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On Construction of the Maximal Parabolic Subgroup P_1 of $E_6(K)$ for Fields K of Characteristic Two

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Abstract. The notion of M-sets has been introduced by the second author to give an elementary construction for the Lie algebras of type E_6 and F_4 and the Chevalley groups $E_6(K)$, $F_4(K)$, and ${}^2E_6(K)$ over fields K of characteristic two. The aim of this article is to use the notion of M-sets to give an elementary and self-contained construction of the maximal parabolic subgroup P_1 of $E_6(K)$ using Levi components and unipotent radical root subgroups of $E_6(K)$.
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1. Introduction

The construction of the finite simple groups $E_6(q)$ and their triple covers (which exist whenever $q \equiv 1 \bmod 3$) goes back over 100 years to the work of Dickson [11]. It was only in the late 1980s, when the maximal subgroup problem came to prominence, that there was renewed interest in Dickson's work. Of particular note are Magaard's unpublished thesis [13] on maximal subgroups of $F_4(q)$ in characteristics at least 5, and the series of papers by Aschbacher [4], [5], [6] and [7] on maximal subgroups of $E_6(q)$. In these papers the 27-dimensional representation of the generic covers reveals much more structure than the 78-dimensional representation on the Lie algebra, and leads to strong restrictions on the shape of a maximal subgroup.

The analysis of maximal subgroups of exceptional groups has a history stretching back to the fundamental work of Dynkin [12], who determined the maximal connected subgroups of the simple algebraic groups G of exceptional type G_2, F_4, E_6, E_7 or E_8 over an algebraically closed field K of characteristic zero. For more information about groups of Lie type, see [9].

It is remarkable to mention that most of the available literature on Lie algebras and Chevalley groups does not deal with fields of characteristic two. Hence this study aims to contribute in this aspect.

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Remark 1.1. It is remarkable to mention that the construction of the embedding $F_{i22} \leq {}^2U_6(2)$ and the D-groups of 3-transpositions is mainly based on this work, [1] and [2]. This will be the subject of a separate paper.

Notation and earlier results

Let V be a 6-dimensional vector space over the Galois field $\mathbb{F}_2$, and Q be a non-degenerate quadratic form on V of minimal Witt index, i.e., $Q : V \to \mathbb{F}_2$, such that:

1. $Q(tx) = t^2 Q(x)$.
2. The map $(x|y) \to Q(x+y) - Q(x) - Q(y)$ is a non-degenerate bilinear form on V.
3. If $U < V$ and $Q(u) = 0$, $\forall u \in U$, then $\dim(U) \leq 2$, i.e., the totally singular subspaces of V are of dimension at most two. This type of quadratic forms is of "$-$" type.

As all quadratic spaces of "$-$" type are isomorphic, we may take as a representative $V = \mathbb{F}_4^3$ considered as a 6-dimensional vector space over $\mathbb{F}_2$, i.e.,

$$V = \{(x, y, z) \mid x, y, z \in \mathbb{F}_4\}$$

and define $Q(x, y, z) = x\overline{x} + y\overline{y} + z\overline{z} = x^3 + y^3 + z^3$, where $\overline{x} = x^2$, $\overline{y} = y^2$ and $\overline{z} = z^2$. For (V, Q), define the automorphism group

$$W = \{g \in GL(V) \mid Q(x^g) = Q(x), \ \forall x \in V\}.$$

By definition $W = G_{O_6^-}(2) = \Omega_6^-(2).2 = U_4(2).2$, a split extension following the Atlas notation [10]. The pair (V, Q) corresponds to a combinatorial geometric object $(\Omega, \mathscr{L})$, where

$$\Omega = \{0 \neq x \in V \mid Q(x) = 0\},$$
$$\mathscr{L} = \{L < V \mid \dim(L) = 2 \text{ and } Q(L) = 0\}.$$

The elements of Ω are called points, Ω is called a quadric, and the elements of $\mathscr{L}$ are called lines.

Observation 1.1 ([3])**.** $|\Omega| = 27$.

Proof. Let $x = (x_1, x_2, x_3) \in V = \mathbb{F}_4^3$, where V is a 6-dimensional vector space over $\mathbb{F}_2$, and Q is a quadratic form of "$-$" type on V as defined above. Then, $Q(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$, as $x_i \in \mathbb{F}_4$ for $i = 1, 2, 3$, this implies $x_i^3 = 1$ if $x_i \neq 0$ and $x_i^3 = 0$ if $x_i = 0$. Hence $Q(x) = 0$ if and only if exactly two components $x_i \neq 0$, i.e., $x = (a, b, 0), x = (a, 0, b)$, or $x = (0, a, b)$ for $a, b \neq 0$ and hence the claim. ∎

Corollary 1.2. *As* $|\Omega| = 27$*, then* $|\{s \in V \mid Q(s) \neq 0\}| = 36$.

Observation 1.2 ([3]). *Let V be a 6-dimensional vector space over $\mathbb{F}_2$, and $V=\mathbb{F}_4^3$, endowed with a quadratic form Q of "$-$" type. Then*

1. *Each point of Ω is contained in exactly 5 lines $L\in\mathscr{L}$.*
2. *$|\mathscr{L}|=45$.*

Proof. Let $v=(1,1,0)\in\Omega$ and $w\in\Omega$, $w\neq 0,v$ such that $\langle v,w\rangle\in\mathscr{L}$. Then, we consider the following cases:
Case 1. $w=(a,b,0)$. This implies $a,b\in\mathbb{F}_4\backslash\mathbb{F}_2$ and hence

$$\langle v,w\rangle=\{(1,1,0),(a,b,0),(a+1,b+1,0)\},$$

where a is unique up to addition by 1. This yields two possibilities and hence two lines.
Case 2. $w=(0,a,b)$ where $a,b\neq 0$. This implies

$$\{v,w,v+w\}=\{(1,1,0),(0,1,b),(1,0,b)\},$$

which yields to three possibilities and hence three lines.
Case 3. $w=(a,0,b)$. This case is covered by Case 2.

Hence v is contained in 5 lines. As the automorphism group $W=\mathrm{Aut}(\Omega,\mathscr{L})$ is transitive on Ω, this holds for all points in Ω. This proves 1.

Counting pairs $\{(P,L)\mid P\in L\in\mathscr{L}\}$ in two ways, one obtains $27\cdot 5=|\mathscr{L}|\cdot 3$. This implies $|\mathscr{L}|=45$. Hence the claim. ∎

Observation 1.3 ([3]). *For $(\Omega,\mathscr{L})$ the "quadrangle" property holds, that is, if L is a line, p a point, $p\notin L$, then there exists a unique point $q\in L$, such that p and q are collinear.*

Proof. Let $L\in\mathscr{L}$, $v\in\Omega$ and $v\notin L$. Then $L\nsubseteq v^{\perp}$, as otherwise $X=\langle L,v\rangle$ would be a totally singular subspace of dimension 3. This contradicts the fact that Q has a minimal Witt index. Hence $L\nsubseteq v^{\perp}$ and this implies $v^{\perp}\cap L$ is a 1-dimensional subspace, i.e., $v^{\perp}\cap L=\langle w\rangle$ and $\langle v,w\rangle\in\mathscr{L}$. ∎

Definition 1.3. For each $s\in V$ with $Q(s)=1$, we define an element of W, the reflection $\sigma_s: x\to(x,(x|s)s)$.

Observation 1.4 ([3]).

1. *The elements σ_s with $Q(s)=1$ generate W.*
2. *The pair $(\Omega,\mathscr{L})$ is called a generalized quadrangle and it is of type $O_6^-(2)$ and has the following combinatorial properties.*
   - *(a) Any two lines intersect in at most one point.*
   - *(b) $p\in\Omega$ is collinear with exactly 10 points, as p is on 5 lines and not collinear with 16 points.*
   - *(c) If $p,q\in\Omega$, $p\neq q$ are not collinear, i.e., $(p,q)=1$, then*

$$|\{x\in\Omega\mid x\text{ is not collinear with } p,q\}|=5.$$

Proof. It is an immediate consequence of the definition of the quadratic form Q, Observation 1.1, Observation 1.2, and Observation 1.3. ■

For a vector space A over a field $\mathbb{F}$, we denote by $\mathrm{End}(A)$ the Lie algebra consisting of all linear transformations of A with Lie product $[X,Y]=XY-YX$.

2. Lie algebras of type E_6

The Lie algebras of type $E_6(K)$ for fields K of characteristic two have been constructed in [1]. To have a self contained paper we present here a brief description of the construction.

Definition 2.1. A subset $\Delta \subset \Omega$ is called a E-set, if $(x|y)=1$ for all distinct $x,y\in\Delta$.

Proposition 2.2 ([1])**.**

1. *E-subsets of Ω have size at most six.*
2. *Ω contains exactly 72 E-subsets of size six. W permutes them transitively with stabilizer S_6.*

Definition 2.3. E-sets of size six are called M-sets.

Remark 2.4. If Δ is an M-set, then $s=\sum_{x\in\Delta} x$ is a non-singular vector. We say that s and σ_s correspond to Δ. Each non-singular vector, or each reflection, corresponds to exactly two M-sets. This is obvious, as W is transitive on M-sets and non-singular vectors. Also, if Δ corresponds to s and σ, then Δ^g corresponds to s^g and σ^g for all $g\in W$.

Proposition 2.5 ([1])**.** *Let Δ be an M-set with corresponding non-singular vector s and reflection $\sigma=\sigma_s$. Let $\tau\neq\sigma$ be a reflection.*

(1) The M-sets corresponding to s are Δ and Δ^σ;

$$\Delta\cup\Delta^\sigma=\{x\in\Omega \mid (x|s)=1\}.$$

(2) If $[\sigma,\tau]=1$, then $\Delta^\tau=\Delta$ and $\Delta^{\sigma\tau}=\Delta^\sigma$.

(3) If $\sigma\tau$ has order 3, then $\Delta\cap\Delta^\tau=\Delta\cap C_V(\tau)$ has size 3 and $\Delta^\sigma\cap\Delta^\tau=\emptyset$, $\Delta\cap\Delta^{\sigma\tau\sigma}=\Delta\cap C_V(\tau^\sigma)$ has size 3 and $\Delta^\sigma\cap\Delta^{\sigma\tau\sigma}=\emptyset$. Hence also $\Delta\cap\Delta^{\sigma\tau}=\Delta\cap\Delta^{\tau\sigma}=\emptyset$.

Corollary 2.6 ([1])**.** *Let Δ, Γ be two M-sets corresponding to two distinct reflections σ and τ. Then:*

1. *If $[\sigma,\tau]=1$, then $|\Delta\cap\Gamma|=|\Delta\cap\Gamma^\sigma|=1$.*
2. *If $[\sigma,\tau]\neq 1$, then $|\Delta\cap\Gamma|=3$ and $\Delta\cap\Gamma^\sigma=\emptyset$ or $|\Delta\cap\Gamma^\sigma|=3$ and $\Delta\cap\Gamma=\emptyset$.*

Let A be a 27-dimensional vector space over $\mathbb{F}_2$ with basis $\{e_x \mid x \in \Omega\}$. For $v \in V$, define $H_v \in \mathrm{End}(A)$ by

$$e_x^{H_v} = (x|v)e_x \qquad (x \in \Omega).$$

and for an M-set Δ with corresponding reflection σ, define $M(\Delta) \in \mathrm{End}(A)$ by

$$e_x^{M(\Delta)} = \begin{cases} e_{x^\sigma}, & x \in \Delta \\ 0, & \text{otherwise} \end{cases} \qquad (x \in \Omega).$$

Remark 2.7. The group W acts on A by $e_x^g = e_{x^g}$. Obviously, $H_v^g = H_{v^g}$ and $M(\Delta)^g = M(\Delta^g)$.

Remark 2.8. This approach can be generalized to arbitrary fields K. Here we define A as a 27-dimensional vector space over K with basis e_x, $x \in \Omega$. For pairs (x, Δ), $x \in \Delta$, Δ an M-set, we choose suitable structure constants $\epsilon_{x,\Delta} = \pm 1$ and define $e_x^{M(\Delta)} = \epsilon_{x,\Delta} e_{x^\sigma}$, if $x \in \Delta$, and $e_x^{M(\Delta)} = 0$, otherwise.

Proposition 2.9 ([1]). *Let Δ be an M-set with corresponding vector s and reflection $\sigma = \sigma_s$. In particular, $x + x^\sigma = s$ for all $x \in \Delta$.*

1. *$H = \{H_v | v \in V\}$ is an Abelian Lie algebra of dimension 6.*
2. *$[H_v, M(\Delta)] = (s|v)M(\Delta)$.*
3. *$[M(\Delta), M(\Delta^\sigma)] = H_s$.*

Let Δ_i be M-sets with corresponding reflections σ_i, $i = 1, 2$. Then

$$e_x^{M(\Delta_1)M(\Delta_2)} = \begin{cases} e_{x^{\sigma_1\sigma_2}}, & x \in \Delta_1 \cap \Delta_2^{\sigma_1} \\ 0, & \text{otherwise} \end{cases} \qquad (x \in \Omega). \tag{1}$$

From this, we get:

Proposition 2.10 ([1]). *Let Δ_i be M-sets with corresponding reflections σ_i, $i = 1, 2$, $\Delta_2 \neq \Delta_1, \Delta_1^{\sigma_1}$. In particular, $\sigma_1 \neq \sigma_2$.*

1. *If $[\sigma_1, \sigma_2] = 1$, then $[M(\Delta_1), M(\Delta_2)] = 0$.*
2. *If $[\sigma_1, \sigma_2] \neq 1$, and $\Delta_2^{\sigma_1} \neq \Delta_1^{\sigma_2}$ then $M(\Delta_1)M(\Delta_2) = [M(\Delta_1), M(\Delta_2)] = 0$.*
3. *If $[\sigma_1, \sigma_2] \neq 1$, and $\Delta_2^{\sigma_1} = \Delta_1^{\sigma_2}$ then $[M(\Delta_1), M(\Delta_2)] = M(\Delta_1^{\sigma_2})$.*

Proposition 2.11 ([1]). *Let Δ_i be M-sets with corresponding reflections σ_i, $i = 1, 2$. If $\sigma_1 \neq \sigma_2$, then:*

1. *$M(\Delta_1)M(\Delta_2)M(\Delta_1) = 0$;*
2. *$[M(\Delta_1), M(\Delta_2)] = 0$ or $[M(\Delta_1), M(\Delta_2^{\sigma_2})] = 0$.*

Theorem 2.12 ([1]). *The transformations H_v, $v \in V$, together with the transformations $M(\Delta)$, Δ an M-set in Ω, generate a Lie algebra $\mathbb{E}$ of dimension 78. $\mathbb{E}$ is of type E_6. H is a Cartan subalgebra of $\mathbb{E}$, the transformations $M(\Delta)$ form a set of 72 roots.*

The relations proved in Propositions 2.9, 2.10 and 2.11, show that $\mathbb{E}$ is in fact a Lie algebra of type E_6 with Weyl group W.

3. The Chevalley group of type $E_6(K)$

Let K be a field of characteristic 2 and A_K be a vector space with basis $\{e_x \mid x \in \Omega\}$, so that $A \leq A_K$, and A_K is obtained from A by extending scalars to K.

The transformations $M(\Delta)$ and H_v defined above induce K-linear transformations of A_K.

Remark 3.1. In the following, brackets $[\ ,\ ]$ are used for commutators of group elements and for Lie multiplication. The respective meaning is clear from the context.

Definition 3.2. Let I be the identity on A_K. For $k \in K$ and an M-set Δ the corresponding root element $r_\Delta(k)$ is defined as

$$r_\Delta(k) = I + k\,M(\Delta).$$

For an M-set Δ the corresponding root subgroup U_Δ is defined as

$$U_\Delta = \{r_\Delta(k) \mid k \in K\} \cong K.$$

The group generated by all root subgroups is denoted by $E(K)$.

Proposition 3.3 ([2]). *Let Δ an M-set with corresponding reflection σ. Then*

$$\langle U_\Delta, U_{\Delta^\sigma}\rangle \cong SL_2(K).$$

Proposition 3.4 ([2]). *Let Δ_i two M-sets with corresponding reflections σ_i such that $\sigma_1 \neq \sigma_2$. For $k_i \in K$, we have*

$$[r_{\Delta_1}(k_1), r_{\Delta_2}(k_2)] = \begin{cases} r_{\Delta_3}(k_1k_2), & \text{if } {\Delta_1}^{\sigma_2} = {\Delta_2}^{\sigma_1} = \Delta_3, \\ 1, & \text{otherwise.} \end{cases}$$

Lemma 3.5 ([2]). *The group $E(K)$ modulo its center is isomorphic to the Chevalley group $E_6(K)$.*

4. Quadratic forms

Remark 4.1. The 27-dimensional vector space A_K over K with basis $\{e_x \mid x \in \Omega\}$ can be turned into a commutative, non-associative algebra. For $x, y \in \Omega$, set

$$e_xe_y = \begin{cases} e_{x+y}, & x \neq y \text{ and } (x|y) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

and extend the multiplication of the basis elements linearly to A_K. Thus $ab = \sum_{x,y\in\Omega} a_x b_y e_x e_y$, where $a = \sum a_x e_x$ and $b = \sum b_y e_y$.

Proposition 4.2. *Let Δ be an M-set with corresponding vector s and reflection $\sigma = \sigma_s$. Set $M = M(\Delta)$. In particular, $M^t = M(\Delta^\sigma)$. For $a, b \in A_K$ we have $ab^M + a^M b = (ab)^{M^t}$.*

Proof. It suffices to prove this relation for base elements $a = e_x$, $b = e_y$ where $x, y \in \Omega$.

Suppose $(e_x e_y)^{M^t} \neq 0$. Then $L = \{x, y, x+y\}$ is a singular line and $x + y \in \Delta^\sigma$. In particular, $(s|x+y) = 1$. We may assume wrong, that $(s|x) = 1$ and $(s|y) = 0$. Hence $e_y^M = 0$ and $x \in \Delta \cup \Delta^\sigma$. As Δ^σ is an M-set containing $x + y \neq x$, we find $x \in \Delta$. Hence, $e_x^M e_y + e_x e_y^M = e_x^M e_y = e_{x+s} e_y = e_{x+y+s} = (e_x e_y)^{M^t}$. Suppose $(e_x e_y)^{M^t} = 0$ and $x^M e_y \neq 0$. In particular, $L = \{x^\sigma, y, x^\sigma + y\}$ is singular and $x \in \Delta$.

If $(s|y) = 0$, then $L^\sigma = \{x, y, x+y\}$ and $x+y \in \Delta \cup \Delta^\sigma$. As $x+y \neq x \in \Delta$ and $(x|x+y) = 0$, this implies $x + y \in \Delta^\sigma$ and $(e_x e_y)^{M^t} \neq 0$, a contradiction.

If $(s|y) = 1$, then $y \in \Delta \cup \Delta^\sigma$. As $x^\sigma \in \Delta^\sigma$ and $(y|x^\sigma) = 0$ it follows $y \in \Delta$. Hence, $e_x^M e_y = e_x e_y^M = e_{x+y+s}$ and $e_x^M e_y + e_x e_y^M = (e_x e_y)^{M^t}$. This completes the proof. ∎

Definition 4.3. For $x \in \Omega$, let Q_x be the uniquely determined quadratic form defined on A_K, such that, for $y, z \in \Omega$, $Q_x(e_y) = 0$ and

$$Q_x(e_y + e_z) = \begin{cases} 1, & \{x, y, z\} \text{ is a singular line,} \\ 0, & \text{otherwise.} \end{cases}$$

and let $\hat{Q}$ the quadratic map from A_K to A_K, defined as $\hat{Q}(a) = \sum_{x\in\Omega} Q_x(a) e_x$.

Proposition 4.4. *Let Δ an M-set with corresponding reflection σ. Set $M = M(\Delta)$, so that $M^t = M(\Delta^\sigma)$. For $a, b \in A_K$ we have:*

1. $\hat{Q}(e_x) = 0$, $\forall x \in \Omega$.
2. $\hat{Q}(a + b) = ab + \hat{Q}(a) + \hat{Q}(b)$.
3. $\hat{Q}(a^M) = 0$.
4. $aa^M = \hat{Q}(a)^{M^t}$.

Proof. (1) and (2) follow immediately from the definition. (2) implies (3), as $\hat{Q}(e_x) = e_x e_y = 0$ for all $x, y \in \Delta^\sigma$, and a^M is a linear combination of such base elements.

Let $x \in \Omega$. Then $e_x e_x^M = 0$, as $e_x^M = 0$ or $(x|x^\sigma) = 1$, if $x \in \Delta$. Let $a = e_x + b$. Then by Proposition 4.2, $aa^M = e_x e_x^M + e_x b^M + b e_x^M + bb^M = e_x b^M + b e_x^M + bb^M = (e_x b)^{M^t} + bb^M$ and $\hat{Q}(a)^{M^t} = (\hat{Q}(e_x) + e_x b + \hat{Q}(b))^{M^t} = (e_x b + \hat{Q}(b))^{M^t}$. Using induction on the weight of a, the number of non-zero coefficients, we get (4). ∎

Definition 4.5. For elements $g \in E(K)$ denote by $g^\star$ the transposed inverse of g with respect to the basis $\{e_x \,|\, x \in \Omega\}$.

Proposition 4.6. *Let Δ an M-set with corresponding reflection σ, and $k \in K$. Then $r_\Delta(k)^\star = r_{\Delta^\sigma}(k)$.*

Proof. Root elements are self-inverse and the transposed of $M(\Delta)$ is $M(\Delta^\sigma)$. ■

Proposition 4.7. *Let $g \in E(K)$ and $a, b \in A_K$. Then $a^g b^g = (ab)^{g^\star}$ and $\hat{Q}(a^g) = \hat{Q}(a)^{g^\star}$.*

Proof. We may assume that $g = r_\Delta(k) = I + kM(\Delta)$ for an M-set Δ and $0 \neq k \in K$, as $E(K)$ is generated by root elements. Using linearity, we may also assume that $k = 1$ and $a, b \in A_K$. Set $M = M(\Delta)$. Proposition 4.2 shows

$$\hat{Q}(a^g) = \hat{Q}(a + a^M) = \hat{Q}(a) + aa^M + \hat{Q}(a^M) = \hat{Q}(a) + \hat{Q}(a)^{M^t} = \hat{Q}(a)^{g^\star}.$$

Hence the claim, as $ab = \hat{Q}(a+b) - \hat{Q}(a) - \hat{Q}(b)$. ■

Remark 4.8. Let

$$V_i = \{U \leq A \mid \dim U = i \text{ and } \hat{Q}(U) = 0\},$$
$$E(K) = \langle r_\Delta(k) \mid \Delta \text{ is an } M\text{-set},\ k \in K\rangle,$$

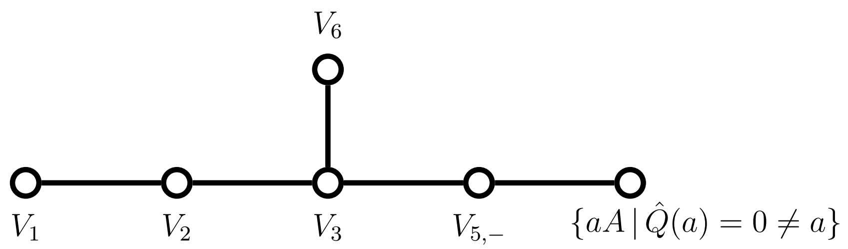
then:

1. $V_i \neq \emptyset$ if $i \leq 6$;
2. $E(K)$ is transitive on V_1, V_2, V_3, V_4, V_6 and has 2 orbits on V_5, namely $V_{5,+}$, $V_{5,-}$, where

$$V_{5,+} = \{U \leq V_5 \mid U \text{ is contained in } V \in V_6\}, \qquad \text{and}$$
$$V_{5,-} = \{U \leq V_5 \mid U \text{ is not contained in } V \in V_6\}.$$

Proof. See [8]. ■

Remark 4.9. The elements of the Tits building of type E_6 are the elements of V_1, V_2, V_3, V_6, $V_{5,-}$ and subspaces of the form aA for $0 \neq a \in A$ with $\hat{Q}(a) = 0$ and diagram



The maximal parabolic subgroups of type E_6 are the stabilizers of the subspaces given in the diagram. The group $\mathbb{E} = E(K)$ acts on the spaces V_i as follows. Let $U \in V_i$, $g \in \mathbb{E}$, then $\dim U^g = i$, and by Proposition 4.6, one has $\hat{Q}(u^g) = \hat{Q}(u)^{g^\star} = 0^{g^\star} = 0$ for all $u \in U$.

Remark 4.10. The maximal parabolic subgroups of type E_6 are all semidirect products $H = X \rtimes \overline{L}$, where $X = O_2(H)$ is the largest normal 2-subgroup in H, called the unipotent radical, and complement $\overline{L}$, usually called Levi complement. If $H = N_{\mathbb{E}}(U)$, $U \in V_i$, then H can be constructed as follows. We determine the sets:

- $\chi = \{\Delta \mid r_\Delta(k) \in H,\ \Delta \text{ is an } M\text{-set},\ k \in K\}$,
- $\chi_0 = \{\Delta \in \chi \mid \Delta^\star \in \chi\}$,
- $\chi_1 = \{\Delta \in \chi \mid \Delta^\star \notin \chi\}$.

Then $R = \langle r_\Delta(k) \mid \Delta \in \chi_1,\ k \in K\rangle = O_2(H)$, $\overline{L} = \langle r_\Delta(k) \mid \Delta \in \chi_0,\ k \in K\rangle$.

5. The maximal parabolic subgroup P_1

Let $p \in \Omega$, $U = \langle e_p\rangle \in V_1$, then the maximal parabolic subgroup P_1 is the stabilizer $H = N_{\mathbb{E}}(U)$. From the above remark, we find $\chi_1 = \{\Delta \mid p \in \Delta^\star\}$, $\chi_0 = \{\Delta \mid p \notin \Delta \cup \Delta^\star\}$. So we prove the following:

Proposition 5.1. *Let Δ be an M-set with corresponding non-singular vector $s = s_\Delta$ and reflection $\sigma = \sigma_s = \sigma_\Delta$. If $x \in \Delta$ and $y \in \Delta^\star$, then $(x|y) = 0$ or $x + y = s$ and $(x|y) = 1$.*

Proof. As $\Delta^\star = \Delta^\sigma = \Delta + s$, then $y = x' + s$ for some $x' \in \Delta$ and $(x|y) = (x|x') + 1$. Hence $(x|y) = 1$ if and only if $x = x'$ by the properties of M-sets. ■

The following lemma is needed.

Lemma 5.2.

1. $|\chi_1| = 16$.
2. $|\chi_0| = 40$.
3. *If $\Delta \in \chi_1$ with corresponding non-singular vector $s = s_\Delta$ and reflection $\sigma = \sigma_s = \sigma_\Delta$, then there is a unique $x \in \Delta$ such that $s = p + x$, $(p|x) = 1$ and $p^\sigma = x$.*

Proof. Let $y \in \Omega$, then $|\{\Delta \mid y \in \Delta\}| = \frac{72 \cdot 6}{27} = 16$. Hence

$$|\chi_1| = |\{\Delta \mid p \in \Delta^\star\}| = 16.$$

This proves (1). As $\chi_0 = \{\Delta \mid p \notin \Delta \cup \Delta^\star\} = \{\Delta \mid (p|s) = 0\}$, this implies that $|\{s \mid Q(s) = 1 \text{ and } (p|s) = 0\}| = \frac{36\cdot 15}{27} = 20$, hence $|\chi_0| = 2 \cdot 20 = 40$. This proves (2).

It is clear that $p^\sigma = x$, to prove the uniqueness. Assume $y \neq x$ and $(p|y) = 1$. As $p \in \Delta^\star$, then $p + s = x \neq y$ by Proposition 5.1, and hence $(x|y) = 0$. This implies that $y \notin \Delta$. This proves (3). ■

Theorem 5.3. *Let $R = \langle r_\Delta(k) \mid \Delta \in \chi_1,\ k \in K\rangle$ be the unipotent radical. Then $R \cong (K^{16}, +)$.*

Proof. Let $A = \langle e_p\rangle \oplus A_1 \oplus A_0$ where

$$A_1 = \langle e_x \mid p \neq x,\ (p|x) = 1\rangle,$$
$$A_0 = \langle e_x \mid p \neq x,\ (p|x) = 0\rangle,$$

$\dim A_1 = 16$, and $\dim A_0 = 10$. If Δ and Γ are two M-sets in χ_1, then $p \in \Delta^\star \cap \Gamma^\star$ and $[r_{\Delta^\star}(k), r_{\Gamma^\star}(m)] = 1$ for all $k, m \in K$, by Proposition 3.4. It follows that R is Abelian of exponent 2. By Lemma 5.2 (3), R can be written as $R = \langle r_\Delta(k) \mid p + s_\Delta \in \Delta \text{ and } p \in \Delta^\star\rangle$ and hence $e_p^{r_\Delta(k)} = e_p$ as $p \notin \Delta$ and $e_{p+s_\Delta}^{r_\Delta(k)} = e_{p+s_\Delta} + k e_{p+s_\Delta}\sigma_\Delta = e_{p+s_\Delta} + k e_p$. If $e_y \in A_1$, $y \neq p + s_\Delta$, then $y \notin \Delta$ by Lemma 5.2 (3), and $e_y^{r_\Delta(k)} = e_y$.

From this argument it follows that R induces the full group of transvections on $\hat{U} = \langle e_p\rangle \oplus A_1$ of dimension 17 with center $\langle e_p\rangle$ and hence $R \cong (K^{16}, +)$. ■

To construct the Levi component $\overline{L}$, we need the following

Remark 5.4. Let $p \in \Omega$ and $P_0 = \{x \in \Omega \mid x \neq p,\ (x|p) = 0\}$, $|P_0| = 10$. Then p lies on 5 lines $\{p, x_i, y_i\}$, $i = 1, 2, \ldots, 5$. Let

$$A_0 = \langle e_{x_1}, e_{y_1}\rangle \oplus \langle e_{x_2}, e_{y_2}\rangle \oplus \cdots \oplus \langle e_{x_5}, e_{y_5}\rangle.$$

For $a = \sum_{i=1}^5 k_i e_{x_i} + \sum_{i=1}^5 m_i e_{y_i}$ with $k_i,\ m_i \in K$, if $\hat{Q}|_{A_0}$ denotes the restriction on A_0, it follows $\hat{Q}(a) = \sum (k_i m_i) e_p$. Hence $(A_0, \hat{Q}|_{A_0})$ is a non-singular orthogonal space of dimension 10 of "+" type, i.e., there exist subspaces $\overline{W}$ of A_0 of dimension 5 with $\hat{Q}(\overline{W}) = 0$. The orthogonal group of $(A_0, \hat{Q}|_{A_0})$ contains root elements or Siegel transformations, i.e., automorphisms γ such that

$$C[A_0, \gamma] = \langle a + a^\gamma \mid a \in A_0\rangle$$

is a totally singular subspace of dimension 2. We call the root elements belonging to the subspaces $\langle e_{x_i}, e_{x_j}\rangle$, $\langle e_{y_i}, e_{y_j}\rangle$, $\langle e_{x_i}, e_{y_j}\rangle$ for $i \neq j$ the canonical root elements (canonical Chevalley generators). These canonical root elements generate a subgroup isomorphic to $\Omega_{10}^+(K)$, see [9].

Now we prove the following:

Theorem 5.5. *Let $A = \langle e_p\rangle \oplus A_0 \oplus A_1$ where A_0 and A_1 are defined above. Then the Levi component $\overline{L}$ induces $\Omega(A_0, \hat{Q}|_{A_0}) = \Omega_{10}^+(K)$.*

Proof. Consider $\overline{L} = \{r_\Delta(k) \mid \Delta \in \chi_0,\ k \in K\}$, then $\overline{L} = \{r_\Delta(k) \mid (p|s_\Delta) = 0\}$. Let Δ be an M-set with corresponding non-singular vector s_Δ and reflection σ_Δ. If $(p|s_\Delta) = 0$, then it follows that $g \in \langle r_\Delta(k) \mid (p|s_\Delta) = 0\rangle$ leaves A_1 and A_0 invariant as $(p|x^{\sigma_\Delta}) = (p|x)$, $\forall x \in \Omega$.

The 5 lines through p may be seen as $\{p, x_i, y_i\}$, $i = 1, 2, \ldots, 5$. If $a \in A_0$, then $a = \sum_{i=1}^5 k_i e_{x_i} + \sum_{i=1}^5 m_i e_{y_i}$ where $k_i, m_i \in K$, and $\hat{Q}(a) = \sum_{i=1}^5 k_i m_i e_p = Q_p(a) e_p$. Hence $(A_0, \hat{Q}|_{A_0})$ is an orthogonal space of "+" type as

$$A_0 = \langle e_{x_1}, e_{x_1+p}\rangle \oplus \langle e_{x_2}, e_{x_2+p}\rangle \oplus \cdots \oplus \langle e_{x_5}, e_{x_5+p}\rangle.$$

This means that $\hat{Q}$ induces a non-degenerate quadratic form on A_0 and $r_\Delta(k)|_{A_0}$ preserves $\hat{Q}$, as $\hat{Q}(a^r) = \hat{Q}(a)^{r^\star} = (Q_p(a)e_p)^{r^\star} = Q_p(a)e_p^{r^\star} = Q_p(a)e_p$, where r denotes $r_\Delta(k) \in \overline{L}$. Furthermore $r|_{A_0}$ preserves $\hat{Q}|_{A_0}$ and induces a Siegel transformation on A_0 as $(p|s_\Delta) = 0$ implies that $s_\Delta = x+y = x+p+y+p$ for x, y with $(x|p) = (y|p) = 0$ and $(x|y) = 1$. This implies $\{x, y+p\} \subset \Delta$ and $\{y, x+p\} \subset \Delta^\star$ and for all $e_z \in A_0 \backslash \{e_x, e_y, e_{x+p}, e_{y+p}\}$, $(z|s_\Delta) = 0$ and hence $e_x^r = e_x + k e_y$ and $e_{x+p}^r = e_{y+p} + k e_{x+p}$. So $r_\Delta(k)$ induces a Siegel transformation (see Remark 5.4) and all Siegel transformations for the hyperplane basis $\{e_{x_i}, e_{x_i+p} \mid i = 1, 2, \ldots, 5\}$ can be thus obtained. This implies $\overline{L} = \Omega_{10}^+(K)$, which acts faithfully on A_0. Hence $\langle R, \overline{L}\rangle$ is a semidirect product $K^{16} \rtimes \Omega_{10}^+(K)$. ■

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