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# Square Integrable Representations of Reductive Lie Groups with Admissible Restriction to $\mathbf{SL_2}(\mathbb{R})$

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**Abstract.** In this note we determine the irreducible square integrable representations of a reductive connected Lie group which admit an $H-$admissible restriction to a subgroup $H$ locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$. We show that such a representation is holomorphic and we determine the essentially unique $H$ with this property as well as multiplicity formulae.
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## 1. Introduction

Let $G$ be a connected reductive Lie group in Harish-Chandra class. Hereafter, we suppose $G$ has compact center and we assume $G$ has a square integrable irreducible unitary representation, or, equivalently, according to Harish-Chandra [HCDS2], we assume that $G$ has a compact Cartan subgroup.

From now on, in this paper, "square integrable representation" will mean "square integrable irreducible unitary representation", and we consider only separable Hilbert spaces.

Let $H \subset G$ be a closed subgroup. If $\pi$ is a unitary representation of $G$, we denote by $\mathrm{res}_H(\pi)$ the representation of $H$ obtained by restriction. Recall [Kb1] that a unitary representation $\sigma$ of $H$ is said to be *admissible* if it is a Hilbertian direct sum of irreducible representations of $H$ occurring with finite multiplicities. If $\mathrm{res}_H(\pi)$ is admissible, we say that $\pi$ is $H$-admissible. Among the many problems that may be formulated about a square integrable representation $\pi$ of $G$, the question of when it is $H$-admissible has attracted a lot of interest (see e.g. [Kb1], [Kb2], [Kb4], [Kb5], [Kb6],[DV]), a solution, when $(G, H)$ is a symmetric pair has been obtained in [GW],[KO1],[KO2].

In this note we determine the pairs $(\pi, H)$ consisting of a square integrable representation $\pi$ of $G$, and of a connected, closed subgroup $H$ locally isomorphic to $SL_2(\mathbb{R})$ such that $\pi$ is $H$-admissible.

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Definition 1.7 establishes what we mean by "holomorphic square integrable representation" of $G$. One of our main results is:

**Theorem 1.1.** *Let $\pi$ be a square integrable representation of $G$, and $H$ a connected closed subgroup locally isomorphic to $SL_2(\mathbb{R})$.*

*1) Assume that $\pi$ is $H$-admissible. Then $\pi$ is holomorphic.*

*2) Assume that $\pi$ is holomorphic. Then there is an $H$ as above such that $\pi$ is $H$-admissible, and all such $H$ are conjugate by inner automorphisms of $G$.*

Note that the conjugacy class of $H$ obtained in the second part of the Theorem depends on $\pi$. Let us comment on that dependence.

We write $\mathfrak{a} := \mathfrak{sl}_2(\mathbb{R})$. We choose a connected Lie group $A$ with Lie algebra $\mathfrak{a}$, with finite center, and such that any morphism $\phi : \mathfrak{a} \to \mathfrak{g}$ integrates to a morphism $\phi : A \to G$. In $\mathfrak{a}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$ we consider the basis $\{E, F, Z\}$ with

$$E = \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}, \ F = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \ Z = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{1.1}$$

The brackets are

$$[E, F] = Z, \ [Z, E] = 2E, \ [Z, F] = -2F. \tag{1.2}$$

We denote by $K_A \subset A$ the one-dimensional torus with Lie algebra $\mathfrak{k}_{\mathfrak{a}} := \mathbb{R}iZ$. Let $\chi$ be a character of $K_A$. If $d\chi(Z) = r$ with $r \in \mathbb{R}$, we write $\chi := \chi_r$. This identifies the set of characters with a discrete subgroup $\Lambda^A$ of $\mathbb{R}$ which contains $2\mathbb{Z}$.

Let $\tau$ be a unitary representation of $K_A$ in a Hilbert space $V$. It is the Hilbertian direct sum of the *weight spaces* $V_r$, in which $K_A$ acts by the character $\chi_r$. We write $\Lambda^A_\tau \subset \Lambda^A$ for the support of $\tau$, that is the set of $r$ such that $V_r \neq \{0\}$.

Suppose that $\sigma$ is a square integrable representation of $A$ in a Hilbert space $V$. We write $\Lambda^A_\sigma$ for the support as a $K_A$-representation. Then it is well known (see [La] page 123, Theorem 8) that one of two following statements holds:

**Proposition 1.2.**

1. *There exists $r > 1$ such that $\Lambda^A_\sigma = r + 2\mathbb{N}$ (in this case we say that $V$ has a lowest weight, and that it is $E$-holomorphic)*
2. *There exists $r < -1$ such that $\Lambda^A_\sigma = r - 2\mathbb{N}$ (in this case we say that $V$ has a highest weight, and that it is $F$-holomorphic)*

We consider a morphism $\phi : A \to G$. If $\pi$ is a representation of $G$, then we write $\mathrm{res}_\phi(\pi) := \pi \circ \phi$ for the corresponding representation of $A$. We say that two such morphisms are conjugate if they are conjugate by inner automorphisms of the target group $G$. Here is a small, but useful, complement to the second part of Theorem 1.1

**Theorem 1.3.** *Let $\pi$ be a holomorphic square integrable representation of $G$.*

*1) There exists a morphism $\phi : A \to G$ such that $\mathrm{res}_\phi(\pi)$ is admissible.*

*2) Let $\phi : A \to G$ be a morphism such that $\mathrm{res}_\phi(\pi)$ is admissible. Then all irreducible factors occurring in $\mathrm{res}_\phi(\pi)$ are of the same type : more precisely, they are all $E$-holomorphic square integrable representations, or all $F$-holomorphic square integrable representations of $A$.*

*3) There exists a morphism $\phi : A \to G$ such that $\mathrm{res}_\phi(\pi)$ is admissible and such that all irreducible factors occurring in $\mathrm{res}_\phi(\pi)$ are $E$-holomorphic. Two such morphisms are conjugate.*

**Remark 1.4.** Consider the external $\kappa$ automorphism of $A$ whose image in $SL_2(\mathbb{R})$ is the conjugacy by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If $\sigma$ is a $E$-holomorphic representation, then $\sigma \circ \kappa$ is a $F$-holomorphic representation. This is the reason why it is more precise to consider morphisms $\phi : A \to G$ than the images $\phi(A) \subset G$.

Since $G$ has compact center, holomorphic square integrable representations exist exactly when the Lie algebra $\mathfrak{g}$ of $G$ is a direct sum of one dimensional ideals, and of simple ideals such that the noncompact simple ones give rise to irreducible Hermitian symmetric space. Index these noncompact simple ideals as $\mathfrak{g}_u$ where $u$ runs in a set with $N$ elements.

The holomorphic square integrable representations of $G$ are assembled in $2^N$ families (in sloppy terms, deciding what is *holomorphic* or *anti-holomorphic* on each simple factor $\mathfrak{g}_u$). In Theorem 1.3, the conjugacy class of $\phi$ depends exactly on the family to which $\pi$ belongs.

**Corollary 1.5.** *Let the notations be as in Theorem 1.3 part 3. The number of conjugacy classes of morphism $\phi : A \to G$ obtained when considering all holomorphic $\pi$ is equal to $2^N$.*

We prove in fact a Theorem more general than the first part of Theorem 1.1. This generalization makes clear what is involved.

**Theorem 1.6.** *Let $\pi$ be a square integrable representations of $G$, and $H$ a connected closed reductive subgroup with commutative maximal compact subgroups. Assume that $\pi$ is $H$-admissible. Then $\pi$ is holomorphic.*

Our next result deals with the description of the conjugacy class of morphisms $\phi : A \to G$ occurring in the third part of Theorem 1.3. For this, we need more notations.

We fix a maximal compact subgroup $K \subset G$, and a Cartan subgroup $T \subset K$. Our assumption on $G$ says that $T$ is a Cartan subgroup of $G$.

The Lie algebra of a Lie group is denoted by the corresponding lowercase Fraktur font and the complexification of a real Lie algebra, or a vector space, is denoted by adding the subscript $\mathbb{C}$.

Denote by $\Phi(\mathfrak{g}, \mathfrak{t})$ (resp. $\Phi(\mathfrak{k}, \mathfrak{t})$) the root system of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$ (resp. $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{k}_\mathbb{C}$). We write also $\Phi := \Phi(\mathfrak{g}, \mathfrak{t})$ and $\Phi_c := \Phi(\mathfrak{k}, \mathfrak{t})$. The set of noncompact roots is defined by $\Phi_n := \Phi \smallsetminus \Phi_c$. We write $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C}$ for the Cartan decomposition corresponding to the pair $(\mathfrak{g}, K)$.

Let $\gamma \in \Phi$. We denote by $Z_\gamma \in i\mathfrak{t}$ the corresponding coroot, and $\mathfrak{g}_\gamma \subset \mathfrak{g}_\mathbb{C}$ the corresponding root space. For $\gamma \in \Phi_c$ we have $\mathfrak{g}_\gamma \subset \mathfrak{k}_\mathbb{C}$, and for $\gamma \in \Phi_n$ we have $\mathfrak{g}_\gamma \subset \mathfrak{p}_\mathbb{C}$

Let $\beta \in \Phi_n$. We choose a morphism

$$\phi_\beta : \mathfrak{a} \to \mathfrak{g} \tag{1.3}$$

such that its complex extension satisfies

$$\phi_\beta(E) \in \mathfrak{g}_\beta, \ \phi_\beta(F) \in \mathfrak{g}_{-\beta}. \tag{1.4}$$

Note that this implies $\phi_\beta(Z) = Z_\beta$, and note that two such morphisms are conjugate by an inner automorphism.

Let $S := \{\beta_1, \dots, \beta_s\} \subset \Phi_n$ a set of noncompact roots consisting of strongly orthogonal roots (that is, for all $i$ and $j$, $\beta_i \pm \beta_j$ is not a root). Then we can consider the diagonal morphism

$$\phi_S : \mathfrak{a} \to \mathfrak{g} \tag{1.5}$$

defined for $x \in \mathfrak{a}$ by

$$\phi_S(x) = \phi_{\beta_1}(x) + \cdots + \phi_{\beta_s}(x). \tag{1.6}$$

Recall that a system of positive roots $\Psi \subset \Phi$ of is said to be *holomorphic* if the sum $\alpha + \beta$ is not a root for every pair $\alpha, \beta \in \Psi_n := \Psi \cap \Phi_n$.

**Definition 1.7.** Let $\Psi \subset \Phi$ be a positive system. We say that a square integrable representation of $G$ is $\Psi$*-holomorphic* if its underlying Harish-Chandra module is a lowest weight module with respect to $\Psi$. We say that a square integrable representation of $G$ is *holomorphic* if it is $\Psi$-holomorphic for some $\Psi$.

If a square integrable representation is $\Psi$-holomorphic, then $\Psi$ is holomorphic. Conversely, when $\Psi$ is holomorphic, Harish-Chandra determined which lowest weight modules correspond to square integrable representations (see the fundamental paper of Harish-Chandra [HC IV], which deals more generally with irreducible unitary representations not necessarily square integrable).

Consider a holomorphic system of positive roots $\Psi \subset \Phi$. Harish-Chandra described in [HC VI] a particular maximal set $S^\Psi_{HC} := \{\beta_1, \dots, \beta_s\} \subset \Psi_n$ of strongly orthogonal roots in $\Psi_n$. Henceforth, we refer to $S^\Psi_{HC}$ as Harish-Chandra set. We denote by $\phi^\Psi_{HC} : \mathfrak{a} \to \mathfrak{g}$ the corresponding morphism. We get the following complement to Theorem 1.3.

**Theorem 1.8.** *Let $\Psi \subset \Phi$ be a holomorphic system of positive roots. We set $\phi := \phi^\Psi_{HC}$. Let $\pi$ be a $\Psi$-holomorphic square integrable representation of $G$. Then, the representation* $\mathrm{res}_\phi(\pi)$ *of $A$ is admissible, and all irreducible factors occurring in* $\mathrm{res}_\phi(\pi)$ *are E-holomorphic square integrable representations of $A$.*

This note is organized as follows. In this introduction (Section 1) we stated our main results except those on multiplicities (which are presented in Section 5) and the explicit description of the subgroups involved in Theorem 1.8 (which are presented in Section 4). In Section 2 and 3 we present proofs of the Theorems stated in the introduction.

## 2. Proof of Theorem 1.6

**2.1. Generalities about restrictions.** In this subsection we recall for reference some mostly well known facts relative to restrictions of unitary representations, which do not depend on the special assumption of Theorem 1.6.

We consider a separable Lie group $G'$, a closed subgroup $H \subset G'$, an unitary representation $\tau$ of $G'$ in a separable Hilbert space, and $\sigma = \mathrm{res}_H(\tau)$ its restriction to $H$. For a proof of the following statements, see [Kb1] Theorem 1.2, [Kb3] Cor. 8.7. respectively.

**Proposition 2.1.** *Suppose that $\sigma$ is admissible. Then $\tau$ is admissible.*

**Proposition 2.2.** *Suppose that $\tau$ is square integrable. Let $\xi$ an irreducible sub-representation of $\sigma$. Then $\xi$ is square integrable.*

We consider now the case of a connected reductive group $H$ with compact center, and of a maximal compact subgroup $L \subset H$. There are many interesting examples of admissible unitary representations of $H$ which are $L$ admissible.

Let us recall a fundamental result of Harish-Chandra. For a proof see [Wa] Theorem 4.5.2.11 and note on page 319.

**Proposition 2.3.** *Let $\sigma$ be a finite direct sum of irreducible unitary representations of $H$. Then $\sigma$ is $L$-admissible.*

In this proposition, it is not possible to replace the word "finite " by "admissible". For instance, if $H$ is not compact, an infinite direct sum of distinct spherical irreducible unitary representations of $H$ is $H$-admissible, but the trivial representation of $L$ occurs with infinite multiplicity.

However, this is true in a particular case which is relevant for this note.

**Proposition 2.4.** *Let $\sigma$ be an admissible Hilbertian direct sum of square integrable representations of $H$. Then $\sigma$ is $L$-admissible.*

**Proof.** We write

$$\sigma = \widehat{\bigoplus}\, m(\xi)\ \xi \tag{2.1}$$

where the $\xi$ are square integrable distinct representations of $H$, and the $m(\xi)$ are positive integers.

Since $L$ is compact, the representation $\mathrm{res}_L(\sigma)$ is a Hilbertian direct sum of irreducible representations of $L$. We have to prove that the multiplicities are finite. We argue by contradiction. So let us suppose that $\tau$ is an irreducible representation of $L$ occurring with infinite multiplicity in $\mathrm{res}_L(\pi)$. By proposition 2.3, the multiplicity of $\tau$ in any $\xi$ is finite; thus, there is an infinite number of $\xi$ such that $\tau$ occurs in $\xi$.

However, this contradicts another Theorem of Harish-Chandra, [HCDS2] Lemma 70, which says that the number of square integrable representations $\xi$ such that $\tau$ occurs in $\xi$ is finite. ∎

We come back to our reductive connected Lie group $G$. Let $H \subset G$ be a closed connected reductive subgroup. We fix a maximal compact subgroup $L \subset H$.

**Corollary 2.5.** *Suppose that $\pi$ is a square integrable representation of $G$. Suppose that $\pi$ is $H$-admissible. Then it is $L$-admissible.*

**Proof.** Proposition 2.2 says that we can apply Proposition 2.4. ■

This result is an important ingredient of our paper [DV]. We provided its simple proof, since it was not included in [DV].

It is an interesting unsolved problem (considered by Kobayashi [Kb5]) whether Corollary 2.5 remains true when $\pi$ is assumed only to be irreducible unitary. To our knowledge, the answer is not known even when $H = SL_2(\mathbb{R})$. On this problem, see also [ZL].

**Corollary 2.6.** *Suppose that $\pi$ is a square integrable representation of $G$. Suppose that $\mathrm{res}_H(\pi)$ is admissible. Then the centralizer of $L$ in $G$ is compact.*

**Proof.** Let $\tilde{B} \subset G$ be the centralizer of $L$, $B$ its connected component, and consider the connected reductive group $D := BL$. Note that since $\tilde{B}/B$ is finite, it is sufficient to prove that $B$ is compact.

By Corollary 2.5, $\mathrm{res}_L(\pi)$ is admissible, and, by 2.1, $\mathrm{res}_D(\pi)$ is admissible. We pick an irreducible representation $\tau$ of $D$ which occurs in $\mathrm{res}_D(\pi)$. It is square integrable by Proposition 2.2. Moreover, $\mathrm{res}_L(\tau)$ is admissible.

Assume that $B$ is not compact. We choose a simple noncompact ideal $\mathfrak{b}' \subset \mathfrak{b}$ of $\mathfrak{b}$, and denote by $\mathfrak{d}'$ the centralizer of $\mathfrak{b}'$ in $\mathfrak{d}$. Denote by $B'$ and $D'$ the corresponding connected groups. The direct product $B' \times D'$ is a finite covering of $D$, and we have $L \subset D'$. The representation $\tau$ is the Hilbertian tensor product $\tau = \tau_1 \otimes \tau_2$ where $\tau_1$ is a square integrable representation of $B'$ and $\tau_2$ a square integrable representation of $D'$. Thus $\mathrm{res}_{D'}(\tau)$ is a Hilbertian direct sum of representations isomorphic to $\tau_2$, and the multiplicity is finite (i.e. $\mathrm{res}_{D'}(\tau)$ is admissible ) if and only if $\tau_1$ is finite dimensional. However, square integrable representations of connected real noncompact reductive groups are infinite dimensional. So $\mathrm{res}_{D'}(\tau)$ is not admissible, and so $\mathrm{res}_L(\tau)$ is not admissible. This is a contradiction, and so, $B$ is compact. ■

### 2.2. Restriction to $T$.

In this subsection we follow the notation of Section 1 and we consider a particular case of Theorem 1.6. We need it in the proof of Theorem 1.6, and moreover it deserves particular attention.

**Theorem 2.7.** *Let $\pi$ be a square integrable representations of $G$. Assume that $\pi$ is $T$-admissible. Then $\pi$ is holomorphic.*

The proof follows the line of arguments given in [Va], which uses ideas from [Vo]. We give the details for completeness. Before going to the proof, we need some more notations.

We denote by $P \subset i\mathfrak{t}^*$ the lattice of weights (the differential of characters of $T$). For $\lambda \in P$ we denote by $\chi_\lambda$ or $e^\lambda$ the corresponding character of $T$. We denote by $R \subset P$ the subgroup generated by $\Phi(\mathfrak{g}, \mathfrak{t})$, and by $R_\mathfrak{k} \subset R$ the subgroup generated by $\Phi(\mathfrak{k}, \mathfrak{t})$.

Let $W_K$ be the Weyl group of $K$, that is the normalizer of $T$ in $K$ (or in $G$, it is the same) divided by $T$. We choose a positive system $\Phi_c^+ \subset \Phi(\mathfrak{k}, \mathfrak{t})$. We denote by $P^+ \subset P$ the set of dominant weights, and for $\mu \in P^+$, by $\tau_\mu$ the irreducible representation of $K$ with highest root $\mu$.

Let $\pi$ be an unitary representation of $G$ in a Hilbert space $V$. Since $K$ is compact, we consider the Hilbertian decomposition of $\mathrm{res}_K(\pi)$ into its isotypic components. We denote by $P_\pi^+ \subset P^+$ the support of the restriction of $\pi$ to $K$, that is the set of $\mu \in P^+$ such that $\tau_\mu$ occurs in $\pi$. We have

$$V = \widehat{\oplus}_{\mu \in P_\pi^+} V_{\tau_\mu} \tag{2.2}$$

where $V_{\tau_\mu}$ is a nonzero $K$-invariant closed subspace in which $K$ acts as a (possibly infinite) multiple of $\tau_\mu$.

In a similar manner, considering the restriction to $T$, we write $P_\pi$ and $P_{\tau_\mu}$ for the support of the restriction of $\pi$ and $\tau_\mu$ to $T$. For $\lambda \in P$, we write $V_\lambda \subset V$ the corresponding weight space. We have the corresponding Hilbertian direct sums (the second one being also the algebraic direct sum since $P_{\tau_\mu}$ is a finite set).

$$V = \widehat{\oplus}_{\lambda \in P_\pi} V_\lambda \tag{2.3}$$

$$V_{\tau_\mu} = \oplus_{\lambda \in P_{\tau_\mu}} V_{\tau_\mu \lambda}. \tag{2.4}$$

We obtain

$$P_\pi = \bigcup_{\mu \in P_\pi^+} P_{\tau_\mu}. \tag{2.5}$$

The set $P_{\tau_\mu}$ is well known: We denote by $\mathrm{conv}(S)$ the convex hull of a subset $S \subset i\mathfrak{t}^*$. We have

$$P_{\tau_\mu} = \mathrm{conv}(W_K \cdot \mu) \cap (\mu + R_\mathfrak{k}). \tag{2.6}$$

We denote by $V_{K-f}$ the algebraic sum

$$V_{K-f} = \oplus_{\mu \in P_\pi^+} V_{\tau_\mu}, \tag{2.7}$$

this is the space of $K$-finite vectors of $V$.

We assume now that $\pi$ is irreducible. Since it is $K$-admissible each $V_{\tau_\mu}$ is finite dimensional. The space $V_{K-f}$ is contained in the space of smooth vectors of $\pi$, and $V_{K-f}$ is stable under the resulting representation of $\mathfrak{g}_\mathbb{C}$ in $V_{K-f}$. We denote by $\pi_{K-f}$ the resulting representation of $\mathfrak{g}_\mathbb{C}$ in $V_{K-f}$. The space $V_{K-f}$, equipped with its $\mathfrak{g}_\mathbb{C}$ and $K$ actions is called the Harish-Chandra module of $\pi$. Harish-Chandra proved that $V_{K-f}$ is an irreducible $\mathfrak{g}_\mathbb{C}$-module.

Let $\nu$ be a representation of $\mathfrak{g}_\mathbb{C}$ in some vector space $W$, and $\mathfrak{b} \subset \mathfrak{g}_\mathbb{C}$ some complex Lie subalgebra. We say that $\nu$ is $\mathfrak{b}$-admissible if $W$ is an algebraic direct sum of simple $\mathfrak{b}$-modules occurring with finite multiplicities. The subject of irreducible $\mathfrak{b}$-admissible modules is a very lively subject.

Suppose that $H \subset G$ is a closed subgroup, and $\nu = \pi_{K-f}$ for some irreducible unitary representation of $G$. Except for results in [Kb2],[Kb3],[Kb4], [DV], it is difficult to compare $\mathfrak{h}_{\mathbb{C}}$-admissibility of $\pi_{K-f}$ and $H$-admissibility of $\pi$, unless $H = L$. However, if $L \subset K$ we have the following simple well known facts.

**Proposition 2.8.** *Let $\pi$ be an irreducible unitary representation of $G$. Let $L \subset K$ be a closed subgroup.*

*1) $\pi$ is $L$-admissible if and only if $\pi_{K-f}$ is $\mathfrak{l}_{\mathbb{C}}$-admissible.*

*2) $\pi$ is $L$-admissible if and only if, for every irreducible unitary representation $\xi$ of $L$, the number of $\mu \in P_\pi^+$ such that $\xi$ occurs in $\tau_\mu$ is finite.*

**Proof of Theorem 2.7.** For the underlying Harish-Chandra module $V_{K-f}$ of $K$-finite vectors in $V$, and $\nu := \pi_{K-f}$ the corresponding representation of $\mathfrak{g}_{\mathbb{C}}$ in $V_{K-f}$. It follows from Proposition 2.8 that $\nu$ is $\mathfrak{t}_{\mathbb{C}}$-admissible, we have

$$V_{K-f} = \oplus_{\lambda \in P_\pi}(V_{K-f})_\lambda, \tag{2.8}$$

where each weight space $(V_{K-f})_\lambda = V_\lambda$ is finite dimensional. We must prove that $V_{K-f}$ is a lowest weight module with respect to some positive holomorphic system $\Psi \subset \Phi$.

For a noncompact root $\beta \in \Phi_n$, we recall the group homomorphism $\phi_\beta : A \to G$ as the lift of (1.3). We denote by $H_\beta \subset G$ its image, and by $\tilde{H}_\beta \subset G$ the group $TH_\beta \subset G$. We write $E_\beta := \phi_\beta(E) \in \mathfrak{p}_{\mathbb{C}}$, $F_\beta := \phi_\beta(F) \in \mathfrak{p}_{\mathbb{C}}$.

By Proposition 2.1, the restriction of $\pi$ to $\tilde{H}_\beta$ is admissible. According to Proposition 2.2, we write

$$V = \widehat{\oplus} V_\sigma \tag{2.9}$$

where $\sigma$ is a square integrable representation of $\tilde{H}_\beta$ occurring in $V$, and where the action of $\tilde{H}_\beta$ in $V_\sigma$ is a finite positive multiple of $\sigma$.

Consider $\lambda \in P_\pi$. We get

$$V_\lambda = \widehat{\oplus}\, V_\sigma \cap V_\lambda. \tag{2.10}$$

This implies that the number of $\sigma$ such that $\lambda$ occurs in $\sigma$ is finite, and we have

$$V_\lambda = \oplus\, V_\sigma \cap V_\lambda. \tag{2.11}$$

Consider a square integrable representation $\sigma$ of $\tilde{H}_\beta$ such that $\lambda$ occurs in $\sigma$. Let $W_\sigma$ be the space of $T$-finite vectors of the representation $\sigma$. We repeat with more details what follows from the properties of square integrable representations of $A$ stated in Proposition 1.2.

- $W_\sigma$ is a lowest weight representation, with lowest weight $\lambda_\sigma$. Then the support $P_\sigma$ is equal to $\{\lambda_\sigma + k\beta \,|\, k \in \mathbb{N}\}$, the weight spaces of $W_\sigma$ are one-dimensional, the action of $E_\beta$ in $W_\sigma$ is injective, and the action of $F_\beta$ in $W_\sigma$ is locally nilpotent. Moreover, we have:

$$1 < \lambda_\sigma(Z_\beta) \leqslant \lambda(Z_\beta). \tag{2.12}$$

- $W_\sigma$ is a highest weight representation, with highest weight $\lambda_\sigma$. Then the support $P_\sigma$ is equal to $\{\lambda_\sigma - k\beta \,|\, k \in \mathbb{N}\}$, the weight spaces of $W_\sigma$ are one-dimensional, the action of $F_\beta$ in $W_\sigma$ is injective, and the action of $E_\beta$ in $W_\sigma$ is locally nilpotent. Moreover, we have :

$$-1 > \lambda_\sigma(Z_\beta) \geqslant \lambda(Z_\beta). \tag{2.13}$$

Following [Vo], for $\beta \in \Phi_n$ we consider the space $V_{K-f}(\beta) \subset V_{K-f}$ of locally $F_\beta$-nilpotent vectors. It is an invariant $\mathfrak{g}_{\mathbb{C}}$-subspace. So, one of the two following statements holds:

- $V_{K-f}(\beta) = 0$ and $V_{K-f}(-\beta,) = V_{K-f}$,
- $V_{K-f}(\beta) = V_{K-f}$ and $V_{K-f}(-\beta) = 0$.

We denote by $\Psi_n$ the set of $\beta \in \Phi_n$ such that the action of $F_\beta$ is locally nilpotent in $U$. Let $\lambda \in P_\pi$. By (2.12), we have

$$\lambda(Z_\beta) > 1 \tag{2.14}$$

for all $\beta \in \Psi_n$. So we can choose a positive system $\Psi \subset \Phi$ such that $\Psi \cap \Phi_n = \Psi_n$. Since $V_{K-f}$ is also a $K$-module, the set $\Psi_n$ is invariant by $W_K$. This implies that $\Psi$ is holomorphic. Thus, for $\beta$ and $\beta'$ in $\Psi_n$, we have $[F_\beta, F_{\beta'}] = 0$.

Let $(V_{K-f})_{min} \subset V_{K-f}$ be the space of elements $u \in V_{K-f}$ such that $F_\beta u = 0$ for all $\beta \in \Psi_n$. It is a nonzero vector subspace, which is invariant by $\mathfrak{k}_{\mathbb{C}}$. So we may find in $(V_{K-f})_{min}$ a nonzero vector $u$ such that $X_{-\beta}\, u = 0$ for all $\beta \in \Psi_c$ and $X_{-\beta} \in \mathfrak{k}_{\mathbb{C}}$ a root vector. So $u$ is a lowest weight vector for $V_{K-f}$ with respect to the holomorphic positive system of roots $\Psi$ In [HC IV] is defined holomorphic discrete series representation to be a lowest weight for some holomorphic system and unitary representation. ∎

Here are some useful complements to Theorem 2.7. We compute the support $P_\pi$ of the representation occurring in Theorem 2.7. For a subset $S \subset \Phi$, we write $\mathbb{N}\, S \subset R$ for the semi-group generated by $S$.

**Theorem 2.9.** *Let $\Psi \subset \Phi$ be a holomorphic system of positive roots. Let $(\pi, V)$ be a square integrable representation such that the Harish-Chandra module $V_{K-f}$ has a nonzero lowest weight vector $u \in V_{K-f}$ with respect to $\Psi$. Up to a scalar, $u$ is unique. We denote by $\lambda_\pi \in P_\pi$ its weight.*

*1) For all $\beta \in \Psi_n$ and all $\lambda \in P_\pi$ we have*

$$\lambda(Z_\beta) > 1 \tag{2.15}$$

*2) Let $(V_{K-f})_{min} \subset V_{K-f}$ be the space of elements $u \in V_{K-f}$ such that $F_\beta u = 0$ for all $\beta \in \Psi_n$. Then it is an irreducible $\mathfrak{k}_{\mathbb{C}}$-module with lowest weight $\lambda_\pi$. We denote by $P_{\pi,min} \subset P_\pi$ the set of weights of $(V_{K-f})_{min}$.*

*3) The set $P_\pi$ is computed in terms of $(V_{K-f})_{min}$ by the formula*

$$P_\pi = \bigcup_{\mu \in P_{\pi,min}} \mu + \mathbb{N}\Psi_n. \tag{2.16}$$

**Proof.** The uniqueness of $u$ is a general fact about irreducible lowest weights $\mathfrak{g}_{\mathbb{C}}$-modules.

Part 1) In [HC IV] Theorem 4 and Corollary page 776 it is shown that the restriction of $\pi$ to $T$ is an admissible representation, now, the argument is just a repetition of (2.14).

Part 2) follows from the fact that any lowest weight (with respect to $\Psi_c$) in $(V_{K-f})_{min}$ is a lowest weight (with respect to $\Psi$) in $V_{K-f}$. So it is unique.

Consider part 3). The inclusion

$$P_\pi \subset \bigcup_{\mu\in P_{\pi,min}} \mu + \mathbb{N}\Psi_n \tag{2.17}$$

follows in a standard way from the fact that $u$ is a lowest weight vector. The equality follows from the following assertion :

Let $\lambda \in P_\pi$. Then $\lambda + \mathbb{N}\Psi_n \subset P_\pi$.

This assertion follows immediately from the fact (recall the definition of $\Psi_n$ in the proof of Theorem 2.7) that $E_\beta$ acts freely in $V_{K-f}$ for all $\beta \in \Psi_n$. ■

**Remark 2.10.** 1) Much more is known about the holomorphic square integrable representations. We just stated what we need below about the support, with its simple proofs.

2) Unfortunately, we do not know simple statements analogous to (2.16) to describe the set of representations of $K$, or more generally of a closed subgroup $L$ with $T \subsetneq L \subset K$, which occur in $\pi$.

Let $U \subset T$ a closed connected subgroup. We study the $U$-admissible square integrable representations of $G$. Since such a representation is $T$-admissible, it is one of the representations considered in Theorem 2.9. We use the notations of Theorem 2.9. In the real vector space $i\mathfrak{t}^*$ we consider the polyhedral cone $\mathbb{R}_+\Psi_n$ generated by $\Psi_n$, and the orthogonal subspace $\mathfrak{u}^\perp$ of elements which are zero on $\mathfrak{u}$.

**Theorem 2.11.** *Let $\pi$ be a $\Psi$-holomorphic square integrable representation of $G$ and $U \subset T$ a closed connected subgroup. The representation $\pi$ is $U$-admissible if and only if we have*

$$\mathbb{R}_+\Psi_n \cap \mathfrak{u}^\perp = \{0\}. \tag{2.18}$$

**Proof.** Consider $i\mathfrak{u}^*$ and the projection $p : i\mathfrak{t}^* \to i\mathfrak{u}^*$ which is obtained by restriction of linear forms from $\mathfrak{t}$ to $\mathfrak{u}$. The kernel of $p$ is $\mathfrak{u}^\perp$. Let $P_U$ be the set of differential of characters of $U$. It is equal to $p(P)$, and $\mathfrak{u}^\perp \cap P$ is the group of $\lambda \in P$ such that the restriction of $\chi_\lambda$ to $U$ is trivial.

Let $\xi \in P_U$. The space $V_\xi$ in which $U$ acts by the character $\chi_\xi$ is the Hilbertian direct sum of the spaces $V_\lambda$ with $\lambda \in P_\pi$ and $p(\lambda) = \xi$. Since the $V_\lambda$ are finite dimensional, we obtain the following assertion ;

The representation $\pi$ is $U$-admissible if and only if, for every $\lambda \in P_\pi$, the number of $\lambda' \in P_\pi$ with $\lambda - \lambda' \in \mathfrak{u}^\perp$ is finite.

Because of the description (2.17) of $P_\pi$, we obtain the following assertion.

The representation $\pi$ is $U$-admissible if and only if, for every $\lambda \in P_\pi$, the number of $\lambda' \in \mathfrak{u}^\perp \cap \lambda + \mathbb{N}\Psi_n$ is finite. It is a standard fact about convex cones that this condition is independent of $\lambda \in P_\pi$, and that it is equivalent to the condition stated in the Theorem. ∎

We use below a special case of this Theorem :

**Corollary 2.12.** *Let $\pi$ be a $\Psi$-holomorphic square integrable representation of $G$ and $U \subset T$ a closed connected one-dimensional subgroup. The representation $\pi$ is $U$-admissible if and only if there exists $Z_U \in i\mathfrak{u}$ such that $\beta(Z_U) > 0$ for all $\beta \in \Psi_n$.*

**Proof of Theorem 1.6.** We use the notations of Theorem 1.6. Let $U \subset H$ be a maximal compact subgroup. Replacing if necessary $H$ by a conjugate, we assume that $U \subset T$. By Proposition 2.4, $\pi$ is $U$-admissible. By Proposition 2.1, $\pi$ is $T$-admissible. By Theorem 2.7, $\pi$ is holomorphic. ∎

## 3. Proof of Theorem 1.3 and Theorem 1.8

We consider a holomorphic system $\Psi \subset \Phi$ and a $\Psi$-holomorphic square integrable representation $\pi$ of $G$.

**Proof of Theorem 1.3, part 2 and 3.** We consider a morphism $\phi : A \to G$. By replacing $\phi$ by a conjugate, we may and do assume that we have $E_\phi := \phi(E) \in \mathfrak{p}_\mathbb{C}$, $F_\phi := \phi(E) \in \mathfrak{p}_\mathbb{C}$ and $Z_\phi := \phi(Z) \in i\mathfrak{t}$. Let $K_\phi := \phi(K_A)$. Then $K_\phi$ is a closed connected one-dimensional subgroup of $T$, and we have $Z_\phi \in i\mathfrak{k}_\phi$.

We assume now that $\mathrm{res}_\phi(\pi)$ is admissible. It follows from Corollary 2.12 that either $\beta(Z_\phi) > 0$ for all $\beta \in \Psi_n$, or $\beta(Z_\phi) < 0$ for all $\beta \in \Psi_n$. Recall the external automorphism $\kappa$ of $A$ defined in remark 1.4. Replacing if necessary $\phi$ by $\phi \circ \kappa$, we may and do assume that we have

$$\beta(Z_\phi) > 0 \text{ for all } \beta \in \Psi_n. \tag{3.1}$$

We denote by $\mathfrak{p}^\Psi \subset \mathfrak{p}_\mathbb{C}$ the space spanned by the root spaces $\mathfrak{g}_\beta$ with $\beta \in \Psi_n$. It is stable under the action of $K_\mathbb{C}$, where $K_\mathbb{C}$ is the analytic subgroup of the adjoint group of $\mathfrak{g}_\mathbb{C}$ with Lie algebra the image of $\mathfrak{k}_\mathbb{C}$.

We write $E_\phi = \sum_{\beta \in \Phi_n} e_\beta$ with well determined $e_\phi \in \mathfrak{g}_\phi$. Since $[Z_\phi, E_\phi] = 2E_\phi$, we see that $e_\beta = 0$ if $\beta(Z_\phi) \neq 2$. It follows from (3.1) that

$$E_\phi \in \mathfrak{p}^\Psi. \tag{3.2}$$

It follows from e.g Proposition III.8 in [HNO] that there exists a subset $S \subset S^\Psi_{HC}$ such that $\phi$ is conjugate to $\phi_S$. So we may and do assume that $\phi = \phi_S$.

We now show that $S = S^\Psi_{HC}$. Suppose that $S \neq S^\Psi_{HC}$. Then there exists $\beta \in \Psi_n$ which is strongly orthogonal to all $\beta_j \in S$. It follows that the centralizer of $Z_\phi$ in $G$ contains the noncompact group $\phi_\beta(A)$. This contradicts Corollary 2.6. ∎

**Proof of Theorem 1.3, part 1, and Theorem 1.8.** Let the notations be as above, and consider $\phi = \phi^{\Psi}_{HC}$ and $Z_\phi$. It follows from Corollary 2.12 that part 1 of Theorem 1.3 as well, as Theorem 1.8, follows from the following:

Claim : We have $\beta(Z_\phi) > 0$ for all $\beta \in \Psi_{\mathfrak{n}}$.

In fact, let $\beta \in \Psi_{\mathfrak{n}}$, since the system $\Psi$ is holomorphic, $\beta + \beta_j$ never is a root. Hence, for every $j$, we have $\beta(Z_{\beta_j}) \geqslant 0$. Since all the roots in the Harish-Chandra set $S^{\Psi}_{HC}$ are long and that roots of distinct length which are orthogonal are strongly orthogonal we have that $\beta$ is not orthogonal to some root in $S^{\Psi}_{HC}$, otherwise $S^{\Psi}_{HC}$ would not be maximal, which shows the claim. ∎

**Remark 3.1.** We recall the Hermitian symmetric space $G/K$ is a *tube domain* if it is biholomorphic to a tube domain. In [KrW] it is shown that $G/K$ is a tube domain if and only if the characteristic vector $Z_\phi$ for $\phi = \phi^{\Psi}_{HC}$, belongs to the center of $\mathfrak{k}_{\mathbb{C}}$. We also set notation for the simple roots in $\Psi$,

$$\{\alpha_1, \dots, \alpha_\ell\} \qquad \text{where}\, \alpha_1, \dots, \alpha_{\ell-1} \in \Phi(\mathfrak{k}, \mathfrak{t}) \text{ and } \alpha_\ell \in \Phi_n. \tag{3.3}$$

We write highest root as $\beta_M = \sum_j c_j \alpha_j$ with $c_j \geqslant 1$ for all $j$ and $c_\ell = 1$. Let $Z_\phi$ for $\phi = \phi^{\Psi}_{HC}$. We have $\alpha_\ell(Z_\phi) = 2$ and $\alpha_j(Z_\phi) = 0$, for all $1 \leqslant j \leqslant \ell - 1$, if and only if $G/K$ is a tube domain. Whenever, $G/K$ is not a tube domain, we have $\alpha_\ell(Z_\phi) = 1$, and $\alpha(Z_\phi) = 0$ for all the compact simple roots but one for which we have $\alpha(Z_\phi) = 1$. Indeed, for a holomorphic system it happens that for any $X$ in the center of $\mathfrak{k}$ we have $\beta(X) = \alpha_\ell(X)$ for any $\beta \in \Psi_n$. Also, by construction, $\beta_M \in S^{\Psi}_{HC}$ which yields $\beta_M(Z_\phi) = 2$. Thus, if $Z_\phi$ belongs to the center of $\mathfrak{k}$ we have $\beta(Z_\phi) = 2$ for every root in $\Psi_n$, which gives $\alpha_j(Z_\phi) = 0$ for every compact simple root. Certainly, the hypothesis $\alpha(Z_\phi) = 0$ for every compact simple root, together with $\Psi$ holomorphic yields $Z_\phi$ lies in the center of $\mathfrak{k}$. The hypothesis $\alpha_\ell(Z_\phi) = 1$, together with $\beta_M(Z_\phi) = 2$, yields $Z_\phi$ is not in the center of $\mathfrak{k}$ which is equivalent to $G/K$ is not a tube domain. When $\alpha_\ell(Z_\phi) = 1$, since $\beta_M(Z_\phi) = 2$ and the multiplicity of $\alpha_\ell$ in $\beta_M$ is one, we obtain that $\alpha_j(Z_\phi) = 1$ for exactly one compact simple root and the root $\alpha_j$ has multiplicity one in the maximal root.

## 4. Explicit examples

In this Section, for each Hermitian symmetric pair, we give the necessary data in order to produce an explicit example of each triple $\{Z_\phi, E_\phi, F_\phi\}$ for $\phi$ as in Theorem 1.8. We also give the values $\alpha(Z_\phi)$ for each simple root for the holomorphic system $\Psi$.

In [Oh] an explicit realization of each of the classical real Lie algebras we are dealing is given as a subalgebra of a convenient $\mathfrak{su}(a, b)$. These realizations have the property that a compactly embedded Cartan subalgebra of the algebras of our interest, consists of the totality of diagonal matrices in the subalgebra. For each classical Lie algebra, we point out the algebra $\mathfrak{t}_{\mathbb{C}}$, a holomorphic system $\Psi$, the Harish-Chandra set $S^{\Psi}_{HC}$, the vector $Z_\phi$, the weights $\alpha_j(Z_\phi), j = 0, \dots, \ell$, for all $\alpha_j$ as in (3.3), the weighted Vogan diagram that corresponds to the $K_{\mathbb{C}}$-orbit of $E_\phi$ (see [Ga]) and the signed Young diagram that corresponds to $E_\phi$.

From the tables in [Dk1], we also present on exceptional Lie algebras the Harish-Chandra set $S$ and the weighted Vogan diagram associated to the orbit of $E_\phi$.

**AIII, $\mathfrak{su}(p,q)$, $p<q$.**
In this case

$$\mathfrak{t}_{\mathbb{C}} = \left\{ D = \operatorname{diag}(h_1,\dots,h_p;k_1,\dots,k_q) \mid \sum h_j + \sum k_s = 0 \right\}.$$

We set $\epsilon_j(D) = h_j, \delta_r(D) = k_r$. Then for a holomorphic system $\Psi$ we choose

$$\Psi_c = \{\epsilon_r - \epsilon_s \mid \delta_i - \delta_j, r<s, i<j\} \quad \Psi_n = \{\epsilon_i - \delta_j \mid 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q\}.$$

The noncompact simple root is $\alpha_p = \epsilon_p - \delta_1$, and another simple root we need is $\alpha_q = \delta_{q-p} - \delta_{q-p+1}$.

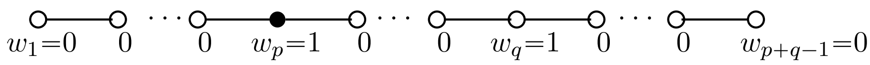
The Harish-Chandra set is

$$S^{\Psi}_{HC} = \{\epsilon_r - \delta_{q-r+1} \mid 1 \leqslant r \leqslant p\}.$$

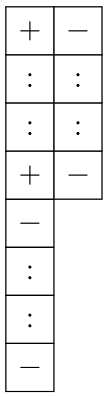
The characteristic vector is

$$Z_\phi = \operatorname{diag}(1,\dots,1;0,\dots,0,-1,\dots,-1) \quad \text{where } \pm 1 \text{ repeats } p \text{ times.}$$

The weights $w_j = \alpha_j(Z_\phi)$ are zero for roots other than $\alpha_p, \alpha_q = \delta_{q-p} - \delta_{q-p+1}$. $w_p = \alpha_p(Z_\phi) = 1$ and $w_q = \alpha_q(Z_\phi)$ are equal one. Whence, the weighted Vogan diagram for the orbit $K_{\mathbb{C}} E_\phi$ is




The signed Young diagram for $E_\phi$ is




Here, there are $p$ rows of length two and $q-p$ rows of length one.

**AIII, $\mathfrak{su}(p,q), p=q$.**

$$\mathfrak{t}_{\mathbb{C}} = \left\{ D = \operatorname{diag}(h_1,\dots,h_p;k_1,\dots,k_p) : \sum h_j + \sum k_s = 0 \right\}.$$

We set $\epsilon_j(D) = h_j, \delta_r(D) = k_r$, with $1 \leqslant j, r \leqslant p$. Then for a holomorphic system $\Psi$ we choose

$$\Psi_c = \{\epsilon_r - \epsilon_s, \delta_i - \delta_j \,|\, r < s, i < j\} \quad \Psi_n = \{\epsilon_i - \delta_j \,|\, 1 \leqslant i, j \leqslant p\}.$$

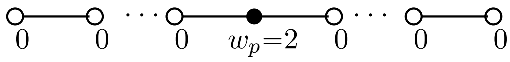
The noncompact simple root is $\alpha_p = \epsilon_p - \delta_1$.
The Harish-Chandra set is

$$S^{\Psi}_{HC} = \{\epsilon_r - \delta_{q-r+1} \,|\, 1 \leqslant r \leqslant p\}.$$

The characteristic vector is

$$Z_\phi = \mathrm{diag}(1, \dots, 1; -1, \dots, -1) \quad \text{where } \pm 1 \text{ repeats } p \text{ times}$$

Thus, the weights are $w_j = \alpha_j(Z_\phi) = 0$ except for $w_p = \alpha_p(Z_\phi) = 2$. So, its weighted Vogan diagram is




The signed Young diagram for $E_\phi$ has $p$ rows of length two.

| | |
|---|---|
| + | − |
| : | : |
| : | : |
| + | − |

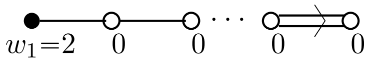
**BDI,** $\mathfrak{so}(2p+1, 2), p \geqslant 1$.

$$\mathfrak{t}_{\mathbb{C}} = \{D = \mathrm{diag}(h_1, \dots, h_p, -h_p, \dots, -h_1, 0, x_1, -x_1)\}.$$

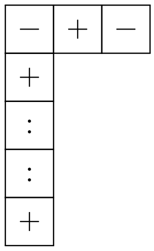
We set $\epsilon_j(D) = h_j, \delta_1(D) = x_1$. We fix the holomorphic system of positive roots,

$$\Psi_c = \{\epsilon_k, \epsilon_i \pm \epsilon_j \,|\, 1 \leqslant k \leqslant p, 1 \leqslant i < j \leqslant p\} \quad \Psi_n = \{\delta_1, \delta_1 \pm \epsilon_j \,|\, 1 \leqslant j \leqslant p\}.$$

The noncompact simple root is $\alpha_1 = \delta_1 - \epsilon_1$.
The Harish-Chandra set is $S^{\Psi}_{HC} = \{\delta_1 + \epsilon_1, \delta_1 - \epsilon_1\}$.
The characteristic vector is $Z_\phi = 2H_{\delta_1} = (0, \dots, 0, 2, -2)$.
The weights of the weighted Vogan diagram are zero except the first one,




The signed Young diagram for $E_\phi$ has $2p$ rows of length one.




**BDI,** $\mathfrak{so}(2p,2), p \geqslant 2$.

This case is similar the previous one.

$$\mathfrak{t}_{\mathbb{C}} = \{D = \operatorname{diag}(h_1,\ldots,h_p,-h_p,\ldots,-h_1,0,x_1,-x_1)\}.$$

We set $\epsilon_j(D) = h_j$, $\delta_1(D) = x_1$. The the holomorphic system we consider is

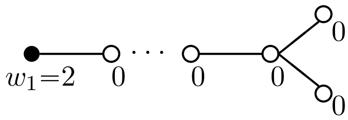
$$\Psi_c = \{\epsilon_i \pm \epsilon_j \,|\, 1 \leqslant i < j \leqslant p\} \quad \Psi_n = \{\, \delta_1 \pm \epsilon_j \,|\, 1 \leqslant j \leqslant p\}.$$

The noncompact simple root is $\alpha_p = \delta_1 - \epsilon_1$.

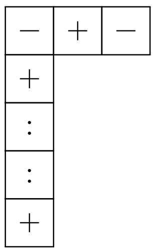
The Harish-Chandra set is $S^{\Psi}_{HC} = \{\delta_1 + \epsilon_1, \delta_1 - \epsilon_1\}$.

The characteristic vector is $Z_\phi = 2H_{\delta_1} = (0,\ldots,0,2,-2)$.

The weights of the weighted Vogan diagram are zero except the first one.




The signed Young diagram for $E_\phi$ has $2p-1$ rows of length one.




**CI,** $\mathfrak{sp}(n,\mathbb{R})$.

$$\mathfrak{t}_{\mathbb{C}} = \{D = \operatorname{diag}(h_1,\ldots,h_n,-h_n,\ldots,-h_1)\}.$$

We set $\epsilon_j(D) = h_j$, $1 \leqslant j \leqslant n$. The holomorphic system we consider is

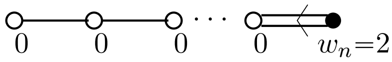
$$\Psi_c = \{\epsilon_i - \epsilon_j \,|\, 1 \leqslant i < j \leqslant n\} \quad \Psi_n = \{\epsilon_k + \epsilon_r \,|\, 1 \leqslant k \leqslant r \leqslant n\}.$$

The noncompact simple root is $\alpha_n = 2\epsilon_n$.

The set $S^{\Psi}_{HC} = \{2\epsilon_1,\ldots,2\epsilon_n\}$.

The characteristic vector is $Z_\phi = (1,\ldots,1,-1,\ldots,-1)$.

The weights of the weighted Vogan diagram are zero except the last one, $w_n = \alpha_n(Z_\phi) = 2$.




The signed Young diagram for $E_\phi$ has $n$ rows of length two.

| | |
|---|---|
| + | − |
| : | : |
| : | : |
| + | − |

**DIII, $\mathfrak{so}^\star(2p)$, $p = 2k$.**

$$\mathfrak{t}_{\mathbb{C}} = \{D = \mathrm{diag}(h_1, \ldots, h_p, -h_p, \ldots, -h_1)\}.$$

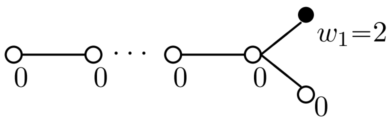
We set $\epsilon_j(D) = h_j$, $1 \leqslant j \leqslant p$. The holomorphic system we consider is

$$\Psi_c = \{\epsilon_i - \epsilon_j \,|\, 1 \leqslant i < j \leqslant p\} \quad \Psi_n = \{\epsilon_s + \epsilon_r \,|\, 1 \leqslant s < r \leqslant p\}$$

The noncompact simple root is $\alpha_p = \epsilon_{p-1} + \epsilon_p$.
The Harish-Chandra set is $S_{HC}^{\Psi} = \{\epsilon_1 + \epsilon_2, \epsilon_3 + \epsilon_4, \ldots, \epsilon_{2k-1} + \epsilon_{2k}\}$.
Characteristic vector is

$$Z_\phi = \sum_{1 \leqslant j \leqslant k} H_{\epsilon_{2j-1}+\epsilon_{2j}} = (1, \ldots, 1, -1, \ldots, -1).$$

The weights $w_j = \alpha_j(Z_\phi)$ are $w_p = 2, w_j = 0$ for $j \neq p$. So the weighted Vogan diagram is the following.




The signed Young diagram for $E_\phi$ has $2k$ rows of length two.

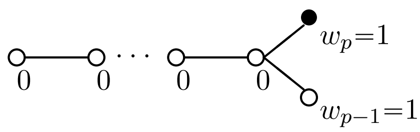
| | |
|---|---|
| + | − |
| : | : |
| : | : |
| + | − |

**DIII, $\mathfrak{so}^\star(2p)$, $p = 2k + 1$.**

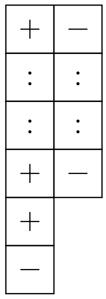
This case is similar to the previous one. The difference is that the characteristic vector is

$$Z_\phi = (1, \ldots, 1, -1, \ldots, -1, 0).$$

Thus, the weights $w_{p-1}, w_p$ are equal to $(\epsilon_{p-1} \pm \epsilon_p)(Z_\phi) = 1$ and the others are zero.




The signed Young diagram for $E_\phi$ has $2k$ rows of length two.

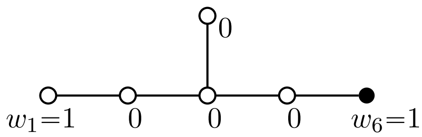



**EIII,** $\mathfrak{e}_{6(-14)}$.

We follow the notation for the simple roots as set for Bourbaki. We fix as noncompact simple root $\alpha_6$. The Harish-Chandra set in this case is

$$S^\Psi_{HC} = \{\beta_1 = 122321, \beta_2 = 101111\}.$$

From table X of [Dk1], we extract that there is only one characteristic vector $Z_\mathfrak{h}$ so that $\alpha_\ell(Z_\mathfrak{h}) > 0$, and we obtain a direct verification of $Z_\mathfrak{h} = Z_\phi$. The weighted Vogan diagram for the nilpotent orbit determined by $E_\phi$ is

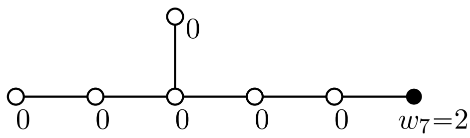



**EVII,** $\mathfrak{e}_{7(-25)}$.

We fix the holomorphic root system such that the noncompact simple root is $\alpha_7$. The Harish-Chandra set is

$$S^\Psi_{HC} = \{\beta_1 = 22343221, \beta_2 = 01122221, \beta_3 = \alpha_7 = 00000001\}$$

From table XIII of [Dk1], we read that the unique $K_\mathbb{C}$-orbit in $\mathfrak{p}_\mathbb{C}$ with characteristic vector $Z_\mathfrak{h}$ so that $\alpha_\ell(Z_\mathfrak{h}) > 0$ is for $Z_\mathfrak{h} = Z_\phi$. The weighted Vogan diagram is




## 5. Multiplicities

In this Section we apply the formula for multiplicities obtained in [DV] to the particular case of the pair $(G, H_0)$ where $H_0 = \phi_S(A)$ and $S = S^\Psi_{HC}$ and a holomorphic square integrable representation. Henceforth, $\Psi$ denotes a holomorphic system of positive roots for $\Phi(\mathfrak{g}, \mathfrak{t})$. To begin with we recall the necessary notation to state the results. In the notation of [DV] the pair $(G, H)$ for our case is $(G, H_0)$. The pair $(K, L = H \cap K)$ is $(K, H_0 \cap K)$, We have $T \subset K \subset G$ as before and $U = T \cap H = L = H_0 \cap K = H_0 \cap T = \exp(\mathbb{R}iZ_\phi)$, $\mathfrak{u} = \mathbb{R}iZ_\phi$, we denote by $\mathfrak{z}_\mathfrak{k} =$ the center of $\mathfrak{k}$. We define $\varphi \in \mathfrak{u}^*$ by $\varphi(Z_\phi) = 1$. Thus $\Phi(\mathfrak{h}_0, \mathfrak{u}) = \{\pm 2\varphi\}$. Let $\mathfrak{k}_\mathfrak{z} = \mathfrak{k}^{Z_\phi}$ denote the centralizer of $Z_\phi$ in $\mathfrak{k}$ and $\Phi_\mathfrak{z}$ the root system for $(\mathfrak{k}_\mathfrak{z}, \mathfrak{t})$. Thus,

$$\Phi_\mathfrak{z} = \{\alpha \in \Phi(\mathfrak{k}, \mathfrak{t}) \mid \alpha(\mathfrak{u}) = 0\}.$$

By Remark 3.1, if the Hermitian symmetric space $G/K$ is a tube domain, then $Z_\phi \in \mathfrak{z}_\mathfrak{k}$, $\Phi_\mathfrak{z} = \Phi(\mathfrak{k}, \mathfrak{t})$ and the analytic subgroup of $G$ with Lie algebra $\mathfrak{k}_\mathfrak{z}$ is $K_\mathfrak{z} = K$. If $G/K$ is not a tube domain, $Z_\phi \notin \mathfrak{z}_\mathfrak{k}$, then owing to $\alpha_j(Z_\phi) = 0$ for all compact simple roots but one, the semisimple factor of $\mathfrak{k}_\mathfrak{z}$ has rank $\ell - 2$. The list of the triples $(\mathfrak{g}, \mathfrak{k}, \mathfrak{k}_\mathfrak{z})$ that corresponds to non-tube domains is:

| | | | |
|---|---|---|---|
| $\mathfrak{g}$ | $\mathfrak{su}(p,q)$, $p < q$ | $\mathfrak{so}^*(2(2k+1))$ | $\mathfrak{e}_{6(-14)}$ |
| $\mathfrak{k}$ | $\mathfrak{su}(p) + \mathfrak{su}(q) + \mathfrak{z}_\mathfrak{k}$ | $\mathfrak{su}(2k+1) + \mathfrak{z}_\mathfrak{k}$ | $\mathfrak{so}(10) + \mathfrak{z}_\mathfrak{k}$ |
| $\mathfrak{k}_\mathfrak{z}$ | $\mathfrak{su}(p) + \mathfrak{su}(q-p) + \mathfrak{su}(p) + \mathfrak{t}$ | $\mathfrak{su}(2k) + \mathfrak{t}$ | $\mathfrak{so}(8) + \mathfrak{t}$ |

The set of equivalence classes of holomorphic square integrable representations of $G$ is parameterized by the set of $\lambda \in i\mathfrak{t}^*$ so that $\lambda + \rho$ lifts to a character of $T$, here $\rho$ is equal to one half of the sum of the elements in $\Psi$, and

$$(\lambda, \alpha) > 0 \text{ for all } \alpha \in \Psi_c \quad \text{and} \quad (\lambda, \beta) > 0 \text{ for all } \beta \in \Psi \cap \Phi_n. \tag{5.1}$$

The set of Harish-Chandra parameters which corresponds to the irreducible square integrable representations of $H_0$ is $P := \{n\varphi \mid n \in \mathbb{Z} \setminus \{0\}\}$.

The set of Harish-Chandra parameters for a compact connected Lie group $R$ is equal to the set of strictly dominant integral weights for $R$, equivalently, the set of Harish-Chandra parameters is equal to the set of infinitesimal character of the set of irreducible representations of $R$. We denote the parametrization by $\mu \leftrightarrow (\pi^R_\mu, V^R_\mu)$.

Hereafter, $(\pi_\lambda, V^G_\lambda)$ denotes a holomorphic irreducible square integrable representation. In Theorem 1.8 we pointed out the restriction $\mathrm{res}_{H_0}(\pi_\lambda)$ of $\pi_\lambda$ to the subgroup $H_0$ is an $H_0-$admissible discretely decomposable representation of $H_0$. For $\mu \in P$, let $(\sigma_\mu, V^{H_0}_\mu)$ denote the irreducible square integrable representation of $H_0$ of Harish-Chandra parameter $\mu$. Let $m(\pi_\lambda, \sigma_\mu) = \mathrm{Hom}_{H_0}(\sigma_\mu, \pi_\lambda)$ denote the multiplicity of $\sigma_\mu$ in $\mathrm{res}_{H_0}(\pi_\lambda)$. Therefore, we have a Hilbertian direct sum

$$\mathrm{res}_{H_0}(\pi_\lambda) = \sum_{\mu \in P} m(\pi_\lambda, \sigma_\mu) V^{H_0}_\mu. \tag{‡}$$

We write the restriction to $K_\mathfrak{z}$ of the lowest $K$-type $\pi^K_{\lambda+\rho_n}$ for $(\pi_\lambda, V^G_\lambda)$ as

$$\mathrm{res}_{K_\mathfrak{z}}(\pi^K_{\lambda+\rho_n}) = \sum_{1 \leqslant j \leqslant s} m(\pi^K_{\lambda+\rho_n}, \pi^{K_\mathfrak{z}}_{\mu_j+\rho_\mathfrak{z}}) V^{K_\mathfrak{z}}_{\mu_j+\rho_\mathfrak{z}}$$

where $\rho_n$ is the half sum of the roots in $\Psi_n$ and $\rho_\mathfrak{z}$ is the half sum of the roots in $\Phi_\mathfrak{z} \cap \Psi$. Under the notation formulated from the beginning of this subsection we have,

**Theorem 5.1.** *Let $(\pi_\lambda, V_\lambda^G)$ be a holomorphic discrete series representation. Let $H_0 := \phi_{S_{HC}^\Psi}(A)$. Then $res_{H_0}(\pi_\lambda)$ is a Hilbertian direct sum of holomorphic discrete series for $H_0$. The description of the elements in the formula (‡) is:*

*(i) For $\mu \in P$, $m(\pi_\lambda, \sigma_\mu) > 0$ if and only if $\mu$ belongs to the set*

$$\{[(\mu_j + \rho_{\mathfrak{z}})(Z_\phi) + n - 1]\varphi \,|\, 1 \leqslant j \leqslant s,\, n \geqslant 0\}$$

*(ii) The multiplicity $m(\pi_\lambda, \sigma_{m\varphi})$ is equal to*

$$\sum_{\substack{j,n \\ \mu_j(Z_\phi)+n=m}} m(\pi^K_{\lambda+\rho_n}, \pi^{K_{\mathfrak{z}}}_{\mu_j+\rho_{\mathfrak{z}}}) \sum_{h=0}^{[\frac{n}{2}]} \binom{n-2h+c-1}{c-1}\binom{h+d-1}{d-1}.$$

*Here $c = |\{\beta \in \Psi_n \,|\, \beta(Z_\phi) = 1\}|$ and $d + 1 = |\{\beta \in \Psi_n \,|\, \beta(Z_\phi) = 2\}|$.*

The proof of the Theorem will take up the rest of this Section. It requires more notation. To start, we consider the restriction $q_{\mathfrak{u}} : \mathfrak{t}^* \to \mathfrak{u}^*$ and the multiset $\Delta(\mathfrak{k}/\mathfrak{l}, \mathfrak{u}) := q_{\mathfrak{u}}(\Psi(\mathfrak{k}, \mathfrak{t}) \smallsetminus \Phi_{\mathfrak{z}})$. So, we have

$$\Delta(\mathfrak{k}/\mathfrak{l}, \mathfrak{u}) = \begin{cases} \varnothing & \text{if } G/K \text{ is a tube domain} \\ \{\underbrace{\varphi, \ldots, \varphi}_{a}\} & \text{if } G/K \text{ is not a tube domain} \end{cases} \tag{5.2}$$

Here $a = |\{\alpha \in \Psi_c \,|\, q_{\mathfrak{u}}(\alpha)(Z_\phi) = 1\}| = \frac{1}{2}\dim(K/K_{\mathfrak{z}})$. In fact, when $G/K$ is a tube domain then $iZ_\phi \in \mathfrak{z}_{\mathfrak{k}}$ and the first claim is obvious. For $G/K$ a non-tube domain Remark 3.1 yields that $\alpha(Z_\phi) = 1$ or $\alpha(Z_\phi) = 0$ for $\alpha \in \Psi_c$. For $w$ in the Weyl group $W_K$ of $K$, we compute the multiset

$$S_w^{H_0} := [q_{\mathfrak{u}}(w\Psi_n) \cup \Delta(\mathfrak{k}/\mathfrak{l}, \mathfrak{u})] \smallsetminus \Phi(\mathfrak{h}_0, \mathfrak{u}).$$

Since, $\Psi$ is holomorphic and $w \in W_K$ it follows that $w\Psi_n = \Psi_n$. Hence, $S_w^{H_0}$ does not depend on $w$. We have,

$$q_{\mathfrak{u}}(\Psi_n) = \{\underbrace{\varphi, \ldots, \varphi}_{c}, \underbrace{2\varphi, \ldots, 2\varphi}_{d+1}\}. \tag{5.3}$$

If $G/K$ is a tube domain, then $c = 0$ and $d + 1 = |\Psi_n|$. Indeed, in Remark 3.1 we show $\beta(Z_\phi) = 2$ for all noncompact positive root. Therefore, from (5.2), (5.3) and the previous computation, for a tube domain we obtain,

$$S_w^{H_0} = \{\underbrace{2\varphi, \ldots, 2\varphi}_{d}\}. \tag{5.4}$$

If $G/K$ is not a tube domain the values of $c$ and $d$ are in the following table.

| $\mathfrak{g}$ | $\mathfrak{su}(p,q)$, $p < q$ | $\mathfrak{so}^*(2(2k+1))$ | $\mathfrak{e}_{6(-14)}$ |
|---|---|---|---|
| $c$ | $(q-p)p$ | $2k$ | 8 |
| $d+1$ | $p^2$ | $k(2k-1)$ | 8 |

Hence,

$$S_w^{H_0} = \{\underbrace{\varphi,\dots,\varphi}_{c},\underbrace{2\varphi,\dots,2\varphi}_{d}\} \cup \Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u}). \tag{5.5}$$

For $\nu \in i\mathfrak{t}^*$ (resp. $\nu \in i\mathfrak{u}^*$), $\delta_\nu$ denotes the Dirac distribution on $i\mathfrak{t}^*$ (resp. on $i\mathfrak{u}^*$) defined by $\nu$. Let $(q_\mathfrak{u})_*(\delta_\nu)$ be the push-forward of $\delta_\nu$ from $i\mathfrak{t}^*$ to $i\mathfrak{u}^*$. Thus, $(q_\mathfrak{u})_*(\delta_\nu) = \delta_\nu$. Let

$$y_\nu = \sum_{n=0}^{\infty} \delta_{n\nu+\frac{\nu}{2}}, \qquad z_\nu = \sum_{n=0}^{\infty} \delta_{n\nu}.$$

For a strict finite set $T = \{\nu_1,\dots,\nu_t\} \subset i\mathfrak{t}^*$ we define

$$y_T = y_{\nu_1} * \cdots * y_{\nu_t} = \mathop{\circledast}_{\nu\in T} y_\nu.$$

Here $*$ means convolution of distributions. Let $\varpi_\mathfrak{z}(\lambda) := \prod_{\alpha\in\Psi\cap\Phi_\mathfrak{z}} \frac{\lambda(\alpha)}{\rho_\mathfrak{z}(\alpha)}$ Then, in [DV] is shown the following equality of distributions on $i\mathfrak{u}^*$,

$$\sum_{\mu\in P} m(\pi_\lambda,\sigma_\mu)\delta_\mu = \sum_{w\in W_\mathfrak{z}\backslash W_K} \epsilon(w)\varpi_\mathfrak{z}(w\lambda)\delta_{q_\mathfrak{u}(w\lambda)} * y_{S_w^{H_0}}. \tag{5.6}$$

where $\epsilon(w)$ is the sign of $w$. The validity of the above equality follows by Theorem 4 in [DV] because Condition (C) is satisfied. We now show Theorem 5.1 for $G$ so that $G/K$ is a tube domain. For a holomorphic system $\Psi$ we always have the equality

$$\varpi_\mathfrak{z}(w\lambda) = \varpi_\mathfrak{z}(w(\lambda+\rho_n)). \tag{5.7}$$

Now $\mathfrak{k}_\mathfrak{z} = \mathfrak{k}$, hence we have $\varpi_\mathfrak{z}(\lambda) = \dim V^K_{\lambda+\rho_n}$ which is equal to the dimension of lowest $K$-type of $\pi_\lambda$. Then, by (5.6), (5.4) together and the above consideration gives

$$\sum_{\mu\in P} m(\pi_\lambda,\sigma_\mu)\delta_\mu = \dim V^K_{\lambda+\rho_n}\, \delta_{\lambda(Z_\phi)\varphi} * y_{2\varphi}^{|\Psi_n|-1}.$$

Obviously, $y_\nu = \delta_{\frac{\nu}{2}} * z_\nu$, $q_\mathfrak{u}(\rho_n) = |\Psi_n|\varphi$ and

$$y_{2\varphi}^{|\Psi_n|-1} = z_{2\varphi}^{|\Psi_n|-1} * \delta_{-\varphi} * \delta_{q_\mathfrak{u}(\rho_n)}.$$

For $r, s$ positive integers, it readily follows that

$$z_{r\,\varphi}^{s} = \sum_{t=0}^{\infty} \binom{t+s-1}{s-1} \delta_{t\,r\,\varphi}. \tag{5.8}$$

Hence, we obtain

$$\sum_{\mu\in P} m(\pi_\lambda,\sigma_\mu)\delta_\mu = \sum_{t\geqslant 0} \dim V^K_{\lambda+\rho_n} \binom{t+|\Psi_n|-2}{|\Psi_n|-2} \delta_{[(\lambda+\rho_n)(Z_\phi)+2t-1]\varphi}.$$

Therefore, whenever $G/K$ is a tube domain, the Harish-Chandra parameters that contribute to $\mathrm{res}_{H_0}(\pi_\lambda)$ are $[(\lambda+\rho_n)(Z_\phi)+2t-1]\varphi = [\lambda(Z_\phi)+d+2t]\varphi$, $t = 0,1,\dots,$

and the respective multiplicities are exactly the numbers $\binom{t+|\Psi_n|-2}{|\Psi_n|-2}\dim V^K_{\lambda+\rho_n} = \binom{t+d-1}{d-1}\dim V^K_{\lambda+\rho_n}$.

To follow, we show the multiplicity formula when $G/K$ is not a tube domain. Hence, $K_{\mathfrak{z}}$ is a proper subgroup of $K$ and $iZ_\phi$ is not in the center of $\mathfrak{k}$. We manipulate on the right hand side of formulae (5.6). Since $\Psi$ is a holomorphic system, $w\rho_n = \rho_n$ for $w \in W_K$. Hence, $q_{\mathfrak{u}}(\rho_n) = [\frac{c}{2} + d + 1]\varphi$ and $y_{q_{\mathfrak{u}}(\Psi_n)\smallsetminus\Phi(\mathfrak{h}_0,\mathfrak{u})} = z_{q_{\mathfrak{u}}(\Psi_n)\smallsetminus\Phi(\mathfrak{h}_0,\mathfrak{u})} * \delta_{[\frac{c}{2}+d]\varphi}$. Hence, the right hand side of (5.6) becomes equal to

$$\sum_{w\in W_{\mathfrak{z}}\backslash W_K} \epsilon(w)\varpi_{\mathfrak{z}}(w(\lambda+\rho_n))\delta_{q_{\mathfrak{u}}(w(\lambda+\rho_n))} * \mathop{\circledast}_{\gamma\in q_{\mathfrak{u}}(\Psi\smallsetminus\Phi_{\mathfrak{z}})} y_\gamma * \mathop{\circledast}_{\beta\in q_{\mathfrak{u}}(\Psi_n)\smallsetminus\Phi(\mathfrak{h}_0)} z_\beta * \delta_{-\varphi}. \tag{5.9}$$

In the language of discrete Heaviside distributions, the restriction of the lowest $K$-type $\pi^K_{\lambda+\rho_n}$ of $\pi_\lambda$ to $U$ is represented by

$$\sum_{w\in W_{\mathfrak{z}}\smallsetminus W_K} \epsilon(w)\varpi_{\mathfrak{z}}(w(\lambda+\rho_n))\delta_{q_{\mathfrak{u}}(w(\lambda+\rho_n))} * \mathop{\circledast}_{\gamma\in q_{\mathfrak{u}}(\Psi_c\smallsetminus\Phi_{\mathfrak{z}})} y_\gamma \tag{5.10}$$

The restriction of $\pi^K_{\lambda+\rho_n}$ to $U$ can be represented as the restriction of $\pi^K_{\lambda+\rho_n}$ to $K_{\mathfrak{z}}$ and then we decompose the resulting representation of $K_{\mathfrak{z}}$ as $U$-module. Let $\mu_1+\rho_{\mathfrak{z}},\dots,\mu_s+\rho_{\mathfrak{z}}$, denote the infinitesimal characters for the irreducible constituents of $\operatorname{res}_{K_{\mathfrak{z}}}(\pi^K_{\lambda+\rho_n})$. Here we take $\mu_j$ dominant with respect to $\Psi\cap\Phi_{\mathfrak{z}}$. Then, we have the equality

$$\operatorname{res}_U(\pi^K_{\lambda+\rho_n}) = \sum_{1\leqslant j\leqslant s} m(\pi^K_{\lambda+\rho_n}, \pi^{K_{\mathfrak{z}}}_{\mu_j+\rho_{\mathfrak{z}}}) V^U_{\mu_j+\rho_{\mathfrak{z}}}.$$

Therefore, we have that (5.10) is equal to

$$\sum_{1\leqslant j\leqslant s} m(\pi^K_{\lambda+\rho_n}, \pi^{K_{\mathfrak{z}}}_{\mu_j+\rho_{\mathfrak{z}}})\delta_{(\mu_j+\rho_{\mathfrak{z}})(Z_\phi)\varphi}. \tag{5.11}$$

Putting together the new expression for (5.10) and (5.9), we obtain that the right hand side of (5.6) is equal to

$$\sum_{1\leqslant j\leqslant s} m(\pi^K_{\lambda+\rho_n}, \pi^{K_{\mathfrak{z}}}_{\mu_j+\rho_{\mathfrak{z}}})\delta_{(\mu_j+\rho_{\mathfrak{z}})(Z_\phi)\varphi} * \mathop{\circledast}_{\gamma\in q_{\mathfrak{u}}(\Psi_n)\smallsetminus\Phi(\mathfrak{h}_0,\mathfrak{u})} z_\gamma * \delta_\varphi.$$

After we recall (5.5) and we apply (5.11) to the previous formula, we obtain

$$\begin{gathered}\sum_{\mu\in P} m(\pi_\lambda,\sigma_\mu)\delta_\mu = \\ \sum_{t\geqslant 0,h\geqslant 0}\sum_j m(\pi^K_{\lambda+\rho_n}, \pi^{K_{\mathfrak{z}}}_{\mu_j+\rho_{\mathfrak{z}}})\binom{t+c-1}{c-1}\binom{h+d-1}{d-1}\delta_{[(\mu_j+\rho_{\mathfrak{z}})(Z_\phi)+t+2h-1]\varphi}.\end{gathered} \tag{5.12}$$

Hence,

$$\begin{gathered}\sum_{\mu\in P} m(\pi_\lambda,\sigma_\mu)\delta_\mu = \\ \sum_{j=1}^{s} m(\pi^K_{\lambda+\rho_n}, \pi^{K_{\mathfrak{z}}}_{\mu_j+\rho_{\mathfrak{z}}})\sum_{n\geqslant 0}\sum_{h=0}^{[\frac{n}{2}]}\binom{n-2h+c-1}{c-1}\binom{h+d-1}{d-1}\delta_{[(\mu_j+\rho_{\mathfrak{z}})(Z_\phi)+n-1]\varphi}.\end{gathered} \tag{5.13}$$

Therefore a Harish-Chandra parameter $m\varphi$ of an irreducible $H_0$-factors for $\mathrm{res}_{H_0}(\pi_\lambda)$ belongs to the set

$$\{[(\mu_j + \rho_{\mathfrak{z}})(Z_\phi) + n - 1]\varphi \,|\, n = 0, 1 \ldots, j = 1, \ldots, s\},$$

and the respective multiplicity is

$$\sum_{\substack{j,n \\ \mu_j(Z_\phi)+n=m}} m(\pi^K_{\lambda+\rho_n}, \pi^{K_{\mathfrak{z}}}_{\mu_j+\rho_{\mathfrak{z}}}) \sum_{h=0}^{[\frac{n}{2}]} \binom{n-2h+c-1}{c-1} \binom{h+d-1}{d-1}.$$

Now, the proof of Theorem 5.1 has been completed.

**Remark 5.2.** In the above formulae for either Harish-Chandra parameters or multiplicities, if we make $c$ equal to zero, we obtain the formula for the tube type case.

**Remark 5.3.** The decomposition of the adjoint representation of $\mathfrak{g}_{\mathbb{C}}$ restricted to $\mathfrak{h}_0$ is,

(i) When $G/K$ is a tube domain,

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_1^{d+1} (\mathbb{C}^3, 2\varphi) \oplus \bigoplus_1^{\dim \mathfrak{k}-d-1} (\mathbb{C}, 0\varphi).$$

(ii) When $G/K$ is not a tube domain,

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_1^{d+1} (\mathbb{C}^3, 2\varphi) \oplus \bigoplus_1^{c} (\mathbb{C}^2, \varphi) \oplus \bigoplus_1^{\dim \mathfrak{k}-d-1} (\mathbb{C}, 0\varphi).$$

Whence, the coefficients $c$ and $d+1$ represent multiplicity of irreducible constituents of the $\mathfrak{h}_0$-module $\mathfrak{g}_{\mathbb{C}}$.

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