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Representations of Hom-Right Symmetric Algebras

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Abstract. In this paper, we study representations of hom-right symmetric algebras, especially the trivial representation and the right and left adjoint representations are studied in detail. We are interested in the understanding of these notions in the setting of the 'operadic' presentation for a multiplicative hom-right symmetric algebra. Derivations, deformations, central extensions of hom-right symmetric algebras are also studied as an application.
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1. Introduction

An algebra $(\mathfrak{g}, \mu)$ over a field $\mathbb{K}$ of 0 characteristic is called right symmetric algebra, if for any $x,\ y,\ z \in \mathfrak{g}$, the following right symmetric identity holds:

$$\mu(x, \mu(y, z)) - \mu(\mu(x, y), z) = \mu(x, \mu(z, y)) - \mu(\mu(x, z), y),$$

see [MS] and the references therein.
Any associative algebra is a right symmetric algebra. For instance, $\mathfrak{gl}_n$, with usual multiplication of matrices is a right symmetric algebra. The algebra of vector fields on $\mathbb{R}^n$, equipped with the multiplication $\mu(u\partial_i, v\partial_j) = v\partial_j(u)\partial_i$ gives a less trivial example of a right symmetric algebra.

The notion of hom-algebras was introduced by Hardwig, Larsson, and Silvestrov in [HLS], as a tool for the study of deformations of Witt an Virasoro algebras. A hom-right symmetric algebra is a vector space $\mathfrak{g}$ equipped whith a bilinear product μ, but the right symmetric identity is twisted by a linear map α. An algebra $(\mathfrak{g}, \mu, \alpha)$ is called hom-right symmetric algebra, if for any $x,\ y,\ z \in \mathfrak{g}$, the hom-right symmetric identity holds:

$$\mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) = \mu(\alpha(x), \mu(z, y)) - \mu(\mu(x, z), \alpha(y)).$$

If α is a morphism for the product μ, the algebra is a multiplicative hom-right symmetric algebra. If moreover α is invertible, the hom-right symmetric algebra

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is regular.

Consider now an arbitrary vector space M, and an arbitrary linear transform A from M to M. A representation of the multiplicative hom-right symmetric algebra $(\mathfrak{g},\mu,\alpha)$ on the vector space M with respect to the linear map A is characterized by two linear maps $R,L:\mathfrak{g}\to\mathfrak{gl}(M)$. Given a representation (R,L) of $\mathfrak{g}$ on M allows to build a new multiplicative hom-right symmetric algebra, the semi-direct product $\mathfrak{g}^+$ of $\mathfrak{g}$ by M. This algebra characterizes the representation (R,L). Each representation gives rise to a sequence of representations (R_s,L_s), just by replacing R by $R_s=R\circ\alpha^s$ and L by $L_s=L\circ\alpha^s$. In particular, the trivial representation is defined by $M=\mathbb{K}$, $R=L=0$, and $A=id$; the adjoint representation is defined by the right and left actions (R,L) on $M=\mathfrak{g}$ ($R(x)y=\mu(y,x)=L(y)x$), and $A=\alpha$.

In this paper, we are interested by an understanding of these notions in the setting of an 'operadic' presentation of a multiplicative hom-right symmetric algebra. More precisely, we are considering here a sequence of comultiplications Δ_r in the space $\mathfrak{g}[1]\otimes S(\mathfrak{g}[1])$. A multiplication μ is thus exactly a bilinear coderivation Q of Δ_1. As usual, the commutator of two coderivations is a coderivation, and the initial multiplication μ is satisfying the hom-right symmetric identity if and only if $[Q,Q]=0$. This is the hom-right symmetric structure equation in this setting. A natural generalization is the notion of up to homotopy hom-right symmetric algebra, for which we do not impose to Q to be bilinear.

A representation (R,L) of $\mathfrak{g}$ on M allows to build a new multiplicative hom-right symmetric algebra, the semi-direct product of $\mathfrak{g}$ by M. The up to homotopy hom-right symmetric structure equation for $\mathfrak{g}^+$ is $[Q_{(R_s,L_s)},Q_{(R_s,L_s)}]=0$. It is easy to refind the hom-right symmetric cohomology by using the coboundary operator $d_s:\varphi\mapsto[Q_{(R_s,L_s)},\varphi]$.

2. hom-right symmetric algebras

Recall that a *hom-right symmetric algebra* is a triple $(\mathfrak{g},\mu,\alpha)$ where $\mathfrak{g}$ is a vector space on a field $\mathbb{K}$ with characteristic 0, μ is a bilinear map from $\mathfrak{g}\times\mathfrak{g}$ into $\mathfrak{g}$, and α is a linear map from $\mathfrak{g}$ to $\mathfrak{g}$, such that for all $x,\ y,\ z\in\mathfrak{g}$ the hom-right symmetric identity holds:

$$\mu(\alpha(x),\mu(y,z))-\mu(\mu(x,y),\alpha(z))=\mu(\alpha(x),\mu(z,y))-\mu(\mu(x,z),\alpha(y))\qquad(1)$$

Let $(x,y,z)_\alpha=\mu(\alpha(x),\mu(y,z))-\mu(\mu(x,y),\alpha(z))$ be the hom-associator of elements $x,\ y,\ z\in\mathfrak{g}$. In terms of hom-associators the hom-right symmetric identity is

$$(x,y,z)_\alpha=(x,z,y)_\alpha,\quad\forall x,\ y,\ z\in\mathfrak{g}.$$

The algebra is called hom-left symmetric algebra, if the hom-left symmetric identity

$$(x,y,z)_\alpha=(y,x,z)_\alpha$$

is holding. In both cases, hom-left or right symmetric, the commutator

$$[x,y]=\mu(x,y)-\mu(y,x)$$

is a hom-Lie bracket: it satisfies the hom-Jacobi identity:

$$[[x,y],\alpha(z)]+[[y,z],\alpha(x)]+[[z,x],\alpha(y)]=0.$$

A *graded* hom-right symmetric algebra is a graded vector space $\mathfrak{g}$ equipped with a degree 0, graded bilinear map μ, and a degree 0 linear map α, such that the graded hom-right symmetric relation holds, *i. e.* the relation (1) holds with the usual Koszul sign rule: each permutation of letters (here y and z) implies multiplication by the sign of the corresponding graded permutation. In the present setting, the graded relation (1) means $\mu(\alpha(x),\mu(y,z))-\mu(\mu(x,y)=$

$$\alpha(z))=(-1)^{\deg(y)\deg(z)}\big(\mu(\alpha(x),\mu(z,y))-\mu(\mu(x,z),\alpha(y))\big).$$

Such an algebra $(\mathfrak{g},\mu,\alpha)$ is *multiplicative* if α is a morphism for the multiplication μ. A multiplicative hom-right symmetric algebra $(\mathfrak{g},\mu,\alpha)$ is said to be *regular* if α is invertible.

Exemple 2.1. The simplest example is given by twisted right symmetric algebras. If $(\mathfrak{g},\mu)$ is a right symmetric algebra, and α a morphism for the multiplication μ, then for $\mathfrak{g}_\alpha=\mathfrak{g}$, and the multiplication

$$\mu_\alpha(x,y)=\alpha(\mu(x,y))=\mu(\alpha(x),\alpha(y)),$$

$(\mathfrak{g}_\alpha,\mu_\alpha,\alpha)$ is a multiplicative hom-right symmetric algebra.

Conversely each regular hom-right symmetric algebra $(\mathfrak{g},\mu,\alpha)$ is the twist of the right symmetric algebra $(\mathfrak{g},\mu_{\alpha^{-1}})$ by α (see [Y]).

Here are two classical examples:

Exemple 2.2. Recall the usual multiplication μ for the matrix algebra $\mathfrak{gl}_n$. This associative algebra is a right symmetric algebra. For any invertible matrix A, put $\alpha(x)=A^{-1}xA$. By twisting through α, this gives rise to the multiplicative $\mu_\alpha=\alpha\circ\mu$, $((\mathfrak{gl}_n)_\alpha,\mu_\alpha,\alpha)$ is a multiplicative hom-right symmetric algebra.

Exemple 2.3. Let $\mathbb{C}[t,t^{-1}]$ the space of Laurent polynomials. The Witt algebra W is the space of derivations of $\mathbb{C}[t,t^{-1}]$, with basis $\{\mathrm{L}_n=t^n\frac{d}{dt},\ \ n\in\mathbb{Z}\}$, it is the right symmetric algebra with multiplication $\mu(\mathrm{L}_n,\mathrm{L}_m)=n\mathrm{L}_{n+m-1}$.

Fix now $q\in\mathbb{C}\setminus\{0,1\}$. The q-derivation of $f\in\mathbb{C}[t,t^{-1}]$ is defined by:

$$(\partial_t f)(t)=\frac{f(t)-f(qt)}{1-q}.$$

Putting $L_n=t^n\partial_t$ the basis q-derivation, we get now the q-deformed Witt algebra, with multiplication

$$\mu_q(L_n,L_m)=\{n\}_qL_{n+m},$$

where the numbers $\{n\}_q$ are:

$$\{n\}_q = \frac{1-q^n}{1-q}.$$

The q-deformed Witt algebra (W_q, μ_q) is no more a right symmetric algebra but it is a hom-right symmetric algebra with the map α by

$$\alpha(L_n) = (1+q^n)L_n.$$

This hom-right symmetric algebra is not multiplicative.

3. Derivation of hom-right symmetric algebras

Let $(\mathfrak{g}, \mu, \alpha)$ be a multiplicative hom-right symmetric algebra. For any non negative integer k, we denote by α^k the k-times composition of α. If $(\mathfrak{g}, \mu, \alpha)$ is regular, we denote by α^{-k} the k-times composition of α^{-1}. Denote also by $\alpha : \otimes^k \mathfrak{g} \to \otimes^k \mathfrak{g}$ the map:

$$\alpha(x_1 \otimes \cdots \otimes x_n) = \alpha(x_1) \otimes \cdots \otimes \alpha(x_n).$$

Definition 3.1. For any non negative integer k, a linear map $D : \mathfrak{g} \to \mathfrak{g}$ is called an α^k-derivation of the multiplicative hom-right symmetric algebra $(\mathfrak{g}, \mu, \alpha)$, if

$$D \circ \alpha = \alpha \circ D,$$

and, for each x, y in $\mathfrak{g}$,

$$D(\mu(x,y)) = \mu(D(x), \alpha^k(y)) + \mu(\alpha^k(x), D(y)).$$

For a regular hom-right symmetric algebra, an α^{-k}-derivation is similarly defined.

Denote by $Der_{\alpha^k}(\mathfrak{g})$ the space of α^k-derivation of $(\mathfrak{g}, \mu, \alpha)$.

Let $(\mathfrak{g}, \mu, \alpha)$ be a multiplicative hom-right symmetric algebra. For any x in $\mathfrak{g}$, satisfying $\alpha(x) = x$, define $D_k : \mathfrak{g} \to \mathfrak{g}$ by:

$$D_k(x)(y) = [x, \alpha^k(y)], \qquad (y \in \mathfrak{g}).$$

Thus $D_k \circ \alpha = \alpha \circ D_k$. Moreover:

$$\begin{aligned} D_k(x)(\mu(y,z)) &= \mu(\alpha(x), \mu(\alpha^k(y), \alpha^k(z))) - \mu(\mu(\alpha^k(y), \alpha^k(z)), \alpha(x) \\ &= \mu(\mu(x, \alpha^k(y)), \alpha^{k+1}(z)) + \mu(\alpha(x), \mu(\alpha^k(z), \alpha^k(y))) - \mu(\mu(x, \alpha^k(z)), \alpha^{k+1}(y)) \\ &\quad - \mu(\alpha^{k+1}(y), \mu(\alpha^k(z), x)) + \mu(\alpha^{k+1}(y), \mu(x, \alpha^k(z))) - \mu(\mu(\alpha^k(y), x), \alpha^{k+1}(z))) \\ &= \mu(D_k(x)y, \alpha^{k+1}(z)) + \mu(\alpha^{k+1}(y), D_k(z)) + (x, \alpha^k(z), \alpha^k(y))_\alpha. \end{aligned}$$

Therefore if x is such that the associator $(x,y,z)_\alpha$ vanishes for all y, z in $\mathfrak{g}$, then $D_k(x)$ is an α^{k+1}-derivation, which we call an inner α^{k+1}-derivation.

Denote $Inn_{\alpha^k}(\mathfrak{g})$ the space of inner α^k-derivations.

Lemma 3.2. *Define the commutator* $[D, D']$ *of two derivations,* $D \in Der_{\alpha^k}(\mathfrak{g})$ *and* $D' \in Der_{\alpha^{k'}}(\mathfrak{g})$*, by* $[D, D'] = D \circ D' - D' \circ D$*. Then* $[D, D']$ *is a* $\alpha^{k+k'}$*-derivation, more precisely,* $[D, D'] \in Der_{\alpha^{k+k'}}(\mathfrak{g})$*.*

Proof. Since $D \circ \alpha = \alpha \circ D$, $D' \circ \alpha = \alpha \circ D'$, clearly $[D, D'] \circ \alpha = \alpha \circ [D, D']$.
Now, for any x, $y \in \mathfrak{g}$,

$$\begin{aligned}
&[D, D'](\mu(x, y)) = D \circ D'(\mu(x, y)) - D' \circ D(\mu(x, y)) \\
&= D\Big(\mu(D'(x), \alpha^{k'}(y)) + \mu(\alpha^{k'}(x), D'(y))\Big) - D'\big(\mu(D(x), \alpha^{k}(y)) + \mu(\alpha^{k}(x), D(y))\big) \\
&= \mu(D \circ D'(x), \alpha^{k'+k}(y)) + \mu(\alpha^{k}(D'(x)), D(\alpha^{k'}(y))) + \mu(D(\alpha^{k'}(x)), \alpha^{k}(D'(y))) \\
&+ \mu(\alpha^{k'+k}(x), D \circ D'(y))) - \mu(D' \circ D(x), \alpha^{k+k'}(y)) - \mu(\alpha^{k'}(D(x)), D'(\alpha^{k}(y))) \\
&- \mu(D'(\alpha^{k}(x)), \alpha^{k'}(D(y))) - \mu(\alpha^{k+k'}(x), D' \circ D(y)).
\end{aligned}$$

Since $D \circ \alpha^{k'} = \alpha^{k'} \circ D$, and $D' \circ \alpha^{k} = \alpha^{k} \circ D'$, this is:

$$[D, D'](\mu(x, y)) = \mu([D, D'](x), \alpha^{k+k'}(y)) + \mu(\alpha^{k+k'}(x), [D, D'](y)).$$ ■

Put

$$Der(\mathfrak{g}) = \oplus_{k \geq 0} Der_{\alpha^k}(\mathfrak{g}).$$

Proposition 3.3. *With the above notation,* $Der(\mathfrak{g})$ *is a Lie algebra for the commutators of derivations are given by* $[D, D'] = D \circ D' - D' \circ D$*.*

Similarly, for a regular hom-right symmetric algebra, the space $\oplus_{k \in \mathbb{Z}} Der_{\alpha^k}(\mathfrak{g})$ is a Lie algebra for the commutators $[D, D']$.

Proof. Let $D \in Der_{\alpha^k}(\mathfrak{g})$, $D' \in Der_{\alpha^{k'}}(\mathfrak{g})$ and $D'' \in Der_{\alpha^{k''}}(\mathfrak{g})$. By definition, the commutators of derivations satisfy:

$$[[D, D'], D''] = (D \circ D') \circ D'' - (D' \circ D) \circ D'' - D'' \circ (D \circ D') + D'' \circ (D' \circ D),$$

$$[[D', D''], D] = (D' \circ D'') \circ D - (D'' \circ D') \circ D - D \circ (D' \circ D'') + D \circ (D'' \circ D')$$

and

$$[[D'', D], D'] = (D'' \circ D) \circ D' - (D \circ D'') \circ D - D' \circ (D'' \circ D) + D' \circ (D \circ D'').$$

Since the operator $\circ$ is associative, then after adding the three last equations, we get the desired result. ■

4. Up to homotopy hom-right symmetric algebras

The definition of up to homotopy hom-right symmetric algebras needs the realization of hom-right symmetric algebra structure equation as a self commutator of a coalgebra codifferential Q.

More precisely, given the graded vector space $(\mathfrak{g}, deg)$ and the degree 0 linear map α, consider $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1]) = \sum_{k\geq 1} \mathfrak{g}[1] \otimes S^{k-1}(\mathfrak{g}[1])$, where $S(\mathfrak{g}[1])$ is the graded symmetric algebra on the graded vector space $(\mathfrak{g}[1], |\ \ |)$, *i. e.* the space $\mathfrak{g}$ with a shift of -1 on the degree: for each $x \in \mathfrak{g}$:

$$|x| = deg(x) - 1.$$

Remark that if $\mathfrak{g}$ is not graded, we put $deg(x) = 0$ for each x, and $g[1] \otimes S(\mathfrak{g}[1])$ is simply the space $g \otimes \bigwedge \mathfrak{g}$ with the degree

$$|x_0 \otimes (x_1 \wedge \cdots \wedge x_{k-1})| = -k.$$

Recall the notion of shuffle permutation. A permutation σ is a $(p-1, 1, k-p-1)$-shuffle ($\sigma \in Sh(p-1, 1, k-p-1)$) if it is a permutation of $\{1, \dots, k-1\}$ such that $\sigma(1) < \cdots < \sigma(p-1)$, and $\sigma(p+1) < \cdots < \sigma(k-1)$.

For each natural number r, extend α^r to $g[1] \otimes S(\mathfrak{g}[1])$ by putting

$$\alpha^r(x_0 \otimes x_1 \dots x_{k-1}) = \alpha^r(x_0) \otimes \alpha^r(x_1) \dots \alpha^r(x_{k-1}).$$

Define also the comultiplication Δ_r on $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$by: $\Delta_r(x_0 \otimes x_1 \dots x_{n-1})$

$$\sum_{\substack{1\leq k\leq n-2 \\ \sigma\in Sh_{k-1,1,n-k-1}}} \alpha^r(x_0 \otimes x_{\sigma(1)} \dots x_{\sigma(k-1)}) \bigotimes \alpha^r(x_{\sigma(k)} \otimes x_{\sigma(k+1)} \dots x_{\sigma(n-1)}).$$

(as usual, the Koszul's sign rule is understood in this formula). For instance, if $\mathfrak{g}$ is not graded, this reads: $\Delta_r(x_0 \otimes x_1 \dots x_{n-1}) =$

$$\sum_{\substack{1\leq k\leq n-1 \\ \sigma\in Sh_{k-1,1,n-k-1}}} \varepsilon(\sigma)\alpha^r(x_0 \otimes x_{\sigma(1)} \dots x_{\sigma(k-1)}) \bigotimes \alpha^r(x_{\sigma(k)} \otimes x_{\sigma(k+1)} \dots x_{\sigma(n-1)}).$$

where $\varepsilon(\sigma)$ is the sign of the permutation σ.

This comultiplication is hom-permutative with respect to α^r, that is the relation:

$$(\alpha^r \bigotimes \Delta_r) \circ \Delta_r = (\Delta_r \bigotimes \alpha^r) \circ \Delta_r = (\alpha^r \bigotimes \tau \circ \Delta_r) \circ \Delta_r,$$

where τ is the twist: $\tau(X \bigotimes Y) = Y \bigotimes X$, holds.

Now a (multiplicative) degree q coderivation Q of Δ_r ($Q \in Coder(\Delta_r)$) is a map $Q : \mathfrak{g}[1] \otimes S(\mathfrak{g}[1]) \to \mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$, such that:

$$Q \circ \alpha = \alpha \circ Q \quad \text{ and } \quad (Q \bigotimes \alpha^r + \alpha^r \bigotimes Q) \circ \Delta_r = \Delta_r \circ Q.$$

Any such coderivation is characterized by its Taylor expansion. More precisely, let p be the projection on $\mathfrak{g}[1]$ parallel to $\sum_{k>0} \mathfrak{g}[1] \otimes S^{k-1}(\mathfrak{g}[1])$, i_k the canonical injection of $\mathfrak{g}[1] \otimes S^{k-1}(\mathfrak{g}[1])$ into $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$. For any coderivation Q, consider the sequence of maps $Q_k = p \circ Q \circ i_k$. Each Q_k extends to a map $\tilde{Q}_k$ defined on $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$ by:

$$\tilde{Q}_k(x_0 \otimes x_1 \dots x_{n-1}) = \sum_{k=1}^{n} \sum_{\sigma \in Sh_{k-1,n-k}} Q_k(x_0 \otimes x_{\sigma(1)} \dots x_{\sigma(k-1)}) \otimes \alpha^r(x_{\sigma(k)} \dots x_{\sigma(n-1)})$$
$$+ \sum_{k=1}^{n} \sum_{\sigma \in Sh_{1,k-1,n-k-1}} (-1)^q \alpha^r(x_0) \otimes Q_k(x_{\sigma(1)} \otimes x_{\sigma(2)} \cdots x_{\sigma(k)}).\alpha^r(x_{\sigma(k+1)} \cdots x_{\sigma(n-1)}). \tag{2}$$

Then $Q = \sum_k \tilde{Q}_k$. Conversely, each sequence of maps $(Q_k : \mathfrak{g}[1] \otimes S^{k-1}(\mathfrak{g}[1]) \to \mathfrak{g}[1])$ $(k > 0)$ defines through extension and sum a coderivation $Q = \sum_k \tilde{Q}_k$.

Lemma 4.1. *Let $Q \in Coder(\Delta_r)$ and $Q' \in Coder(\Delta_{r'})$. Then the commutator $[Q, Q'] = Q \circ Q' - (-1)^{qq'} Q' \circ Q$ (with the Koszul's rule) is a degree $q + q'$ coderivation:*

$$[Q, Q'] \in Coder(\Delta_{r+r'}).$$

Proof. It is enough to prove this for $Q = \tilde{Q}_k$ and $Q' = \tilde{Q}'_{k'}$. For $I = \{i_1 < \dots < i_{k-1}\}$, put $x_I = x_{i_1} \dots x_{i_{k-1}}$. With this notation, for each $n \geq k$,

$$Q(x_0 \otimes x_{[1,n-1]}) = \sum_{\substack{I \sqcup J = [1,n-1] \\ \#I = k-1}} Q_k(x_0 \otimes x_I) \otimes \alpha^r(x_J) + \sum_{\substack{\{j\} \sqcup I \sqcup J = [1,n-1] \\ \#I = k-1}} (-1)^q \alpha^r(x_0) \otimes Q_k(x_j \otimes x_I).\alpha^r(x_J).$$

For simplicity, in the following computation, the cardinality of subsets in $[1, n-1]$ are not indicated: in each expression, this cardinality is evident. Thus

$$Q \circ Q'(x_0 \otimes x_{[1,n-1]}) = \sum_{I \sqcup J \sqcup K = [1,n-1]} Q_k(Q'_{k'}(x_0 \otimes x_I) \otimes \alpha^{r'}(x_J)) \otimes \alpha^{r+r'}(x_K)$$
$$+ \sum_{\{j\} \sqcup I \sqcup J \sqcup K = [1,n-1]} (-1)^{qk'+qq'} \alpha^r(Q'_{k'}(x_0 \otimes x_I)) \otimes Q_k(\alpha^{r'}(x_j \otimes x_J)).\alpha^{r+r'}(x_K)$$
$$+ \sum_{\{j\} \sqcup I \sqcup J \sqcup K = [1,n-1]} (-1)^{q'k} Q_k(\alpha^{r'}(x_0 \otimes x_J)) \otimes \alpha^r(Q'_{k'}(x_j \otimes x_I)).\alpha^{r+r'}(x_K)$$
$$+ \sum_{\{j\} \sqcup I \sqcup J \sqcup K = [1,n-1]} (-1)^{q'} Q_k(\alpha^{r'}(x_0) \otimes Q'_{k'}(x_j \otimes x_I).\alpha^{r'}(x_J)) \otimes \alpha^{r+r'}(x_K)$$
$$+ \sum_{\{j\} \sqcup I \sqcup J \sqcup K = [1,n-1]} (-1)^{q+q'} \alpha^{r'+r}(x_0) \otimes Q_k(Q'_{k'}(x_j \otimes x_I) \otimes \alpha^{r'}(x_J)).\alpha^{r+r'}(x_K)$$
$$+ \sum_{\{j\} \sqcup I \sqcup J \sqcup K = [1,n-1]} (-1)^{q+q'} \alpha^{r'+r}(x_0) \otimes Q_k(\alpha^{r'}(x_j \otimes x_J)).\alpha^r(Q'_{k'}(x_j \otimes x_I)).\alpha^{r+r'}(x_K)$$
$$+ \sum_{\{j\} \sqcup \{s\} \sqcup I \sqcup J \sqcup K = [1,n-1]} (-1)^{q+q'+q'} \alpha^{r'+r}(x_0) \otimes Q_k(\alpha^{r'}(x_j) \otimes Q'_{k'}(x_s \otimes x_I)\alpha^{r'}(x_J)).\alpha^{r+r'}(x_K).$$

Since $\alpha^r(Q'_{k'}(x_I)) = Q'_{k'}(\alpha^r(x_I))$, some terms disappear in the commutator and there remains:

$$[Q,Q'](x_0\otimes x_{[1,n-1]}) = \sum_{I\sqcup J\sqcup K=[1,n-1]} Q_k(Q'_{k'}(x_0\otimes x_I)\otimes \alpha^{r'}(x_J))\otimes \alpha^{r+r'}(x_K)$$
$$-\sum_{I\sqcup J\sqcup K=[1,n-1]} (-1)^{qq'} Q'_k(Q_k(x_0\otimes x_J)\otimes \alpha^{r'}(x_I))\otimes \alpha^{r+r'}(x_K)$$
$$+\sum_{\{j\}\sqcup I\sqcup J\sqcup K=[1,n-1]} (-1)^{q'} Q_k(\alpha^{r'}(x_0)\otimes Q'_{k'}(x_j\otimes x_I).\alpha^{r'}(x_J))\otimes \alpha^{r+r'}(x_K)$$
$$-\sum_{\{j\}\sqcup I\sqcup J\sqcup K=[1,n-1]} (-1)^{qq'+q} Q'_{k'}(\alpha^{r}(x_0)\otimes Q_k(x_j\otimes x_J).\alpha^{r}(x_I))\otimes \alpha^{r+r'}(x_K)$$
$$+\sum_{\{j\}\sqcup I\sqcup J\sqcup K=[1,n-1]} (-1)^{q+q'} \alpha^{r'+r}(x_0)\otimes Q_k(Q'_{k'}(x_j\otimes x_I)\otimes \alpha^{r'}(x_J))\otimes .\alpha^{r}(x_I))\otimes \alpha^{r+r'}(x_K)$$
$$-\sum_{\{j\}\sqcup I\sqcup J\sqcup K=[1,n-1]} (-1)^{qq'+q+q'} \alpha^{r'+r}(x_0)\otimes Q'_{k'}(Q_k(x_j\otimes x_J)\otimes \alpha^{r'}(x_J))\otimes .\alpha^{r}(x_I))\otimes \alpha^{r+r'}(x_K)$$
$$+\sum_{\{s\}\sqcup\{j\}\sqcup I\sqcup J\sqcup K=[1,n-1]} (-1)^{q+q'+q'} \alpha^{r'+r}(x_0)\otimes Q_k(\alpha^{r'}(x_j)\otimes Q'_{k'}(x_s\otimes x_I)\alpha^{r'}(x_J)).\alpha^{r+r'}(x_K)$$
$$-\sum_{\{s\}\sqcup\{j\}\sqcup I\sqcup J\sqcup K=[1,n-1]} (-1)^{qq'+q+q'+q} \alpha^{r'+r}(x_0)\otimes Q'_{k'}(\alpha^{r}(x_j)\otimes Q_k(x_s\otimes x_J)\alpha^{r}(x_I)).\alpha^{r+r'}(x_K).$$

Put thus $k''=k+k'-1$, with the convention that the cardinality of subsets I, J, K are well chosen in each term $[Q,Q']_{k''}(x_0\otimes x_L)=$

$$\sum_{I\sqcup J=L} Q_k(Q'_{k'}(x_0\otimes x_I)\otimes \alpha^{r'}(x_J)) - (-1)^{qq'} Q'_k(Q_k(x_0\otimes x_J)\otimes \alpha^{r'}(x_I))$$
$$+\sum_{\{j\}\sqcup I\sqcup J=L} (-1)^{q'} Q_k(\alpha^{r'}(x_0)\otimes Q'_{k'}(x_j\otimes x_I).\alpha^{r'}(x_J))$$
$$-(-1)^{qq'+q} Q'_{k'}(\alpha^{r}(x_0)\otimes Q_k(x_j\otimes x_J).\alpha^{r}(x_I)).$$

Then:

$$[Q,Q'](x_0\otimes x_{[1,n-1]}) = \sum_{L\sqcup K=[1,n-1]} [Q,Q']_{k''}(x_0\otimes x_L)\otimes \alpha^{r+r'}(x_K)$$
$$+\sum_{\{j\}\sqcup L\sqcup K=[1,n-1]} (-1)^{q+q'} \alpha^{r+r'}(x_0)\otimes [Q,Q']_{k''}(x_j\otimes x_L).\alpha^{r+r'}(x_K).$$

This relation proves that $[Q,Q']$ is a coderivation of $\Delta_{r+r'}$. ■

Corollary 4.2. *Define the coderivation space as* $Coder=\oplus_{r\geq 0} Coder(\Delta_r)$. *Equipped with the commutator, this space is a graded Lie algebra.*

Let μ be a multiplicative bilinear map on $\mathfrak{g}$, with degree 0. Put $Q_2(x,y)=\mu(x,y)$. Then Q_2 is a degree 1 map $\mathfrak{g}[1]\otimes\mathfrak{g}[1]\to\mathfrak{g}[1]$, and if $r=r'=1$, and $Q=\tilde{Q}_2$,

$$[Q,Q]_3(x_0\otimes x_1.x_2) = 2\big(\mu(\mu(x_0\otimes x_1)\otimes\alpha(x_2)) - \mu(\mu(x_0\otimes x_2)\otimes x_1)$$
$$-\mu(\alpha(x_0)\otimes\mu(x_1\otimes x_2)] + \mu(\alpha(x_0)\otimes\mu(x_2\otimes x_1))\big).$$

In other words, $(\mathfrak{g}, \mu, \alpha)$ is a multiplicative hom-right symmetric algebra if and only if the structure equation: $[Q, Q] = 0$ holds.

Definition 4.3. A up to homotopy hom-right symmetric algebra $(\mathfrak{g}, Q, \alpha)$ is a (graded) vector space $\mathfrak{g}$, a degree 0 linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ and a degree 1 coderivation $Q \in Coder(\Delta_1)$ of $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$ such that:

$$[Q, Q] = 0.$$

Exemple 4.4. The usual example is a differential multiplicative hom-right symmetric algebra. Let $(\mathfrak{g}, \mu, \alpha)$ be a multiplicative hom-right symmetric algebra. A differential d of this algebra is a map $d : \mathfrak{g} \to \mathfrak{g}$ such that:

$$d\left(\mu(x, y)\right) = \mu(d(x), \alpha(y)) + \mu(\alpha(x), d(y)), \qquad d \circ d = 0, \qquad d \circ \alpha = \alpha \circ d.$$

Putting $Q_1 = d$, $Q_2(x, y) = \mu(x, y)$, $Q = \tilde{Q}_1 + \tilde{Q}_2$ gives that $(\mathfrak{g}, \mu, \alpha, d)$ is a differential multiplicative hom-right symmetric algebra if and only if $(\mathfrak{g}, Q, \alpha)$ is a up to homotopy hom-right symmetric algebra.

5. Representations, semi-direct products and cohomology

In this section we study the representations of multiplicative hom-right symmetric algebras, the cohomology of these algebras with value in a module, and give the corresponding coboundary operators. We also prove that one can form semi-direct products of multiplicative hom-right symmetric algebras. Please see [Dz] for more details about right symmetric algebras and their cohomology. Let $(\mathfrak{g}, \mu, \alpha)$ be a multiplicative hom-right symmetric algebra and M an arbitrary vector space. Let $A \in \mathfrak{gl}(M)$ be an arbitrary linear transformation from M to M.

Definition 5.1. A representation of the multiplicative hom-right symmetric algebra $(\mathfrak{g}, \mu, \alpha)$ is defined by two linear maps $R, L : \mathfrak{g} \to \mathfrak{gl}(M)$, such that, for each x, $y \in \mathfrak{g}$,

$$R(\alpha(x)) \circ A = A \circ R(x), \qquad L(\alpha(x)) \circ A = A \circ L(x),$$

and

$$R([x, y]) \circ A - R(\alpha(y)) \circ R(x) + R(\alpha(x)) \circ R(y) = 0,$$

and

$$R(\alpha(y)) \circ L(x) - L(\alpha(x)) \circ R(y) - L(\mu(x, y)) \circ A + L(\alpha(x)) \circ L(y) = 0.$$

Observe that, for any $r = 0, 1, 2, \ldots$, if $R_r = R \circ \alpha^r$, $L_r = L \circ \alpha^r$, (R_r, L_r) is also a representation of $(\mathfrak{g}, \mu, \alpha)$. If α is invertible, the same result holds for $r = -1, -2, \ldots$.

An example of representation is the *trivial representation*: put $M = \mathbb{K}$, $R = L = 0$, $A = id$, then $(\mathbb{K}, 0, 0, id)$ is a representation of $(\mathfrak{g}, \mu, \alpha)$.

Exemple 5.2. A fundamental example is the α^0-adjoint representation of $(\mathfrak{g}, \mu, \alpha)$: choose $M = \mathfrak{g}$, $R_0(x)y = \mu(y, x)$, $L_0(x)y = \mu(x, y)$, $A = \alpha$, then (R_0, L_0) is a representatation of $(\mathfrak{g}, \mu, \alpha)$. As above, for each natural number, denote by $R_r(x), L_r(x)$ the maps $R_r(x) = R_0(\alpha^r(x))$, $L_r(x) = L_0(\alpha^r(x))$, then $(\mathfrak{g}, R_r, L_r, \alpha)$ is the α^r-adjoint a representatation of $(\mathfrak{g}, \mu, \alpha)$.

Lemma 5.3. *Let $(\mathfrak{g}, \mu, \alpha)$ be a multiplicative hom-right symmetric algebra. Let M be a vector space, R, L two linear maps $R, L : \mathfrak{g} \to \mathfrak{gl}(M)$ and $A \in \mathfrak{gl}(M)$. Put:*

$$\mu_{(R,L)}\big((x, u), (y, v)\big) = (\mu(x, y), L(x)v + R(y)u), \qquad (\alpha + A)(x, u) = (\alpha(x), Au).$$

Then $(\mathfrak{g} \times M, \mu_{(R,L)}, \alpha + A)$ is a multiplicative hom-right symmetric algebra if and only if (M, R, L, A) is a representatation of $(\mathfrak{g}, \mu, \alpha)$.

Proof. First observe that $(\alpha + A) \circ \mu_{(R,L)} = \mu_{(R,L)} \circ (\alpha + A) \otimes (\alpha + A)$ if and only if

$$R(\alpha(x)) \circ A = A \circ R(x), \quad \text{and} \quad L(\alpha(x)) \circ A = A \circ L(x).$$

On the other hand, the hom-right symmetric identity with product $\mu_{(R,L)}$ is equivalent to the identities

$$R([x, y]) \circ A - R(\alpha(y)) \circ R(x) + R(\alpha(x)) \circ R(y) = 0,$$

and

$$R(\alpha(y)) \circ L(x) - L(\alpha(x)) \circ R(y) - L(\mu(x, y)) \circ A + L(\alpha(x)) \circ L(y) = 0. \qquad \blacksquare$$

In the situation of this Lemma, we say that $(\mathfrak{g} \times M, \mu_{(R,L)}, \alpha + A)$ is the *semi-direct product* of $\mathfrak{g}$ by M.

Definition 5.4. Let (M, R, L, A) be a representation of a multiplicative hom-right symmetric algebra $(\mathfrak{g}, \mu, \alpha)$. A n-cochain C, with value in M, is a n-linear map $C : \mathfrak{g} \otimes \wedge^{n-1}\mathfrak{g} \to M$ such that $A \circ C = C \circ \alpha$. Denote $\mathcal{C}^n(\mathfrak{g}, M)$ the space of n-cochains C.
A 0-cochain is a vector $m \in M$ such that

$$A(m) = m, \quad \text{and} \quad R(\alpha(y)) \circ R(x)m = R(\mu(x, y))m$$

for all $x, y \in \mathfrak{g}$.

Fix a natural number $n \geq 1$, the n-coboundary operator is the map d_n from $\mathcal{C}^n(\mathfrak{g}, M)$ into $\mathcal{C}^{n+1}(\mathfrak{g}, M)$ defined by:

$$(d_n C)(x_0 \otimes x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^{i-1} (d_n^i C)(x_0 \otimes x_1, \ldots, x_n)$$

and:

$$\begin{aligned}(d_n^i C)(x_0\otimes x_1,\dots,x_n) &= L(\alpha^n(x_0))C(x_i\otimes x_1,\dots\hat{\imath}\dots,x_n)\\ &+ R(\alpha^n(x_i))C(x_0\otimes x_1,\dots\hat{\imath}\dots,x_n)\\ &- C(\mu(x_0,x_i)\otimes\alpha(x_1),\dots\hat{\imath}\dots,\alpha(x_n))\\ &+\sum_{i<j} C(\alpha(x_0)\otimes\alpha(x_1),\dots,\hat{\imath},\dots,\alpha(x_{j-1}),[x_i,x_j],\dots,\alpha(x_n)).\end{aligned}$$

Let m be a 0-cochain , the 0-coboundary operator is the map $(d_0m)(x) = L(x)m - R(x)m$.

Let m be a 0-cochain

$$\begin{aligned}d_1\circ d_0(m)(x_0\otimes x_1) &= L(\alpha(x_0))(d_0m)(x_1) + R(\alpha(x_1)(d_0m)(x_0) - (d_0m)(\mu(x_0,x_1))\\ &= L(\alpha(x_0))\circ L(x_1)m - L(\alpha(x_0))\circ R(x_1)m + R(\alpha(x_1))\circ L(x_0)m\\ &- R(\alpha(x_1))\circ R(x_0)m + R(\mu(x_0,x_1))m - L(\mu(x_0,x_1))m.\end{aligned}$$

Since $A(m) = m$, and $R(\alpha(y))\circ R(x)m = R(\mu(x,y))m$ for all $x,y\in\mathfrak{g}$, it follows that $d_1\circ d_0 = 0$.

In light of Lemma (5.3), we can define the multiplicative hom-right symmetric algebra $(\mathfrak{g}\times M,\mu_{(R,L)},\alpha+A)$ and its coalgebra $((\mathfrak{g}\times M)[1]\otimes S((\mathfrak{g}\times M)[1]),\Delta_n)$. Then, for each $n\geq 1$, there is a natural bijection ψ_n from $\mathcal{C}^n(\mathfrak{g},M)$ onto a subspace of $Coder(\Delta_n)$, namely: let $C\in\mathcal{C}^n(\mathfrak{g},M)$,

1. Consider C as a n-degree map $C:(\mathfrak{g}\times M)[1]\otimes S^{n-1}((\mathfrak{g}\times M)[1])\to(\mathfrak{g}\times M)[1]$, vanishing if one of the argument is in M.

2. Extend this map to $\tilde{C}\in Coder(\Delta_n)$ as in (2). Put $\psi_n(C)=\tilde{C}$.

Conversely, for each coderivation $\Gamma\in Coder(\Delta_n)$ vanishing if one of its argument is in M, and with values in M, there is $C_n\in\mathcal{C}^n(\mathfrak{g},M)$, such that $\Gamma=\psi_n(\Gamma)$, namely:

$$C_n(x_0\otimes x_1\dots x_{n-1}) = \psi_n^{-1}(\Gamma)(x_0,\dots,x_{n-1}) = \Gamma_n((x_0,0),\dots,(x_{n-1},0)).$$

Proposition 5.5. *Let $Q_{(R,L)}\in Coder(\Delta_1)$ the coderivation defined by $Q_{(R,L)}=\mu_{(R,L)}$. Then $\psi_{n+1}^{-1}([Q_{(R,L)},\psi_n(C)]) = (-1)^n d_nC$.*

Proof. By definition, the commutator $C' = [Q_{(R,L)},\psi_n(C)]$ is vanishing if one of its argument is in M and it is explicitly given by:

$$\begin{aligned} C'(x_0\otimes x_1\dots x_n) &= (Q_{(R,L)}\circ C-(-1)^{(n)\cdot 1}C\circ Q_{(R,L)})(x_0\otimes x_1\dots x_n)\\ &= \sum_{\sigma\in Sh(n-1,1)}\varepsilon_\sigma R(\alpha^n(x_{\sigma(n)}))C(x_0\otimes x_{\sigma(1)}\dots x_{\sigma(n-1)})+\\ &\quad +(-1)^n\sum_{\sigma\in Sh(1,n-1)}\varepsilon_\sigma L(\alpha^n(x_0))C(x_{\sigma(1)}\otimes x_{\sigma(2)}\dots x_{\sigma(n)})-\\ &\quad -(-1)^n\sum_{\sigma\in Sh(n-1,1)}\varepsilon_\sigma C(\mu(x_0,x_{\sigma(1)})\otimes\alpha(x_{\sigma(2)})\dots x_{\sigma(n)})\\ &\quad +(-1)^n\sum_{\sigma\in Sh(1,1,n-2)}\varepsilon_\sigma C(\alpha(x_0)\otimes\mu(x_{\sigma(1)},x_{\sigma(2)}).\alpha(x_{\sigma(3)})\dots\alpha(x_{\sigma(n)}))\\ &= (-1)^n\sum_i(-1)^{i-1}(d_n^iC)(x_0\otimes x_1,\dots,x_n). \end{aligned}$$

This proves the proposition. ∎

Since the commutator of coderivations is a Lie bracket and $[Q_{(R,L)},Q_{(R,L)}]=0$,

Corollary 5.6. *The operators d_n satisfy:*

$$d_{n+1}\circ d_n=0.$$

Putting $Z^n_{(R,L)}=\{C\in\mathcal{C}^n(\mathfrak{g},M),\ d_nC=0\}$ the space of closed n-cochain, and $B^n_{(R,L)}=d_{n-1}(\mathcal{C}^{n-1}(\mathfrak{g},M))$ the space of exact n-cochain , and $H^0(\mathfrak{g},M)=Z^0_{(R,L)}$, $H^n(\mathfrak{g},M)=Z^n_{(R,L)}/B^n_{(R,L)}$ the space of n-cohomology, this defines the cohomologies of $(\mathfrak{g},\mu,\alpha)$, with value in the module (M,R,L,A).

6. The trivial Representation

In this section we study the trivial representation $(\mathbb{R},0,0,id)$ of a multiplicative hom-right symmetric algebra $(\mathfrak{g},\mu,\alpha)$. As above, the cohomology associated to this representation is defined on cochains, elements of $\mathcal{C}^0(\mathfrak{g},\mathbb{R})=\mathbb{R}$, or $\mathcal{C}^n(\mathfrak{g},\mathbb{R})=\{C:\mathfrak{g}\otimes\wedge^{n-1}\mathfrak{g}\to\mathbb{R},\ \ C=C\circ\alpha\}$ $(n\geq 1)$.

For instance, $H^0(\mathfrak{g},\mathbb{R})=Z^0=\mathbb{R}$, and for any $C\in C^1(\mathfrak{g},\mathbb{R})$, C is closed if and only if $C(\mu(x_0,x_1))=0$, and there is no non trivial exact cochain. Thus

$$H^1(\mathfrak{g},\mathbb{R})=\{C\in C^1(\mathfrak{g},\mathbb{R}),\quad C(\mu(x_0,x_1))=0\}.$$

Let us now prove that the second cohomology group classifies the central extensions of $(\mathfrak{g},\mu,\alpha)$.

Let $(\mathfrak{g},\mu,\alpha)$ be a multiplicative hom-right symmetric algebra. A central extension of $(\mathfrak{g},\mu,\alpha)$ is a multiplicative hom-right symmetric algebra $(\mathfrak{g}^+,\mu^+,\alpha^+)$ with $\mathfrak{g}^+=\mathfrak{g}\oplus\mathbb{R}$, $\mu^+\big((x,s),(y,k)\big)=(\mu(x,y),\ \ C(x,y))$ and $\alpha^+(x,s)=(\alpha(x),s)$.

Proposition 6.1. *Let C be a 2-cochain. Put $\mathfrak{g}^+ = \mathfrak{g} \oplus \mathbb{R}$, $\alpha^+(x,s) = (\alpha(x), s)$ and $\mu^+\big((x,s),(y,k)\big) = (\mu(x,y), C(x,y))$. Then $(\mathfrak{g}^+, \mu^+, \alpha^+)$ is a central extension of $(\mathfrak{g}, \mu, \alpha)$ if and only if $d_2 C = 0$.*

It is a direct computation:

$$\mu^+\big(\mu^+((x,s),(y,r)), \alpha^+(z,t)\big) - \mu^+\big(\alpha^+((x,s)), \mu^+((y,r)),(z,t))\big) = \\ = (\mu(\mu(x,y),\alpha(z)) - \mu(\alpha(x),\mu(y,z)), C(\mu(x,y),\alpha(z)) - C(\alpha(x),\mu(y,z))).$$

Thus $(\mathfrak{g}^+, \mu^+, \alpha^+)$ is a hom-right-symmetric algebra if and only if

$$C(\mu(x,y),\alpha(z)) - C(\alpha(x),\mu(y,z))) = C(\mu(x,z),\alpha(y)) - C(\alpha(x),\mu(z,y))),$$

which exactly means that $d_2 C(x \otimes y.z) = 0$.

Two central extensions $(\mathfrak{g}_1^+, \mu_1^+, \alpha_1^+)$ and $(\mathfrak{g}_2^+, \mu_2^+, \alpha_2^+)$ are isomorphic if there is a hom-right symmetric algebras isomorphism $f^+ : \mathfrak{g}_1^+ \to \mathfrak{g}_2^+$, of the form $f^+(x,s) = (x, s + f(x))$, with $f : \mathfrak{g} \to \mathbb{R}$.

Proposition 6.2. *For C_1 , $C_2 \in Z^2$, the two coresponding central extensions $(\mathfrak{g}_1^+, \mu_1^+, \alpha_1^+)$ and $(\mathfrak{g}_2^+, \mu_2^+, \alpha_2^+)$ are isomorphic if and only if $C_1 - C_2$ is exact.*

The cochains C_1 and C_2 differs by an exact cochain if and only if there is $f : \mathfrak{g} \to \mathbb{R}$ such that

$$C_1(x \otimes y) - C_2(x \otimes y) = d_1 f(x \otimes y) = -f(\mu(x,y)).$$

This is equivalent to say that $f^+(x,s) = (x, s + f(x))$ is a hom-right-symmetric algebra isomorphism.

7. The α^0-adjoint representation

Let $(\mathfrak{g}, \mu, \alpha)$ by a multiplicative hom-right-symmetric algebra. Consider now the α^0-adjoint representation of $\mathfrak{g}$ on itself, defined by $R_0(x)y = \mu(y,x)$ and $L_0(x)y = \mu(x,y)$, $A(y) = \alpha(y)$.

Then the spaces of 0 and 1-cochains are by definition:

$$\mathcal{C}^0 = \{y \in \mathfrak{g}, \quad \alpha(y) = y, \quad \mu(y, \mu(x,z)) = \mu(\mu(y,x), \alpha(z)), \forall x, \ z \in \mathfrak{g}\},$$
$$\mathcal{C}^1 = \{f : \mathfrak{g} \to \mathfrak{g}, \quad \alpha \circ f = f \circ \alpha\}.$$

Each 0-cochain y defines an α-derivation D of $(\mathfrak{g}, \alpha, \mu)$, by $D(x) = [x,y]$. Such a derivation is called inner. Denote by $Inn_\alpha(\mathfrak{g})$ the space of inner derivations.

Proposition 7.1. *The first cohomology groups of the α^0-adjoint representation are*

$$H^0 = \{y \in \mathcal{C}^0, \quad [y,x] = 0, \forall x \in \mathfrak{g}\},$$

and

$$H^1 = Der_\alpha(\mathfrak{g})/Inn_\alpha(\mathfrak{g}).$$

Since $d_0 y(x) = [x,y] = 0$, the first point is clear.
For any 1-cochain f,

$$d_1 f(x,y) = \mu(\alpha(x),y) + \mu(x,\alpha(y)) - f(\mu(x,y)).$$

Therefore, the set of closed 1-cochain is exactly the space $Der_\alpha(\mathfrak{g})$. Furthermore, an exact 1-cochain has the form $f(x) = d_0 y(x)$ for some 0-cochain y. Thus $H^1 = Der_\alpha(\mathfrak{g})/Inn_\alpha(\mathfrak{g})$.

8. The α^{-1}-adjoint Representation

Let $(\mathfrak{g},\mu,\alpha)$ be a regular multiplicative hom-right-symmetric algebra. Consider now the α^{-1}-adjoint representation of $\mathfrak{g}$ on itself, with $A = \alpha$, $R_{-1}(x)y = \mu(y,\alpha^{-1}(x))$ and $L_{-1}(x)y = \mu(\alpha^{-1}(x),y)$.

Then the spaces of 0 and 1-cochains are as above:

$$\begin{aligned}
\mathcal{C}^0 &= \{y \in \mathfrak{g}, \quad \alpha(y) = y, \quad \mu(y,\alpha^{-1}(\mu(x,z))) = \mu(\mu(y,\alpha^{-1}(x)),z), \forall x, \ z \in \mathfrak{g}\} \\
&= \{y \in \mathfrak{g}, \quad \alpha(y) = y, \quad \mu(y,\mu(x,z)) = \mu(\mu(y,x),\alpha(z)), \forall x, \ z \in \mathfrak{g}\}, \\
\mathcal{C}^1 &= \{f : \mathfrak{g} \to \mathfrak{g}, \quad \alpha \circ f = f \circ \alpha\}.
\end{aligned}$$

Each 0-cochain y defines an α^0-derivation D of $(\mathfrak{g},\alpha,\mu)$, by $D(x) = [\alpha^{-1}(x),y]$. Such a derivation is called inner. Denote by $Inn_{\alpha^0}(\mathfrak{g})$ this space of inner derivations.

Proposition 8.1. *The first cohomology groups of the α^{-1}-adjoint representation are*

$$H^0 = \{y \in \mathcal{C}^0, \ [\alpha^{-1}(x),y] = 0, \ \forall x \in \mathfrak{g}\} = \{y \in \mathcal{C}^0, \ [x,y] = 0, \ \forall x \in \mathfrak{g}\}.$$

And

$$H^1 = Der_{\alpha^0}(\mathfrak{g})/Inn_{\alpha^0}(\mathfrak{g}).$$

The proof is the same as the one of Proposition (7.1), and will be omitted.

Recall that an infinitesimal deformation of the regular multiplicative hom-right-symmetric algebra $(\mathfrak{g},\mu,\alpha)$, is a one-parameter family of multiplications $t \mapsto \mu_t$ of the form

$$\mu_t(x,y) = \mu(x,y) + tC(x,y),$$

such that C is a 2-cochain and the hom-right-symmetric relation:

$$\mu_t(\alpha(x),\mu_t(y,z))-\mu_t(\mu_t(x,y),\alpha(z))-\mu_t(\alpha(x),\mu_t(z,y))+\mu_t(\mu_t(x,z),\alpha(y))=0$$

holds modulo terms in t^2. If the hom-right-symmetric relation holds for any t, the hom-right-symmetric algebra $(\mathfrak{g},\mu_t,\alpha)$ is called a true deformation of $(\mathfrak{g},\mu,\alpha)$.

Since C commutes with α, α is a morphism for each product μ_t. Now:

$$\begin{aligned}&\mu_t(\alpha(x),\mu_t(y,z))-\mu_t(\mu_t(x,y),\alpha(z))-\mu_t(\alpha(x),\mu_t(z,y))+\mu_t(\mu_t(x,z),\alpha(y))=\\&\quad=td_2C(x,y,z)+\\&\quad+t^2\left[C(\alpha(x),C(y,z))-C(C(x,y),\alpha(z))-C(\alpha(x),C(z,y))+C(C(x,z),\alpha(y))\right].\end{aligned}$$

Therefore, μ_t is an infinitesimal deformation of μ if and only if C is a closed cochain, and a true deformation if furthermore C itself is a hom-right symmetric algebra structure on $\mathfrak{g}$.

An infinitesimal deformation μ_t is trivial if there is a linear map $\theta\in C(\mathfrak{g})$ such that $\varphi_t(x)=x+t\theta(x)$ satisfies

$$\mu(\varphi_t(x),\varphi_t(y))=\varphi_t(\mu_t(x,y))$$

modulo terms in t^2. This condition is equivalent to $C=d_1\theta$. Thus, we say that $(\mathfrak{g},\mu,\alpha)$ is infinitesimally rigid if the second cohomology space of the α^{-1}-adjoint representation vanishes.

For any invertible $n\times n$ matrix A, define $\alpha:\mathfrak{gl}_n\to\mathfrak{gl}_n$ by $\alpha(x)=A^{-1}xA$. Let us now study the natural hom-right symmetric algebra $((\mathfrak{gl}_n)_\alpha,\mu_\alpha,\alpha)$ defined by the twisting of usual matrix multiplication through α.

Theorem 8.2. *The twisted hom-right symmetric algebra $((\mathfrak{gl}_n)_\alpha,\mu_\alpha,\alpha)$ is not rigid.*

The claim follows from the observation that α is invertible. Let C be a 2-cochain the coboundary d_2C is

$$\begin{aligned}d_2C(x,y,z)&=\mu_\alpha(\alpha(x),C(y,z))+\mu_\alpha(C(x,z),\alpha(y))-C(\mu_\alpha(x,y),\alpha(z))\\&+C(\alpha(x),[y,z]_\alpha)-\mu_\alpha(C(x,y),\alpha(z))-\mu_\alpha(\alpha(x),C(z,y))\\&+C(\mu_\alpha(x,z),\alpha(y))\\&=\alpha^2\big[\mu(x,\alpha^{-1}\circ C(y,z))+\mu(\alpha^{-1}\circ C(x,z),y)-\alpha^{-1}\circ C(\mu(x,y),z)\\&+\alpha^{-1}\circ C(x,[y,z])-\mu(\alpha^{-1}\circ C(x,y),z)-\mu(x,\alpha^{-1}\circ C(z,y))\\&+\alpha^{-1}\circ C(\mu(x,z),y)\big]\\&=\alpha^2(d_2'(\alpha^{-1}\circ C)(x,y,z)).\end{aligned}$$

therefore, the set of closed 2-cochains is exactly $Ker(d_2)=\alpha(Ker(d_2'))$.

Let C be a 1- cochain the cobondary

$$\begin{aligned} d_1C(x,y) =& \mu_\alpha(x, C(y)) + \mu_\alpha(C(x), y) - C(\mu_\alpha(x,y)) \\ =&\alpha\big[\mu(x, C(y)) + \mu(C(x), y) - C(\mu(x,y))\big] \\ =&\alpha(d'_1(C)(x,y)). \end{aligned}$$

Therefore, the set of exact 2-cochains is exactly $Im(d_1) = \alpha(Im(d'_1))$.

Let a be an arbitrary matrix which commute with A. Consider now the 2-cochain

$$f(x,y) = \mathrm{tr}(y).[x,a]$$

By direct computation, we obtain

$$\begin{aligned} d'2f(x,y,z) &= x.f(y,z) + f(x,z).y - f(x.y,z) + f(x,[y,z]) \\ &- f(x,y).z - x.f(z,y) + f(x.z,y) \\ &= \mathrm{tr}(z).x.[y,a] + \mathrm{tr}(z).[x,a].y - \mathrm{tr}(z).[x.y,a] + \mathrm{tr}([y,z]).[x,a] \\ &- \mathrm{tr}(y).[x,a].z - \mathrm{tr}(z).x.[z,a] + \mathrm{tr}(y).[x.z,a] \\ &= \mathrm{tr}(z).(x.[y,a] + [x,a].y - [x.y,a]) \\ &- \mathrm{tr}(y).(x.[z,a] + [x,a].z - [x.z,a]) \\ &= 0, \end{aligned}$$

thus f is a closed cochain.

If f were exact, there would be a linear map c such that:

$$f(x,y) = d'_1c(x,y) = xc(y) - c(xy) + c(x)y.$$

But, in this case, f is an usual Hochschild coboundary. Thus:

$$\begin{aligned} 0 = d'_H f(x,y,z) &= x.f(y,z) - f(x.y,z) + f(x,y.z) \\ &- f(x,y).z \\ &= [a,x].(tr(y)z - tr(yz)I_n + tr(z)y). \end{aligned}$$

But this is impossible: if $y = z = I_n$ and $a \neq I_n$, then $d'_H f(x, I_n, I_n) = n[a,x]$ is not nul. We conclude that the hom-right symmetric algebra $((\mathfrak{gl}_n)_\alpha, \mu_\alpha, \alpha)$ is not rigid.

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