A Weyl-Type Character Formula for PDC Modules of $\mathfrak{gl}(m|n)$

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Abstract.   In 1994, Kac and Wakimoto suggested a generalization of Bernstein and Leites' character formula for basic Lie superalgebras, and the natural question was raised: to which simple highest weight modules does it apply? In this paper, we prove a similar formula for a large class of finite-dimensional simple modules over the Lie superalgebra $\mathfrak{gl}(m|n)$, which we call piecewise disconnected modules, or PDC. The class of PDC modules naturally includes totally connected modules and totally disconnected modules, the two families for which similar character formulas were proven by Su and Zhang as special cases of their general formula. This paper is part of our program for the pursuit of elegant character formulas for Lie superalgebras.
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1.   Introduction

It is well known that the theory of character formulas for simple finite-dimensional modules over a Lie superalgebra is a nontrivial extension of the classical case. The problem originates from the existence of the so called *atypical* roots. In the absence of these roots, Kac proved in 1977 that the Weyl character formula generalizes in a straightforward fashion to finite-dimensional basic classical Lie superalgebras [K2, K3]. In 1980, an elegant Weyl-type character formula was proven by Bernstein and Leites [BL] for simple highest weight modules of atypicality 1 (see Section 2) over finite-dimensional basic classical Lie superalgebras. Let $L(\lambda)$ be a finite-dimensional simple module of highest weight λ and with atypical root β, then

$$e^{\rho} R \cdot \operatorname{ch} L(\lambda) = \sum_{w \in W} (-1)^{l(w)} w \left(\frac{e^{\lambda + \rho}}{1 + e^{-\beta}} \right),$$

where R is the Weyl denominator. In 1988, this formula was generalized by Kac and Wakimoto to a large class of modules over infinite-dimensional symmetrizable Kac-Moody superalgebras [KW1].

Great efforts were invested in generalizing this formula to all finite-dimensional modules of $\mathfrak{gl}(m|n)$. It was shown in [VHKT] that such a formula does not hold for all modules but does hold for various families of modules, such as the covariant and contravariant modules. In [KW2], Kac and Wakimoto stated a similar formula for the case when all of the atypical roots are simple. This was proven by the authors in [CHR] for $\mathfrak{gl}(m|n)$- modules, and for modules over other Lie superalgebras in [CK1, GK]. In [SZ], Su and Zhang gave a closed character formula for all finite-dimensional $\mathfrak{gl}(m|n)$-modules, based on the work of Serganova [S1, S2] and Brundan [B]. However, this formula is rather intricate and difficult to apply. For modules of atypicality r, the Su-Zhang formula consists of an alternating sum of up to $r! \cdot 2^{r-1}$ terms, each of which resembles the Kac-Wakimoto formula. Therefore, it is still a major goal to find classes of modules which satisfy a simpler Weyl-type character formula.

In this paper, we present a class of $\mathfrak{gl}(m|n)$-modules for which we prove a Weyl-type character formula. Our formula consists of only one Kac-Wakimoto term. We call these modules *piecewise disconnected*, or PDC. This class of modules naturally extends the two classes previously known to admit the Kac-Wakimoto formula: the totally connected modules and totally disconnected modules (see [SZ, Corollaries 4.13 ,4.15]). The PDC modules are the modules whose highest weight splits into components, each of which resembles a totally connected module while the relation between these components resembles a totally disconnected module (see Definition 3.5).

The class of totally connected modules includes the covariant and contravariant modules and was shown by the authors in [CHR] to be precisely the same class of modules for which the Kac-Wakimoto formula was originally conjectured in [KW2, Section 3], [KW3, Conjecture 3.6]. Totally connected modules and totally disconnected modules were also studied over the queer Lie superalgebras in [CK2] where closed character formulas for these classes were derived and proven.

Our main result is as follows. Let $L(\lambda)$ be a PDC $\mathfrak{gl}(m|n)$-module of highest weight λ with respect to the standard choice of simple roots. We prove the following character formula for $L(\lambda)$:

$$e^\rho R \cdot \operatorname{ch} L(\lambda) = \frac{(-1)^{|(\lambda^\rho)^\Uparrow - \lambda^\rho|_{S_\lambda}}}{t_\lambda} \sum_{w \in W} (-1)^{l(w)} w \left(\frac{e^{(\lambda^\rho)^\Uparrow}}{\prod_{\beta \in S_\lambda} (1 + e^{-\beta})} \right), \quad (1)$$

where S_λ is a maximal orthogonal set of atypical roots; the weight $(\lambda^\rho)^\Uparrow$ is obtained by adding certain atypical roots to $\lambda + \rho$; the exponent $|(\lambda^\rho)^\Uparrow - \lambda^\rho|_{S_\lambda}$ is the number of such roots added; and t_λ is a positive integer determined by the lengths of the atypical components of λ (see Definitions 3.4, 3.15 and 3.18).

When the defect of $\mathfrak{gl}(m|n)$ (i.e. $\min\{m, n\}$) is less than or equal to 2, then all modules are PDC and so the character formula above applies. This follows from the fact that the defect is equal to the maximal possible atypicality of a weight. The standard module of $\mathfrak{g} = \mathfrak{gl}(m|n)$ is totally connected, and hence PDC (see Example 3.10). If $\mathfrak{g} = \mathfrak{gl}(m|n)$ has defect greater than or equal to 3, then the non-trivial simple subquotient of the adjoint module of $\mathfrak{g}$ is not PDC (see Example 3.12). However, the module $\mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ obtained by tensoring

$n-1$ copies of the adjoint module contains a simple subquotient that is PDC but is neither totally connected nor totally disconnected (see Example 3.14).

It is natural to ask about character formulas in other choices of simple roots. Using the techniques of the authors from [CHR] for totally connected modules one can use odd reflections on the right hand side of the character formula (1) to obtain character formulas for other choices of simple roots.

Totally connected modules (also known as Kostant modules) admit character formula (1), and these modules were shown to possess a BGG-type resolution by Brundan and Stroppel in [BS2, BS4]. It would be interesting to determine whether or not PDC modules possess a BGG-type resolution.

Our proof of the above character formula uses Brundan's algorithm for computing Kazhdan-Lusztig Polynomials [B] in the language of [BS1] and is based on ideas from [SZ]. In particular, the right hand side of (1) is expanded as a geometric series parameterized using paths given by the algorithm of Brundan and Stroppel. Unlike the totally connected and totally disconnected cases, for a general piecewise disconnected weight λ, the weight $(\lambda^\rho)^{\Uparrow}$ appearing in formula (1) does not correspond to a highest weight vector for any choice of simple roots.

2. Preliminaries

2.1. The general linear Lie superalgebra.

Let $\mathfrak{g}$ denote the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ over the complex field $\mathbb{C}$. As a vector space, $\mathfrak{g}$ can be identified with the endomorphism algebra $\mathrm{End}(V_{\bar{0}} \oplus V_{\bar{1}})$ of a $\mathbb{Z}_2$-graded vector space $V_{\bar{0}} \oplus V_{\bar{1}}$ with $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$. Then $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where

$$\mathfrak{g}_{\bar{0}} = \mathrm{End}(V_{\bar{0}}) \oplus \mathrm{End}(V_{\bar{1}}) \quad \text{and} \quad \mathfrak{g}_{\bar{1}} = \mathrm{Hom}(V_{\bar{0}}, V_{\bar{1}}) \oplus \mathrm{Hom}(V_{\bar{1}}, V_{\bar{0}}).$$

A homogeneous element $x \in \mathfrak{g}_{\bar{0}}$ has degree 0, denoted $\deg(x) = 0$, while $x \in \mathfrak{g}_{\bar{1}}$ has degree 1, denoted $\deg(x) = 1$. We define a bilinear operation on $\mathfrak{g}$ by letting

$$[x, y] = xy - (-1)^{\deg(x)\deg(y)} yx$$

on homogeneous elements and then extending linearly to all of $\mathfrak{g}$.

By fixing a basis of $V_{\bar{0}}$ and $V_{\bar{1}}$, we can realize $\mathfrak{g}$ as the set of $(m+n) \times (m+n)$ matrices, where

$$\mathfrak{g}_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in M_{m,m},\ B \in M_{n,n} \right\},$$

$$\mathfrak{g}_{\bar{1}} = \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \mid C \in M_{m,n},\ D \in M_{n,m} \right\},$$

and $M_{r,s}$ denotes the set of $r \times s$ matrices.

2.2. Root space decomposition and choice of simple roots.

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the set of diagonal matrices, and it has a natural basis

$$\{E_{1,1}, \ldots, E_{m,m}; E_{m+1,m+1}, \ldots, E_{m+n,m+n}\},$$

where E_{ij} denotes the matrix whose ij-entry is 1 and all other entries are 0. Fix the dual basis $\{\varepsilon_1, \ldots, \varepsilon_m; \delta_1, \ldots, \delta_n\}$ for $\mathfrak{h}^*$. We define a bilinear form on $\mathfrak{h}^*$ by $(\varepsilon_i, \varepsilon_j) = \delta_{ij} = -(\delta_i, \delta_j)$ and $(\varepsilon_i, \delta_j) = 0$.

Then $\mathfrak{g}$ has a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_{\bar{0}}} \mathfrak{g}_\alpha \right) \oplus \left(\bigoplus_{\alpha \in \Delta_{\bar{1}}} \mathfrak{g}_\alpha \right)$, where the set of roots of $\mathfrak{g}$ is $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$, with

$$\Delta_{\bar{0}} = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq m\} \cup \{\delta_k - \delta_l \mid 1 \leq k \neq l \leq n\},$$
$$\Delta_{\bar{1}} = \{\pm(\varepsilon_i - \delta_k) \mid 1 \leq i \leq m, \ 1 \leq k \leq n\},$$

and $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}, \ \mathfrak{g}_{\delta_k - \delta_l} = \mathbb{C}E_{m+k,m+l}, \ \mathfrak{g}_{\varepsilon_i - \delta_k} = \mathbb{C}E_{i,m+k}, \ \mathfrak{g}_{\delta_k - \varepsilon_i} = \mathbb{C}E_{m+k,i}.$

The Weyl group of $\mathfrak{g}$ is $W = Sym(m) \times Sym(n)$, where $Sym(k)$ denotes the symmetric group of degree k. The group W acts on $\mathfrak{h}^*$ by permuting the indices of the ε's and by permuting the indices of the δ's. In particular, the even reflection $s_{\varepsilon_i - \varepsilon_j}$ interchanges the i and j indices of the ε's and fixes all other indices, while $s_{\delta_k - \delta_l}$ interchanges the k and l indices of the δ's and fixes all other indices.

A set of simple roots $\pi \subset \Delta$ determines a decomposition of Δ into positive and negative roots, $\Delta = \Delta^+ \cup \Delta^-$. There is a corresponding triangular decomposition of $\mathfrak{g}$ given by $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{n}^\pm = \oplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$. Let $\Delta_{\bar{d}}^+ = \Delta_{\bar{d}} \cap \Delta^+$ for $d \in \{0, 1\}$. For the rest of the paper, we fix the standard choice of simple roots

$$\pi = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \delta_1, \delta_1 - \delta_2 \ldots, \delta_{n-1} - \delta_n\}.$$

The corresponding decomposition $\Delta = \Delta^+ \cup \Delta^-$ is given by

$$\Delta_{\bar{0}}^+ = \{\varepsilon_i - \varepsilon_j\}_{1 \leq i < j \leq m} \cup \{\delta_k - \delta_l\}_{1 \leq k < l \leq n} \quad \text{and} \quad \Delta_{\bar{1}}^+ = \{\varepsilon_i - \delta_k\}_{1 \leq i \leq m, \ 1 \leq k \leq n}. \quad (2)$$

The standard choice of simple roots has the unique property that W fixes $\Delta_{\bar{1}}^+$. Moreover, it contains a basis for $\Delta_{\bar{0}}^+$, which we denote by $\pi_{\bar{0}}$.

Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_{\bar{0}}^+} \alpha - \frac{1}{2} \sum_{\alpha \in \Delta_{\bar{1}}^+} \alpha$. Then for $\alpha \in \pi$, we have $(\rho, \alpha) = (\alpha, \alpha)/2$.

We define the root lattice as $Q = \sum_{\alpha \in \pi} \mathbb{Z}\alpha$ and the positive root lattice as $Q^+ = \sum_{\alpha \in \pi} \mathbb{N}\alpha$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. A partial order is defined on $\mathfrak{h}^*$ by $\mu > \nu$ when $\mu - \nu \in Q^+$.

2.3. Finite dimensional modules for $\mathfrak{g} = \mathfrak{gl}(\mathfrak{m}|\mathfrak{n})$.

For each weight $\lambda \in \mathfrak{h}^*$, the *Verma module* of highest weight λ is the induced module $M(\lambda) := \mathrm{Ind}_{\mathfrak{n}^+ \oplus \mathfrak{h}}^{\mathfrak{g}} \mathbb{C}_\lambda$,

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where $\mathbb{C}_\lambda$ is the one-dimensional module such that $h \in \mathfrak{h}$ acts by scalar multiplication of $\lambda(h)$ and $\mathfrak{n}^+$ acts trivially. The Verma module $M(\lambda)$ has a unique simple quotient, which we denote $L(\lambda)$.

For each $\lambda \in \mathfrak{h}^*$, let $L_{\bar{0}}(\lambda)$ denote the simple highest weight $\mathfrak{g}_{\bar{0}}$-module with respect to $\pi_{\bar{0}}$. The *Kac module* of highest weight λ with respect to π is the induced module

$$K(\lambda) := \mathrm{Ind}_{\mathfrak{g}_{\bar{0}} \oplus \mathfrak{n}_{\bar{1}}^+}^{\mathfrak{g}} L_{\bar{0}}(\lambda)$$

defined by letting $\mathfrak{n}_{\bar{1}}^+ := \oplus_{\alpha \in \Delta_{\bar{1}}^+} \mathfrak{g}_\alpha$ act trivially on the $\mathfrak{g}_{\bar{0}}$-module $L_{\bar{0}}(\lambda)$. The unique simple quotient of $K(\lambda)$ is $L(\lambda)$.

Let $\mathfrak{h}_{\mathbb{R}}^* = \sum_{\alpha \in \pi} \mathbb{R}\alpha$. A weight $\nu \in \mathfrak{h}_{\mathbb{R}}^*$ is called integral (resp. dominant; strictly dominant) if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ (resp. $\langle \lambda, \alpha \rangle \geq 0$; $\langle \lambda, \alpha \rangle > 0$) for all $\alpha \in \Delta_{\bar{0}}^+$, where $\langle \lambda, \alpha \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$.

For a proof of the following proposition see for example [M, 14.1.1]. Given $\lambda \in \mathfrak{h}^*$, we use the abbreviation $\lambda^\rho := \lambda + \rho$.

Proposition 2.1. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\lambda \in \mathfrak{h}^*$. Then, $L(\lambda)$ is a finite dimensional $\mathfrak{g}$-module iff $L_{\bar{0}}(\lambda)$ is a finite dimensional $\mathfrak{g}_{\bar{0}}$-module iff the Kac module $K(\lambda)$ is a finite dimensional iff λ is a dominant integral weight iff λ^ρ is a strictly dominant integral weight.*

An element $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ is called *regular* if $(\nu, \varepsilon_i) \neq (\nu, \varepsilon_j)$ and $(\nu, \delta_i) \neq (\nu, \delta_j)$ for all $i \neq j$. An element $\nu \in \mathfrak{h}_{\mathbb{R}}^*$ is regular if and only if there exists $w \in W$ such that $w(\nu)$ is strictly dominant.

2.4. Atypical modules.

Let $L(\lambda)$ be a finite dimensional $\mathfrak{g}$-module. We call $\beta \in \Delta_{\bar{1}}$ atypical if $(\lambda^\rho, \beta) = (\beta, \beta) = 0$. The *atypicality* of $L(\lambda)$ is the maximal number of linearly independent roots $\beta_1, ..., \beta_r$ such that $(\beta_i, \beta_j) = 0$ and $(\lambda^\rho, \beta_i) = 0$ for $i, j = 1, \ldots, r$. Such a set $S_\lambda = \{\beta_1, \ldots, \beta_r\}$ is called a λ^ρ-*maximal isotropic set*, and we assume that the elements of S_λ are ordered so that $\beta_i = \varepsilon_{p_i} - \delta_{q_i}$ and $q_i < q_{i+1}$. As in [KW2], we denote the atypicality of $L(\lambda)$ by $\operatorname{atp}(\lambda^\rho) = r$. The module $L(\lambda)$ is called *typical* if this set is empty, and *atypical* otherwise. For the standard choice of simple roots the set S_λ is uniquely determined.

Let P denote the set of integral weights, P^+ the set of dominant integral weights, and define

$$\mathbb{P}^+ = \{\mu \in P^+ \mid (\mu, \varepsilon_i) \in \mathbb{Z}, \ (\mu_\pi, \delta_j) \in \mathbb{Z}\}.$$

Remark 2.2. When studying the characters of simple finite dimensional atypical modules, we may restrict without loss of generality to the case that $\lambda \in \mathbb{P}^+$. See Remark 8 in [CHR].

2.5. Weight diagrams and cap diagrams.

Diagrams encoding the weights of a module (among other things) were introduced by Brundan and Stroppel in [BS1] and were shown to have numerous applications to the representation theory of $\mathfrak{gl}(m|n)$. In particular, Brundan and Stroppel show that in some cases the corresponding Khovanov algebra gives rise to a category equivalence with representations of $GL(m|n)$ [BS4]. Similar diagrams for $\mathfrak{osp}(m|2n)$ were used to study character formulas and give certain decomposition numbers for $SOSP(m|2n)$-modules by Gruson and Serganova in [GS1, GS2] and for $OSP(m|2n)$-modules by Ehrig and Stroppel in [ES1, ES2]. These diagrams were also used in [CH] to classify indecomposable summands in tensor powers of the standard $OSP(m|2n)$-module.

Let us introduce the weight diagrams and cap diagrams from [BS1]. Let $\lambda \in \mathbb{P}^+$ and write

$$\lambda^\rho = \sum_{i=1}^m a_i \varepsilon_i - \sum_{j=1}^n b_j \delta_j. \tag{3}$$

On the $\mathbb{Z}$-lattice, put $\vee$ above t if $t \in \{a_i\} \cap \{b_j\}$, put $\times$ above t if $t \in \{a_i\} \setminus \{b_j\}$, and put $\circ$ above t if $t \in \{b_i\} \setminus \{a_j\}$. If $t \notin \{a_i\} \cup \{b_j\}$, then put $\wedge$. We refer to such a diagram as a *weight diagram.*

Note that each $\vee$ corresponds to some atypical root β_i. We number the $\vee$'s left to right, and this is consistent our chosen ordering for the set S_λ.

We obtain a *cap diagram* from a weight diagram as follows. Going from right to left and starting with the rightmost $\vee$, we draw a cap connecting $\vee$ to the first $\wedge$ to its right which is *"unmarked"*, and we say that this $\wedge$ is *marked* by $\vee$. By construction, the caps in our diagrams do not intersect.

Example 2.3. If $\lambda^\rho = 10\varepsilon_1 + 9\varepsilon_2 + 8\varepsilon_3 + 5\varepsilon_4 + 4\varepsilon_5 - \delta_1 - 4\delta_2 - 6\delta_3 - 8\delta_4 - 10\delta_5$, then the corresponding cap diagram D_λ is

$$\tag{4}$$
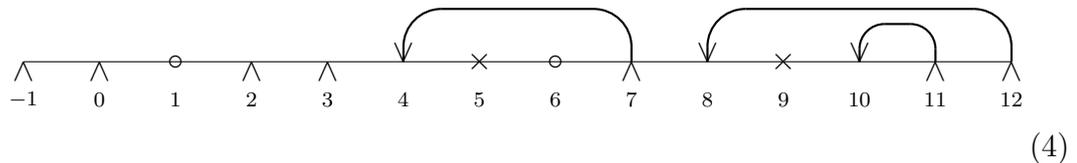
2.6. Characters and category $\mathcal{O}$.

Let M be a module from the BGG category $\mathcal{O}$ [M, 8.2.3]. Then M has a weight space decomposition $M = \oplus_{\mu \in \mathfrak{h}^*} M_\mu$, where
$$M_\mu = \{x \in M \mid h.x = \mu(h)x \text{ for all } h \in \mathfrak{h}^*\},$$
and the *character of M* is by definition ch $M = \sum_{\mu \in \mathfrak{h}^*} \dim M_\mu \, e^\mu$.

Denote by $\mathcal{E}$ the algebra $\mathbb{Q}(e^\nu, \nu \in \mathfrak{h}^*)$ of functions on $\mathfrak{h}$ which are rational in the variables e^ν. The group W acts on $\mathcal{E}$ by mapping e^ν to $e^{w(\nu)}$. For $\beta \in \Delta_{\bar{1}}^+$, we identify elements of the form $\frac{1}{1+e^{-\beta}}$ with their expansion as geometric series in the domain $|e^{-\beta}| < 1$. Since $\Delta_{\bar{1}}^+$ is fixed by W, expanding commutes with the action of W.

The *Weyl denominator of $\mathfrak{g}$* is defined to be

$$R = \frac{\Pi_{\alpha \in \Delta_{\bar{0}}^+}(1 - e^{-\alpha})}{\Pi_{\alpha \in \Delta_{\bar{1}}^+}(1 + e^{-\alpha})}.$$

Then $e^\rho R$ is W-skew-invariant, i.e. $w(e^\rho R) = (-1)^{l(w)} e^\rho R$, and ch $L(\lambda)$ is W-invariant for $\lambda \in P^+$. The character of a Verma module $M(\lambda)$ with $\lambda \in \mathfrak{h}^*$ is ch $M(\lambda) = e^\lambda R^{-1}$. The character of the Kac module $K(\lambda)$ with $\lambda \in P^+$ is

$$\text{ch } K(\lambda) = \frac{1}{e^\rho R} \sum_{w \in W} (-1)^{l(w)} w(e^{\lambda^\rho}). \tag{5}$$

For more details on character formulas see for example [M, Section 8.4].

For $X \in \mathcal{E}$, we define

$$\mathcal{F}_W(X) := \sum_{w \in W} (-1)^{l(w)} w(X). \tag{6}$$

We shall use the following lemma (see for example [G, 4.1.1]).

Lemma 2.4. *If $\nu \in \mathfrak{h}_\mathbb{R}^*$ is not regular, then $\mathcal{F}_W(e^\nu) = 0$.*

2.7. Kazhdan-Lusztig polynomials and character formulas.

Generalized Kazhdan-Lusztig polynomials $K_{\lambda,\mu}(q)$ were introduced in [S1] by Serganova to give an algorithmic character formula for finite-dimensional irreducible representations of $\mathfrak{gl}(m|n)$. Brundan gave a new algorithm in [B] for computing the generalized Kazhdan-Lusztig polynomials for $\mathfrak{gl}(m|n)$ which can be described in terms of paths. This algorithm was made more explicit by Brundan and Stroppel in [BS1] using the language of cap diagrams.

We begin by recalling the algorithm for computing $K_{\lambda,\mu}(q)$ using cap diagrams. We define a *right move* map from the set of (labeled) weight diagrams to itself in two steps.

Definition 2.5. Let D_μ be a weight diagram for $\mu \in \mathbb{P}^+$. The right move R_i is defined by exchanging $\vee_i$ with the $\wedge$ that it marks, that is, we switch $\vee_i$ with the $\wedge$ to which it is connected to by a cap in the corresponding cap diagram of μ.
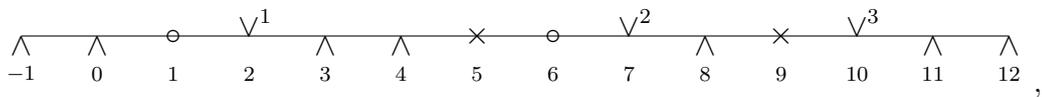
Example 2.6. Let D_μ be the weight diagram for

$$\mu^\rho = 8\varepsilon_1 + 7\varepsilon_2 + 6\varepsilon_3 + 2\varepsilon_4 + 1\varepsilon_5 - 2\delta_1 - 5\delta_2 - 7\delta_3 - 8\delta_4 - 9\delta_5,$$
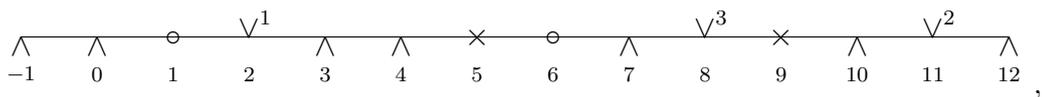
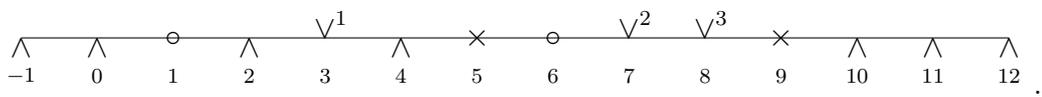and consider the corresponding cap diagram:
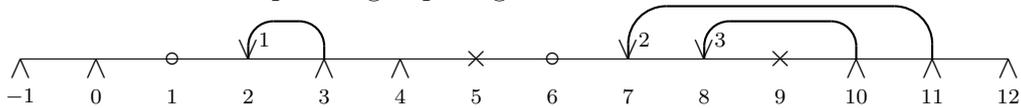
In this case, we see that $\vee_1$ marks 3, $\vee_2$ marks 11, and $\vee_3$ marks 10. Hence $R_3(D_\mu)$ is

$R_2(D_\mu)$ is

and $R_1(D_\mu)$ is

Definition 2.7.   Let $\lambda, \mu \in \mathbb{P}^+$. Label the $\vee$'s in the diagram D_μ from left to right with $1, \ldots, r$. A *right path* from D_μ to D_λ is a sequence of right moves $\theta = R_{i_1} \circ \cdots \circ R_{i_k}$ where $i_1 \leq \ldots \leq i_k$ and $\theta(D_\mu) = D_\lambda$. The length of the path is $l(\theta) := k$.

Example 2.8.   Let D_μ be as in the previous example. Then $R_1 \circ R_1 \circ R_2 \circ R_3 (D_\mu)$ is the diagram D_λ of Example 2.3. Note that after R_3 was applied to D_μ, $\vee_2$ marks 8 since the spot became empty.

Define a partial order on P by $\mu^\rho \preceq \lambda^\rho$ if and only if λ^ρ and μ^ρ have the same typical entries, $\mathrm{atp}(\lambda^\rho) = \mathrm{atp}(\mu^\rho)$ and the i-th atypical entry of μ^ρ is less than or equal to the i-th atypical entry of λ^ρ

Remark 2.9.   For each $\mu, \lambda \in \mathbb{P}^+$, there exists a path from D_μ to D_λ if and only if $\mu^\rho \preceq \lambda^\rho$ [B].

Let $P_{\lambda,\mu}$ denote the set of paths from D_μ to D_λ. If $P_{\lambda,\mu}$ is non-empty, it contains a unique longest path, which sends the i-th $\vee$ of μ^ρ to the location of the i-th $\vee$ of λ^ρ. We call this path the *trivial path* from D_μ to D_λ and denote its length by $l_{\lambda,\mu}$.

Lemma 2.10 (Brundan, [B, Lemma 3.42]).   *For all $\lambda, \mu \in \mathbb{P}^+$ and $\theta \in P_{\lambda,\mu}$, $l(\theta) \equiv l_{\lambda,\mu} \pmod{2}$.*

We are now ready to state the result of Brundan and Serganova.

Theorem 2.11 (Serganova [S1], Brundan [B]).   *For each $\lambda \in \mathbb{P}^+$,*

$$ch\, L(\lambda) = \sum_{\mu \in \mathbb{P}^+} K_{\lambda,\mu}(-1)\, ch\, \overline{L}(\mu).$$

where

$$K_{\lambda,\mu}(q) = \sum_{\theta \in P_{\lambda,\mu}} q^{l(\theta)}$$

and $P_{\lambda,\mu}$ is the set of paths from D_μ to D_λ and $l(\theta)$ denotes the length of the path θ.

The following is a corollary of Theorem 2.11, Lemma 2.10 and Equation (5).

Corollary 2.12.   *Let $\lambda \in \mathbb{P}^+$, and let $P_\lambda = \{\mu \in \mathbb{P}^+ \mid P_{\lambda,\mu} \text{ is non-empty}\}$. Then*

$$e^\rho R \cdot ch\, L(\lambda) = \sum_{\mu \in P_\lambda} d_{\lambda,\mu} \cdot (-1)^{l_{\lambda,\mu}} \mathcal{F}_W\left(e^{\mu^\rho}\right) \tag{7}$$

where $d_{\lambda,\mu}$ is the number of paths from D_μ to D_λ.

Remark 2.13.   Note that in [BS1, Formula 5.12] the notation $d_{\lambda,\mu}$ denotes the dual Kazdhan-Lusztig coefficient.

3.   Piecewise disconnected weights

3.1. Piecewise disconnected weights.

We will see that some simple highest weight modules have particularly nice character formulas. In this section we characterize their highest weights.

The following definition is equivalent to that of [SZ, Section 3.7].

Definition 3.1.   A weight $\lambda \in \mathbb{P}^+$ is called *totally connected* if in the weight diagram D_λ between any two $\vee$'s there is no $\wedge$. A weight $\lambda \in \mathbb{P}^+$ is called *totally disconnected* if the diagram D_λ contains at least one $\wedge$ between any two $\vee$'s.

Remark 3.2.   The cap diagram for a totally connected weight looks like a rainbow. In particular, a weight is totally connected if and only if its cap diagram satisfies the property that if a cap A is below cap B, then all the caps that are below B are either above or below A. Whereas, a weight is totally disconnected if and only if its cap diagram satisfies the property that no cap is below another cap.

Remark 3.3.   A weight $\lambda \in \mathbb{P}^+$ is totally connected if and only if for every $\mu \in \mathbb{P}^+$ the only possible path from D_μ to D_λ is the trivial path, whereas it is totally disconnected if and only if there exists $\mu \in \mathbb{P}^+$ with $r!$ paths from D_μ to D_λ, where $r = \mathrm{atp}(\lambda^\rho)$.

Definition 3.4.   Let $\lambda \in \mathbb{P}^+$. We call a nonempty continuous subsection of the weight diagram D_λ an *atypical component* if it contains an $\vee$, does not contain any $\wedge$'s and is maximal with this property. If $\vee_j$ and $\vee_k$ belong to the same atypical component then we *write $j \sim k$*. Enumerate the atypical components of D_λ left to right $T_1, \ldots, T_N$, and let t_i be the number of $\vee$'s contained in T_i for $i = 1, \ldots, N$. We define $t_\lambda = t_1! t_2! \cdots t_N!$.

Definition 3.5.   We call a weight $\lambda \in \mathbb{P}^+$ and the corresponding weight diagram D_λ *piecewise disconnected (or PDC)* if $t_i \leq s_i$, where s_i is the number of $\wedge$'s between T_i and T_{i+1}, for $i = 1, \ldots, N - 1$.

Remark 3.6.   A weight is piecewise disconnected if and only if its cap diagram satisfies the property that whenever two caps A and B are both below the same cap C, then either A is below B, or B is below A.

Remark 3.7.   In the language of [HW, Section 14], a weight is piecewise disconnected if and only if the forest of λ is a disjoint union of lines. In this case, t_λ is equal to the forest factorial.

Example 3.8.   The weight diagram D_λ in Example 2.3 is piecewise discon-

nected, but is neither totally connected nor totally disconnected. It has two atypical components, namely, $T_1 = \{4, 5, 6\}$, $T_2 = \{8, 9, 10\}$, and $t_1 = 1$, $t_2 = 2$, $s_1 = 1$.

The following lemma is a corollary of the definition.

Lemma 3.9.   *Any weight of atypicality 1 or 2 is either totally connected or totally disconnected, and hence is piecewise disconnected.*

Example 3.10.   The highest weight $\lambda = \varepsilon_1$ of the standard module V of $\mathfrak{gl}(m|n)$ is totally connected, as is the highest weight $\lambda = -\delta_n$ of the dual module V^*. For example, for $\mathfrak{gl}(3|3)$ we have that $\lambda^\rho = 4\varepsilon_1 + 2\varepsilon_2 + 1\varepsilon_3 - 1\delta_1 - 2\delta_2 - 3\delta_3$ and the cap diagram D_λ is
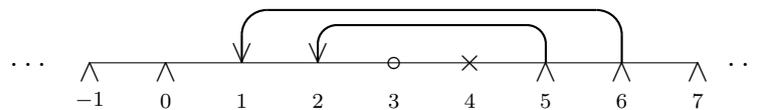
Example 3.11.   The highest weight λ of the non-trivial subquotient of the adjoint module of $\mathfrak{gl}(2|2)$ is totally disconnected. Indeed, $\lambda^\rho = 3\varepsilon_1 + 1\varepsilon_2 - 1\delta_1 - 3\delta_2$ and the cap diagram D_λ is
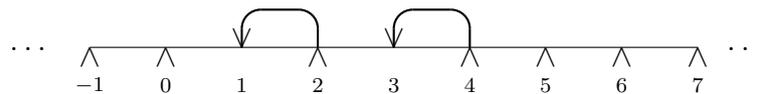

Example 3.12.   For $n \geq 3$, the highest weight $\lambda = \varepsilon_1 - \delta_n$ of the non-trivial subquotient of the adjoint module of $\mathfrak{gl}(n|n)$ is not piecewise disconnected. For example, for $\mathfrak{gl}(3|3)$ we have that $\lambda^\rho = 4\varepsilon_1 + 2\varepsilon_2 + 1\varepsilon_3 - 1\delta_1 - 2\delta_2 - 4\delta_3$ and the corresponding cap diagram D_λ is
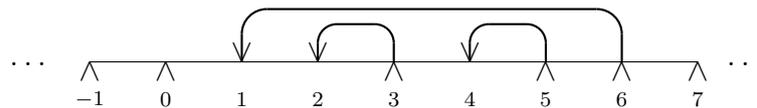
If $\mathfrak{g} = \mathfrak{gl}(n|n)$, then the adjoint module of $\mathfrak{g}$ has a unique non-trivial simple subquotient $L(\nu)$. The highest weight $\nu = \varepsilon_1 - \delta_n$ is not piecewise disconnected.

Remark 3.13.   It is known that the character of the simple quotient of the adjoint module of $\mathfrak{gl}(3|3)$ does not satisfy a Weyl-type character formula.

Example 3.14.   If $\mathfrak{g} = \mathfrak{gl}(n|n)$ then the module $\mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ obtained by tensoring $n - 1$ copies of the adjoint module has a maximal weight $\mu = (n-1)\nu = (n-1)\varepsilon_1 - (n-1)\delta_n$ and a simple subquotient $L(\mu)$. The weight ν is piecewise disconnected, but is neither totally connected nor totally disconnected. In particular if $\mathfrak{g} = \mathfrak{gl}(3|3)$, then $L(\nu)$ with $\nu = 2\varepsilon_1 - 2\delta_n$ is a simple subquotient of the module $\mathfrak{g} \otimes \mathfrak{g}$. So $\nu^\rho = 5\varepsilon_1 + 2\varepsilon_2 + 1\varepsilon_3 - 1\delta_1 - 2\delta_2 - 5\delta_3$ and the cap diagram for D_ν is
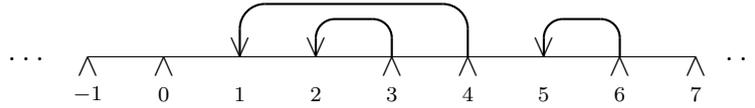
3.2. Definition of $(\lambda^\rho)^\Uparrow$.

The integral weight $(\lambda^\rho)^\Uparrow$ is a modification of λ^ρ which shall replace λ^ρ in the character formula (Theorem 4.1). Let $\lambda \in \mathbb{P}^+$ and write λ^ρ as in (3). We refer to the coefficient a_i (resp. b_j) as the ε_i-*entry* (resp. δ_j-entry). If $\pm(\varepsilon_k - \delta_l) \in S_\lambda$, then we call the ε_k and δ_l entries *atypical*. Otherwise, an entry is called *typical*.

Definition 3.15. If $\lambda \in \mathbb{P}^+$ is piecewise disconnected, we denote by $(\lambda^\rho)^\Uparrow$ the element obtained from λ^ρ by replacing each atypical entry with the maximal atypical entry in the atypical component to which it belongs.

Remark 3.16. If $\lambda \in \mathbb{P}^+$ is totally disconnected then $(\lambda^\rho)^\Uparrow = \lambda^\rho$, whereas if $\lambda \in \mathbb{P}^+$ is totally connected then all the atypical entries of $(\lambda^\rho)^\Uparrow$ equal the maximal atypical entry of λ^ρ.

Example 3.17. If λ^ρ is as in Example 2.3, then

$$(\lambda^\rho)^\Uparrow \;=\; 10\varepsilon_1 + 9\varepsilon_2 + 10\varepsilon_3 + 5\varepsilon_4 + 4\varepsilon_5 - \delta_1 - 4\delta_2 - 6\delta_3 - 10\delta_4 - 10\delta_5.$$

Definition 3.18. If $\nu \in \mathfrak{h}^*$ can be written as $\nu = \sum_{\alpha \in S_\lambda} k_\alpha \alpha$, then we define

$$|\nu|_{S_\lambda} := \sum_{\alpha \in S_\lambda} k_\alpha.$$

Observe that $|(\lambda^\rho)^\Uparrow - \lambda^\rho|_{S_\lambda}$ is a non-negative integer.

4. Main theorem

The main theorem of this paper is as follows.

Theorem 4.1. *Let* $\lambda \in \mathbb{P}^+$ *be a piecewise disconnected weight. Then*

$$e^\rho R \cdot ch\, L\,(\lambda) = \frac{(-1)^{|(\lambda^\rho)^\Uparrow - \lambda^\rho|_{S_\lambda}}}{t_\lambda} \sum_{w \in W} (-1)^{l(w)} w \left(\frac{e^{(\lambda^\rho)^\Uparrow}}{\prod_{\beta \in S_\lambda} \left(1 + e^{-\beta}\right)} \right), \quad (8)$$

where $t_\lambda = t_1! t_2! \cdots t_N!$ *(see Definition 3.4) and* S_λ *is the (unique)* λ^ρ-*maximal isotropic set of roots.*

Remark 4.2. A totally connected weight λ is piecewise disconnected with $N = 1$ and $t_\lambda = r!$. A totally disconnected weight λ is piecewise disconnected with $N = r$ and $t_\lambda = 1$. Here $r = \text{atp}(\lambda^\rho)$.

4.1. A map from the set of paths to $Sym(r)$.

    One of the ideas of the proof is to translate the character formula given in terms of paths in (7) to a formula in terms of the Weyl group. For each $\lambda, \mu \in \mathbb{P}^+$, we give an injective map from the set of paths $P_{\lambda,\mu}$ to $Sym(r)$, where r is the atypicality of λ. We shall later embed $Sym(r)$ in W. We describe the image of this map when λ is piecewise disconnected. The image of such a map for general λ was described by Su and Zhang in [SZ, Section 3.8].

    For $\lambda, \mu \in \mathbb{P}^+$, number the $\vee$'s of D_μ left to right $\vee_1, \ldots, \vee_r$ and number the $\check{\vee}$'s of D_λ left to right $\check{\vee}_1, \ldots, \check{\vee}_r$. Then a path $\theta \in P_{\lambda,\mu}$ determines uniquely an element of $Sym(r)$ given by the assignment

$$\vee_k \mapsto \check{\vee}_{\sigma_\theta(k)}.$$

In this way, we define the map $\Theta_{\lambda,\mu} : P_{\lambda,\mu} \to Sym(r)$. The map $\Theta_{\lambda,\mu}$ is injective, since a path is determined by this ordering. The image of the trivial path is the identity element of $Sym(r)$.

Example 4.3.   Let D_μ be

and let D_λ be

There are two paths from D_μ to D_λ, namely, the trivial path and the path $R_1 R_1 R_1 R_2 R_2 R_2 R_3 R_3$ which can be computed as follows.

The image of this non-trivial path under the map $\Theta_{\lambda,\mu}$ is the cycle (23). There are no other paths, because if positions 4 and 5 were filled before position 7, then position 7 would be marked by the $\vee$ in position 4, making the path impossible to complete.

    For an element $\nu \in P$ with $\mathrm{atp}(\nu) = r$ let $S_\nu = \{\varepsilon_{m_1} - \delta_{n_1}, \ldots, \varepsilon_{m_r} - \delta_{n_r}\}$ be such that $n_1 < \ldots < n_r$. We denote $\nu_i := (\nu, \delta_{n_i})$. Then $\vee_k = (\mu^\rho)_k$ and that $\check{\vee}_k = (\lambda^\rho)_k$.

    In the following lemma we describe the image of $\Theta_{\lambda,\mu}$ for an arbitrary piecewise disconnected weight.

Lemma 4.4.   *If $\lambda \in \mathbb{P}^+$ is piecewise disconnected, then*

$$Im\ \Theta_{\lambda,\mu} = \left\{\sigma \in Sym(r) \mid \sigma(\mu^\rho) \preceq \lambda^\rho,\ and\ \sigma^{-1}(j) < \sigma^{-1}(k)\ if\ j < k\ and\ j \sim k\right\},$$

where $j \sim k$ when j and k label $\check{\vee}$'s from the same atypical component of λ.

Proof.   Let $\theta \in P_{\lambda,\mu}$. Since the $\vee$'s move in order from left to right to their respective destinations, we have that $\vee_k \le \check{\vee}_{\sigma_\theta(k)}$. This ensures that $\sigma(\mu^\rho) \preceq \lambda^\rho$. When an $\vee$ reaches its destination, it marks the next $\wedge$ after it. Hence, the $\vee$'s must go in order into each atypical component so that every spot can be filled, that is, if $j < k$ and $j \sim k$ then $\sigma_\theta^{-1}(j) < \sigma_\theta^{-1}(k)$. Hence, we always have inclusion. When λ is piecewise disconnected, these conditions on $\sigma \in Sym(r)$ are sufficient to define a path θ from D_μ to D_λ which satisfies $\vee_k \mapsto \check{\vee}_{\sigma_\theta(k)}$. Indeed, the number of $\wedge$'s following an atypical component and preceding the next is greater than or equal to the number of $\vee$'s in a given atypical component, so an $\vee$ does not hold an $\check{\vee}$ spot.   ∎

Remark 4.5.   If λ is not piecewise disconnected then Lemma 4.4 does not hold. See [SZ, Section 3.8] for a description of the image in the general case.

In the following lemma we change the defining conditions of the set from Lemma 4.4 by replacing λ^ρ with $(\lambda^\rho)^\Uparrow$, and then we show that this does not change the set.

Lemma 4.6.   *If $\lambda \in \mathbb{P}^+$ is piecewise disconnected, then*

$$Im\ \Theta_{\lambda,\mu} = \left\{\sigma \in Sym(r) \ \middle|\ \begin{array}{c} \sigma(\mu^\rho) \preceq (\lambda^\rho)^\Uparrow,\ and \\ \sigma^{-1}(j) < \sigma^{-1}(k)\ if\ j < k\ and\ j \sim k \end{array}\right\}. \quad (9)$$

Proof.   Let $A_{\lambda,\mu} = LHS$ and $B_{\lambda,\mu} = RHS$. By Lemma 4.4, $A_{\lambda,\mu} \subseteq B_{\lambda,\mu}$. Now suppose towards a contradiction that $\sigma \in B_{\lambda,\mu} \setminus A_{\lambda,\mu}$. Choose s maximal such that $(\lambda^\rho)_{\sigma(s)} < (\mu^\rho)_s \le (\lambda^\rho)^\Uparrow_{\sigma(s)}$. By definition $(\lambda^\rho)^\Uparrow_{\sigma(s)} = (\lambda^\rho)_k$, where k is the index of the maximal atypical entry in the atypical component containing $(\lambda^\rho)_{\sigma(s)}$. Thus $(\mu^\rho)_s = (\lambda^\rho)_j$ for some $\sigma(s) < j \leqslant k$, since the atypical components of λ^ρ are connected and μ^ρ is regular with the same typical entries as λ^ρ. Thus $s < \sigma^{-1}(j)$ since $\sigma(s) \sim j$. Then since μ^ρ is strictly dominant we have that $(\lambda^\rho)_j = (\mu^\rho)_s < (\mu^\rho)_{\sigma^{-1}(j)}$. Note that we also have $(\mu^\rho)_{\sigma^{-1}(j)} \le (\lambda^\rho)^\Uparrow_j$ since $\sigma \in B_{\lambda,\mu}$. This contradicts the maximality of s, since $\sigma^{-1}(j)$ is larger and satisfies the required properties. Hence $A_{\lambda,\mu} = B_{\lambda,\mu}$.   ∎

4.2. A bijection of indexing sets.

In this section, we change the indexing set of the character formula in (7) from P_λ to a particular subset of $(\lambda^\rho - \mathbb{N}S_\lambda)$.

Fix $\lambda \in \mathbb{P}^+$. For each $\mu \in P_\lambda$, the W orbit of μ^ρ intersects $(\lambda^\rho - \mathbb{N}S_\lambda)$. We denote by $\overline{\mu}$ the unique maximal element of this intersection with respect to

the standard order on $\mathfrak{h}^*$. We define

$$C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}} := \{\overline{\mu} \in \lambda^\rho - \mathbb{N}S_\lambda \mid \mu \in P_\lambda\}.$$

Since $P_\lambda \subset \mathbb{P}^+$, this defines a bijection between the sets P_λ and $C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}}$. Recall that $S_\lambda = \{\beta_1, \ldots, \beta_r\}$ is ordered so that $\beta_i = \varepsilon_{p_i} - \delta_{q_i}$ and $q_i < q_{i+1}$. For $\nu \in (\lambda^\rho)^\Uparrow - \mathbb{N}S_\lambda$ and $i = 1, \ldots, r$, define

$$\nu_{\beta_i} = (\nu, \delta_{q_i}).$$

Lemma 4.7.   *One has*

$$C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}} = \{\nu \in \lambda^\rho - \mathbb{N}S_\lambda \mid \nu_{\beta_1} < \nu_{\beta_2} < \ldots < \nu_{\beta_r} \text{ and } \nu \text{ is regular}\}.$$

Proof.   Clearly we have $\subseteq$, since μ^ρ is strictly dominant. The reverse inclusion follows from Remark 2.9 since for regular $\nu \in \lambda^\rho - \mathbb{N}S_\lambda$ and $w \in W$ with $w(\nu)$ strictly dominant, $w(\nu) \preceq \lambda^\rho$ by definition. ∎

Definition 4.8.   For $\bar{\mu} \in C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}}$, define $\bar{d}_{\lambda,\bar{\mu}}$ to be the number of paths from D_μ to D_λ, where μ is the unique dominant element in the W orbit of $\bar{\mu}$.

The following lemma is proven using techniques from [SZ, Section 4.1]. Recall the definition of $\mathcal{F}_W$ given in (6).

Lemma 4.9.   *One has*

$$e^\rho R \cdot ch\, L(\lambda) = \sum_{\bar{\mu} \in C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}}} \bar{d}_{\lambda,\bar{\mu}} (-1)^{|\lambda^\rho - \bar{\mu}|_{S_\lambda}} \mathcal{F}_W(e^{\bar{\mu}}).$$

Proof.   By Corollary 2.12 it suffices to show that for each $\mu \in P_\lambda$,

$$(-1)^{l_{\lambda,\mu}} \mathcal{F}_W(e^{\mu^\rho}) = (-1)^{|\lambda^\rho - \overline{\mu}|_{S_\lambda}} \mathcal{F}_W(e^{\bar{\mu}}).$$

Let $w' \in W$ such that $w'(\mu^\rho) = \overline{\mu}$. To complete the proof it is sufficient to show that $|\lambda^\rho - \overline{\mu}|_{S_\lambda} = l_{\lambda,\mu} + l(w')$. The number $|\lambda^\rho - \overline{\mu}|_{S_\lambda}$ is the sum of the differences between the atypical entries of λ^ρ and $\overline{\mu}$. This is equal to the number of moves in the trivial path $l_{\lambda,\mu}$ plus the number of spots being skipped. We will show that $l(w')$ is exactly the number of spots skipped in the trivial path.

The element $w' \in W$ for which $w'(\mu^\rho) = \overline{\mu}$ can be described explicitly in terms of the trivial path θ. Denote $\theta = R_{i_1} \circ \cdots \circ R_{i_N}$, then $w' = w_1 \cdot \cdots \cdot w_N$ where each w_j is defined as follows. Suppose that the move R_{i_j} moved the $\vee$ at n_j to an $\wedge$ at $n_j + k_j + 1$, namely, it skipped over k_j spots with $\times$'s and $\circ$'s. Then $w_j = s_1 \cdot \cdots \cdot s_{k_j-1}$ where s_i is of the form $s_{\varepsilon_l - \varepsilon_{l+1}}$ if the i-th skip is over the $\times$ of ε_l and is of the form $s_{\delta_l - \delta_{l+1}}$ if it is over the $\circ$ of δ_l. It is easy to see that the expression is reduced, so $l(w_j) = k_j$ is the number of spots skipped in the move R_{i_j}. Also $l(w') = \sum l(w_i)$, so $l(w')$ is exactly the number of spots skipped in the trivial path. ∎

4.3. Paths and permutations for piecewise disconnected weights.

In this section, we show that if $\lambda \in \mathbb{P}^+$ is a piecewise disconnected weight, then for each $\mu \in P_\lambda$ there exists a t_λ to 1 map from the set of paths from μ to λ to a certain subset of the Weyl group. This is a crucial step in the proof of the main theorem.

Let W_r be the subgroup of W that permutes S_λ. Then $W_r \cong Sym(r)$ and is generated by elements of the form $s_{\varepsilon_i - \varepsilon_j} s_{\delta_{i'} - \delta_{j'}}$ where $\varepsilon_i - \delta_{i'}, \varepsilon_j - \delta_{j'} \in S_\lambda$. So $|W_r| = r!$ and all $w \in W_r$ have positive sign.

Fix $\lambda \in \mathbb{P}^+$, and recall the notation of Section 3. We define a subgroup of W_r that preserves the atypical components of λ^ρ, that is,

$$W_r(t_\lambda) = \left\langle s_{\varepsilon_i - \varepsilon_j} s_{\delta_{i'} - \delta_{j'}} \mid i \sim j \right\rangle. \tag{10}$$

So $w \in W_r(t_\lambda)$ and $\lambda_\beta \in T_i$ imply that $\lambda_{w(\beta)} \in T_i$. Clearly,

$$W_r(t_\lambda) \cong Sym(t_1) \vee \cdots \vee Sym(t_N)$$

and hence $W_r(t_\lambda)$ has cardinality t_λ.

Definition 4.10. For each $\nu \in C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}}$, let

$$W_r(\lambda, \nu) := \left\{ w \in W_r \mid w(\nu) \in (\lambda^\rho)^\Uparrow - \mathbb{N}S_\lambda \right\},$$

and let $c_{\lambda,\nu} = |W_r(\lambda, \nu)|$.

Then

$$\mathcal{F}_W \left(\sum_{w \in W_r(\lambda,\nu)} e^{w(\nu)} \right) = c_{\lambda,\nu} \cdot \mathcal{F}_W \left(e^\nu \right). \tag{11}$$

Proposition 4.11. *Let $\lambda \in \mathbb{P}^+$ be a piecewise disconnected weight. Then for every $\mu \in P_\lambda$, the number of paths from D_μ to D_λ equals $\frac{1}{t_\lambda}|W_r(\lambda, \bar\mu)|$. Hence, for each $\nu \in C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}}$, we have that $\frac{\bar{d}_{\lambda,\nu}}{c_{\lambda,\nu}} = \frac{1}{t_\lambda}$.*

Proof. First, we observe that there is a natural bijection between the sets $W_r(\lambda, \bar\mu)$ and

$$\widetilde{B}_{\lambda,\mu} = \left\{ \sigma \in Sym(r) \mid \sigma(\mu^\rho) \preceq (\lambda^\rho)^\Uparrow \right\},$$

since the bijective map $P_\lambda \to C_{\lambda,\mathrm{reg}}^{\mathrm{Lex}}$ defined by $\mu^\rho \mapsto \bar\mu$ preserves the relative order of the atypical roots. So we may in fact identify $W_r(\lambda, \bar\mu)$ with $\widetilde{B}_{\lambda,\mu}$ under this correspondence.

Now by Lemma 4.6, $d_{\lambda,\mu} := |P_{\lambda,\mu}|$ equals the cardinality of the set in (9), which we denote by $B_{\lambda,\mu}$. We claim that there is a bijection of sets $W_r(t_\lambda) \vee B_{\lambda,\mu} \cong \widetilde{B}_{\lambda,\mu}$ defined by $(w, \sigma) \mapsto w\sigma$. Now by definition, $(\lambda^\rho)_j^\Uparrow = (\lambda^\rho)_k^\Uparrow$ when λ_j^ρ and λ_k^ρ belong to the same atypical component, that is, when $j \sim k$. Since $W_r(t_\lambda)$ preserves each atypical component, the map is well-defined, that is, $\sigma(\mu^\rho) \preceq (\lambda^\rho)^\Uparrow$ implies that $w\sigma(\mu^\rho) \preceq (\lambda^\rho)^\Uparrow$ for any $w \in W_r(t_\lambda)$.

If $\sigma \in B_{\lambda,\mu}$, then the atypical entries of each atypical component of $\sigma(\mu^\rho)$ are in increasing order and distinct, since $\sigma \in B_{\lambda,\mu}$ satisfies: $\sigma^{-1}(j) < \sigma^{-1}(k)$ when $j < k$ and $j \sim k$. It is not difficult to show that the map defined above is bijective. Indeed, given $\sigma' \in \widetilde{B}_{\lambda,\mu}$ there exists a unique $w \in W_r(t_\lambda)$ such that the atypical entries of each atypical component of $w^{-1}\sigma'(\mu^\rho)$ are in increasing order, that is, such that $w^{-1}\sigma' \in B_{\lambda,\mu}$. Therefore, $W_r(t_\lambda) \vee B_{\lambda,\mu} \cong \widetilde{B}_{\lambda,\mu}$ and $t_\lambda \cdot d_{\lambda,\mu} = c_{\lambda,\overline{\mu}}$. ∎

Example 4.12. If $\lambda \in \mathbb{P}^+$ is not piecewise disconnected, then the ratio $\frac{\bar{d}_{\lambda,\nu}}{c_{\lambda,\nu}}$ is not necessarily constant. Consider the weight λ from Example 3.12. If μ is the weight from Example 4.3 then $\bar{d}_{\lambda,\bar{\mu}} = 2$ and $c_{\lambda,\bar{\mu}} = 6$, whereas, if $\mu = \lambda$ then $\bar{d}_{\lambda,\bar{\mu}} = 1$ and $c_{\lambda,\bar{\mu}} = 2$.

4.4. Enlarging the indexing set.

In this section, we enlarge the indexing set $C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}}$ by adding non-regular elements, namely, we define

$$\overline{C_\lambda^{\mathrm{Lexi}}} = \left\{ \nu \in (\lambda^\rho)^{\Uparrow} - \mathbb{N}S_\lambda \ \mid \ \nu_{\beta_1} < \nu_{\beta_2} < \ldots < \nu_{\beta_r} \right\}.$$

Lemma 4.13. *If* $\nu \in \overline{C_\lambda^{\mathrm{Lexi}}} \setminus C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}}$, *then* ν *is not regular.*

Proof. Let j be such that $\lambda_{\beta_j}^\rho < \nu_{\beta_j} \leq (\lambda^\rho)_{\beta_j}^{\Uparrow}$ and $\nu_{\beta_i} \leq \lambda_{\beta_i}^\rho$ for all $i > j$. By definition of $(\lambda^\rho)^{\Uparrow}$, all the integers between $\lambda_{\beta_j}^\rho + 1$ and $((\lambda^\rho)^{\Uparrow})_{\beta_j}$ are entries of λ^ρ. The typical entries of ν are the same as of λ^ρ and there are $r - j + 1$ atypical entries which are strictly greater than $\lambda_{\beta_j}^\rho$. This implies that there must be equal entries of the same type, and hence ν is not regular. ∎

Lemma 4.14. *Let* $\mathfrak{C}_\lambda = \left\{ w(\nu) \in (\lambda^\rho)^{\Uparrow} - \mathbb{N}S_\lambda \mid w \in W_r, \ \nu \in \overline{C_\lambda^{\mathrm{Lexi}}} \right\}$ *and*

$$\mathfrak{D}_\lambda = \left\{ \nu \in (\lambda^\rho)^{\Uparrow} - \mathbb{N}S_\lambda \ \mid \ \nu_{\beta_i} \neq \nu_{\beta_j} \text{ for any } i \neq j \right\}.$$

Then $\mathfrak{C}_\lambda = \mathfrak{D}_\lambda$ *as multisets, and hence elements of* $((\lambda^\rho)^{\Uparrow} - \mathbb{N}S_\lambda) \setminus \mathfrak{C}_\lambda$ *are not regular.*

Proof. Clearly we have $\mathfrak{C}_\lambda \subseteq \mathfrak{D}_\lambda$ as sets. Since there is a unique element in the W_r orbit of any $\nu \in \overline{C_\lambda^{\mathrm{Lexi}}}$ that satisfies $\nu_{\beta_1} < \nu_{\beta_2} < \ldots < \nu_{\beta_r}$, the orbits of distinct elements from $\overline{C_\lambda^{\mathrm{Lexi}}}$ do not intersect. Hence, we have an inclusion of multisets. For the reverse inclusion, suppose that $\nu \in \mathfrak{D}_\lambda$. Take $\sigma \in W_r$ such that $\sigma^{-1}(\nu)$ satisfies $\nu_{\beta_{\sigma(1)}} < \nu_{\beta_{\sigma(2)}} < \cdots < \nu_{\beta_{\sigma(r)}}$. Since

$$\nu_{\beta_{\sigma(i)}} \leq \max\{\nu_{\beta_1}, \ldots, \nu_{\beta_i}\} \leq (\lambda^\rho)_{\beta_i}^{\Uparrow}$$

we have that $\sigma^{-1}(\nu) \in (\lambda^\rho)^{\Uparrow} - \mathbb{N}S_\lambda$. Hence $\sigma^{-1}(\nu) \in \overline{C_\lambda^{\mathrm{Lexi}}}$ and $\nu = \sigma(\sigma^{-1}(\nu)) \in \mathfrak{C}_\lambda$. ∎

4.5. Proof of the main theorem.

Proof of Theorem 4.1. By Lemma 4.9, we have that

$$e^{\rho} R \cdot \operatorname{ch} L(\lambda) = \sum_{\nu \in C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}}} \bar{d}_{\lambda,\nu} \cdot (-1)^{|\lambda^{\rho} - \nu|_{S_{\lambda}}} \mathcal{F}_W(e^{\nu})$$

which by (11) equals

$$(-1)^{|(\lambda^{\rho})^{\Uparrow} - (\lambda^{\rho})|_{S_{\lambda}}} \sum_{\nu \in C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}}} \frac{\bar{d}_{\lambda,\nu}}{c_{\lambda,\nu}} (-1)^{\left|(\lambda^{\rho})^{\Uparrow} - \nu\right|_{S_{\lambda}}} \mathcal{F}_W \left(\sum_{w \in W_r(\lambda,\nu)} e^{w(\nu)} \right).$$

Then by Proposition 4.11 the latter is equal to

$$(-1)^{|(\lambda^{\rho})^{\Uparrow} - (\lambda^{\rho})|_{S_{\lambda}}} \sum_{\nu \in C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}}} \frac{1}{t_{\lambda}} (-1)^{\left|(\lambda^{\rho})^{\Uparrow} - \nu\right|_{S_{\lambda}}} \mathcal{F}_W \left(\sum_{w \in W_r(\lambda,\nu)} e^{w(\nu)} \right)$$

and so by Lemma 4.13 and Lemma 2.4 we have that it is equal to

$$\frac{(-1)^{|(\lambda^{\rho})^{\Uparrow} - (\lambda^{\rho})|_{S_{\lambda}}}}{t_{\lambda}} \sum_{\nu \in \overline{C_{\lambda}^{\mathrm{Lexi}}}} (-1)^{\left|(\lambda^{\rho})^{\Uparrow} - \nu\right|_{S_{\lambda}}} \mathcal{F}_W \left(\sum_{w \in W_r(\lambda,\nu)} e^{w(\nu)} \right).$$

Then Lemma 4.14 and Lemma 2.4, it is equal to

$$\frac{(-1)^{|(\lambda^{\rho})^{\Uparrow} - (\lambda^{\rho})|_{S_{\lambda}}}}{t_{\lambda}} \sum_{\nu \in (\lambda^{\rho})^{\Uparrow} - \mathbb{N} S_{\lambda}} (-1)^{\left|(\lambda^{\rho})^{\Uparrow} - \nu\right|_{S_{\lambda}}} \mathcal{F}_W(e^{\nu})$$

which can be rewritten as

$$\frac{(-1)^{|(\lambda^{\rho})^{\Uparrow} - (\lambda^{\rho})|_{S_{\lambda}}}}{t_{\lambda}} \mathcal{F}_W \left(\sum_{\nu \in (\lambda^{\rho})^{\Uparrow} - \mathbb{N} S_{\lambda}} (-1)^{\left|(\lambda^{\rho})^{\Uparrow} - \nu\right|_{S_{\lambda}}} e^{\nu} \right)$$

$$= \frac{(-1)^{|(\lambda^{\rho})^{\Uparrow} - (\lambda^{\rho})|_{S_{\lambda}}}}{t_{\lambda}} \mathcal{F}_W \left(\frac{e^{(\lambda^{\rho})^{\Uparrow}}}{\prod_{\beta \in S_{\lambda}} (1 + e^{-\beta})} \right). \qquad \blacksquare$$

References

[BL]   Bernstein, I. N., and D. A. Leites, *A formula for the characters of the irreducible finite-dimensional representations of Lie superalgebras of series* $\mathfrak{gl}$ *and* $\mathfrak{sl}$, C. R. Acad. Bulgare Sci. **33** (1980) 1049–1051.

[B]   Brundan, J., *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra* $\mathfrak{gl}(m|n)$, J. Amer. Math. Soc. **16** (2003), 185–231.

[BS1]   Brundan, J., and C. Stroppel, *Highest weight categories arising from Khovanov's diagram algebra I: Cellularity*, Mosc. Math. J. **11** (2011), 685–722.

[BS2]   —, *Highest weight categories arising from Khovanov's diagram algebra 2: Koszulity,* Transformation Groups **15** (2010), 1–45.

[BS4]   —, *Highest weight categories arising from Khovanov's diagram algebra IV: the general linear supergroup,* J. Eur. Math. Soc. **14** (2012), 373–419.

[CH]   Comes, J. and T. Heidersdorf, "Thick Ideals in Deligne's category $\underline{\mathrm{Rep}}(O_\delta)$," arXiv:1507.06728.

[CHR]   Chmutov, M. , C. Hoyt, and S. Reif, *Kac-Wakimoto character formula for the general linear Lie superalgebra*, Algebra and Number Theory **9** (2015), 1419–1452.

[CK1]   Cheng, S.-J., and J.-H. Kwon, *Kac-Wakimoto character formula for orthosymplectic Lie superalgebras*, Adv. Math. **304** (2017), 1296–1329.

[CK2]   —, *Finite-Dimensional Half-Integer Weight Modules over Queer Lie Superalgebras*, Commun. Math. Phys. **346** (2015), 945–965.

[CWZ]   Cheng, S.-J., W. Wang, and R. B. Zhang, *Super duality and Kazhdan-Lusztig polynomials*, Trans. Amer. Math. Soc. **360** (2008), 5883–5924.

[ES1]   Ehrig, M., and C. Stroppel, *Diagrammatic description for the categories of perverse sheaves on isotropic Grassmannians*, Selecta Math. (N.S.) **22** (2016), 1455–1536.

[ES2]   —, *On the category of finite-dimensional representations of* $OSp(r|2n)$*: Part I,* in: Representation theory—current trends and perspectives, EMS Series of Congress Reports (2017), 109–170.

[G]   Gorelik, M., *Weyl denominator identity for finite-dimensional Lie superalgebras*, in: Highlights in Lie algebraic methods. New York: Birkhäuser–Springer; Progr. Math. **295** (2012), 167–188.

[GK]   Gorelik, M., and V. G. Kac, *Characters of (relatively) integrable modules over affine Lie superlagebras*, Japanese Journal of Mathematics **10** (2015), 135–235.

[GS1] Gruson, C., and V. Serganova, *Cohomology of generalized supergrassmannians and character formulae for basic classical Lie superalgebras*, Proc. Lond. Math. Soc. **101** (2010), 852–892.

[GS2] —, *Bernstein-Gelfand-Gelfand reciprocity and indecomposable projectiv emodules for classical algebraic supergroups*, Mosc. Math. J. **13** (2013), 281–313.

[HW] Heidersdorf, T., and R. Weissauer, "Cohomological Tensor Functors on Representations of the General Linear Supergroup," arXiv:1406.0321.

[K1] Kac, V. G., *Lie superalgebras*, Adv. Math. **26** (1977), 8–96.

[K2] —, *Characters of typical representations of classical Lie superalgebras*, Comm. Algebra **5** (1977), 889–897.

[K3] —, *Representations of classical Lie superalgebras*, Lecture Notes in Math. **676** (1978) 597–626.

[KW1] Kac, V. G., and M. Wakimoto, *Modular invariant representations of infinite-dimesional Lie algebras and superalgebras*, Proc. Natl. Acad. Sci. USA **85** (1988), 4956–4960.

[KW2] —, *Integrable highest weight modules over affine superalgebras and number theory*, in: Lie theory and geometry, J.-L. Brylinski et al. eds., Progr. Math. **123** (1994), 415–456.

[KW3] —, *Representations of affine superalgebras and mock theta functions*, Transformation Groups **19** (2014), 383–455.

[M] Musson, I., "Lie superalgebras and enveloping algebras," Graduate Studies in Mathematics, Springer, New York, 2012.

[MS] Musson, I., and V. Serganova, *Combinatorics of Character Formulas for the Lie Superalgebra $\mathfrak{gl}(m|n)$*, Transform. Groups **16** (2011), 555–578.

[MV] Moens, E. M., and J. Van der Jeugt, *A characer formula for atypical critical $\mathfrak{gl}(m|n)$ representations labeled by composite partitions*, J. Phys. A **37** (2004), 12019–12039.

[S1] Serganova, V., *Kazhdan-Lusztig polynomials and character formula for the Lie superalgebra $\mathfrak{gl}(m|n)$*, Selecta Math. (N.S.) **2** (1996), 607–651.

[S2] —, *Characters of irreducible representations of simple Lie superalgebras*, in: Proceedings of the International Congress of Mathematicians **II**, Berlin, Doc. Math., J. Deutsch. Math.-Verein. (1998), 583–593.

[SZ] Su, Y., and R. B. Zhang, *Character and dimension formulae for general linear superalgebra*, Advances in Mathematics **211** (2007), 1–33.

[VHKT] Van der Jeugt, J., J. W. B. Hughes, R. C. King, and J. Thierry-Mieg, *Character formulas for irreducible modules of the Lie superalgebras* $\mathfrak{sl}(m|n)$, J. Math. Phys. **31** (1990) 2278–2304.

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