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Commutators and Cartan Subalgebras in Lie Algebras of Compact Semisimple Lie Groups

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Abstract. Short proofs are given of the following facts concerning the Lie algebra $\mathfrak{g}$ of a compact semisimple Lie group.
1) Any element in $\mathfrak{g}$ is a commutator bracket of some two elements of $\mathfrak{g}$.
2) Given a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there exists a Cartan subalgebra $\mathfrak{h}'$ which is orthogonal to $\mathfrak{h}$.
Moreover, as a Corollary, we obtain the known fact that any element in $\mathfrak{g}$ is conjugate to some element in $\mathfrak{h}^{\perp}$.
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1. Introduction

Unless stated otherwise, we denote in this article by $\mathfrak{g}$ the Lie algebra of a compact connected semisimple Lie group G. We fix a maximal torus T of G whose Lie algebra is a fixed Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$.

For background material on Lie groups and Lie algebras, the reader may consult several excellent references, which include [2], [3], [4], [7], [8] and [12]. The Lie algebra version of the Gotô Theorem [6] for compact semisimple Lie groups is not as well known as it ought to be. It says that

Theorem A. *Every element x in $\mathfrak{g}$ can be written as $x = [y, z]$ for some $y, z \in \mathfrak{g}$.*

The first proof of Theorem A, (i.e. the additive version of Gotô's Theorem) was given by Neeb [9], p. 653 and communicated to the authors of [9]. Neeb's proof was based on Kostant's Convexity Theorem, [11] Theorem 8.1. An alternate proof was obtained by D'Andrea and Maffei in [5], Corollary 1.4, based on some classification tables explained right after Theorem B below.

In Section 2 we prove Theorem A directly by using any Coxeter transformation of the Weyl group $W(\mathfrak{g}, \mathfrak{h})$ for any Cartan subalgebra (i.e, any maximal toral subalgebra) $\mathfrak{h}$ of $\mathfrak{g}$. We remark that certain Coxeter transformations were

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used to prove the Gotô Theorem (on the group level) as in Corollary, Section 4 of Chapter 9 in Bourbaki's book [3], or in Corollary 6.56 in Hofmann and Morris's book [9]. To the best of our knowledge our direct proof of the Lie algebra version of Gotô's Theorem is new.

In Section 3, we shall prove the following theorem, for which we recall that the negative Cartan-Killing form on a compact semisimple Lie algebra $\mathfrak{g}$ is positive definite.

Theorem B. *For each Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$*, its orthogonal complement* $\mathfrak{h}^\perp$ *contains a Cartan subalgebra* $\mathfrak{h}'$*.*

This interesting result was proved recently by d'Andrea and Maffei in [5], Lemma 1.2, via the classification tables of fundamental weights for all types of simple Lie algebras. Specifically, their proof uses the tables in [3] to verify that in any root system of $(\mathfrak{g},\mathfrak{h})$ where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, the *highest root* is either equal to or twice some fundamental weight in all simple Lie algebras except type A_n). However, our proof is a simple consequence of the method we propose in Section 2.

For the following corollary, we recall that G acts on $\mathfrak{g}$ via the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}\,\mathfrak{g}$ (see e.g. [8], Theorem 5.44), and that for each Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ we have $\mathfrak{g} = \bigcup_{g\in G} \mathrm{Ad}(g)\mathfrak{h}$ (see e.g. [8], Theorem 6.27 (25)).

Corollary C. *Let* $\mathfrak{h}$ *be a Cartan subalgebra of* $\mathfrak{g}$ *. Then* $\mathfrak{g} = \bigcup_{g\in G} \mathrm{Ad}(g)\mathfrak{h}^\perp$.

We note that Corollary C was essentially the key step in Neeb's proof of Theorem A involving Kostant's Convexity Theorem, [11], Theorem 8.1.

2. Gotô's Theorem on the Lie Algebra Level

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of a compact semisimple Lie group G and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let $\Phi \subseteq \mathfrak{h}^*$ (the dual of $\mathfrak{h}$) be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We also fix an ordering of the roots, with Φ^+ being the set of positive roots, and let $\{a_1,\dots,a_n\}$ be the corresponding set of simple roots.

Lemma 2.1. *Let* $\{s_1,\dots,s_n\}$ *be the Weyl reflections corresponding to the simple positive roots* $\{a_1,\dots,a_n\}$*. Then, for any permutation of* $\{1,\dots,n\}$*, the Coxeter transformation* $c = s_1 s_2\cdots s_n$ *has no fixed points, other than* 0*. That is,* 1 *is not an eigenvalue of* c*.*

See the very short proof in [10], p. 76. Since c is in the Weyl group of (G,T), where $\mathfrak{h}$ is the Lie algebra of T, the theory of the Weyl group provides us with an element $g \in N(T)$, the normalizer of T in G such that $c = \mathrm{Ad}(g)|\mathfrak{h}$. (See e.g. [8], Theorem 6.52.)

Lemma 2.2. (Cf. [8], Lemma 6.53) *Let* c *be any Coxeter transformation with respect to* $(\mathfrak{g},\mathfrak{h})$ *as in* Lemma 2.1 *and let* $g\in G$ *be such that* $c = \mathrm{Ad}(g)|\mathfrak{h}$*. Then*

$$L := \mathrm{Ad}(g) - \mathrm{Id}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$$

restricts to a bijection on $\mathfrak{h}$.

Proof. Given the theory of the Weyl group, one derives this from Lemma 2.1. ■

Proof of Theorem A. Since the exponential map is surjective in compact connected Lie groups (see e.g. [8], Theorem 6.30(26)), for the element $g \in N(T)$ of Lemma 2.2, there is an element $y \in \mathfrak{g}$ such that $g = \exp(y)$. In view of [8], Theorem 5.44(i)(20), we get

$$L = e^{\mathrm{ad}(y)} - \mathrm{Id}_{\mathfrak{g}} \tag{1}$$

and thus, using Lemma 2.1, we obtain

$$\mathfrak{h} \subseteq L(\mathfrak{g}) = (e^{\mathrm{ad}(y)} - \mathrm{Id}_{\mathfrak{g}})(\mathfrak{g}) \tag{2}$$

Note that (1) may be expressed equivalently as

$$L = \mathrm{ad}(y) \circ f \text{ where } f = \mathrm{Id}_{\mathfrak{g}} + \sum_{m=2}^{\infty} \frac{1}{m!}\,\mathrm{ad}(y)^{m-1}$$

and that from (2) we get $\mathfrak{h} \subseteq [y, f(\mathfrak{g})] \subseteq [y, \mathfrak{g}]$. Since $\mathfrak{g} = \bigcup_{k\in G} \mathrm{Ad}(k)\mathfrak{h}$ by [8], Theorem 6.27, we have $\mathfrak{g} \subseteq \bigcup_{k\in G}[\mathrm{Ad}(k)y, \mathfrak{g}]$, which proves Theorem A. ■

Let $\mathfrak{g}$ be a finite-dimensional vector space and κ a nondegenerate symmetric bilinear form on $\mathfrak{g}$. For an endomorphism ϕ of $\mathfrak{g}$ let ϕ^* denote the adjoint defined by

$$(\forall x, y \in \mathfrak{g}) \quad \kappa(\phi^*(x), y) = \kappa(x, \phi(y)) \tag{3}$$

Then $y \in \ker\phi$ iff $(\forall x \in \mathfrak{g})\, \kappa(x, \phi(y)) = 0$ iff $(\forall x \in \mathfrak{g})\, \kappa(\phi^*(x), y) = 0$ iff $y \in \mathrm{im}(\phi^*)^{\perp}$. Equivalently,

$$(\ker\phi)^{\perp} = \mathrm{im}\,\phi^* \tag{4}$$

A nondegenerate bilinear form κ on a finite-dimensional Lie algebra is called *invariant* if $\mathrm{ad}(x)^* = -\,\mathrm{ad}(x)$, for all $x \in \mathfrak{g}$. In that case, for $\phi = \mathrm{ad}(x)$, relation (4) reads

$$(\forall x \in \mathfrak{g}) \quad Z(x)^{\perp} = [x, \mathfrak{g}],$$

where $Z(x)$ denotes the centralizer of x in $\mathfrak{g}$.

Since a finite-dimensional real Lie algebra permits an invariant nondegenerate (symmetric) bilinear form if it is the Lie algebra of a compact Lie group by [8], Theorem 6.6, or if it is the Lie algebra of a semisimple Lie group by [7] or [12], we have the following observation:

Lemma 2.3. *For each element x in the Lie algebra of a compact Lie group, or a semisimple Lie group, the following relation holds*

$$Z(x)^{\perp} = [x, \mathfrak{g}].$$

We note that Lemma 2.3 is a slight generalization of both Theorem 4.1.6 (ii) in [12] and Lemma 2.1 in [1]. Next we slightly strengthen Theorem A.

Theorem 2.4. *In the setting of Theorem A, there exists a regular element y' of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq [y', \mathfrak{g}]$. In particular, every element x of $\mathfrak{g}$ can be written as $x = [y_1, z_1]$ where y_1 can be chosen to be regular.*

Proof. By Theorem A, $\mathfrak{h} \subseteq [y, \mathfrak{g}]$. By [8], Theorem 6.27, y is contained in a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$, so $\mathfrak{h}' \subseteq Z(y)$. Now $\mathfrak{h}' = Z(y')$ for some regular element y' of $\mathfrak{g}$. (Cf. [8], Lemma 6.24.) So $Z(y') \subseteq Z(y)$. Hence $Z(y)^{\perp} \subseteq Z(y')^{\perp}$. Then $Z(y)^{\perp} = [y, \mathfrak{g}]$ and $Z(y')^{\perp} = [y', \mathfrak{g}]$ by Lemma 2.3. Hence $[y, \mathfrak{g}] \subseteq [y', \mathfrak{g}]$. But we have $\mathfrak{h} \subseteq [y, \mathfrak{g}]$. Hence $\mathfrak{h} \subseteq [y', \mathfrak{g}]$ as desired. ■

3. Orthogonal Cartan subalgebras

In this section, we prove Theorem B more directly than in [5], Lemma 2.2.

Proof of Theorem B. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then by Theorem 2.4, $\mathfrak{h} \subseteq [y', \mathfrak{g}]$ for some regular element y' of $\mathfrak{g}$. Since y' is regular in $\mathfrak{g}$, $Z(y')$ is a Cartan subalgebra of $\mathfrak{g}$. Moreover, $Z(y')^{\perp} = [y', \mathfrak{g}]$ by Lemma 2.3. Hence $\mathfrak{h} \subseteq Z(y')^{\perp}$. Thus the Cartan subalgebras $\mathfrak{h}$ and $Z(y')$ are orthogonal. This proves the theorem. ■

K. H. Hofmanm has kindly pointed out that Theorem B can be obtained directly from Theorem A (thus avoiding the use of regular elements) as follows.

By the proof of Theorem A, we have $\mathfrak{h} \subseteq [y, \mathfrak{g}]$. But $Z(y)^{\perp} = [y, \mathfrak{g}]$ by Lemma 2.3. Hence $\mathfrak{h} \subseteq Z(y)^{\perp}$. By [8], Corollary 6.32 (28bL),

$$Z(y) = \bigcup \{\mathfrak{k} \mid \mathfrak{k} \text{ is a Cartan subalgebra containing } y\}.$$

Accordingly,

$$Z(y)^{\perp} = \bigcap \{\mathfrak{k}^{\perp} \mid \mathfrak{k} \text{ is a Cartan subalgebra containing } y\}.$$

Taking these pieces of information together we obtain

$$\mathfrak{h} \subseteq [y, \mathfrak{g}] = Z(y)^{\perp} \subseteq \mathfrak{k}^{\perp}$$

for each Cartan subalgebra $\mathfrak{k}$ containing y. Thus we have

Theorem 3.1. *Let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of a compact Lie group. Then there is an element $y \in \mathfrak{g}$ such that every Cartan subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ containing y is orthogonal to $\mathfrak{h}$.*

Since the element y is certainly contained in *some* Cartan subalgebra (by [8], Theorem 6.27), this theorem implies Theorem B.

Proof of Corollary C. By Theorem B, there exists a Cartan subalgebra $\mathfrak{h}'$ which is orthogonal to $\mathfrak{h}$. For every element x in $\mathfrak{g}$, we know that there exist a g in G such that $\mathrm{Ad}(g)x \in \mathfrak{h}'$. Hence $\mathrm{Ad}(g)x$ is orthogonal to $\mathfrak{h}$, so $\mathrm{Ad}(g)x$ belongs to $\mathfrak{h}^{\perp}$. Thus, $\mathfrak{h}^{\perp}$ intersects every G-orbit in $\mathfrak{g}$. (Or equivalently, $\mathfrak{g}$ is the union of all $\mathrm{Ad}(g)\mathfrak{h}^{\perp}$ as g varies over G). ■

As noted in the introduction, Corollary C can be obtained almost immediately from Kostant's Convexity Theorem, Theorem 8.1 in [11].

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