Three-Term Recurrence Relations
of Minimal Affinizations of Type $\mathbf{G_2}$

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Abstract.   Minimal affinizations introduced by Chari form a class of modules of quantum affine algebras. In this paper, we introduce a system of equations satisfied by the q-characters of minimal affinizations of type G_2 which we call the M-system of type G_2. The M-system of type G_2 contains all minimal affinizations of type G_2 and only contains minimal affinizations. The equations in the M-system of type G_2 are three-term recurrence relations. The M-system of type G_2 is much simpler than the extended T-system of type G_2 obtained by Mukhin and the second author. We also interpret the three-term recurrence relations in the M-system of type G_2 as exchange relations in a cluster algebra constructed by Hernandez and Leclerc.
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1.   Introduction

Let $\mathfrak{g}$ be a simple Lie algebra and $U_q\widehat{\mathfrak{g}}$ the corresponding quantum affine algebra. Minimal affinizations are simple modules of $U_q\widehat{\mathfrak{g}}$ which were introduced by Chari in [1]. The family of minimal affinizations contains the celebrated Kirillov-Reshetikhin modules. Minimal affinizations are studied intensively in recent years, see for example, [3], [2], [16], [20], [21], [22], [23], [24], [25], [26], [27], [28].

The aim of this paper is to study three-term recurrence relations satisfied by the q-characters of minimal affinizations of type G_2. The set of minimal affinizations of type G_2 can be divided into two sets X_1, X_2 according to their highest l-weights. The minimal affinizations in X_1 have highest l-weight monomials of

the form (see Section 2)

$$T_{k,l}^{(s)} = \left(\prod_{i=0}^{k-1} 1_{s+6i}\right)\left(\prod_{j=0}^{l-1} 2_{s+6k+2j+1}\right)$$

and the minimal affinizations in X_2 have highest l-weight monomials of the form

$$\widetilde{T}_{k,l}^{(s)} = \left(\prod_{i=0}^{l-1} 2_{-s-6k-2i-1}\right)\left(\prod_{j=0}^{k-1} 1_{-s-6j}\right).$$

We introduce a system of equations which we call the M-system of type G_2 and prove that the equations in the M-system of type G_2 are satisfied by the q-characters of minimal affinizations.

The equations in the first part of the M-system of type G_2 are three-term recurrence relations:

$$[\mathcal{T}_{k,l}^{(s)}][\mathcal{T}_{k,0}^{(s+6)}] = [\mathcal{T}_{k+1,0}^{(s)}][\mathcal{T}_{k-1,l}^{(s+6)}] + [\mathcal{T}_{0,3k+l}^{(s)}] \quad (k \in \mathbb{Z}_{\geq 1}, l \in \{1,2,3\}), \quad (1.1)$$

$$[\mathcal{T}_{k,l+3}^{(s)}][\mathcal{T}_{k,l}^{(s+6)}] = [\mathcal{T}_{k+1,l}^{(s)}][\mathcal{T}_{k-1,l+3}^{(s+6)}] + [\mathcal{T}_{0,l}^{(s+6k+6)}][\mathcal{T}_{0,3k+l+3}^{(s)}] \quad (k,l \in \mathbb{Z}_{\geq 1}),(1.2)$$

see Theorem 3.1. They are satisfied by the q-characters of the minimal affinizations in X_1. Here we use $\mathcal{T}$ to denote a module with highest l-weight T.

The equations in the second part of the M-system of type G_2 are three-term recurrence relations:

$$[\widetilde{\mathcal{T}}_{k,l}^{(s)}][\widetilde{\mathcal{T}}_{k,0}^{(s+6)}] = [\widetilde{\mathcal{T}}_{k+1,0}^{(s)}][\widetilde{\mathcal{T}}_{k-1,l}^{(s+6)}] + [\widetilde{\mathcal{T}}_{0,3k+l}^{(s)}] \quad (k \in \mathbb{Z}_{\geq 1}, l \in \{1,2,3\}),$$

$$[\widetilde{\mathcal{T}}_{k,l+3}^{(s)}][\widetilde{\mathcal{T}}_{k,l}^{(s+6)}] = [\widetilde{\mathcal{T}}_{k+1,l}^{(s)}][\widetilde{\mathcal{T}}_{k-1,l+3}^{(s+6)}] + [\widetilde{\mathcal{T}}_{0,l}^{(s+6k+6)}][\widetilde{\mathcal{T}}_{0,3k+l+3}^{(s)}] \quad (k,l \in \mathbb{Z}_{\geq 1}),$$

see Theorem 3.2. They are satisfied by the q-characters of the minimal affinizations in X_2.

The extended T-system of type G_2 obtained by Mukhin and the second author in [20] contains all minimal affinizations of type G_2 and some other modules which are not minimal affinizations. The M-system of type G_2 also contains all minimal affinizations of type G_2. But unlike the extended T-system of type G_2, the M-system of type G_2 contains only minimal affinizations of type G_2. The M-system of type G_2 is much simpler than the extended T-system of type G_2.

The equations the M-system of type G_2 can be interpreted as exchange relations in a certain cluster algebra $\mathscr{A}$ constructed by Hernandez and Leclerc in [19], see Section 4. In the paper [19], the equations in the usual T-systems are interpreted as exchange relations in some cluster algebras. The T-system of type G_2 and the M-system of type G_2 are special cases of exchange relations in the cluster algebra $\mathscr{A}$.

We also used the M-system of type G_2 to compute the decomposition of a minimal affinization of type G_2 as a $U_q\mathfrak{g}$-module into simple $U_q\mathfrak{g}$-modules. This helps us to obtain the general decomposition formula in [21].

We show that the modules associated to the summands on the right hand side of each equation in the M-system are simple.

The paper is organized as follows. In Section 2, we give some background information about finite-dimensional representations of quantum affine algebras and cluster algebras. In Section 3, we describe the M-system of type G_2. In Section 4, we interpret the equations in the M-system of type G_2 as exchange relations. In Section 5 and 6 we prove Theorem 3.1. In Section 7, we prove Theorem 3.2.

2. Background

2.1. The quantum affine algebra of type G_2.

In this paper, we take $\mathfrak{g}$ to be the complex simple Lie algebra of type G_2 and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let $I = \{1, 2\}$. We choose simple roots α_1, α_2 and scalar product $(\cdot, \cdot)$ such that

$$(\alpha_1, \alpha_1) = 6, \ (\alpha_1, \alpha_2) = -3, \ (\alpha_2, \alpha_2) = 2.$$

Therefore α_1 is the long simple root and α_2 is the short simple root. Let $\{\alpha_1^\vee, \alpha_2^\vee\}$ and $\{\omega_1, \omega_2\}$ be the sets of simple coroots and fundamental weights respectively. Let $C = (C_{ij})_{i,j\in I}$ denote the Cartan matrix, where $C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. Let $r_1 = 3, r_2 = 1$, $D = \mathrm{diag}(r_1, r_2)$ and $B = DC$. Then

$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}.$$

Let Q (resp. Q^+) and P (resp. P^+) denote the $\mathbb{Z}$-span (resp. $\mathbb{Z}_{\geq 0}$-span) of the simple roots and fundamental weights respectively. Let $\leq$ be the partial order on P in which $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda \in Q^+$.

Let $\widehat{\mathfrak{g}}$ denote the untwisted affine algebra corresponding to $\mathfrak{g}$. Fix a $q \in \mathbb{C}^\times$, not a root of unity. Let $q_i = q^{r_i}, i = 1, 2$. Let $\mathcal{P}$ the free abelian multiplicative group of monomials in infinitely many formal variables $(Y_{i,a})_{i\in I, a\in\mathbb{C}^\times}$.

The quantum affine algebra $U_q\widehat{\mathfrak{g}}$ in Drinfeld's new realization, see [9], is generated by $x_{i,n}^\pm$ $(i \in I, n \in \mathbb{Z})$, $k_i^{\pm 1}$ $(i \in I)$, $h_{i,n}$ $(i \in I, n \in \mathbb{Z}\backslash\{0\})$ and central elements $c^{\pm 1/2}$, subject to certain relations.

The quantum affine algebra $U_q\widehat{\mathfrak{g}}$ contains two standard quantum affine algebras of type A_1. The first one is $U_{q_1}\widehat{\mathfrak{sl}}_2$ generated by $x_{1,n}^\pm$ $(n \in \mathbb{Z})$, $k_1^{\pm 1}$, $h_{1,n}$ $(n \in \mathbb{Z}\backslash\{0\})$ and central elements $c^{\pm 1/2}$. The second one is $U_{q_2}\widehat{\mathfrak{sl}}_2$ generated by $x_{2,n}^\pm$ $(n \in \mathbb{Z})$, $k_2^{\pm 1}$, $h_{2,n}$ $(n \in \mathbb{Z}\backslash\{0\})$ and central elements $c^{\pm 1/2}$.

The subalgebra of $U_q\widehat{\mathfrak{g}}$ generated by $(k_i^\pm)_{i\in I}, (x_{i,0}^\pm)_{i\in I}$ is a Hopf subalgebra of $U_q\widehat{\mathfrak{g}}$ and is isomorphic as a Hopf algebra to $U_q\mathfrak{g}$. Therefore $U_q\widehat{\mathfrak{g}}$-modules restrict to $U_q\mathfrak{g}$-modules.

2.2. Finite-dimensional representations of $U_q\widehat{\mathfrak{g}}$ and q-characters.

In this section, we recall the standard facts about finite-dimensional $U_q\widehat{\mathfrak{g}}$-modules and q-characters of these representations, see [5], [6], [11], [24].

A representation V of $U_q\widehat{\mathfrak{g}}$ is of type 1 if $c^{\pm 1/2}$ acts as the identity on V and

$$V = \bigoplus_{\lambda \in P} V_\lambda, \ V_\lambda = \{v \in V : k_i v = q^{(\alpha_i, \lambda)} v\}. \tag{2.1}$$

In the following, all representations will be assumed to be finite-dimensional and of type 1 without further comment. The decomposition (2.1) of a finite-dimensional representation V into its $U_q\mathfrak{g}$-weight spaces can be refined by decomposing it into the Jordan subspaces of the mutually commuting operators $\phi_{i,\pm r}^{\pm}$, see [11]:

$$V = \bigoplus_{\gamma} V_{\gamma}, \ \gamma = (\gamma_{i,\pm r}^{\pm})_{i \in I, r \in \mathbb{Z}_{\geq 0}}, \ \gamma_{i,\pm r}^{\pm} \in \mathbb{C}, \tag{2.2}$$

where

$$V_{\gamma} = \{v \in V : \exists k \in \mathbb{N}, \forall i \in I, m \geq 0, (\phi_{i,\pm m}^{\pm} - \gamma_{i,\pm m}^{\pm})^k v = 0\}.$$

Here $\phi_{i,n}^{\pm}$'s are determined by the formula

$$\phi_i^{\pm}(u) := \sum_{n=0}^{\infty} \phi_{i,\pm n}^{\pm} u^{\pm n} = k_i^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m} u^{\pm m}\right). \tag{2.3}$$

If $\dim(V_{\gamma}) > 0$, then γ is called an l-weight of V. Let γ be the l-weight of a finite dimensional $U_q\widehat{\mathfrak{g}}$-module. In [11], it is shown γ satisfies

$$\gamma_i^{\pm}(u) := \sum_{r=0}^{\infty} \gamma_{i,\pm r}^{\pm} u^{\pm r} = q_i^{\deg Q_i - \deg R_i} \frac{Q_i(uq_i^{-1})R_i(uq_i)}{Q_i(uq_i)R_i(uq_i^{-1})}, \tag{2.4}$$

where the right hand side is to be treated as a formal series in positive (resp. negative) integer powers of u, and Q_i, R_i are polynomials of the form

$$Q_i(u) = \prod_{a \in \mathbb{C}^{\times}} (1 - ua)^{w_{i,a}}, \ R_i(u) = \prod_{a \in \mathbb{C}^{\times}} (1 - ua)^{x_{i,a}}, \tag{2.5}$$

for some $w_{i,a}, x_{i,a} \in \mathbb{Z}_{\geq 0}, i \in I, a \in \mathbb{C}^{\times}$. Let $\mathcal{P}$ denote the free abelian multiplicative group of monomials in infinitely many formal variables $(Y_{i,a})_{i \in I, a \in \mathbb{C}^{\times}}$. There is a bijection γ from $\mathcal{P}$ to the set of l-weights of finite-dimensional modules such that for the monomial $m = \prod_{i \in I, a \in \mathbb{C}^{\times}} Y_{i,a}^{w_{i,a} - x_{i,a}}$, the l-weight $\gamma(m)$ is given by (2.4), (2.5).

Let $\mathbb{Z}\mathcal{P} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^{\times}}$ be the group ring of $\mathcal{P}$. For $\chi \in \mathbb{Z}\mathcal{P}$, we write $m \in \mathcal{P}$ if the coefficient of m in χ is non-zero.

The q-character of a $U_q\widehat{\mathfrak{g}}$-module V is defined by

$$\chi_q(V) = \sum_{m \in \mathcal{P}} \dim(V_m) m \in \mathbb{Z}\mathcal{P},$$

where $V_m = V_{\gamma(m)}$, see [11].

Let $\mathrm{Rep}(U_q\widehat{\mathfrak{g}})$ be the Grothendieck ring of finite-dimensional $U_q\widehat{\mathfrak{g}}$-modules and $[V] \in \mathrm{Rep}(U_q\widehat{\mathfrak{g}})$ the class of a finite-dimensional $U_q\widehat{\mathfrak{g}}$-module V. The q-character map defines an injective ring homomorphism, see [11],

$$\chi_q : \mathrm{Rep}(U_q\widehat{\mathfrak{g}}) \to \mathbb{Z}\mathcal{P}.$$

For any finite-dimensional $U_q\widehat{\mathfrak{g}}$-module V, we use $m \in \chi_q(V)$ to denote that m is a monomial in $\chi_q(V)$. For each $j \in I$, a monomial $m = \prod_{i\in I, a\in\mathbb{C}^\times} Y_{i,a}^{u_{i,a}}$, where $u_{i,a}$ are some integers, is said to be j-*dominant* (resp. j-*anti-dominant*) if and only if $u_{j,a} \geq 0$ (resp. $u_{j,a} \leq 0$) for all $a \in \mathbb{C}^\times$. A monomial is called *dominant* (resp. *anti-dominant*) if and only if it is j-dominant (resp. j-anti-dominant) for all $j \in I$. Let $\mathcal{P}^+ \subset \mathcal{P}$ denote the set of all dominant monomials.

Let V be a $U_q\widehat{\mathfrak{g}}$-module and $m \in \chi_q(V)$ a monomial. A non-zero vector $v \in V_m$ is called a *highest l-weight vector* with *highest l-weight* $\gamma(m)$ if

$$x_{i,r}^+ \cdot v = 0, \ \phi_{i,\pm t}^\pm \cdot v = \gamma(m)_{i,\pm t}^\pm v, \ \forall i \in I, r \in \mathbb{Z}, t \in \mathbb{Z}_{\geq 0}.$$

The module V is called a *highest l-weight module* if $V = U_q\widehat{\mathfrak{g}} \cdot v$ for some highest l-weight vector $v \in V$.

In [5], [6], it is shown that there is a one to one correspondence between dominant l-weights and finite-dimensional simple $U_q\widehat{\mathfrak{g}}$-modules. Therefore for every $m_+ \in \mathcal{P}^+$, there is a unique finite-dimensional simple $U_q\widehat{\mathfrak{g}}$-module $L(m_+)$. We use $\chi_q(m_+)$ to denote $\chi_q(L(m_+))$.

Let p_1, p_2 be two polynomials in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^\times}$. If m is a monomial in the polynomial, then we write $m \in p_1$. If $m \in p_1$ and $m \in p_2$, then we write $m \in p_1 \cap p_2$. If all monomials in p_1 are in p_2, then we write $p_1 \subseteq p_2$.

The following lemma is well-known.

Lemma 2.1. *Let m_1, m_2 be two dominant monomials. Then $L(m_1 m_2)$ is a sub-quotient of $L(m_1) \otimes L(m_2)$. In particular, $\chi_q(m_1 m_2) \subseteq \chi_q(m_1)\chi_q(m_2)$.* ∎

For $b \in \mathbb{C}^\times$, define the shift of spectral parameter map $\tau_b : \mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{P}$ to be a homomorphism of rings sending $Y_{i,a}^{\pm 1}$ to $Y_{i,ab}^{\pm 1}$. Let $m_1, m_2 \in \mathcal{P}^+$. If $\tau_b(m_1) = m_2$, then

$$\tau_b \chi_q(m_1) = \chi_q(m_2). \tag{2.6}$$

A finite-dimensional $U_q\widehat{\mathfrak{g}}$-module V is said to be *special* if and only if $\chi_q(V)$ contains exactly one dominant monomial. It is called *anti-special* if and only if $\chi_q(V)$ contains exactly one anti-dominant monomial. It is said to be *prime* if and only if it is not isomorphic to a tensor product of two non-trivial $U_q\widehat{\mathfrak{g}}$-modules, see [8]. Clearly, if a module is special or anti-special, then it is simple.

Define $A_{i,a} \in \mathcal{P}, i \in I, a \in \mathbb{C}^\times$, by

$$A_{1,a} = Y_{1,aq^3} Y_{1,aq^{-3}} Y_{2,aq^{-2}}^{-1} Y_{2,a}^{-1} Y_{2,aq^2}^{-1}, \qquad A_{2,a} = Y_{2,aq} Y_{2,aq^{-1}} Y_{1,a}^{-1}.$$

When $a \in \mathbb{C}^\times$ is fixed, we write $A_{i,s} = A_{i,aq_i^s}$.

Let $\mathcal{Q}$ be the subgroup of $\mathcal{P}$ generated by $A_{i,a}, i \in I, a \in \mathbb{C}^\times$. Let $\mathcal{Q}^\pm$ be the monoids generated by $A_{i,a}^{\pm 1}, i \in I, a \in \mathbb{C}^\times$. There is a partial order $\leq$ on $\mathcal{P}$ in which

$$m \leq m' \text{ if and only if } m'm^{-1} \in \mathcal{Q}^+. \tag{2.7}$$

For all $m_+ \in \mathcal{P}^+$, $\chi_q(m_+) \subset m_+\mathcal{Q}^-$, see [10].

Definition 2.2 ([10]). Let m be a monomial. Suppose that for all $a \in \mathbb{C}^\times$ and $i \in I$, we have the property: if the power of $Y_{i,a}$ in m is non-zero and the power of Y_{j,aq^k} in m is zero for all $j \in I, k \in \mathbb{Z}_{>0}$, then the power of $Y_{i,a}$ in m is negative. Then the monomial m is called *right negative*.

Lemma 2.3 ([10], [16]). *For $i \in I, a \in \mathbb{C}^\times$, $A_{i,a}^{-1}$ is right-negative. A product of right-negative monomials is right-negative. If m is right-negative and $m' \leq m$, then m' is right-negative.*

Lemma 2.4 (Lemma 4.4, [15]). *All monomials in the q-character of a Kirillov-Reshetikhin module is right-negative except the highest l-weight monomial.*

We need the following result from [10], [18].

Proposition 2.5 (Proposition 5.3, [18]). *Let V, W be two $U_q\widehat{\mathfrak{g}}$-modules. If $\chi_q(V)$ and $\chi_q(W)$ have the same dominant monomials with the same multiplicities, then $\chi_q(V) = \chi_q(W)$.*

2.3. Minimal affinizations of $U_q\mathfrak{g}$-modules. Let $\lambda = k\omega_1 + l\omega_2$. A simple $U_q\widehat{\mathfrak{g}}$-module $L(m_+)$ is a *minimal affinization* of $V(\lambda)$ if and only if m_+ is one of the following monomials

$$\left(\prod_{i=0}^{k-1} Y_{1,aq^{6i}}\right)\left(\prod_{i=0}^{l-1} Y_{2,aq^{6k+2i+1}}\right), \qquad \left(\prod_{i=0}^{l-1} Y_{2,aq^{-6k-2i-1}}\right)\left(\prod_{j=0}^{k-1} Y_{1,aq^{-6j}}\right),$$

for some $a \in \mathbb{C}^\times$, see [7].

From now on, we fix an $a \in \mathbb{C}^\times$ and denote $i_s = Y_{i,aq^s}$, $i \in I$, $s \in \mathbb{Z}$. Without loss of generality, we may assume that a simple $U_q\widehat{\mathfrak{g}}$-module $L(m_+)$ is a minimal affinization of $V(\lambda)$ if and only if m_+ is one of the following monomials

$$T_{k,l}^{(s)} = \left(\prod_{i=0}^{k-1} 1_{s+6i}\right)\left(\prod_{j=0}^{l-1} 2_{s+6k+2j+1}\right), \quad \widetilde{T}_{k,l}^{(s)} = \left(\prod_{i=0}^{l-1} 2_{-s-6k-2i-1}\right)\left(\prod_{j=0}^{k-1} 1_{-s-6j}\right).$$

2.4. q-characters of $U_q\widehat{\mathfrak{sl}}_2$-modules and the Frenkel-Mukhin algorithm. We recall the results of the q-characters of $U_q\widehat{\mathfrak{sl}}_2$-modules and Frenkel-Mukhin algorithm, see [4], [11], [10], [14], [17].

Let $W_k^{(a)}$ be the simple $U_q\widehat{\mathfrak{sl}}_2$-module with highest weight monomial

$$X_k^{(a)} = \prod_{i=0}^{k-1} Y_{aq^{k-2i-1}},$$

where $Y_a = Y_{1,a}$. Then the q-character of $W_k^{(a)}$ is given by

$$\chi_q(W_k^{(a)}) = X_k^{(a)} \sum_{i=0}^{k} \prod_{j=0}^{i-1} A_{aq^{k-2j}}^{-1}, \tag{2.8}$$

where $A_a = Y_{aq^{-1}} Y_{aq}$.

For $a \in \mathbb{C}^\times, k \in \mathbb{Z}_{\geq 1}$, the set $\Sigma_k^{(a)} = \{aq^{k-2i-1}\}_{i=0,\dots,k-1}$ is called a q-*string*. Two q-strings $\Sigma_k^{(a)}$ and $\Sigma_{k'}^{(a')}$ are said to be in *general position* if the union $\Sigma_k^{(a)} \cup \Sigma_{k'}^{(a')}$ is not a q-string or $\Sigma_k^{(a)} \subset \Sigma_{k'}^{(a')}$ or $\Sigma_{k'}^{(a')} \subset \Sigma_k^{(a)}$.

Denote by $L(m_+)$ the simple $U_q\widehat{\mathfrak{sl}}_2$-module with highest weight monomial m_+. Let $m_+ \neq 1$ and $\in \mathbb{Z}[Y_a]_{a\in\mathbb{C}^\times}$ be a dominant monomial. Then m_+ can be uniquely (up to permutation) written in the form

$$m_+ = \prod_{i=1}^{s} \left(\prod_{b \in \Sigma_{k_i}^{(a_i)}} Y_b \right),$$

where s is an integer, $\Sigma_{k_i}^{(a_i)}, i = 1, \dots, s$, are q-strings which are pairwise in general position and

$$L(m_+) = \bigotimes_{i=1}^{s} W_{k_i}^{(a_i)}, \qquad \chi_q(L(m_+)) = \prod_{i=1}^{s} \chi_q(W_{k_i}^{(a_i)}). \tag{2.9}$$

Let $i \in I$. We call $n = Y_{i,a} Y_{i,aq_i^2} \cdots Y_{i,aq_i^{2k-2}}$ a q_i-string in a monomial m if n is a factor of m. We say that two q_i-strings n_1 and n_2 are in general position if $n_1 n_2$ is not a q_i-string or n_1 is a factor of n_2 or n_2 is a factor of n_1.

The Frenkel-Mukhin algorithm is very powerful to compute q-characters of simple $U_q\mathfrak{g}$-modules, [10]. Let m_+ be a dominant monomial. Roughly speaking, when the Frenkel-Mukhin algorithm computes $\chi_q(m_+)$, the algorithm starts with m_+ and gradually expand it in all possible $U_{q_i}\widehat{\mathfrak{sl}}_2$-directions ($i \in I$).

Although in some cases the algorithm may fail, it works for a large family of modules. In particular, if a module $L(m_+)$ is special, then we can use Frenkel-Mukhin algorithm to compute its q-character, see [10].

Theorem 2.6 (Theorem 3.8, [16], Proposition 7.1, Theorem 7.2, [20]). *The minimal affinizations in the first (resp. second) part of the M-system of type G_2 are special (resp. anti-special). Therefore we can use the Frenkel-Mukhin algorithm to compute the q-characters of the minimal affinizations in the first part of the M-system of type G_2.*

We will need the following result from Section 5 of [18]. Let m be an i-dominant monomial and $\varphi_i(m)$ a polynomial defined as follows. Let $\overline{m}$ be the monomial obtained from m by replacing $Y_{j,a}$ by Y_a if $j = i$ and by 1 if $j \neq i$. Then the q-character $\chi_q(L(\overline{m}))$ of the $U_q\widehat{\mathfrak{sl}}_2$-module $L(\overline{m})$ is given by (2.8), (2.9). Write $\chi_q(L(\overline{m})) = \overline{m}(1 + \sum_p \overline{M}_p)$, where the $\overline{M}_p$ are monomials in the variables A_a^{-1} ($a \in \mathbb{C}^\times$). Let $\varphi_i(m) := m(1 + \sum_p M_p)$ where each M_p is obtained from the corresponding $\overline{M}_p$ by replacing each variable A_a^{-1} by $A_{i,a}^{-1}$.

Theorem 2.7 (Section 5.3, [18]). *Let m be a dominant monomial and let mM be a monomial of $\chi_q(L(m))$, where M is a monomial in $A_{j,a}^{-1}$ ($j \in I$). If M*

contains no $A_{i,a}^{-1}$, then mM is i-dominant and $\varphi_i(mM)$ is contained in $\chi_q(L(m))$. In particular, $\varphi_i(m)$ is contained in $\chi_q(L(m))$.

Let $i \in I$ and $\beta_i : \mathcal{P} \to \mathcal{P}$ be a map such that $\beta_i(m)$ is obtained from $m \in \mathcal{P}$ by replacing all $Y_{j,a}$ by 1, $j \neq i$. For example, $\beta_1(1_0 1_6 1_{12}^{-1} 2_1 2_3) = 1_0 1_6 1_{12}^{-1}$.

By the Frenkel-Mukhin algorithm [10] and the formulas (2.8), (2.9), we have the following result which is used frequently in our proof.

Lemma 2.8. *Let m_+ be a dominant monomial. Then every monomial in $\chi_q(m_+)$ is a monomial in some $\varphi_i(m)$, where $i \in I$ and m is an i-dominant monomial in $\chi_q(m_+)$. The l-weights of the monomials in $\varphi_i(m)$ are less or equal to the l-weight of m.*

Suppose that $\beta_i(m) = i_s i_{s+2r_i} \cdots i_{s+2kr_i - 2r_i}$ ($k \in \mathbb{Z}_{\geq 1}$) is a q_i-string and $m' \in \varphi_i(m)$. If $i_{s+2kr_i-2r_i}$ is a factor of m', then $\beta_i(m') = \beta_i(m)$ and hence $m A_{i,s+2jr_i-r_i}^{-1}$ ($j \in \{1, \ldots, k\}$) is not a monomial in $\chi_q(m_+)$.

For example, $1_0 2_7 2_9 A_{2,aq^8}^{-1} = 1_0 1_9$ is not in $\chi_q(1_0 2_7 2_9)$.

2.5. Cluster algebras.

Cluster algebras are invented by Fomin and Zelevinsky in [12]. Let $\mathbb{Q}$ be the field of rational numbers and $\mathcal{F} = \mathbb{Q}(x_1, x_2, \cdots, x_n)$ the field of rational functions. A seed in $\mathcal{F}$ is a pair $\Sigma = (\mathbf{y}, Q)$, where $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ is a free generating set of $\mathcal{F}$, and Q is a quiver with vertices labeled by $\{1, 2, \cdots, n\}$. Assume that Q has neither loops nor 2-cycles. For $k = 1, 2, \cdots, n$, one defines a mutation μ_k by $\mu_k(\mathbf{y}, Q) = (\mathbf{y}', Q')$. Here $\mathbf{y}' = (y_1', \ldots, y_n')$, $y_i' = y_i$, for $i \neq k$, and

$$y_k' = \frac{\prod_{i \to k} y_i + \prod_{k \to j} y_j}{y_k}, \tag{2.10}$$

where the first (resp. second) product in the right hand side is over all arrows of Q with target (resp. source) k, and Q' is obtained from Q by

   (i) adding a new arrow $i \to j$ for every existing pair of arrow $i \to k$ and $k \to j$;

  (ii) reversing the orientation of every arrow with target or source equal to k;

 (iii) erasing every pair of opposite arrows possible created by (i).

The mutation class $\mathcal{C}(\Sigma)$ is the set of all seeds obtained from Σ by a finite sequence of mutation μ_k. If $\Sigma' = ((y_1', y_2', \cdots, y_n'), Q')$ is a seed in $\mathcal{C}(\Sigma)$, then the subset $\{y_1', y_2', \cdots, y_n'\}$ is called a *cluster*, and its elements are called *cluster variables*. The *cluster algebra* $\mathscr{A}_\Sigma$ as the subring of $\mathcal{F}$ generated by all cluster variables. *Cluster monomials* are monomials in the cluster variables supported on a single cluster.

In this paper, the initial seed in the cluster algebra we use is of the form $\Sigma = (\mathbf{y}, Q)$, where $\mathbf{y}$ is an infinite set and Q is an infinite quiver.

Definition 2.9 (Definition 3.1, [13]). Let Q be a quiver without loops or 2-cycles and with a countably infinite number of vertices labelled by all integers

$i \in \mathbb{Z}$. Furthermore, for each vertex i of Q let the number of arrows incident with i be finite. Let $\mathbf{y} = \{y_i \mid i \in \mathbb{Z}\}$. An infinite initial seed is the pair $(\mathbf{y}, Q)$. By finite sequences of mutation at vertices of Q and simultaneous mutation of the set $\mathbf{y}$ using the exchange relation (2.10), one obtains a family of infinite seeds. The sets of variables in these seeds are called the infinite clusters and their elements are called the cluster variables. The cluster algebra of infinite rank of type Q is the subalgebra of $\mathbb{Q}(\mathbf{y})$ generated by the cluster variables.

3. The M-system of type G_2

In this section, we describe the M-system of type G_2.

3.1. The M-system of type G_2. We use $\mathcal{T}_{k,l}^{(s)}$ to denote the simple finite-dimensional $U_q\widehat{\mathfrak{g}}$-module with highest l-weight $T_{k,l}^{(s)}$. Here $T_{k,l}^{(s)}$ is defined in Section 2. Let $[\mathcal{T}]$ be the equivalence class of the $U_q\widehat{\mathfrak{g}}$-module $\mathcal{T}$ in the Grothendieck ring $\mathrm{Rep}(U_q\widehat{\mathfrak{g}})$.

Theorem 3.1. *For $s \in \mathbb{Z}$, we have the following system of equations:*

$$[\mathcal{T}_{k,l}^{(s)}][\mathcal{T}_{k,0}^{(s+6)}] = [\mathcal{T}_{k+1,0}^{(s)}][\mathcal{T}_{k-1,l}^{(s+6)}] + [\mathcal{T}_{0,3k+l}^{(s)}] \quad (k \in \mathbb{Z}_{\geq 1}, l \in \{1,2,3\}), \quad (3.1)$$

$$[\mathcal{T}_{k,l+3}^{(s)}][\mathcal{T}_{k,l}^{(s+6)}] = [\mathcal{T}_{k+1,l}^{(s)}][\mathcal{T}_{k-1,l+3}^{(s+6)}] + [\mathcal{T}_{0,3k+l+3}^{(s)}][\mathcal{T}_{0,l}^{(s+6k+6)}] \quad (k,l \in \mathbb{Z}_{\geq 1}). (3.2)$$

Moreover, every module in the summands on the right hand side of the above equations corresponds to simple modules.

This is the first part of the M-system of type G_2. The equations in Theorem 3.1 will be proved in Section 5 and the simplicity of the modules in the summands on the right hand side of the equations in Theorem 3.1 will be proved in Section 6.

Theorem 3.2. *For $s \in \mathbb{Z}$, we have the following system of equations:*

$$[\widetilde{\mathcal{T}}_{k,l}^{(s)}][\widetilde{\mathcal{T}}_{k,0}^{(s+6)}] = [\widetilde{\mathcal{T}}_{k+1,0}^{(s)}][\widetilde{\mathcal{T}}_{k-1,l}^{(s+6)}] + [\widetilde{\mathcal{T}}_{0,3k+l}^{(s)}] \quad (k \in \mathbb{Z}_{\geq 1}, l \in \{1,2,3\}),$$

$$[\widetilde{\mathcal{T}}_{k,l+3}^{(s)}][\widetilde{\mathcal{T}}_{k,l}^{(s+6)}] = [\widetilde{\mathcal{T}}_{k+1,l}^{(s)}][\widetilde{\mathcal{T}}_{k-1,l+3}^{(s+6)}] + [\widetilde{\mathcal{T}}_{0,l}^{(s+6k+6)}][\widetilde{\mathcal{T}}_{0,3k+l+3}^{(s)}] \quad (k,l \in \mathbb{Z}_{\geq 1}).$$

Moreover, every module in the summands on the right hand side of the above equations corresponds to simple modules.

This is the second part of the M-system of type G_2. Theorem 3.2 will be proved in Section 7.

The M-system gives more efficient recursive procedure for computing the q-characters of minimal affinizations than the extended T-systems from [20].

The equations in Theorem 3.1 are equivalent to the following equations.

$$\chi_q(\mathcal{T}_{k,l}^{(s)})\chi_q(\mathcal{T}_{k,0}^{(s+6)}) = \chi_q(\mathcal{T}_{k+1,0}^{(s)})\chi_q(\mathcal{T}_{k-1,l}^{(s+6)}) + \chi_q(\mathcal{T}_{0,3k+l}^{(s)}) \quad (k \in \mathbb{Z}_{\geq 1}, l \in \{1,2,3\}),$$

$$\chi_q(\mathcal{T}_{k,l+3}^{(s)})\chi_q(\mathcal{T}_{k,l}^{(s+6)}) = \chi_q(\mathcal{T}_{k+1,l}^{(s)})\chi_q(\mathcal{T}_{k-1,l+3}^{(s+6)}) + \chi_q(\mathcal{T}_{0,3k+l+3}^{(s)})\chi_q(\mathcal{T}_{0,l}^{(s+6k+6)}) \quad (k,l \in \mathbb{Z}_{\geq 1}).$$

Example 3.3. The following are some examples of equations in the M-system of type G_2.

$$[1_{-7}2_0][1_{-1}] = [1_{-7}1_{-1}][2_0] + [2_{-6}2_{-4}2_{-2}2_0],$$
$$[1_{-9}2_{-2}2_0][1_{-3}] = [1_{-9}1_{-3}][2_{-2}2_0] + [2_{-8}2_{-6}2_{-4}2_{-2}2_0],$$
$$[1_{-11}2_{-4}2_{-2}2_0][1_{-5}] = [1_{-11}1_{-5}][2_{-4}2_{-2}2_0] + [2_{-10}2_{-8}2_{-6}2_{-4}2_{-2}2_0],$$
$$[1_{-13}2_{-6}2_{-4}2_{-2}2_0][1_{-7}2_0] = [1_{-13}1_{-7}2_0][2_{-6}2_{-4}2_{-2}2_0] + [2_0][2_{-12}2_{-10}\cdots 2_{-2}2_0],$$
$$[1_{-33}1_{-27}2_{-20}\cdots 2_{-2}2_0][1_{-27}1_{-21}2_{-14}\cdots 2_{-2}2_0] = [1_{-33}1_{-27}1_{-21}2_{-14}\cdots 2_{-2}2_0][1_{-27}2_{-20}\cdots 2_{-2}2_0] + [2_{-14}2_{-12}\cdots 2_{-2}2_0][2_{-32}2_{-30}\cdots 2_{-2}2_0].$$

Example 3.4. The following are some examples of equations in the second part of the M-system of type G_2.

$$[2_0 1_7][1_1] = [1_1 1_7][2_0] + [2_0 2_2 2_4 2_6],$$
$$[2_0 2_2 1_9][1_3] = [1_3 1_9][2_0 2_2] + [2_0 2_2 2_4 2_6 2_8],$$
$$[2_0 2_2 2_4 1_{11}][1_5] = [1_5 1_{11}][2_0 2_2 2_4] + [2_0 2_2 2_4 2_6 2_8 2_{10}],$$
$$[2_0 2_2 2_4 2_6 1_{13}][2_0 1_7] = [2_0 1_7 1_{13}][2_0 2_2 2_4 2_6] + [2_0][2_0 2_2 \cdots 2_{10} 2_{12}],$$
$$[2_0 2_2 \cdots 2_{20} 1_{27} 1_{33}][2_0 2_2 \cdots 2_{14} 1_{21} 1_{27}] = [2_0 2_2 \cdots 2_{14} 1_{21} 1_{27} 1_{33}][2_0 2_2 \cdots 2_{20} 1_{27}] + [2_0 2_2 \cdots 2_{12} 2_{14}][2_0 2_2 \cdots 2_{30} 2_{32}].$$

3.2. The m-system of type G_2. For $k, l \in \mathbb{Z}_{\geq 0}$, let $m_{k,l} = \mathrm{Res}(\mathcal{T}_{k,l}^{(0)})$ (resp. $\widetilde{m}_{k,l} = \mathrm{Res}(\widetilde{\mathcal{T}}_{k,l}^{(0)})$) be the restriction of $\mathcal{T}_{k,l}^{(0)}$ (resp. $\widetilde{\mathcal{T}}_{k,l}^{(0)}$) to $U_q\mathfrak{g}$. Let $\chi(M)$ be the character of a $U_q\mathfrak{g}$-module M. By Theorem 3.1, we have the following result.

Corollary 3.5. *We have*

$$\begin{aligned}
\chi(m_{k,l})\chi(m_{k,0}) &= \chi(m_{k+1,0})\chi(m_{k-1,l}) + \chi(m_{0,3k+l}) \quad (k \in \mathbb{Z}_{\geq 1}, l \in \{1,2,3\}),\\
\chi(m_{k,l+3})\chi(m_{k,l}) &= \chi(m_{k+1,l})\chi(m_{k-1,l+3}) + \chi(m_{0,l})\chi(m_{0,3k+l+3}) \quad (k,l \in \mathbb{Z}_{\geq 1}),\\
\chi(\widetilde{m}_{k,l})\chi(\widetilde{m}_{k,0}) &= \chi(\widetilde{m}_{k+1,0})\chi(\widetilde{m}_{k-1,l}) + \chi(\widetilde{m}_{0,3k+l}) \quad (k \in \mathbb{Z}_{\geq 1}, l \in \{1,2,3\}),\\
\chi(\widetilde{m}_{k,l+3})\chi(\widetilde{m}_{k,l}) &= \chi(\widetilde{m}_{k+1,l})\chi(\widetilde{m}_{k-1,l+3}) + \chi(\widetilde{m}_{0,l})\chi(\widetilde{m}_{0,3k+l+3}) \quad (k,l \in \mathbb{Z}_{\geq 1}).
\end{aligned}$$

We call the above system of equations the m-system of type G_2.

4. Interpretation of the equations in the M-system of type G_2 as exchange relations

In this section, we interpret the equations in the M-system of type G_2 as exchange relations in certain cluster algebra constructed by Hernandez and Leclerc in [19].

4.1. The cluster algebra $\mathscr{A}$ constructed by Hernandez and Leclerc in [19]. Let $S = \{-2n + 1 \mid n \in \mathbb{Z}_{\geq 1}\}$, $S' = \{-2n + 2 \mid n \in \mathbb{Z}_{\geq 1}\}$, and $V = (\{1\} \times S)\bigcup(\{2\} \times S')$. Let Q be a quiver with the vertex set V. The arrows of Q are given by the following rules. For $s_1, s_2 \in S, s'_1, s'_2 \in S'$, there is an arrow from $(1, s_1)$ to $(1, s_2)$ if and only if $s_2 = s_1 + 6$, there is an arrow from $(2, s'_1)$ to $(2, s'_2)$ if and only if $s'_2 = s'_1 + 2$, there is an arrow from $(1, s_1)$ to $(2, s'_1)$ if and only if $s'_1 = s_1 - 5$, and there is an arrow from $(2, s'_2)$ to $(1, s_2)$ if and only if $s_2 = s'_2 - 1$. The quiver Q is the quiver G^- of type G_2 in [19].

Let $\mathbf{t} = \{t_{k,0}^{(s_1)}, t_{0,l}^{(s_2)} \mid s_1, s_2 \in S, k, l \in \mathbb{Z}_{\geq 1}\}$. Let $\mathscr{A}$ be the cluster algebra defined by the initial seed $(\mathbf{t}, Q)$. By Definition 2.9, $\mathscr{A}$ is the $\mathbb{Q}$-subalgebra of the field of rational functions $\mathbb{Q}(\mathbf{t})$ generated by all the elements obtained from some elements of $\mathbf{t}$ via a finite sequence of seed mutations.

4.2. Interpretation of the first part of the M-system of type G_2 as exchange relations. We use "C_1" to denote the column of vertices $(1, -1)$, $(1, -7)$, ..., $(1, -6n + 5)$, $\cdots$ in the quiver Q. We use "C_2" to denote the column of vertices $(1, -3)$, $(1, -9)$, ..., $(1, -6n + 3)$, $\cdots$ in Q. We use "C_3" to denote the column of vertices $(1, -5)$, $(1, -11)$, ..., $(1, -6n + 1)$, $\cdots$ in Q. We use "C_4" to denote the column of vertices $(2, 0)$, $(2, -2)$, ..., $(1, -2n + 2)$, $\cdots$ in Q.

By saying that mutate at the column C_i, $i \in \{1, 2, 3, 4\}$, we mean that we mutate the vertices of C_i as follows. First we mutate at the first vertex in the column C_i, then the second vertex, an so on until the vertex at infinity. By saying that we mutate $C_{i_1}, C_{i_2}, \ldots$, where $i_j \in \{1, 2, 3, 4\}, j = 1, 2, \ldots, n$, we mean that we first mutate the column C_{i_1}, then the column C_{i_2}, an so on.

The variables $t_{k,0}^{(s_1)}$, $t_{0,l}^{(s_2)}$, $s_1, s_2 \in S$, are the cluster variables in the initial seed of $\mathscr{A}$ defined in Section 4. For convenience, we write $t_{\lceil -s_1/6 \rceil, 0}^{(s_1)}$ at the vertex $(1, s_1)$ and write $t_{0, (-s_2+1)/2}^{(s_2)}$ at the vertex $(2, s_2)$ in the initial quiver Q, $s_1, s_2 \in S$. Then we obtain the quiver (a) in Figure 1.

We define some variables $t_{k,l}^{(s)}$ ($k, l \in \mathbb{Z}_{\geq 1}$, $s \in S$) recursively as follows. Let Seq_i, $i = 1, 2, 3$, be the mutation sequence $C_i, C_i, C_i, \ldots$.

We define

$$
\begin{aligned}
t_{k,l}^{(s)} &= (t_{k,0}^{(s+6)})', \quad k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\}, s \equiv 2l + 3 \pmod 6, \\
t_{k,l+3}^{(s)} &= (t_{k,l}^{(s+6)})', \quad k, l \in \mathbb{Z}_{\geq 1},
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
(t_{k,0}^{(s+6)})' &= \frac{t_{k+1,0}^{(s)} t_{k-1,l}^{(s+6)} + t_{0,3k+l}^{(s)}}{t_{k,0}^{(s+6)}}, \quad k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\}, \\
(t_{k,l}^{(s+6)})' &= \frac{t_{k+1,l}^{(s)} t_{k-1,l+3}^{(s+6)} + t_{0,3k+l+3}^{(s)} t_{0,l}^{(s+6k+6)}}{t_{k,l}^{(s+6)}}, \quad k, l \in \mathbb{Z}_{\geq 1}.
\end{aligned}
\tag{4.2}
$$

are exchange relations which occur when we mutate Seq_i, $i \in \{1, 2, 3\}$. The variables (4.1) are defined in the order according to the mutation sequence Seq_i. In this order, every variable in (4.1) is defined by an equation of (4.2) using variables in $\mathbf{t}$ and those variables in (4.1) which are already defined.

Figure 1 is the first few mutations in the mutation sequence Seq_1.

The exchange relations in (4.2) coincides with the equations in the first part of the M-system of type G_2 in Theorem 3.1. Therefore the equations in the first part of the M-system of type G_2 can be interpreted as exchange relations in the cluster algebra $\mathscr{A}$. The cluster variables $t_{k,l}^{(s)}$ corresponds to the minimal affinizations $\mathcal{T}_{k,l}^{(s)}$, $k, l \in \mathbb{Z}_{\geq 0}$.

Using the mutation sequence Seq_i, $i \in \{1, 2, 3\}$, we obtain minimal affinizations

$$
\mathcal{T}_{k,l}^{(-6k-2l+1)}, \ k, l \in \mathbb{Z}_{\geq 1}, \ l \equiv i \pmod 3.
$$

4.3. Interpretation of the second part of the M-system of type G_2 as exchange relations. We can also interpret the second part of the M-system of type G_2 as exchange relations in the cluster algebra $\mathscr{A}$ defined in Section 4. Let Seq_i, $i = 1, 2, 3$, be the mutation sequence $C_i, C_i, C_i, \ldots$. The cluster variables $t_{k,l}^{(s)}$ corresponds to the minimal affinizations $\widetilde{\mathcal{T}}_{k,l}^{(s)}$, $k, l \in \mathbb{Z}_{\geq 0}$. Using the mutation sequence Seq_i, $i \in \{1, 2, 3\}$, we obtain minimal affinizations

$$\widetilde{\mathcal{T}}_{k,l}^{(-6k-2l+1)}, \ k, l \in \mathbb{Z}_{\geq 1}, \ l \equiv i \pmod 3.$$

5. Proof of the equations in Theorem 3.1

In this section, we prove the equations in Theorem 3.1.

Using the Frenkel-Mukhin algorithm, one can easily compute the q-characters of the fundamental modules.

Lemma 5.1. *The fundamental q-characters for $U_q\widehat{\mathfrak{g}}$ of type G_2 are given by*

$$
\begin{aligned}
\chi_q(1_0) &= 1_0 + 2_1 2_3 2_5 1_6^{-1} + 2_1 2_3 2_7^{-1} + 2_1 2_5^{-1} 2_7^{-1} 1_4 + 2_3^{-1} 2_5^{-1} 2_7^{-1} 1_2 1_4 \\
&\quad + 2_1 2_9 1_{10}^{-1} + 1_4 1_8^{-1} + 2_3^{-1} 2_9 1_2 1_{10}^{-1} + 2_5 2_7 2_9 1_8^{-1} 1_{10}^{-1} + 2_1 2_{11}^{-1} \\
&\quad + 2_3^{-1} 2_{11}^{-1} 1_2 + 2_5 2_7 2_{11}^{-1} 1_8^{-1} + 2_5 2_9^{-1} 2_{11}^{-1} + 2_7^{-1} 2_9^{-1} 2_{11}^{-1} 1_6 + 1_{12}^{-1}, \\
\chi_q(2_0) &= 2_0 + 2_2^{-1} 1_1 + 2_4 2_6 1_7^{-1} + 2_4 2_8^{-1} + 2_6^{-1} 2_8^{-1} 1_5 + 2_{10} 1_{11}^{-1} + 2_{12}^{-1}.
\end{aligned}
$$

5.1. Classification of dominant monomials in the summands on both sides of the M-system. By Theorem 3.8 in [16] (see also Theorem 3.3 in [20]), the modules $\mathcal{T}_{k,l}^{(s)}$ ($s \in \mathbb{Z}, k, l \in \mathbb{Z}_{\geq 0}$) are special. Therefore we can use the Frenkel-Mukhin algorithm to compute the q-characters of $\mathcal{T}_{k,l}^{(s)}$ ($s \in \mathbb{Z}, k, l \in \mathbb{Z}_{\geq 0}$). Now we use the Frenkel-Mukhin algorithm to classify dominant monomials in the summands on both sides of the M-system.

Lemma 5.2. *We have the following cases.*

(1) Let

$$M = T_{k,l}^{(s)} T_{k,0}^{(s+6)} \ (k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\}).$$

Then the dominant monomials in $\chi_q(T_{k,l}^{(s)}) \chi_q(T_{k,0}^{(s+6)})$ $(k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\})$ are M and

$$M_i = M \prod_{j=0}^{i-1} A_{1, s+6k-6j-3}^{-1}, \quad i = 1, 2, \ldots, k,$$

with multiplicity 1.

The dominant monomials in $\chi_q(T_{k+1,0}^{(s)}) \chi_q(T_{k-1,l}^{(s+6)})$ $(k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\})$ are M, M_1, ..., M_{k-1}, with multiplicity 1.

The only dominant monomial in $T_{0,3k+l}^{(s)}$ $(k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\})$ is M_k with multiplicity 1.
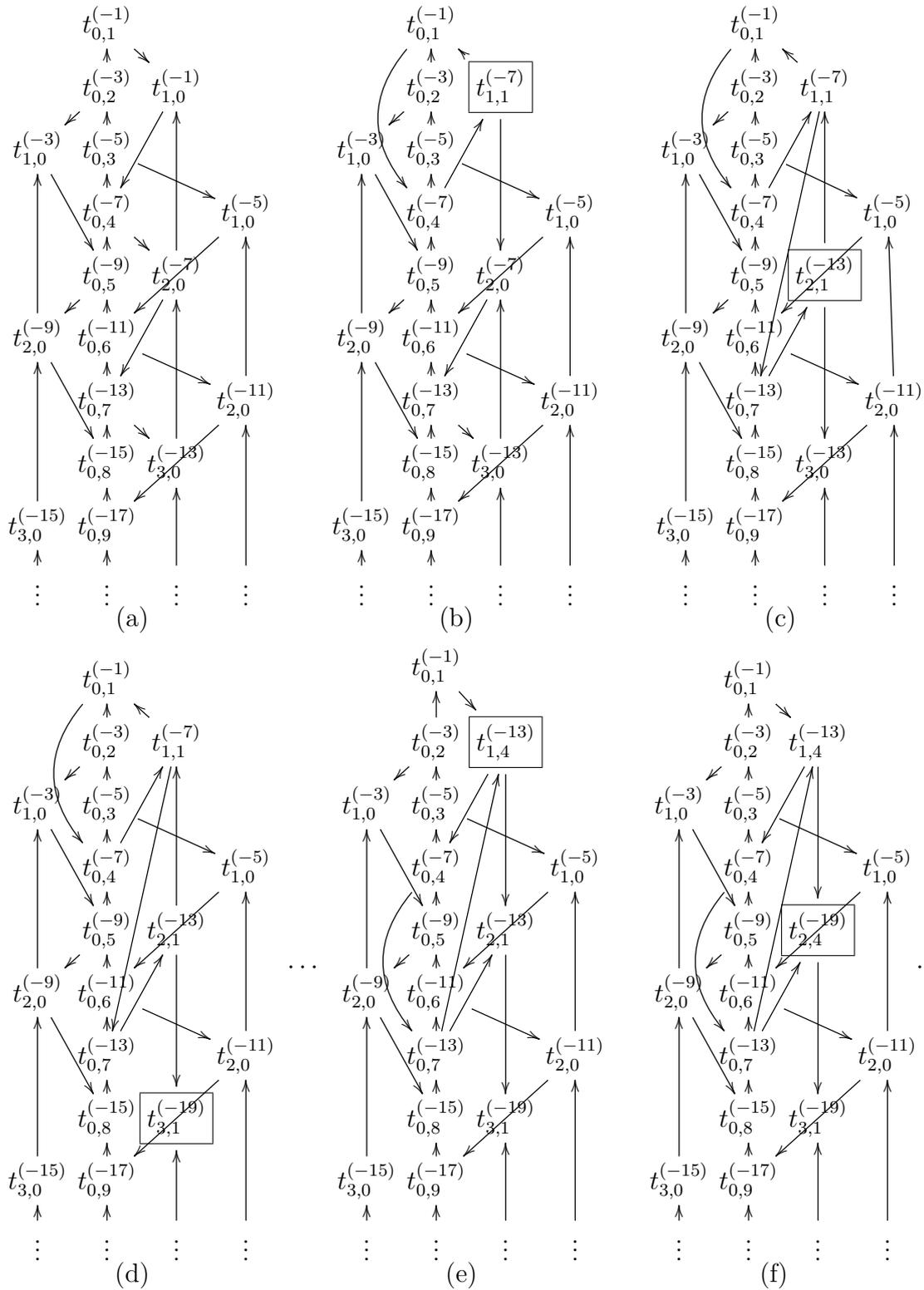
Figure 1: The mutation sequence $C_1, C_1, C_1, \ldots$

(2) Let

$$M = T^{(s)}_{k,l+3} T^{(s+6)}_{k,l} \quad (k,l \in \mathbb{Z}_{\geq 1}).$$

Then the dominant monomials in $\chi_q(T^{(s)}_{k,l+3})\chi_q(T^{(s+6)}_{k,l})$ $(k,l \in \mathbb{Z}_{\geq 1})$ are M and

$$M_i = M \prod_{j=0}^{i-1} A^{-1}_{1,s+6k-6j-3}, \quad i = 1, 2, \ldots, k,$$

with multiplicity 1.

The dominant monomials in $\chi_q(T^{(s)}_{k+1,l})\chi_q(T^{(s+6)}_{k-1,l+3})$ $(k,l \in \mathbb{Z}_{\geq 1})$ are M, M_1, ..., M_{k-1}, with multiplicity 1.

The only dominant monomial in $\chi_q(T^{(s)}_{0,3k+l+3})\chi_q(T^{(s+6k+6)}_{0,l})$ $(k,l \in \mathbb{Z}_{\geq 1})$ is M_k with multiplicity 1.

Proof.   We will prove Part (2). Part (1) is similar. In the proof, we use Lemma 2.8 frequently to show that some monomials cannot occur in a q-character.

Classify the dominant monomials in $\chi_q(\mathcal{T}^{(s)}_{k,l+3})\chi_q(\mathcal{T}^{(s+6)}_{k,l})$.

Let $m'_1 = T^{(s)}_{k,l+3}$, $m'_2 = T^{(s+6)}_{k,l}$. Without loss of generality, we may assume that $s = 0$. Then

$$m'_1 = (1_0 1_6 \cdots 1_{6k-6})(2_{6k+1} 2_{6k+3} \cdots 2_{6k+2l+5}),$$
$$m'_2 = (1_6 1_{12} \cdots 1_{6k})(2_{6k+7} 2_{6k+9} \cdots 2_{6k+2l+5}).$$

By Theorem 2.6, we can use Frenkel-Mukhin algorithm to compute $\chi_q(m'_1)$ and $\chi_q(m'_2)$.

We want to classify all dominant monomials $m = m_1 m_2$, $m_i \in \chi_q(m'_i), i = 1, 2$. Let $m = m_1 m_2$ be a dominant monomial, where $m_i \in \chi_q(m'_i), i = 1, 2$. We denote

$$m_3 = 2_{6k+1} 2_{6k+3} \cdots 2_{6k+2l+5},$$
$$m_4 = 2_{6k+7} 2_{6k+9} \cdots 2_{6k+2l+5}.$$

We have the following cases.
Case 1.

$$m_1 \in \chi_q(m'_1) \cap \chi_q(1_0 1_6 \cdots 1_{6k-6})(\chi_q(m_3) - m_3),$$
$$m_2 \in \chi_q(m'_2) \cap \chi_q(1_6 1_{12} \cdots 1_{6k})(\chi_q(m_4) - m_4).$$

We have $m_1 = xy$, $x \in \chi_q(1_0 1_6 \cdots 1_{6k-6})$, $y \in \chi_q(m_3) - m_3$. By Lemma 2.4, y is right negative since $L(m_3)$ is a Kirillov-Reshetikhin module. If $x = 1_0 1_6 \cdots 1_{6k-6}$, then $m_1 = xy$ must be right negative because the largest index in x is $6k - 6$ and 1_{6k-6} cannot cancel the negative factors in y (all indices of the factors in y are larger than $6k - 6$). If $x \in \chi_q(1_0 1_6 \cdots 1_{6k-6}) - 1_0 1_6 \cdots 1_{6k-6}$, then x is right negative since $L(1_0 1_6 \cdots 1_{6k-6})$ is a Kirillov-Reshetikhin module.

By Lemma 2.3, the product of two right negative monomials are right negative. Therefore $m_1 = xy$ is right negative.

Similarly, m_2 is right negative. It follows that $m = m_1 m_2$ is right negative and hence m is not dominant. This contradicts our assumption.

Case 2.

$$m_1 \in \chi_q(m_1') \cap \chi_q(1_0 1_6 \cdots 1_{6k-6})(\chi_q(m_3) - m_3),$$
$$m_2 \in \chi_q(m_2') \cap \chi_q(1_6 1_{12} \cdots 1_{6k})m_4.$$

In this case, the indices of the negative factors in m_1 are larger than $6k + 2l + 5$. By Lemma 5.3, the largest index in m_2 is $6k + 2l + 5$. It follows that the negative factor with largest index in m_1 cannot be canceled by the factors in m_2. Therefore $m = m_1 m_2$ is right negative and hence m is not dominant. This contradicts our assumption.

Case 3.

$$m_1 \in \chi_q(m_1') \cap \chi_q(1_0 1_6 \cdots 1_{6k-6})m_3,$$
$$m_2 \in \chi_q(m_2') \cap \chi_q(1_6 1_{12} \cdots 1_{6k})(\chi_q(m_4) - m_4).$$

By using the same argument as Case 2, we have that $m = m_1 m_2$ is right negative and hence m is not dominant. This contradicts our assumption.

Case 4.

$$m_1 \in \chi_q(m_1') \cap \chi_q(1_0 1_6 \cdots 1_{6k-6})m_3,$$
$$m_2 \in \chi_q(m_2') \cap \chi_q(1_6 1_{12} \cdots 1_{6k})m_4.$$

We need the following lemma.

Lemma 5.3.   *1. Suppose that*

$$m_1 \in \chi_q(m_1') \cap \chi_q(1_0 1_6 \cdots 1_{6k-6})m_3.$$

Then m_1 is one of the following monomials:

m_1',

$n_1 = m_1' A_{1,6k-3}^{-1} = 1_0 1_6 \cdots 1_{6k-12} 1_{6k}^{-1} 2_{6k-5} 2_{6k-3} \cdots 2_{6k+2l+5}$,

$n_2 = m_1' A_{1,6k-3}^{-1} A_{1,6k-9}^{-1} = 1_0 1_6 \cdots 1_{6k-18} 1_{6k-6}^{-1} 1_{6k}^{-1} 2_{6k-11} 2_{6k-9} \cdots 2_{6k+2l+5}$,

$\cdots$

$n_k = m_1' A_{1,6k-3}^{-1} A_{1,6k-9}^{-1} \cdots A_{1,3}^{-1} = 1_6^{-1} \cdots 1_{6k-6}^{-1} 1_{6k}^{-1} 2_1 2_3 \cdots 2_{6k+2l+5}$.

2. Suppose that

$$m_2 \in \chi_q(m_2') \cap \chi_q(1_6 1_{12} \cdots 1_{6k})m_4.$$

Then m_2 is one of the following monomials:

m_2',

$m_2' A_{1,6k+3}^{-1} = 1_6 \cdots 1_{6k-6} 1_{6k+6}^{-1} 2_{6k+1} 2_{6k+3} \cdots 2_{6k+2l+5}$,

$m_2' A_{1,6k+3}^{-1} A_{1,6k-3}^{-1} = 1_6 \cdots 1_{6k-12} 1_{6k}^{-1} 1_{6k+6}^{-1} 2_{6k-5} 2_{6k-3} \cdots 2_{6k+2l+5}$,

$\cdots$

$m_2' A_{1,6k+3}^{-1} A_{1,6k-3}^{-1} \cdots A_{1,9}^{-1} = 1_{12}^{-1} 1_{18}^{-1} \cdots 1_{6k}^{-1} 1_{6k+6}^{-1} 2_7 2_9 \cdots 2_{6k+2l+5}$.

Proof. We will prove Part 1. Part 2 can be proved similarly. Suppose that $m_1 \in \chi_q(m_1') \cap \chi_q(1_0 1_6 \cdots 1_{6k-6}) m_3$. We have

$$m_1 \in \chi_q(m_1') \cap \chi_q(1_0 1_6 \cdots 1_{6k-12}) 1_{6k-6} m_3$$

or

$$m_1 \in \chi_q(m_1') \cap \chi_q(1_0 1_6 \cdots 1_{6k-12})(\chi_q(1_{6k-6}) - 1_{6k-6}) m_3.$$

If $m_1 \in \chi_q(m_1') \cap \chi_q(1_0 1_6 \cdots 1_{6k-12}) 1_{6k-6} m_3$, then $m_1 \in \varphi_1(m_1')$, where the map φ_1 is defined before Theorem 2.7. By Lemma 2.8, we have $m_1 = m_1'$ since $\beta_1(1_0 1_6 \cdots 1_{6k-12} 1_{6k-6} m_3) = 1_0 1_6 \cdots 1_{6k-6}$ is a q_1-string in m_1' and 1_{6k-6} is a factor of m_1.

If $m_1 \in \chi_q(m_1') \cap \chi_q(1_0 1_6 \cdots 1_{6k-12})(\chi_q(1_{6k-6}) - 1_{6k-6}) m_3$, then

$$m_1 \in \chi_q(m_1') \cap \chi_q(1_0 1_6 \cdots 1_{6k-12}) 1_{6k}^{-1} 2_{6k-5} 2_{6k-3} 2_{6k-1} m_3$$

since $2_{6k-5} 2_{6k-3} 2_{6k-1} m_3 = 2_{6k-5} 2_{6k-3} 2_{6k-1} \cdots 2_{6k+2l+5}$ is a q_2-string and $2_{6k+2l+5}$ is a factor of m_1.

By the same argument, since $2_{6k-5} 2_{6k-3} 2_{6k-1} \cdots 2_{6k+2l+5}$ is a q_2-string and $2_{6k+2l+5}$ is a factor of m_1, by Lemma 2.8 we have that $m_1 = m_1'$ or

$$m_1 = n_1 = m_1' A_{1,6k-3}^{-1} = 1_0 1_6 \cdots 1_{6k-12} 1_{6k}^{-1} 2_{6k-5} 2_{6k-3} \cdots 2_{6k+2l+5},$$

or

$$m_1 \in \chi_q(m_1') \cap \chi_q(1_0 1_6 \cdots 1_{6k-18}) 1_{6k-6}^{-1} 1_{6k}^{-1} 2_{6k-11} 2_{6k-9} \cdots 2_{6k+2l+5}.$$

Using the same argument, we have that m_1 must be one of the following monomials:

m_1',

$n_1 = m_1' A_{1,6k-3}^{-1} = 1_0 1_6 \cdots 1_{6k-12} 1_{6k}^{-1} 2_{6k-5} 2_{6k-3} \cdots 2_{6k+2l+5}$,

$n_2 = m_1' A_{1,6k-3}^{-1} A_{1,6k-9}^{-1} = 1_0 1_6 \cdots 1_{6k-18} 1_{6k-6}^{-1} 1_{6k}^{-1} 2_{6k-11} 2_{6k-9} \cdots 2_{6k+2l+5}$,

$\cdots$

$n_k = m_1' A_{1,6k-3}^{-1} A_{1,6k-9}^{-1} \cdots A_{1,3}^{-1} = 1_6^{-1} \cdots 1_{6k-6}^{-1} 1_{6k}^{-1} 2_1 2_3 \cdots 2_{6k+2l+5}$. ■

In this case, we have $m_1 \in \chi_q(1_0 1_6 \cdots 1_{6k-6}) m_3$ and $m_2 \in \chi_q(1_6 1_{12} \cdots 1_{6k}) m_4$. Since $m = m_1 m_2$ is dominant, by Lemma 5.3 we have that $m = m_1 m_2$ is one of the following dominant monomials

$$M = m_1' m_2', \ M_1 = n_1 m_2' = M A_{1,6k-3}^{-1}, \ M_2 = n_2 m_2' = M \prod_{j=0}^{1} A_{1,6k-6j-3}^{-1}, \ \cdots,$$

$$M_{k-1} = n_{k-1} m_2' = M \prod_{j=0}^{k-2} A_{1,6k-6j-3}^{-1}, \ M_k = n_k m_2' = M \prod_{j=0}^{k-1} A_{1,6k-6j-3}^{-1},$$

and every monomial above has multiplicity 1 in $\chi_q(\mathcal{T}_{k,l+3}^{(0)}) \chi_q(\mathcal{T}_{k,l}^{(6)})$.

Classify the dominant monomials in $\chi_q(T_{k+1,l}^{(s)})\chi_q(T_{k-1,l+3}^{(s+6)})$**.**

Let $m_1' = T_{k+1,l}^{(s)}$, $m_2' = T_{k-1,l+3}^{(s+6)}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_1' = (1_0 1_6 \cdots 1_{6k})(2_{6k+7} 2_{6k+9} \cdots 2_{6k+2l+5}),$$
$$m_2' = (1_6 1_{12} \cdots 1_{6k-6})(2_{6k+1} 2_{6k+3} \cdots 2_{6k+2l+5}).$$

Let $m = m_1 m_2$ be a dominant monomial, where $m_i \in \chi_q(m_i'), i = 1, 2$. By the same argument as above, we have $m_1 = m_1'$ and m_2 is one of the following monomials.

$$p_1 = m_2' A_{1,6k-3}^{-1} = 1_0 1_6 \cdots 1_{6k-12} 1_{6k}^{-1} 2_{6k-5} 2_{6k-3} \cdots 2_{6k+2l+5},$$
$$p_2 = m_2' A_{1,6k-3}^{-1} A_{1,6k-9}^{-1} = 1_0 1_6 \cdots 1_{6k-18} 1_{6k-6}^{-1} 1_{6k}^{-1} 2_{6k-11} 2_{6k-9} \cdots 2_{6k+2l+5},$$
$$\cdots$$
$$p_{k-1} = m_2' A_{1,6k-3}^{-1} A_{1,6k-9}^{-1} \cdots A_{1,9}^{-1} = 1_{12}^{-1} \cdots 1_{6k-6}^{-1} 1_{6k}^{-1} 2_7 2_9 \cdots 2_{6k+2l+5}.$$

It follows that the dominant monomials in $\chi_q(T_{k+1,l}^{(0)})\chi_q(T_{k-1,l+3}^{(6)})$ are

$$M = m_1' m_2', \ M_1 = m_1' p_1 = M A_{1,6k-3}^{-1}, \ M_2 = m_{1'} p_2 = M \prod_{j=0}^{1} A_{1,6k-6j-3}^{-1}, \ \cdots,$$

$$M_{k-1} = m_1' p_{k-1} = M \prod_{j=0}^{k-2} A_{1,6k-6j-3}^{-1},$$

and every dominant monomial has multiplicity one in $\chi_q(T_{k+1,l}^{(0)})\chi_q(T_{k-1,l+3}^{(6)})$.

Classify the dominant monomials in $\chi_q(T_{0,3k+l+3}^{(s)})\chi_q(T_{0,l}^{(s+6k+6)})$**.**

Let $m_1' = T_{0,3k+l+3}^{(s)}$, $m_2' = T_{0,l}^{(s+6k+6)}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_1' = 2_1 2_3 \cdots 2_{6k+2l+5},$$
$$m_2' = 2_{6k+7} 2_{6k+9} \cdots 2_{6k+2l+5}.$$

Let $m = m_1 m_2$ be a dominant monomial, where $m_i \in \chi_q(m_i'), i = 1, 2$. By Lemma 2.4, if $m_1 \neq m_1'$, then m_1 is right negative. The index of the negative factor in m_1 with largest index is greater than $6k + 2l + 5$. If $m_2 = m_2'$, then the negative factor with largest index in m_1 cannot be canceled by m_2. Therefore $m = m_1 m_2$ is not dominant which contradicts our assumption. Hence $m_2 \neq m_2'$. Therefore by Lemma 2.4, m_2' is right negative. It follows that $m = m_1 m_2$ is right negative since both of m_1 and m_2 are right negative. This is a contradiction. Therefore $m_1 = m_1'$.

If $m_2 \neq m_2'$, then m_2 is right negative and $m = m_1 m_2$ is right negative. This is a contradiction. Therefore $m_2 = m_2'$. It follows that the only dominant monomial in $\chi_q(T_{0,3k+l+3}^{(0)})\chi_q(T_{0,l}^{(6k+6)})$ is $T_{0,3k+l+3}^{(0)} T_{0,l}^{(6k+6)}$ and $T_{0,3k+l+3}^{(0)} T_{0,l}^{(6k+6)}$ has multiplicity one in $\chi_q(T_{0,3k+l+3}^{(0)})\chi_q(T_{0,l}^{(6k+6)})$. ∎

5.2. Proof of the equations in Theorem 3.1.   By Lemma 5.2, the dominant monomials in the q-characters of the left hand side and of the right hand side of every equation in Theorem 3.1 are the same and have the same multiplicities. Therefore by Proposition 2.5, the theorem is true.

6. Proof of the simplicity of the modules in the summands on the right hand side of the equations in Theorem 3.1

By Lemma 5.2, the modules corresponding to the second summand of every equation in Theorem 3.1 are special and hence they are simple. We only need to show that the modules in the first summand corresponding to every equation in Theorem 3.1 are simple. Let $\mathcal{S}$ be a module corresponding to the first summand corresponding to an equation in Theorem 3.1. It suffices to prove that for each non-highest dominant monomial M in $\mathcal{S}$, we have $\chi_q(L(M)) \not\subseteq \chi_q(\mathcal{S})$, see [15], [24].

Lemma 6.1.   *We consider the same cases as in Lemma* 5.2*. In each case* M_i *are the dominant monomials described by that Lemma* 5.2*.*

(1) For $k \in \mathbb{Z}_{\geq 1}$, $l \in \{1, 2, 3\}$, *let*

$$t_i = M_i A_{1,s+6k-6i+3}^{-1}, \quad i = 1, 2, \ldots, k-1.$$

Then for $i = 1, 2, \cdots, k-1$, $t_i \in \chi_q(M_i)$ *and* $t_i \notin \chi_q(\mathcal{T}_{k+1,0}^{(s)})\chi_q(\mathcal{T}_{k-1,l}^{(s+6)})$.

(2) For $k, l \in \mathbb{Z}_{\geq 1}$, *let*

$$t_i = M_i A_{1,s+6k-6i+3}^{-1}, \quad i = 1, 2, \ldots, k-1.$$

Then for $i = 1, 2, \cdots, k-1$, $t_i \in \chi_q(M_i)$ *and* $t_i \notin \chi_q(\mathcal{T}_{k+1,l}^{(s)})\chi_q(\mathcal{T}_{k-1,l+3}^{(s+6)})$.

Proof.   We will prove Part (2). Part (1) is similar. Without loss of generality, we may assume that $s = 0$. By definition, we have

$$T_{k+1,l}^{(0)} = 1_0 1_6 \cdots 1_{6k-6} 1_{6k} 2_{6k+7} 2_{6k+9} \cdots 2_{6k+2l+5},$$
$$T_{k-1,l+3}^{(6)} = 1_6 1_{12} \cdots 1_{6k-6} 2_{6k+1} 2_{6k+3} \cdots 2_{6k+2l+5}.$$

Let $i \in \{1, 2, \ldots, k-1\}$. Then

$$M_i = M \prod_{j=0}^{i-1} A_{1,6k-6j-3}^{-1}$$

$$= T_{k+1,l}^{(0)} T_{k-1,l+3}^{(6)} \prod_{j=0}^{i-1} A_{1,6k-6j-3}^{-1}$$

$$= 1_0 1_6^2 \cdots 1_{6k-6i-6}^2 1_{6k-6i} 2_{6k-6i+1} 2_{6k-6i+3} \cdots 2_{6k+5} 2_{6k+7}^2 \cdots 2_{6k+2l+5}^2.$$

By Theorem 2.7, the monomial

$$M_i A_{1,6k-6i+3}^{-1}$$
$$= 1_0 1_6^2 \cdots 1_{6k-6i-6}^2 1_{6k-6i+6}^{-1} 2_{6k-6i+1}^2 2_{6k-6i+3}^2 2_{6k-6i+5}^2 2_{6k-6i+7} 2_{6k-6i+9} \cdots 2_{6k+5} 2_{6k+7}^2 \cdots 2_{6k+2l+5}^2$$

is in $\chi_q(M_i)$.

We have

$$t_i = M_i A_{1,6k-6i+3}^{-1}$$
$$= \left(T_{k+1,l}^{(0)} T_{k-1,l+3}^{(6)} \prod_{j=0}^{i-1} A_{1,6k-6j-3}^{-1} \right) A_{1,6k-6i+3}^{-1}$$
$$= \left(T_{k+1,l}^{(0)} A_{1,6k-6i+3}^{-1} \right) \left(T_{k-1,l+3}^{(6)} \prod_{j=0}^{i-1} A_{1,6k-6j-3}^{-1} \right).$$

By Theorem 2.7, the monomial

$$T_{k-1,l+3}^{(6)} \prod_{j=0}^{i-1} A_{1,6k-6j-3}^{-1}$$

$$= 1_6 1_{12} \cdots 1_{6k-6} 2_{6k+1} 2_{6k+3} \cdots 2_{6k+2l+5} \prod_{j=0}^{i-1} A_{1,6k-6j-3}^{-1}$$

$$= 1_6 1_{12} \cdots 1_{6k-6i-12} 1_{6k-6i-6} 1_{6k-6i+6}^{-1} 1_{6k-6i}^{-1} \cdots 1_{6k}^{-1} 2_{6k-6i+1} 2_{6k-6i+3} \cdots 2_{6k+2l+5}$$

is in $\chi_q(T_{k-1,l+3}^{(6)})$. Since 1_{6k-6i} is not a factor of $T_{k-1,l+3}^{(6)} \prod_{j=0}^{i-1} A_{1,6k-6j-3}^{-1}$ (this monomial is in $\chi_q(T_{k-1,l+3}^{(6)})$), we have that the monomial

$$\left(T_{k-1,l+3}^{(6)} \prod_{j=0}^{i-1} A_{1,6k-6j-3}^{-1} \right) A_{1,6k-6i+3}^{-1}$$

is not in $\chi_q(T_{k-1,l+3}^{(6)})$ by the Frenkel-Mukhin algorithm.

Therefore if

$$t_i = \left(T_{k+1,l}^{(0)} A_{1,6k-6i+3}^{-1} \right) \left(T_{k-1,l+3}^{(6)} \prod_{j=0}^{i-1} A_{1,6k-6j-3}^{-1} \right)$$

were in $\chi_q(\mathcal{T}_{k+1,l}^{(0)})\chi_q(\mathcal{T}_{k-1,l+3}^{(6)})$, then $T_{k+1,l}^{(0)} A_{1,6k-6i+3}^{-1}$ would be in $\chi_q(T_{k+1,l}^{(0)})$. This implies that $T_{k+1,l}^{(0)} A_{1,6k-6i+3}^{-1} \in \varphi_1(T_{k+1,l}^{(0)})$, where the map φ_1 is defined before Theorem 2.7, which contradicts Lemma 2.8: $\beta_1(T_{k+1,l}^{(0)}) = 1_0 1_6 \cdots 1_{6k}$ is a q_1-string in $T_{k+1,l}^{(0)}$, 1_{6k} is a factor of $T_{k+1,l}^{(0)} A_{1,6k-6i+3}^{-1}$, but $\beta_1(T_{k+1,l}^{(0)} A_{1,6k-6i+3}^{-1}) \neq \beta_1(T_{k+1,l}^{(0)})$. Therefore t_i is not in $\chi_q(\mathcal{T}_{k+1,l}^{(0)})\chi_q(\mathcal{T}_{k-1,l+3}^{(6)})$. ∎

7. Proof of Theorem 3.2

In this section, we prove Theorem 3.2.

Theorem 7.1 (Theorem 7.2, [20]).   *The module $\widetilde{\mathcal{T}}_{k,l}^{(s)}$, $s \in \mathbb{Z}, k, l \in \mathbb{Z}_{\geq 0}$ are anti-special.*

Lemma 7.2 (Lemma 7.3, [20]).   *Let $\iota : \mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{P}$ be a homomorphism of rings such that $Y_{1,aq^s} \mapsto Y_{1,aq^{12-s}}^{-1}$, $Y_{2,aq^s} \mapsto Y_{2,aq^{12-s}}^{-1}$ for all $a \in \mathbb{C}^{\times}, s \in \mathbb{Z}$. Then*

$$\chi_q(\widetilde{\mathcal{T}}_{k,l}^{(s)}) = \iota(\chi_q(\mathcal{T}_{k,l}^{(s)})).$$

Proof of Theorem 3.2.   The lowest weight monomial of $\chi_q(\mathcal{T}_{k,l}^{(s)})$ is obtained from the highest weight monomial of $\chi_q(\mathcal{T}_{k,l}^{(s)})$ by the substitutions: $1_s \mapsto 1_{12+s}^{-1}$, $2_s \mapsto 2_{12+s}^{-1}$. After we apply ι to $\chi_q(\mathcal{T}_{k,l}^{(s)})$, the lowest weight monomial of $\chi_q(\mathcal{T}_{k,l}^{(s)})$ becomes the highest weight monomial of $\iota(\chi_q(\mathcal{T}_{k,l}^{(s)}))$. Therefore the highest weight monomial of $\iota(\chi_q(\mathcal{T}_{k,l}^{(s)}))$ is obtained from the lowest weight monomial of $\chi_q(\mathcal{T}_{k,l}^{(s)})$ by the substitutions: $1_s \mapsto 1_{12-s}^{-1}$, $2_s \mapsto 2_{12-s}^{-1}$. It follows that the highest weight monomial of $\iota(\chi_q(\mathcal{T}_{k,l}^{(s)}))$ is obtained from the highest weight monomial of $\chi_q(\mathcal{T}_{k,l}^{(s)})$ by the substitutions: $1_s \mapsto 1_{-s}$, $2_s \mapsto 2_{-s}$. Therefore the second part of the M-system is obtained from the first part of the M-system by applying ι to both sides of every equation in the first part of the M-system.

The simplicity of every module corresponding to the summands on the right hand side of every equation in Theorem 3.2 follows from the simplicity of the modules corresponding to the summands on the right hand side of the equations in Theorem 3.1 and Lemma 7.2. ∎

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