Journal of Lie Theory
Volume **27** (2017) 1057–1068

On Lie Algebras Consisting of Locally Nilpotent Derivations

Anatoliy Petravchuk and Kateryna Sysak

Communicated by A. Fialowski

Abstract. Let K be an algebraically closed field of characteristic zero and A an integral K-domain. The Lie algebra $\mathrm{Der}_K(A)$ of all K-derivations of A contains the set $\mathrm{LND}(A)$ of all locally nilpotent derivations. The structure of $\mathrm{LND}(A)$ is of great interest, and the question about properties of Lie algebras contained in $\mathrm{LND}(A)$ is still open. An answer to it in the finite dimensional case is given. It is proved that any subalgebra of finite dimension (over K) of $\mathrm{Der}_K(A)$ consisting of locally nilpotent derivations is nilpotent. In the case $A = K[x, y]$, it is also proved that any subalgebra of $\mathrm{Der}_K(A)$ consisting of locally nilpotent derivations is conjugate by an automorphism of $K[x, y]$ with a subalgebra of the triangular Lie algebra.
Mathematics Subject Classification 2010: 17B66, 17B05, 13N15.
Key Words and Phrases: Lie algebra, vector field, triangular, locally nilpotent derivation.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and A an associative commutative $\mathbb{K}$-algebra that is a domain. A derivation $D : A \to A$ is called locally nilpotent if for any element $a \in A$ there exists a positive integer $n = n(a)$ such that $D^n(a) = 0$. The study of locally nilpotent derivations is an important problem in differential algebra because the exponents of such derivations are automorphisms of the associative algebra A and they carry information about A. Many papers and a few monographs are devoted to locally nilpotent derivations (see, for example, [10], [4], [7], [9], [3], [6], etc). One of unsolved problems is to describe all Lie algebras contained in the set $\mathrm{LND}(A)$ of all locally nilpotent derivations on the algebra A (see Problem 11.6 in [4]). In this paper, it is proved that every finite dimensional (over $\mathbb{K}$) subalgebra of the Lie algebra $\mathrm{Der}_{\mathbb{K}}(A)$ consisting of locally nilpotent derivations is nilpotent (Theorem 2.5). In the case $A = \mathbb{K}[x, y]$, the polynomial ring in two variables, it is proved that every subalgebra $L \subseteq \mathrm{LND}(A)$ of $\mathrm{Der}_{\mathbb{K}}(A)$ is conjugated with a subalgebra of the triangular Lie algebra $u_2(\mathbb{K})$ by an automorphism of $\mathbb{K}[x, y]$. By Rentschler's Theorem [10], the structure of

ISSN 0949–5932 / \$2.50

$\mathrm{LND}(\mathbb{K}[x,y])$ is as follows:

$$\mathrm{LND}(\mathbb{K}[x,y]) = \bigcup_{\theta\in Aut(\mathbb{K}[x,y])} \theta u_2(\mathbb{K})\theta^{-1}.$$

It is proved that every Lie algebra lying in $\mathrm{LND}(\mathbb{K}[x,y])$ is contained entirely in at least one of subalgebras conjugated with $u_2(\mathbb{K})$ (Theorem 3.11).

We use standard notations. The ground field $\mathbb{K}$ is algebraically closed of characteristic zero. The quotient field of the integral domain A under consideration is denoted by R. The set of all locally nilpotent derivations of A is denoted by $\mathrm{LND}(A)$. Any derivation D of A can be uniquely extended to a derivation of R by the rule: $D(a/b) = (D(a)b - aD(b))/b^2$. If F is a subfield of the field R and $r_1, \dots, r_k \in R$, then the set of all linear combinations of these elements with coefficients in F is denoted by $F\langle r_1, \dots, r_k\rangle$; it is a subspace of the F-space R. The set $\mathrm{Der}_{\mathbb{K}}(A)$ of all $\mathbb{K}$-derivations of A is a Lie algebra over $\mathbb{K}$ relative to the Lie bracket: $[D_1, D_2] = D_1D_2 - D_2D_1$ for $D_1, D_2 \in \mathrm{Der}_{\mathbb{K}}(A)$. The Lie algebra $\mathrm{Der}_{\mathbb{K}}(A)$ is also an A-module in a natural way: given $a \in A, D \in \mathrm{Der}_{\mathbb{K}}(A)$, the derivation aD sends any element $x \in A$ to $a \cdot D(x)$. In case $A = \mathbb{K}[x,y]$, we denote the Lie algebra $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[x,y])$ by $W_2(\mathbb{K})$, any element $D \in W_2(\mathbb{K})$ is of the form $D = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$ for some $f, g \in \mathbb{K}[x,y]$.

The triangular subalgebra $u_2(\mathbb{K})$ of the Lie algebra $W_2(\mathbb{K})$ consists of all the derivations on the ring $\mathbb{K}[x,y]$ of the form $D = \alpha\frac{\partial}{\partial x} + \beta(x)\frac{\partial}{\partial y}$, where $\alpha \in \mathbb{K}, \beta(x) \in \mathbb{K}[x]$ (about properties of triangular Lie algebras see [1]). Recall that a subalgebra B of an associative commutative algebra A is factorially closed in A if the relations $a_1a_2 \in B, a_1 \neq 0, a_2 \neq 0$, imply $a_1 \in B$ and $a_2 \in B$. A polynomial $a = a(x,y) \in \mathbb{K}[x,y]$ is called coordinate if there exists a polynomial $b = b(x,y) \in \mathbb{K}[x,y]$ such that $\mathbb{K}[x,y] = \mathbb{K}[a,b]$. Then the polynomials a and b form a coordinate pair (a,b). If $f \in \mathbb{K}[x,y]$, then f induces the Jacobian derivation D_f of the ring $\mathbb{K}[x,y]$ by the rule: $D_f(h) = \det J(f,h)$ for any $h \in \mathbb{K}[x,y]$, where $\det J(f,h)$ is the Jacobian determinant of the polynomials f and h. For any derivation $D = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} \in \mathrm{Der}(\mathbb{K}[x,y])$ we denote by $\operatorname{div} D$ the divergence of D: $\operatorname{div} D = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. The Lie algebra $W_2(\mathbb{K})$ is a free module over the ring $\mathbb{K}[x,y]$ (of rank 2), so for any subalgebra $L \subseteq W_2(\mathbb{K})$ one can define rank of L over the ring $\mathbb{K}[x,y]$.

2. Finite dimensional Lie algebras consisting of locally nilpotent derivations

Throughout this section, A denotes an integral $\mathbb{K}$-domain and R the field of fractions for the algebra A. Recall that a Lie algebra L over a field $\mathbb{K}$ is locally finite (or locally finite dimensional) if every its finitely generated subalgebra is of finite dimension over $\mathbb{K}$. A Lie algebra L is locally nilpotent if every its finitely generated subalgebra is nilpotent. We need also some properties of locally nilpotent derivations, they are pointed out in the next two lemmas.

Lemma 2.1. *Let D be a locally nilpotent derivation of the algebra A and δ its extension on the fraction field R of A. Then:*

(a) *[4, Principle 1]* $\operatorname{Ker} D$ *is a factorially closed subring of* A.

(b) *[4, Corollary 1.23]* $\operatorname{Ker}\delta = \operatorname{Frac}(\operatorname{Ker} D)$.

(c) *[4, Principle 11(e)] Transcendence degree of the field* R *over the subfield* $\operatorname{Ker}\delta$ *equals* 1.

(d) *[4, Corollary 1.20] If* $D(a) = ab$ *for some* $a, b \in A$, *then* $D(a) = 0$.

Lemma 2.2. *[4, Principle 12] Let* D_1, D_2 *be locally nilpotent derivations of the algebra* A *such that* $\operatorname{Ker} D_1 = \operatorname{Ker} D_2 = B$. *Then there exist nonzero elements* $a, b \in B$ *such that* $aD_1 = bD_2$.

Lemma 2.3. *(see, for example, [5], p.54). Let* L *be a finite dimensional Lie algebra over an algebraically closed field* $\mathbb{K}$. *Then the algebra* L *is nilpotent if and only if every two-dimensional subalgebra of* L *is abelian.*

Lemma 2.4. *Let* $D_1, D_2 \in \operatorname{Der}_{\mathbb{K}} A$ *and* $a, b \in R$. *Then*

(a) $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$.

(b) *If* $a, b \in \operatorname{Ker}D_1 \cap \operatorname{Ker}D_2$, *then* $[aD_1, bD_2] = ab[D_1, D_2]$.

Proof. Straightforward check. ■

Theorem 2.5. *Let* $\mathbb{K}$ *be an algebraically closed field of characteristic zero and* A *an integral* $\mathbb{K}$*-domain. If* L *is a finite dimensional (over* $\mathbb{K}$*) subalgebra of the Lie algebra* $\operatorname{Der}_{\mathbb{K}} A$ *and every element of* L *is a locally nilpotent derivation on* A, *then the Lie algebra* L *is nilpotent.*

Proof. Let M be any two-dimensional subalgebra of the Lie algebra L. Our aim is to prove that M is abelian. Suppose this is not the case and choose a basis $\{D_1, D_2\}$ of the non-abelian subalgebra M such that $[D_1, D_2] = D_2$. Let us show that $\operatorname{Ker} D_1 \subseteq \operatorname{Ker} D_2$. Take any element $f \in \operatorname{Ker} D_1$. Then

$$D_2(f) = [D_1, D_2](f) = (D_1D_2 - D_2D_1)(f) = D_1(D_2(f)) - D_2(D_1(f)) = D_1(D_2(f)),$$

because $D_2(D_1(f)) = D_2(0) = 0$. Since $D_1 \in \operatorname{LND}(A)$ and $D_1(D_2(f)) = D_2(f)$, then $D_2(f) = 0$ (by Lemma 2.1(d)) and $f \in \operatorname{Ker} D_2$. But then $\operatorname{Ker} D_1 \subseteq \operatorname{Ker} D_2$, because the element $f \in \operatorname{Ker} D_1$ is arbitrarily chosen.

Let δ_1, δ_2 be extensions of derivations D_1 and D_2 respectively on the fraction field $R = \operatorname{Frac}(A)$, and let R_1, R_2 be subfields of constants for δ_1 and δ_2 respectively in R. The nonzero derivations D_1, D_2 are locally nilpotent on A, so by Lemma 2.1(b) we get equalities $\operatorname{Frac}(\operatorname{Ker} D_1) = R_1$ and $\operatorname{Frac}(\operatorname{Ker} D_2) = R_2$. The inclusion $\operatorname{Ker} D_1 \subseteq \operatorname{Ker} D_2$ implies $R_1 \subseteq R_2$. Note that by Lemma 2.1(a,c) the subfields R_1, R_2 are algebraically closed in the field R and $\operatorname{tr.deg}_{R_1} R = \operatorname{tr.deg}_{R_2} R = 1$. Then one can easily show that $R_1 = R_2$ and therefore $\operatorname{Ker} D_1 = \operatorname{Ker} D_2$. Denote $B = \operatorname{Ker} D_1 = \operatorname{Ker} D_2$. Using Lemma 2.2, we see that there exist nonzero elements $a, b \in B$ such that $aD_1 = bD_2$. But then we get

$$[aD_1, bD_2] = ab[D_1, D_2] = abD_2 = 0$$

by Lemma 2.4(b). Since $D_2 \neq 0$ we have $ab = 0$. This is impossible, because A is an integral domain. This contradiction shows that every two-dimensional subalgebra of the finite dimensional Lie algebra L is abelian. Therefore, L is nilpotent by Lemma 2.3. ■

Corollary 2.6. *Let L be a locally finite subalgebra of the Lie algebra* $\mathrm{Der}_{\mathbb{K}}(A)$. *If $L \subseteq \mathrm{LND}(A)$, then the Lie algebra L is locally nilpotent.*

3. On subalgebras of $W_2(\mathbb{K})$ consisting of locally nilpotent derivations.

In this section, $A = \mathbb{K}[x, y]$ is the polynomial ring in two variables over the field $\mathbb{K}$ and $R = \mathbb{K}(x, y)$, the field of rational functions. $W_2(\mathbb{K})$ denotes the Lie algebra $\mathrm{Der}_{\mathbb{K}} A$ of all $\mathbb{K}$-derivations of A.

Lemma 3.1. *([4], Corollary 4.7) Let D be a derivation of the ring $A = \mathbb{K}[x, y]$. Then D is locally nilpotent if and only if $D = D_{f(a)} = f'(a)D_a$ for a coordinate polynomial $a \in A$ and some $f \in \mathbb{K}[t]$.*

Lemma 3.2. *Let D_f, D_g be Jacobian derivations of the ring $A = \mathbb{K}[x, y]$. Then $[D_f, D_g] = D_{[f,g]}$, where $[f, g] = \det J(f, g)$ is the Jacobian determinant of polynomials $f, g \in A$.*

Proof. Straightforward check. ■

Corollary 3.3. *Let L be a subalgebra of the Lie algebra $W_2(\mathbb{K})$. If $L \subseteq \mathrm{LND}(A)$, then L satisfies the Engel condition, i.e. for any $D_1, D_2 \in L$ there exists an integer $k \geq 1$ (depending on D_1, D_2) such that $[D_1, \underbrace{D_2, \ldots, D_2}_{k}] = 0$.*

Proof. Take arbitrary $D_1, D_2 \in L$. Since $D_1, D_2 \in \mathrm{LND}(A)$, Lemma 3.1 implies $D_1 = D_f$ and $D_2 = D_g$ for some $f, g \in A$. It follows from Lemma 3.2 that $[D_1, D_2] = D_{[f,g]} = -D_g(f)$, where $[f, g] = \det J(f, g)$ is the Jacobian determinant of $f, g \in A$. Further,

$$[D_1, \underbrace{D_2, \ldots, D_2}_{k}] = D_h,$$

where $h = [\ldots[[f, \underbrace{g], g], \ldots, g}_{k}] = [f, \underbrace{g, g, \ldots, g}_{k}]$. It is easy to check that

$$[f, \underbrace{g, g, \ldots, g}_{k}] = (-1)^k D_g^k(f).$$

Since D_g is locally nilpotent, we get $D_g^k(f) = 0$ for a sufficiently large k. The latter means that $[D_1, \underbrace{D_2, \ldots, D_2}_{k}] = 0$. ■

Recall that the Lie algebra $W_2(\mathbb{K}) = \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[x,y])$ being a vector space over the field $\mathbb{K}$ is also a $\mathbb{K}[x,y]$-module. So we may say about linear dependence over $\mathbb{K}[x,y]$ for elements of $W_2(\mathbb{K})$ and about rank $\mathrm{rk}_A L$ over $A=\mathbb{K}[x,y]$ for any subalgebra $L \subseteq W_2(\mathbb{K})$ (we consider L as a set of elements of the A-module $W_2(\mathbb{K})$).

Lemma 3.4. *Let D_1, D_2 be locally nilpotent derivations of the ring $A = \mathbb{K}[x,y]$.*

(1) *If D_1 and D_2 are linearly dependent over A, then there exists a coordinate polynomial $a \in A$ such that $D_1 = D_{f(a)}$, $D_2 = D_{g(a)}$ for some $f, g \in \mathbb{K}[t]$.*

(2) *If D_1 and D_2 are linearly independent over A and $[D_1, D_2] = 0$, then there exists a coordinate pair (a,c) such that $D_1 = D_a$, $D_2 = D_c$.*

Proof. Since $D_1 \in \mathrm{LND}(A)$, Lemma 3.1 implies that $D_1 = D_{f(a)}$ for a coordinate pair (a,b) and some $f \in \mathbb{K}[t]$. Similarly, since $D_2 \in \mathrm{LND}(A)$, there exists a coordinate pair (c,d) such that $D_2 = D_{g(c)}$ for some $g \in \mathbb{K}[t]$.

(1) Let $r_1D_1 + r_2D_2 = 0$ for some $r_1, r_2 \in A$, and at least one of r_1, r_2 is nonzero. Without loss of generality, one can assume that $D_1 \neq 0$ and $D_2 \neq 0$. Then it obviously holds the equality $\mathrm{Ker}\, D_1 = \mathrm{Ker}\, D_2$. Since $\mathrm{Ker}\, D_1 = \mathbb{K}[a]$ and $\mathrm{Ker}\, D_2 = \mathbb{K}[c]$, we get $\mathbb{K}[a] = \mathbb{K}[c]$. Then $c = \varphi(a)$ for some $\varphi \in \mathbb{K}[t]$, and $D_2 = D_{g(c)} = D_{g(\varphi(a))} = D_{g_1(a)}$.

(2) Let $D_1, D_2 \in \mathrm{LND}(A)$ be linearly independent over A and $[D_1, D_2] = 0$. By Lemma 3.2, we have

$$[D_1, D_2] = [D_{f(a)}, D_{g(c)}] = D_{[f(a),g(c)]} = 0,$$

where $[f(a), g(c)] = \det J(f(a), g(c))$. It is easy to check that $[f(a), g(c)] = f'(a)g'(c)[a,c]$. Then we get

$$D_{[f(a),g(c)]} = D_{f'(a)g'(c)[a,c]} = 0,$$

and hence $f'(a)g'(c)[a,c] \in \mathbb{K}$.

Let us show that $f'(a)g'(c)[a,c] \in \mathbb{K}^*$. Indeed, if $f'(a)g'(c)[a,c] = 0$ then $[a,c] = 0$, because $f'(a) \neq 0$, $g'(c) \neq 0$ (note that $\deg f \geq 1$ and $\deg g \geq 1$). The equality $[a,c] = 0$ implies that D_a and D_c are linearly dependent over A (see [9, Corollary 7.2.10]). This contradicts our assumption.

Therefore, $f'(a)g'(c)[a,c] \in \mathbb{K}^*$ and especially $[a,c] \in \mathbb{K}^*$. Since (a,b) is a coordinate pair, there exists a polynomial $p(u,v) \in \mathbb{K}[u,v]$ such that $c = p(a,b)$. It follows

$$[a,c] = [a, p(a,b)] = \frac{\partial}{\partial b}(p(a,b))[a,b] \in \mathbb{K}^*.$$

Thus, $\frac{\partial}{\partial b}(p(a,b) \in \mathbb{K}^*$. This implies that $c = p(a,b) = \mu b + q(a)$ for some $\mu \in \mathbb{K}^*$ and $q \in \mathbb{K}[t]$. Since the polynomials a and $\mu b + q(a)$ form a coordinate pair in $A = \mathbb{K}[x,y]$, we get that (a,c) is also a coordinate pair in A. Furthermore, from the relation $f'(a)g'(c)[a,c] \in \mathbb{K}^*$ we get $\deg f = \deg g = 1$. Write $f(t) = \alpha t + \beta$,

$g(t) = \gamma t + \delta$ for $\alpha, \beta, \gamma, \delta \in \mathbb{K}$, $\alpha, \gamma \neq 0$. Then $D_1 = D_{\alpha a+\beta}$ and $D_2 = D_{\gamma b+\delta}$. Clearly, $\alpha a+\beta$ and $\gamma b+\delta$ form a coordinate pair in A. Without loss of generality, we may denote $\alpha a + \beta$ by a, $\gamma b + \delta$ by c, and get $D_1 = D_a$, $D_2 = D_c$, where (a, c) is a coordinate pair in A. ∎

Lemma 3.5. *Let L be an abelian subalgebra of the Lie algebra $W_2(\mathbb{K})$ (not necessarily finite dimensional over $\mathbb{K}$). If $L \subseteq \operatorname{LND}(A)$, then L is one of the following algebras:*

(1) $L = \mathbb{K}\langle\{f_i(a)D_a\}_{i\in I}\rangle$, where $\{f_i(t) \in \mathbb{K}[t], i \in I\}$ is a finite or a countable infinite set of polynomials that are linearly independent over $\mathbb{K}$, and $a \in A$ is a coordinate polynomial.

(2) $L = \mathbb{K}\langle D_a, D_b\rangle$, where (a, b) is a coordinate pair in A.

Proof. Let $\operatorname{rk}_A L = 1$ and $D \in L$ be nonzero. By Lemma 3.1, there exists a coordinate polynomial $a \in A$ such that $D = D_{f(a)} = f'(a)D_a$ for some $f \in \mathbb{K}[t]$. Take an arbitrary $D_1 \in L$. Then D_1 and D are linearly dependent over A. By Lemma 3.4(1), there exists $g \in \mathbb{K}[t]$ such that $D_1 = D_{g(a)} = g'(a)D_a$. Thus, $L \subseteq \mathbb{K}[a]D_a$. Since $\mathbb{K}[a]D_a$ has a countable basis over $\mathbb{K}$, we can find a finite or a countable infinite basis $\{f_i(a)D_a\}_{i\in I}$ of the Lie algebra L. We see that L is of type 1).

Now let $\operatorname{rk}_A L = 2$. Take arbitrary $D_1, D_2 \in L$ that are linearly independent over A. Since the Lie algebra L is abelian, $[D_1, D_2] = 0$. By Lemma 3.4(2), there exists a coordinate pair $(a, b) \in A$ such that $D_1 = D_a$, $D_2 = D_b$. Then for every $D = fD_a + gD_b \in L$, where $f, g \in A$, we have

$$0 = [D_a, D] = D_a(f)D_a + D_a(g)D_b,$$

$$0 = [D_b, D] = D_b(f)D_a + D_b(g)D_b.$$

Since D_a and D_b are linearly independent over A, we have $D_a(f) = D_a(g) = 0$ and $D_b(f) = D_b(g) = 0$. These equalities imply that $f, g \in \operatorname{Ker} D_a \cap \operatorname{Ker} D_b = \mathbb{K}$. Therefore, $L = \mathbb{K}\langle D_a, D_b\rangle$. ∎

Lemma 3.6. *Let L be a subalgebra of the Lie algebra $W_2(\mathbb{K})$ with $\operatorname{rk}_A L = 2$. If $L \subseteq \operatorname{LND}(A)$, then there exist linearly independent (over A) elements $D_1, D_2 \in L$ such that $[D_1, D_2] = 0$. Moreover, there exists a coordinate pair $(a, b) \in A$ such that $D_1 = D_a, D_2 = D_b$.*

Proof. Take any elements $D_1, D_2 \in L$ that are linearly independent over A. Consider inductively defined elements $D_{k+1} = [D_k, D_1] \in L$ for $k \geq 2$. By Corollary 3.3, there exists the least number s, $s \geq 2$, such that $D_{s+1} = 0$. If D_1 and D_s are linearly independent over A, then we denote D_s by D_2 and all is done. Assume that D_1, D_s are linearly dependent over A. By Lemma 3.4(1), there exists a coordinate polynomial $a \in A$ such that $D_1 = D_{f(a)}$, $D_s = D_{h(a)}$ for some $f, h \in \mathbb{K}[t]$. Since $D_{s-1} \in \operatorname{LND}(A)$, Lemma 3.1 implies that there exists a coordinate polynomial $c \in A$ such that $D_{s-1} = D_{g(c)}$ for some $g \in \mathbb{K}[t]$. Note that D_1 and D_{s-1} are linearly independent over A. Indeed, in the opposite case $D_1 = D_{f_1(d)}$ and $D_{s-1} = D_{g_1(d)}$ for some coordinate polynomial $d \in A$ and $f_1, g_1 \in \mathbb{K}[t]$ (see Lemma 3.4). By Lemma 3.2,

$$[D_{g_1(d)}, D_{f_1(d)}] = D_{g_1'(d)f_1'(d)[d,d]} = 0,$$

and thus $[D_{s-1}, D_1] = D_s = 0$. This contradicts our choice of D_s.

It follows from linear independence of D_1 and D_{s-1} that $\det J(a,c) = [a,c] \neq 0$ (see, for example, [9, Corollary 7.2.10]). Further, we have

$$D_s = [D_{s-1}, D_1] = [D_{g(c)}, D_{f(a)}] = D_{[g(c),f(a)]} = D_{g'(c)f'(a)[c,a]},$$

and since $D_s = D_{h(a)}$ we get

$$g'(c)f'(a)[c,a] = h(a) + \gamma. \tag{1}$$

for some $\gamma \in \mathbb{K}$. The field $\mathbb{K}$ is algebraically closed, so we have

$$h(a) + \gamma = \mu(a - \alpha_1)(a - \alpha_2)\dots(a - \alpha_k),$$

where $\mu \in \mathbb{K}^\star$ and $\alpha_1, \alpha_2, \dots, \alpha_n$ are all the roots of the polynomial $h(a) + \gamma$. Rewrite the equality (1) in the form

$$g'(c)f'(a)[c,a] = \mu(a - \alpha_1)(a - \alpha_2)\dots(a - \alpha_k). \tag{2}$$

The polynomial a is coordinate and thus all the polynomials $a - \alpha_i$, $i = 1, 2, \dots, k$ are irreducible.

Let us show that $[c,a] \in \mathbb{K}^*$. It was mentioned above that $[c,a] \neq 0$. Assume that $[c,a] \in \mathbb{K}[x,y] \setminus \mathbb{K}$. It follows from (2) that $[c,a]$ is divided by some $a - \alpha_i$, assume by $a - \alpha_1$. Then $[c,a] = D_c(a) = (a - \alpha_1)u(x,y)$ for some $u(x,y) \in \mathbb{K}[x,y]$. It is obvious that $D_c(a - \alpha_1) = (a - \alpha_1)u(x,y)$ and hence $D_c(a - \alpha_1) = 0 = [c,a]$ (see, Lemma 2.1(d)). Contradiction. Thus $[c,a] \in \mathbb{K}^\star$. It follows from this relation that $[D_c, D_a] = D_{[c,a]} = 0$ and D_a and D_c are linearly independent over A. By Lemma 3.4, we see that (a,c) is a coordinate pair for $A = \mathbb{K}[x,y]$.

We now find an element $\widetilde{D}$ such that D_{s-1} and $\widetilde{D}$ are linearly independent over A and $[\widetilde{D}, D_{s-1}] = 0$. It follows from the equality (2) that $g'(c)$ is a nonzero constant, because $\mu(a - \alpha_1)(a - \alpha_2)\dots(a - \alpha_k)$ is divided by $g'(c)$ and the polynomials a, c are algebraically independent over $\mathbb{K}$. Thus $g(c) = \beta c + \sigma$ for some $\beta \in \mathbb{K}^*$, $\sigma \in \mathbb{K}$ and $D_{s-1} = D_{g(c)} = D_{\beta c + \sigma}$. Without loss of generality, we may assume that $D_{s-1} = D_c$ and $[a,c] = 1$. Denote $\deg f(t)$ by m (recall that $D_1 = D_{f(a)}$). Then put

$$\widetilde{D} = [D_1, \underbrace{D_{s-1}, \dots, D_{s-1}}_{m-1}] = D_{f^{(m-1)}(a)[a,c]} \in L,$$

where $f^{(m-1)}(t)$ is the $(m-1)$-th derivative of $f(t)$. Since $\deg f(t) = m$, we get $\widetilde{D} = D_{\delta a + \tau}$ for some $\delta \in \mathbb{K}^*$, $\tau \in \mathbb{K}$ and may assume that $\widetilde{D} = D_a$. Therefore, D_{s-1} and $\widetilde{D}$ are linearly independent over A. Moreover, $[\widetilde{D}, D_{s-1}] = D_\delta = 0$. Thus by the denoting $D_1 = \widetilde{D}$ and $D_2 = D_{s-1}$, we get desired derivations, and (a,c) is the desired coordinate pair in A. ■

Lemma 3.7. *Let L be a subalgebra of the Lie algebra $W_2(\mathbb{K})$ with $\mathrm{rk}_A L = 2$. If $L \subseteq \mathrm{LND}(A)$ and $\dim_{\mathbb{K}} L \geq 3$, then there exists an automorphism θ of the ring A such that $\theta L \theta^{-1}$ contains the elements $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}$.*

Proof. The Lie algebra L contains elements D_a, D_b for a coordinate pair (a, b), by Lemma 3.6. These elements are linearly independent over A and $[D_a, D_b] = 0$. Define an automorphism $\varphi \in Aut(A)$ by the rule: $\varphi(a) = x, \varphi(b) = y$. Then φ induces an automorphism $\widetilde{\varphi}$ of the Lie algebra $W_2(\mathbb{K})$, namely: $\widetilde{\varphi}(D) = \varphi D \varphi^{-1}$ for any $D \in W_2(\mathbb{K})$ (see, for example, [1]). One can easily see that $\varphi D_a \varphi^{-1} = \frac{\partial}{\partial y}$ and $\varphi D_b \varphi^{-1} = -\frac{\partial}{\partial x}$. Denote $L_1 = \varphi L \varphi^{-1}$. It is a subalgebra of the Lie algebra $W_2(\mathbb{K})$. The Lie algebra L_1 consists of locally nilpotent derivations of the ring A and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \in L_1$. Let us show that L_1 contains an element D of the form $D = p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y}$ with $\deg p \leq 1, \deg q \leq 1$ and at least one of these polynomials is nonconstant. Take any element $D_1 \in L_1 \setminus \mathbb{K}\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ (such an element does exist because $\dim_{\mathbb{K}} L \geq 3$). Let $D_1 = u(x, y)\frac{\partial}{\partial x} + v(x, y)\frac{\partial}{\partial y}$, where $u(x, y), v(x, y) \in \mathbb{K}[x, y]$. Without loss of generality we may assume that $\deg u \geq \deg v$ and $\deg u \geq 1$. Using the following relations

$$[\frac{\partial}{\partial x}, D_1] = u'_x \frac{\partial}{\partial x} + v'_x \frac{\partial}{\partial y}, \quad [\frac{\partial}{\partial y}, D_1] = u'_y \frac{\partial}{\partial x} + v'_y \frac{\partial}{\partial y},$$

one can easily show that for some $s, k, \ s \geq k$ we have

$$\frac{\partial^s u}{\partial x^k y^{s-k}} \frac{\partial}{\partial x} + \frac{\partial^s v}{\partial x^k y^{s-k}} \frac{\partial}{\partial y} \in L_1,$$

where the polynomial $\frac{\partial^s u}{\partial x^k \partial y^{s-k}}$ is of degree 1 and $\frac{\partial^s v}{\partial x^k \partial y^{s-k}}$ is of degree ≤ 1. Thus, one may assume that L_1 contains an element $D = p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y}$, where $\deg p \leq 1, \ \deg q \leq 1$ and $D \notin \mathbb{K}\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$. Since $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \in L_1$, this element D can be chosen in the form

$$D = (\alpha_{11}x + \alpha_{12}y)\frac{\partial}{\partial x} + (\alpha_{21}x + \alpha_{22}y)\frac{\partial}{\partial y},$$

where $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \in \mathbb{K}$ and at least one of them is nonzero.

The locally nilpotent derivation D has zero divergence (see, for example, [4, Corollary 3.16]), so we have $\alpha_{11} + \alpha_{22} = \mathrm{div} D = 0$. Then

$$D = (\alpha_{11}x + \alpha_{12}y)\frac{\partial}{\partial x} + (\alpha_{21}x - \alpha_{11}y)\frac{\partial}{\partial y}$$

and hence $D = D_h$ for the polynomial $h = \alpha_{21}x^2/2 - \alpha_{11}xy - \alpha_{12}y^2/2$. There exists (by Lemma 3.1) a coordinate polynomial $c \in A$ such that $h = g(c)$ for some polynomial $g(t) \in \mathbb{K}[t]$. If $\deg g = 1$, then h is a coordinate polynomial. This is impossible, because h is reducible as a homogeneous polynomial in two variables. Hence $\deg g = 2$ and $\deg c = 1$. A straightforward check shows that there exist $\mu, \nu \in \mathbb{K}$ such that $h = (\mu x + \nu y)^2$. Choose a polynomial $\mu_1 x + \nu_1 y$ $(\mu_1, \nu_1 \in \mathbb{K})$ in such a way that $\mu\nu_1 - \mu_1\nu = 1$. The polynomials $\mu x + \nu y, \ \mu_1 x + \nu_1$ form a coordinate pair in $\mathbb{K}[x, y]$, and thus there exists an automorphism ψ of the ring A defined by the rule: $\psi(\mu x + \nu y) = x, \psi(\mu_1 x + \nu_1 y) = y$. Denote $L_2 = \psi L_1 \psi^{-1}$.

One can easily check that

$$\psi D_{\mu x+\nu y}\psi^{-1} = \frac{\partial}{\partial y}, \quad \psi D_{\mu_1 x+\nu_1 y}\psi^{-1} = -\frac{\partial}{\partial x}.$$

Since $D_{\mu x+\nu y}$, $D_{\mu_1 x+\nu_1 y} \in L_1$, we obtain that $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \in L_2$. It follows from the equality $\psi D_h \psi^{-1} = 2x\frac{\partial}{\partial y}$ that $x\frac{\partial}{\partial y} \in L_2$. Thus $L_2 = \theta L \theta^{-1}$ and $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}\} \subseteq L_2$, where $\theta = \psi\varphi \in \mathrm{Aut}A$. ■

Lemma 3.8. *Let L be a subalgebra of the Lie algebra $W_2(\mathbb{K})$ such that $L \subseteq \mathrm{LND}(A)$. If $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}\} \subseteq L$, then every element $D = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}$ of L with $\max\{\deg p, \deg q\} \le 1$ belongs to the Lie subalgebra $\mathbb{K}\langle\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}\rangle$.*

Proof. Since $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \in L$, one can assume, without loss of generality, that $D = (\alpha_{11}x + \alpha_{12}y)\frac{\partial}{\partial x} + (\alpha_{21}x + \alpha_{22}y)\frac{\partial}{\partial y}$, where $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \in \mathbb{K}$ and at least one of them is nonzero. The derivation D is locally nilpotent, so we have $\mathrm{div}D = \alpha_{11} + \alpha_{22} = 0$ (see [4, Corollary 3.16]). Then

$$[x\frac{\partial}{\partial y}, D] = \alpha_{12}(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) - 2\alpha_{11}x\frac{\partial}{\partial y} \in L.$$

Since $x\frac{\partial}{\partial y} \in L$ we get $\alpha_{12}(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) \in L$. But $(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) \notin \mathrm{LND}(A)$ and therefore $\alpha_{12} = 0$. Thus $D = \alpha_{11}(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + \alpha_{21}x\frac{\partial}{\partial y}$ and by the same reason, we have $\alpha_{11} = 0$. Hence $D = \alpha_{21}x\frac{\partial}{\partial y}$ and $D \in \mathbb{K}\langle\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}\rangle$. ■

Lemma 3.9. *Let L be a subalgebra of the Lie algebra $W_2(\mathbb{K})$ such that $L \subseteq \mathrm{LND}(A)$. If $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}\} \subseteq L$, then for any $D = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y} \in L$ with $\max\{\deg p, \deg q\} \ge 1$ the following is true:*

1) $\deg p < \deg q$.

2) the highest homogeneous component of the polynomial $q = q(x,y)$ depends only on x.

Proof. Suppose there exists $D = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y} \in L$ satisfying conditions of the lemma such that $\deg p \ge \deg q$. Denote $m = \deg p$, then $m \ge 1$ by conditions of the lemma. Since

$$[\frac{\partial}{\partial x}, D] = p'_x\frac{\partial}{\partial x} + q'_x\frac{\partial}{\partial y} \in L \text{ and } [\frac{\partial}{\partial y}, D] = p'_y\frac{\partial}{\partial x} + q'_y\frac{\partial}{\partial y} \in L,$$

it is easy to show that for any nonnegative integers k, s, $k \le s$, we have

$$\frac{\partial^s p}{\partial x^k \partial y^{s-k}}\frac{\partial}{\partial x} + \frac{\partial^s q}{\partial x^k \partial y^{s-k}}\frac{\partial}{\partial y} \in L.$$

Denote by $p_m(x,y)$ the highest homogeneous component of the polynomial $p(x,y)$. Let $p_m(x,y) = \sum_{i=0}^m \alpha_{i,m-i}x^i y^{m-i}$ for $\alpha_{ij} \in \mathbb{K}$ and let, for example, $\alpha_{k,m-k} \ne 0$. First, let $k > 0$. Then as above

$$D_1 := \frac{\partial^{m-1} p}{\partial x^{k-1}\partial y^{m-k}}\frac{\partial}{\partial x} + \frac{\partial^{m-1} q}{\partial x^{k-1}\partial y^{m-k}}\frac{\partial}{\partial y} \in L,$$

and D_1 is of the form $D_1 = (\alpha_1 x + \beta_1 y + \gamma_1)\frac{\partial}{\partial x} + (\delta_1 x + \mu_1 y + \nu_1)\frac{\partial}{\partial y}$ with all the coefficients in $\mathbb{K}$. Since $\alpha_{k,m-k} \neq 0$, we have $\alpha_1 \neq 0$. The latter is impossible by Lemma 3.8. Further, if $k = 0$, i.e. $\alpha_{0,m} \neq 0$, we get

$$D_2 := \frac{\partial^{m-1} p}{\partial y^{m-1}} \frac{\partial}{\partial x} + \frac{\partial^{m-1} q}{\partial y^{m-1}} \frac{\partial}{\partial y} \in L,$$

D_2 is of the form $D_2 = (\alpha_2 x + \beta_2 y + \gamma_2)\frac{\partial}{\partial x} + (\delta_2 x + \mu_2 y + \nu_2)\frac{\partial}{\partial y}$ with all the coefficients in $\mathbb{K}$, and $\alpha_2 \neq 0$ because $\alpha_{0,m} \neq 0$. This is also impossible by the same reason. Therefore $\deg p < \deg q$ for any derivation $D = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y} \in L$.

Denote $n = \deg q(x,y)$ and let $q_n(x,y)$ be the highest homogeneous component of the polynomial q. Suppose $\deg_y q_n(x,y) = l \geq 1$. Then as above

$$D_3 := \frac{\partial^{n-1} p}{\partial x^{n-l} \partial y^{l-1}} \frac{\partial}{\partial x} + \frac{\partial^{n-1} q}{\partial x^{n-l} \partial y^{l-1}} \frac{\partial}{\partial y} \in L$$

and D_3 is of the form $D_3 = \alpha\frac{\partial}{\partial x} + (\beta x + \gamma y + \delta)\frac{\partial}{\partial y}$ with $\gamma \neq 0$ because $\deg_y q_n = l$. Since $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}\} \subseteq L$ we get $y\frac{\partial}{\partial y} \in L$. The latter is impossible because $y\frac{\partial}{\partial y} \notin \mathrm{LND}(A)$. The obtained contradiction shows that $\deg_y q_n(x,y) = 0$ and therefore $q_n = q_n(x)$. ∎

Lemma 3.10. *Let L be a subalgebra of the Lie algebra $W_2(\mathbb{K})$ such that $L \subseteq \mathrm{LND}(A)$. If $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}\} \subseteq L$, then every element $D \in L$ is of the form $D = \alpha\frac{\partial}{\partial x} + q(x)\frac{\partial}{\partial y}$, where $\alpha \in \mathbb{K}$ and $q(t) \in \mathbb{K}[t]$.*

Proof. Suppose L contains elements D of the form $D = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}$ with $p(x,y) \in \mathbb{K}[x,y] \setminus \mathbb{K}$. Choose among such elements an element $D = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}$ with a minimum $\deg q$. Let us show that $p(x,y)$ is a polynomial only in x. Suppose to the contrary that $p'_y(x,y) \neq 0$. Then

$$D_1 := [x\frac{\partial}{\partial y}, D] = [x\frac{\partial}{\partial y}, p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y}] = xp'_y\frac{\partial}{\partial x} + (-p + xq'_y)\frac{\partial}{\partial y}$$

and $D_1 \in L$. Since $xp'_y \neq \mathrm{const}$, we have that $\deg(-p + xq'_y) \geq \deg q$ by the choice of the polynomial q. By Lemma 3.9(1), $\deg p < \deg q$, and by Lemma 3.9(2), $\deg q'_y \leq \deg q - 2$. Thus $\deg(-p + xq'_y) < \deg q$. This contradicts our choice of D and therefore $p'_y = 0$, i.e. $p = p(x)$.

Further

$$[\frac{\partial}{\partial x}, D] = p'_x\frac{\partial}{\partial x} + q'_x(x,y)\frac{\partial}{\partial y} \in L$$

and $\deg q'_x < \deg q$. By the choice of D, we get $p'_x \in \mathbb{K}$. Since $p(x) \in \mathbb{K}[x,y] \setminus \mathbb{K}$, we have $p(x) = \alpha x + \beta$ for some $\alpha \in \mathbb{K}^\star, \beta \in \mathbb{K}$. Without loss of generality, one can assume that $\beta = 0$ because $\frac{\partial}{\partial x} \in L$. We have $D = \alpha x\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}$. Then

$$[\frac{\partial}{\partial x}, D] = \alpha\frac{\partial}{\partial x} + q'_x(x,y)\frac{\partial}{\partial y} \in L$$

and hence $q'_x(x,y)\frac{\partial}{\partial y} \in L$. The inclusion $L \subseteq \mathrm{LND}(A)$ implies that every element of L has zero divergence and therefore $q''_{xy}(x,y) = 0$. One can easily show that $q(x,y) = u(y) + r(x)$ for some univariate polynomials $u(t), r(t) \in \mathbb{K}[t]$. The latter means that $D = \alpha x\frac{\partial}{\partial x} + (u(y) + r(x))\frac{\partial}{\partial y}$ and since $\mathrm{div}D = \alpha + u'(y) = 0$, we get $u(y) = -\alpha y + \delta$ for some $\delta \in \mathbb{K}$. Since $\frac{\partial}{\partial y} \in L$, we may assume that $\delta = 0$. Thus $D = \alpha(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + r(x)\frac{\partial}{\partial y}$. By Lemma 3.1, $D = D_{-\alpha xy+s(x)}$, where $s(x)$ is a polynomial such that $s'(x) = r(x)$. By the same lemma, we have $-\alpha xy+s(x) = f(a)$ for a coordinate polynomial $a = a(x,y)$ and some $f(t) \in \mathbb{K}[t]$. Note that $a'_y(x,y) \neq 0$ because in the other case $\alpha = 0$ which contradicts the hypothesis on α (recall $\alpha \in \mathbb{K}^*$).

Since this fact, if $\deg f(t) \geq 2$, then we get $\deg_y f(a) \geq 2$. The latter is impossible because of equality $f(a) = -\alpha xy + s(x)$. Therefore, $\deg f(t) = 1$ and $f(a) = \alpha xy+s(x)$ is a coordinate polynomial of the ring $\mathbb{K}[x,y]$. Denote by s_0 the constant term of the polynomial $s(x)$. We see that $f(a) - s_0 = -\alpha xy - s(x) - s_0$ is also a coordinate polynomial. But the polynomial $\alpha xy - s(x) - s_0$ divides by x and thus is reducible. The obtained contradiction shows that every element $D \in L$ is of the form $D = \alpha\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}, \alpha \in \mathbb{K}$. Since $\mathrm{div}D = q'_y(x,y) = 0$, we get that q depends only on x. Therefore, $D = \alpha\frac{\partial}{\partial x} + q(x)\frac{\partial}{\partial y}$. ■

Theorem 3.11. *Let L be a subalgebra of the Lie algebra $W_2(\mathbb{K})$. If L consists of locally nilpotent derivations of the ring $\mathbb{K}[x,y]$, then there exists an automorphism $\varphi : \mathbb{K}[x,y] \to \mathbb{K}[x,y]$ such that $L_1 = \varphi L\varphi^{-1}$ is a subalgebra of the triangular Lie algebra $u_2(\mathbb{K})$.*

Proof. If L is abelian, then the statement of the theorem follows from Lemma 3.5. Let L be nonabelian. Then $\mathrm{rk}_A(L) = 2$ and $\dim_{\mathbb{K}} L \geq 3$. By Lemma 3.7, there exists an automorphism φ of the polynomial ring $\mathbb{K}[x,y]$ such that the Lie algebra $L_1 = \varphi L\varphi^{-1}$ contains the elements $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}$. By Lemma 3.10, every element $D \in L_1$ is of the form $D = \alpha\frac{\partial}{\partial x} + q(x)\frac{\partial}{\partial y}$. The latter means that $L_1 \subseteq u_2(\mathbb{K})$. ■

Corollary 3.12. *Every maximal (by inclusion) subalgebra of the Lie algebra $W_2(\mathbb{K})$ which is contained in the set $\mathrm{LND}(\mathbb{K}[x,y])$ is either $u_2(\mathbb{K})$ or one of its conjugated by automorphisms of $\mathbb{K}[x,y]$ subalgebras.*

Remark 3.13. By the known Miyanishi's Theorem [8] and a result of D. Daigle [2], every nonzero $D \in \mathrm{LND}(\mathbb{K}[x,y,z])$ is of the form $D = hD_{(a,b)}$, where $a, b \in \mathbb{K}[x,y,z]$ such that $\mathrm{Ker}\, D = \mathbb{K}[a,b]$, $D_{(a,b)}$ is a jacobian derivation and $h \in \mathrm{Ker}\, D = \mathbb{K}[a,b]$. Unfortunately, this fact does not enable to prove that every Lie algebra $L \subseteq \mathrm{LND}(\mathbb{K}[x,y,z])$ is Engelian (as in the case $\mathrm{LND}(\mathbb{K}[x,y])$, in Corollary 3.3).

References

[1] Bavula, V., *Lie algebras of triangular polynomial derivations and an isomorphism criterion for t heir Lie factor algebras*, Izv. RAN. Ser. Mat. **77** (2013), 3–44.

[2] Daigle, D., *On some properties of locally nilpotent derivations*, J. Pure Appl. Algebra **114** (1997), 221–230.

[3] Essen van den, A., "Polynomial Automorphisms and the Jacobian Conjecture," Progr. Math. **190**, Birkhäuser, Boston, 2000.

[4] Freudenburg, G., "Algebraic theory of locally nilpotent derivations," Encyclopaedia of Math. Sciences **136**, Springer-Verlag, Berlin etc., 2006.

[5] Jacobson, N., "Lie algebras," Interscience Publishers, New York, London,1962.

[6] Kaliman, S., *Polynomials with general* $\mathbb{C}^2$*-fibers are variables*, Pacific J. Math. **203** (2002), 161–189.

[7] Makar-Limanov, L., *Locally nilpotent derivations of affine domains*, MPIM Preprint Series **92** (2004).

[8] Miyanishi, M., *Normal affine subalgebras of a polynomial ring*, in: Algebraic and Topological Theories—in memory of Dr. Takehiko Miyata, 1985, 37–51.

[9] Nowicki, A., "Polynomial Derivations and their Rings of Constants," Uniwersytet Mikolaja Kopernika, Torun, 1994.

[10] Rentschler, R., *Operations du groupe additif sur le plan affine*, C. R. Acad. Sc. Paris **267** (1968), 384–387.

Anatoliy P. Petravchuk
Department of Algebra
and Mathematical Logic
Faculty of Mechanics and Mathematics
Taras Shevchenko
National University of Kyiv
64 Volodymyrska Street
Kyiv, 01033, Ukraine
aptr@univ.kiev.ua
apetrav@gmail.com

Kateryna Ya. Sysak
Department of Algebra
and Mathematical Logic
Faculty of Mechanics and Mathematics
Taras Shevchenko
National University of Kyiv
64 Volodymyrska Street
Kyiv, 01033, Ukraine
sysakkya@gmail.com

Received September 27, 2016
and in final form April 17, 2017