep the assumptions from 3.1. We need one more special case:

Lemma. *The claim in Proposition 2 from the introduction holds for any $\chi \in \mathfrak{g}^*$ such that C_χ has only one eigenvalue.*

Proof.   Let $a \in K$ denote the only eigenvalue of C_χ. We may assume that C_χ has Jordan normal form. This implies that $C_\chi = aI + N$ where I is the identity matrix in $\mathfrak{g}$ and where N is a nilpotent matrix in Jordan normal form. The Jordan blocks of N and of C_χ have the same sizes.

Denote by $\psi \in \mathfrak{g}^*$ the linear form with $N = C_\psi$. Then $C_\chi = aI + C_\psi$ implies $\chi = a \operatorname{tr} + \psi$. Choose $b \in K$ with $b^p - b = a$. Then the one dimensional $\mathfrak{g}$–module K_b (as in 2.2) has p–character equal to $a \operatorname{tr}$.

Now the functor $M \mapsto M \otimes K_b$ induces an equivalence from the category of all $U_\psi(\mathfrak{g})$–modules to the category of all $U_\chi(\mathfrak{g})$–modules; an inverse is given by $N \mapsto N \otimes K_{-b}$.

We have $(M \otimes K_b) \otimes K_r \simeq (M \otimes K_r) \otimes K_b$ for any $r \in \mathbf{F}_p$. If M is simple, this shows that M has type B if and only if $M \otimes K_b$ has type B. So the number of isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules of type B is equal to the corresponding number for $U_\psi(\mathfrak{g})$. Now the claim follows from Proposition 4.5 applied to ψ in stead of χ and from the comparison of the Jordan blocks of C_χ and of $N = C_\psi$. ∎

5.2.   Let now $\chi \in \mathfrak{g}^*$ be arbitrary. We assume that C_χ has Jordan normal form. So its (additive) Jordan decomposition has the form $C_\chi = S + N$ where S is the diagonal matrix with the same entries in the diagonal as C_χ and where N is a strictly upper triangular matrix with the same off-diagonal entries as C_χ.

Denote the centraliser in $\mathfrak{g}$ of the diagonal matrix S by $\mathfrak{m}$. This is a restricted Lie subalgebra of $\mathfrak{g}$ that is the Levi factor of a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}$ of $\mathfrak{g}$ with nilradical $\mathfrak{u}$.

Lemma. *There is a bijection from the set of isomorphism classes of simple $U_\chi(\mathfrak{m})$–modules L such that $L \otimes K_r \simeq L$ for all $r \in \mathbf{F}_p$ to the set of isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules of type B.*

Proof.   There exists an equivalence of categories (cf. [5], Prop. 7.4) from $U_\chi(\mathfrak{m})$–modules to $U_\chi(\mathfrak{g})$–modules constructed as follows: We first extend any $U_\chi(\mathfrak{m})$–module to an ("inflated") $\mathfrak{p}$–module $\inf V$ letting any element in $\mathfrak{u}$ act as 0. It turns out that $\inf V$ is a $U_\chi(\mathfrak{p})$–module. Our equivalence of categories takes now V to $\operatorname{ind}_\chi(\inf V)$ with ind_χ as in 1.2 with $\mathfrak{p}$ in stead of $\mathfrak{g}'$. This functor induces a bijection between the isomorphism classes of simple modules. (This goes back to Weisfeiler and Kac in [7] and Friedlander and Parshall in [3].)

Let E be a simple $U_\chi(\mathfrak{g})$–module and let L be a simple $U_\chi(\mathfrak{m})$–module with $E \simeq \operatorname{ind}_\chi(\inf L)$. The tensor identity (e.g., [6], 1.12(1)) implies for all $r \in \mathbf{F}_p$ that

$$E \otimes K_r \simeq \operatorname{ind}_\chi((\inf V) \otimes K_r) \simeq \operatorname{ind}_\chi(\inf(L \otimes K_r))$$

where we use at the second step that the nilpotent matrices in $\mathfrak{u}$ have trace 0, hence act as 0 on K_r. It follows that E has type B if and only if $L \otimes K_r \simeq L$ for all $r \in \mathbf{F}_p$. ∎

5.3. Let $a_1, a_2, \ldots, a_s \in K$ be the distinct eigenvalues of C_χ. So the characteristic polynomial of C_χ has the form $\det(T\,I - C_\chi) = \prod_{i=1}^{s} (T - a_i)^{r(i)}$ with suitable integers $r(i) > 0$. We may and shall assume that we have arranged the Jordan blocks of C_χ such that the first $r(1)$ entries in the diagonal are equal to a_1, the next $r(2)$ entries in the diagonal equal to a_2, and so on.

Then the centraliser $\mathfrak{m}$ of S has a direct product decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \times \mathfrak{m}_2 \times \cdots \times \mathfrak{m}_s \tag{1}$$

where $\mathfrak{m}_1$ is the span of all E_{ij} with $1 \le i, j \le r(1)$, and $\mathfrak{m}_2$ the span of all E_{ij} with $r(1) + 1 \le i, j \le r(1) + r(2)$, and so on. We have thus a natural identification

$$\varphi_i : \mathfrak{gl}_{r(i)}(K) \xrightarrow{\sim} \mathfrak{m}_i \qquad \text{for each } i,\ 1 \le i \le s. \tag{2}$$

We can choose the parabolic subalgebra $\mathfrak{p}$ in 5.2 such that $\mathfrak{u}$ is the span of all E_{ij} with $1 \le i < j \le n$ such that E_{ij} does not belong to $\mathfrak{m}$.

Lemma. *The set of isomorphism classes of tuples $(L_1, L_2, \ldots, L_s)$ where each L_i is a simple $U_\chi(\mathfrak{m}_i)$–module with $L_i \otimes K_r \simeq L_i$ for all $r \in \mathbf{F}_p$ is in bijection with the set of isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules of type B.*

Proof. The direct product decomposition in (1) implies that there is a bijection from the set of isomorphism classes of tuples $(L_1, L_2, \ldots, L_s)$ where each L_i is a simple $U_\chi(\mathfrak{m}_i)$–module to the set of isomorphism classes of simple $U_\chi(\mathfrak{m})$–modules induced by $(L_1, L_2, \ldots, L_s) \mapsto L_1 \otimes L_2 \otimes \cdots \otimes L_s$. Note that any $X_i \in \mathfrak{m}_i$ acts on the right hand side as $\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes X_i \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}$. We have then for each $r \in \mathbf{F}_p$ an isomorphism

$$(L_1 \otimes L_2 \otimes \cdots \otimes L_s) \otimes K_r \simeq (L_1 \otimes K_r) \otimes (L_2 \otimes K_r) \otimes \cdots \otimes (L_s \otimes K_r),$$

where K_r on the left hand side is its restriction to $\mathfrak{m}$ and on the right hand side its restriction to the appropriate $\mathfrak{m}_i$. Because of the bijection induced by (3) we see that $(L_1 \otimes L_2 \otimes \cdots \otimes L_s) \otimes K_r \simeq L_1 \otimes L_2 \otimes \cdots \otimes L_s$ if and only if $L_i \otimes K_r \simeq L_i$ for each i. Now the claim follows from Lemma 5.2. ∎

5.4. Then the matrix C_χ has block form with square blocks $C_i \in \mathfrak{gl}_{r(i)}(K)$, $1 \le i \le s$, along the diagonal and all off-diagonal blocks equal to 0. Each C_i has Jordan normal form with all diagonal entries equal to a_i.

Denote by $\chi_i \in \mathfrak{gl}_{r(i)}(K)^*$ the linear form with $C_{\chi_i} = C_i$. We get then $\chi_i = \chi \circ \varphi_i$ since $\chi_i(X) = \mathrm{tr}(C_i X) = \mathrm{tr}(C_\chi \varphi_i(X)) = \chi \circ \varphi_i(X)$ for all $X \in \mathfrak{gl}_{r(i)}(K)$.

This shows: If we regard a $U_\chi(\mathfrak{m}_i)$–module via φ_i as a $\mathfrak{gl}_{r(i)}(K)$–module, then we get a $U_{\chi_i}(\mathfrak{gl}_{r(i)}(K))$–module. Since φ_i is an isomorphism, we get thus a bijection between isomorphism classes of simple modules.

If we regard any K_r via φ_i as a $\mathfrak{gl}_{r(i)}(K)$–module, then we get the analogue of K_r for $\mathfrak{gl}_{r(i)}(K)$ since $\mathrm{tr}(X) = \mathrm{tr}(\varphi_i(X))$ for all $X \in \mathfrak{gl}_{r(i)}(K)$. Therefore φ_i induces a bijection from the set of isomorphism classes of simple $U_\chi(\mathfrak{m}_i)$–module

with $L_i \otimes K_r \simeq L_i$ for all $r \in \mathbf{F}_p$ to the set of isomorphism classes of simple $U_\chi(\mathfrak{gl}_{r(i)}(K))$–modules of type B.

We can apply Lemma 5.1 to $\mathfrak{gl}_{r(i)}(K)$ and χ_i. Any integer is the size of a Jordan block of C_χ if and only if it is the size of a Jordan block of some C_i. Therefore Proposition 2 in the introduction follows now from Lemma 5.3.

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