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Reconstruction from Representations: Jacobi via Cohomology

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Abstract. A subalgebra of a Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ determines an $\mathfrak{h}$-representation ρ on $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$. In this note we discuss how to reconstruct $\mathfrak{g}$ from $(\mathfrak{h}, \mathfrak{m}, \rho)$. In other words, we find all the ingredients for building non-reductive Klein geometries. The Lie algebra cohomology plays a decisive role here.
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Introduction

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra. Let $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ be the quotient $\mathfrak{h}$-module with representation $\rho\colon \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$. We address the reconstruction problem for the Lie algebra structure of $\mathfrak{g}$ from the data $(\mathfrak{h}, \mathfrak{m}, \rho)$.

In this note the spaces $(\mathfrak{h}, \mathfrak{m})$ are finite-dimensional. We do not assume the existence of an embedding $\mathfrak{m} \subset \mathfrak{g}$ as a reductive ($\mathfrak{h}$-invariant) complement to $\mathfrak{h}$, as such embeddings exist in general only if $H^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h}) = 0$.

We show that the Lie algebra cohomology $H^1(\mathfrak{h}, \mathbb{V})$ for $\mathfrak{h}$-modules $\mathbb{V}$ (see Appendix) plays a key role in the reconstruction. Parametrizing Lie brackets on $\mathfrak{h} \oplus \mathfrak{m}$, the Jacobi identity constrains the parameters. Our cohomological approach allows to single out linear equations, and significantly reduce the amount of quadratic constraints.

Klein geometries are homogeneous spaces G/H, and such have been extensively studied for reductive subgroups H. Our method allows to effectively handle non-reductive Klein geometries via symbolic computations.

Some other approaches to the reconstruction in the case of filtered algebras via the deformation technique can be found in [12, 4, 3]. We also mention Cartan's procedure for construction of homogeneous models of a given geometry type [2].

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1. The main result

Since $\mathfrak{h}$ is a subalgebra, the bracket $\Lambda^2\mathfrak{h} \to \mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra structure on $\mathfrak{h}$, but since $\mathfrak{m}$ is not a reductive complement the other brackets on $\mathfrak{g}$ are:

$$\mathfrak{h} \otimes \mathfrak{m} \to \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad [h, u] = \varphi(h, u) + h \cdot u;$$
$$\Lambda^2\mathfrak{m} \to \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad [u_1, u_2] = \theta_\mathfrak{h}(u_1, u_2) + \theta_\mathfrak{m}(u_1, u_2),$$

where $h \in \mathfrak{h}$, $u, u_1, u_2 \in \mathfrak{m}$, $\varphi \in \mathfrak{h}^* \otimes \mathfrak{m}^* \otimes \mathfrak{h}$, $\theta_\mathfrak{h} \in \Lambda^2\mathfrak{m}^* \otimes \mathfrak{h}$, $\theta_\mathfrak{m} \in \Lambda^2\mathfrak{m}^* \otimes \mathfrak{m}$, and we denote (here and in what follows) $h \cdot u = \rho(h)u$.

The cohomology class of φ is known from the splitting theory for modules [6]. We interpret it as the complete obstruction to the existence of a reductive complement.

Proposition 1.1. *The element φ is closed in the complex $\Lambda^\bullet\mathfrak{h}^* \otimes \mathfrak{m}^* \otimes \mathfrak{h}$: $d\varphi = 0$, and it changes by an exact element when $\mathfrak{m} \subset \mathfrak{g}$ varies. Thus we get $[\varphi] \in H^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h})$ an invariant.*

Proof. The Jacobi identity with 2 arguments from $\mathfrak{h}$ is

$$[h_1, [h_2, u]] + [h_2, [u, h_1]] + [u, [h_1, h_2]] = 0.$$

The $\mathfrak{h}$-part of this is the representation property of ρ, while the $\mathfrak{m}$-part yields

$$[h_1, \varphi(h_2, u)] + \varphi(h_1, h_2 \cdot u) - [h_2, \varphi(h_1, u)] - \varphi(h_2, h_1 \cdot u) - \varphi([h_1, h_2], u) = 0, \text{ i.e.}$$
$$(h_1 \cdot \varphi)(h_2, u) - (h_2 \cdot \varphi)(h_1, u) + \varphi([h_1, h_2], u) = 0, \text{ i.e.}$$

$$d\varphi(h_1, h_2)(u) = 0 \quad \Leftrightarrow \quad \varphi \in Z^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h}).$$

If we change $\mathfrak{m}$ to $\tilde{\mathfrak{m}} = \text{graph}(\sigma) = \{\sigma(u) + u \in \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\}$ for $\sigma \in \mathfrak{m}^* \otimes \mathfrak{h}$ then φ is changed to $\tilde{\varphi} = \varphi + d\sigma$, so $[\varphi] \in Z^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h})/B^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h})$. ∎

Let us introduce the linear operator δ as the composition

$$\Lambda^i\mathfrak{h}^* \otimes \mathfrak{m}^* \otimes \mathfrak{h} \xrightarrow{\rho} \Lambda^i\mathfrak{h}^* \otimes \mathfrak{m}^* \otimes \mathfrak{m}^* \otimes \mathfrak{m} \xrightarrow{\text{alt}_{2,3}} \Lambda^i\mathfrak{h}^* \otimes \Lambda^2\mathfrak{m}^* \otimes \mathfrak{m}$$

where $\text{alt}_{2,3}$ is the skew-symmetrization by the corresponding arguments, e.g.

$$\delta\varphi(h)(u_1, u_2) = \varphi(h, u_1) \cdot u_2 - \varphi(h, u_2) \cdot u_1.$$

Lemma 1.2. *For the differential d in the complex $\Lambda^\bullet\mathfrak{h}^* \otimes \mathfrak{m}^* \otimes \mathfrak{h}$ we have: $[\delta, d] = 0$, whence $d\delta\varphi = 0$ and $\delta(\varphi + d\sigma) = \delta\varphi + d\delta\sigma$.*

Proof. This is because δ is obtained from an $\mathfrak{h}$-homomorphism of the coefficients module. ∎

Given $\theta_\mathfrak{m} \in \Lambda^2\mathfrak{m}^* \otimes \mathfrak{m}$ satisfying $\delta\varphi = d\theta_\mathfrak{m}$, let us introduce the nonlinear operator

$$Q : \mathfrak{h}^* \otimes \mathfrak{m}^* \otimes \mathfrak{h} \to \mathfrak{h}^* \otimes \Lambda^2\mathfrak{m}^* \otimes \mathfrak{h}$$

by $Q\varphi = \mathrm{Sq}_2(\varphi,\varphi) - \varphi\circ(\mathbf{1}_{\mathfrak{h}}\wedge\theta_{\mathfrak{m}})$, where Sq_2 is the composition

$$(\mathfrak{h}^*\otimes\mathfrak{m}^*\otimes\mathfrak{h})^{\otimes 2} = \mathfrak{h}^*\otimes\mathfrak{m}^*\otimes\mathfrak{h}\otimes\mathfrak{h}^*\otimes\mathfrak{m}^*\otimes\mathfrak{h} \xrightarrow{\mathrm{contr}_{3,4}} \mathfrak{h}^*\otimes\mathfrak{m}^*\otimes\mathfrak{m}^*\otimes\mathfrak{h} \xrightarrow{\mathrm{alt}_{2,3}} \mathfrak{h}^*\otimes\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h}$$

and $\mathrm{contr}_{3,4}$ is the contraction by the corresponding arguments. In other words,

$$Q\varphi(h)(u_1,u_2) = \varphi(\varphi(h,u_1),u_2) - \varphi(\varphi(h,u_2),u_1) - \varphi(h,\theta_{\mathfrak{m}}(u_1,u_2)).$$

Lemma 1.3. *We have $d(Q\varphi)=0$, i.e. $[Q\varphi]\in H^1(\mathfrak{h},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h})$.*

Proof. This follows by a straightforward computation: Let

$$\begin{aligned}\Xi(h_1,h_2,u_1,u_2) =& h_1\cdot\varphi(\varphi(h_2,u_1),u_2) - \varphi(\varphi(h_2,h_1\cdot u_1),u_2) - \varphi(\varphi(h_2,u_1),h_1\cdot u_2),\\ =&(h_1\cdot\varphi)(\varphi(h_2,u_1),u_2) + \varphi((h_1\cdot\varphi)(h_2,u_1),u_2).\end{aligned}$$

Then using $d\varphi=0$ and $d\theta_{\mathfrak{m}}=\delta\varphi$ (and $h_1\cdot h_2=[h_1,h_2]$) we get

$$\begin{aligned}d(Q\varphi)(h_1,h_2)(u_1,u_2) &= \Xi(h_1,h_2,u_1,u_2) - \Xi(h_1,h_2,u_2,u_1) - \Xi(h_2,h_1,u_1,u_2) + \Xi(h_2,h_1,u_2,u_1)\\ &\quad -(h_1\cdot\varphi)(h_2,\theta_{\mathfrak{m}}(u_1,u_2)) - \varphi(h_2,(h_1\cdot\theta_{\mathfrak{m}})(u_1,u_2))\\ &\quad +(h_2\cdot\varphi)(h_1,\theta_{\mathfrak{m}}(u_1,u_2)) + \varphi(h_1,(h_2\cdot\theta_{\mathfrak{m}})(u_1,u_2))\\ &\quad +\varphi(\varphi([h_1,h_2],u_1),u_2) - \varphi(\varphi([h_1,h_2],u_2),u_1) - \varphi([h_1,h_2],\theta_{\mathfrak{m}}(u_1,u_2))\\ &= \Xi(h_1,h_2,u_1,u_2) - \Xi(h_1,h_2,u_2,u_1) - \Xi(h_2,h_1,u_1,u_2) + \Xi(h_2,h_1,u_2,u_1)\\ &\quad +\varphi(h_1,\varphi(h_2,u_1)\cdot u_2) - \varphi(h_1,\varphi(h_2,u_2)\cdot u_1) - \varphi(h_2,\varphi(h_1,u_1)\cdot u_2) + \varphi(h_2,\varphi(h_1,u_2)\cdot u_1)\\ &\quad +\varphi(\varphi([h_1,h_2],u_1),u_2) - \varphi(\varphi([h_1,h_2],u_2),u_1)\\ &= (\varphi(h_2,u_1)\cdot\varphi)(h_1,u_2) + \varphi(\varphi(h_2,u_1)\cdot h_1,u_2) + \varphi(h_1,\varphi(h_2,u_1)\cdot u_2)\\ &\quad -(\varphi(h_2,u_2)\cdot\varphi)(h_1,u_1) - \varphi(\varphi(h_2,u_2)\cdot h_1,u_1) - \varphi(h_1,\varphi(h_2,u_2)\cdot u_1)\\ &\quad -(\varphi(h_1,u_1)\cdot\varphi)(h_2,u_2) - \varphi(\varphi(h_1,u_1)\cdot h_2,u_2) - \varphi(h_2,\varphi(h_1,u_1)\cdot u_2)\\ &\quad +(\varphi(h_1,u_2)\cdot\varphi)(h_2,u_1) + \varphi(\varphi(h_1,u_2)\cdot h_2,u_1) + \varphi(h_2,\varphi(h_1,u_2)\cdot u_1)\\ &= \varphi(h_2,u_1)\cdot\varphi(h_1,u_2) - \varphi(h_2,u_2)\cdot\varphi(h_1,u_1)\\ &\quad -\varphi(h_1,u_1)\cdot\varphi(h_2,u_2) + \varphi(h_1,u_2)\cdot\varphi(h_2,u_1) = 0.\end{aligned}$$

■

Let us define the linear operators $q:\mathfrak{m}^*\otimes\mathfrak{h}\to\mathfrak{h}^*\otimes\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h}$, $\sigma\mapsto q_\sigma$, and $p:(\Lambda^2\mathfrak{m}^*\otimes\mathfrak{m})^{\mathfrak{h}}\to\mathfrak{h}^*\otimes\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h}$, $\nu\mapsto p_\nu$, by the formulae

$$\begin{aligned}q_\sigma(h)(u_1,u_2) =& d\sigma(\varphi(h,u_1),u_2) - d\sigma(\varphi(h,u_2),u_1) + \varphi(d\sigma(h,u_1),u_2)\\ &- \varphi(d\sigma(h,u_2),u_1) + d\sigma(d\sigma(h,u_1),u_2) - d\sigma(d\sigma(h,u_2),u_1)\\ &- \varphi(h,\delta\sigma(u_1,u_2)) - d\sigma(h,\theta_{\mathfrak{m}}(u_1,u_2)) - d\sigma(h,\delta\sigma(u_1,u_2));\\ p_\nu(h)(u_1,u_2) =& \varphi(h,\nu(u_1,u_2)).\end{aligned}$$

For $\sigma\in\mathfrak{m}^*\otimes\mathfrak{h}$ and $\varphi,\theta_{\mathfrak{m}}$ as above define the elements $\phi_i\in\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h}$, $1\le i\le 4$, so

$$\begin{aligned}\phi_1(u_1,u_2) &= \varphi(\sigma(u_1),u_2) - \varphi(\sigma(u_2),u_1), &\quad \phi_2(u_1,u_2) &= [\sigma(u_1),\sigma(u_2)],\\ \phi_3(u_1,u_2) &= \sigma((\sigma(u_1)\cdot u_2 - (\sigma(u_2)\cdot u_1), &\quad \phi_4(u_1,u_2) &= \sigma(\theta_{\mathfrak{m}}(u_1,u_2).\end{aligned}$$

Lemma 1.4. *We have $d\,p_\nu = 0$ and $q_\sigma = d(\phi_1+\phi_2-\phi_3-\phi_4)$, whence $d\,q_\sigma = 0$.*

Proof. Here the computations are a bit more involved:

$$\begin{aligned}
d\phi_1(h)(u_1,u_2) &= (h\cdot\varphi)(\sigma(u_1),u_2)+\varphi(h\cdot\sigma(u_1),u_2)-\varphi(\sigma(h\cdot u_1),u_2)\\
&\quad -(h\cdot\varphi)(\sigma(u_2),u_1)-\varphi(h\cdot\sigma(u_2),u_1)+\varphi(\sigma(h\cdot u_2),u_1)\\
&= (\sigma(u_1)\cdot\varphi)(h,u_2)-(\sigma(u_2)\cdot\varphi)(h,u_1)-\varphi(\sigma(h\cdot u_1),u_2)+\varphi(\sigma(h\cdot u_2),u_1),\\
d\phi_2(h)(u_1,u_2) &= h\cdot(\sigma(u_1)\cdot\sigma(u_2))-\sigma(h\cdot u_1)\cdot\sigma(u_2)+\sigma(h\cdot u_2)\cdot\sigma(u_1),\\
d\phi_3(h)(u_1,u_2) &= h\cdot\sigma(\sigma(u_1)\cdot u_2)-\sigma(\sigma(h\cdot u_1)\cdot u_2)-\sigma(\sigma(u_1)\cdot(h\cdot u_2))\\
&\quad -h\cdot\sigma(\sigma(u_2)\cdot u_1)+\sigma(\sigma(h\cdot u_2)\cdot u_1)+\sigma(\sigma(u_2)\cdot(h\cdot u_1)).
\end{aligned}$$

Therefore (we again use $d\varphi=0$ and $d\theta_{\mathfrak{m}}=\delta\varphi$ and also the Jacobi identity)

$$\begin{aligned}
q_\sigma(h)(u_1,u_2) &= \varphi(h,u_1)\cdot\sigma(u_2)-\sigma(\varphi(h,u_1)\cdot u_2)-\varphi(h,u_2)\cdot\sigma(u_1)+\sigma(\varphi(h,u_2)\cdot u_1)\\
&\quad +\varphi(h\cdot\sigma(u_1),u_2)-\varphi(\sigma(h\cdot u_1),u_2)-\varphi(h\cdot\sigma(u_2),u_1)+\varphi(\sigma(h\cdot u_2),u_1)\\
&\quad +(h\cdot\sigma(u_1))\cdot\sigma(u_2)-\sigma(h\cdot u_1)\cdot\sigma(u_2)-\sigma((h\cdot\sigma(u_1))\cdot u_2)+\sigma(\sigma(h\cdot u_1)\cdot u_2)\\
&\quad -(h\cdot\sigma(u_2))\cdot\sigma(u_1)+\sigma(h\cdot u_2)\cdot\sigma(u_1)+\sigma((h\cdot\sigma(u_2))\cdot u_1)-\sigma(\sigma(h\cdot u_2)\cdot u_1)\\
&\quad -\varphi(h,\sigma(u_1)\cdot u_2)+\varphi(h,\sigma(u_2)\cdot u_1)-h\cdot\sigma(\theta_{\mathfrak{m}}(u_1,u_2))+\sigma(h\cdot\theta_{\mathfrak{m}}(u_1,u_2))\\
&\quad -h\cdot\sigma(\sigma(u_1)\cdot u_2)+\sigma(h\cdot(\sigma(u_1)\cdot u_2))+h\cdot\sigma(\sigma(u_2)\cdot u_1)-\sigma(h\cdot(\sigma(u_2)\cdot u_1))\\
&= [\varphi(h,u_1)\cdot\sigma(u_2)-\varphi(h,u_2)\cdot\sigma(u_1)+\varphi(h\cdot\sigma(u_1),u_2)-\varphi(\sigma(h\cdot u_1),u_2)\\
&\quad -\varphi(h\cdot\sigma(u_2),u_1)+\varphi(\sigma(h\cdot u_2),u_1)-\varphi(h,\sigma(u_1)\cdot u_2)+\varphi(h,\sigma(u_2)\cdot u_1)]\\
&\quad +[(h\cdot\sigma(u_1))\cdot\sigma(u_2)-(h\cdot\sigma(u_2))\cdot\sigma(u_1)-\sigma(h\cdot u_1)\cdot\sigma(u_2)+\sigma(h\cdot u_2)\cdot\sigma(u_1)]\\
&\quad -[h\cdot\sigma(\sigma(u_1)\cdot u_2)-h\cdot\sigma(\sigma(u_2)\cdot u_1)-\sigma(\sigma(h\cdot u_1)\cdot u_2)+\sigma(\sigma(h\cdot u_2)\cdot u_1)\\
&\quad +\sigma((h\cdot\sigma(u_1))\cdot u_2)-\sigma(h\cdot(\sigma(u_1)\cdot u_2))-\sigma((h\cdot\sigma(u_2))\cdot u_1)+\sigma(h\cdot(\sigma(u_2)\cdot u_1))]\\
&\quad -[\sigma(\varphi(h,u_1)\cdot u_2)-\sigma(\varphi(h,u_2)\cdot u_1)+(h\cdot(\sigma\circ\theta_{\mathfrak{m}}))(u_1,u_2)-\sigma((h\cdot\theta_{\mathfrak{m}})(u_1,u_2))]\\
&= d\phi_1(h)(u_1,u_2)+d\phi_2(h)(u_1,u_2)-d\phi_3(h)(u_1,u_2)-d\phi_4(h)(u_1,u_2).
\end{aligned}$$

The other computation is easy:

$$\begin{aligned}
d(p_\nu)(h_1,h_2)(u_1,u_2) &= d\varphi(h_1,h_2)(\nu(u_1,u_2))\\
&\quad -\varphi(h_1)((h_2\cdot\nu)(u_1,u_2))+\varphi(h_2)((h_1\cdot\nu)(u_1,u_2))=0.
\end{aligned}$$ ■

Thus $\mathrm{Im}(q_\sigma)\subset B^1(\mathfrak{h},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h})$, $\mathrm{Im}(p_\nu)\subset Z^1(\mathfrak{h},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h})$. Let us denote $\Pi_\varphi=\mathrm{Im}(p_\nu)\bmod B^1(\mathfrak{h},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h})\subset H^1(\mathfrak{h},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h})$.

We are now ready to state our main result.

Theorem 1.5. *The Jacobi identity* $\mathrm{Jac}(v_1,v_2,v_3)=0$ *with 1 argument from* $\mathfrak{h}$ *and the others from* $\mathfrak{m}$ *gives the following constraints on the cohomology* $[\varphi]\in H^1(\mathfrak{h},\mathfrak{m}^*\otimes\mathfrak{h})$*: (1)* $[\delta\varphi]=0\in H^1(\mathfrak{h},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{m})$*, whence* $\delta\varphi=d\theta_{\mathfrak{m}}$*;*
(2) $[Q\varphi]\equiv 0\in H^1(\mathfrak{h},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h})\bmod\Pi_\varphi$*, so* $Q\varphi=d\theta_{\mathfrak{h}}$ *for some choices of* $\varphi,\theta_{\mathfrak{m}}$*.*

Proof. Consider the Jacobi identity with 1 argument from $\mathfrak{h}$:

$$[h,[u_1,u_2]]+[u_1,[u_2,h]]+[u_2,[h,u_1]]=0.$$

Taking $\mathfrak{m}$-part of this identity (this is canonical: projection along $\mathfrak{h}$), we obtain

$$\delta\varphi(h)(u_1,u_2)=(h\cdot\theta_{\mathfrak{m}})(u_1,u_2)\quad\Leftrightarrow\quad\delta\varphi=d\theta_{\mathfrak{m}}\in B^1(\mathfrak{h},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{m}).$$

This implies (1).

Two remarks are in order. First, changing $\varphi\mapsto\varphi+d\sigma$ we do not alter the property $[\delta\varphi]=0$. Second, $\theta_{\mathfrak{m}}$ is determined by constraint (1) modulo

$(\Lambda^2\mathfrak{m}^*\otimes\mathfrak{m})^{\mathfrak{h}}$. This changes $Q\varphi$ by an element $p_\nu\in Z^1(\mathfrak{h},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{m})$ due to Lemma 1.4.

To obtain (2) consider the $\mathfrak{h}$-part of the Jacobi identity. Note that if we change $\varphi\mapsto\varphi+d\sigma$, then $\theta_{\mathfrak{m}}\mapsto\theta_{\mathfrak{m}}+\delta\sigma$, so $Q\varphi\mapsto Q\varphi+q_\sigma$. This leaves $Q\varphi$ closed by Lemma 1.3 and does not change the cohomology class by Lemma 1.4. However, the latter is influenced by the change of $\theta_{\mathfrak{m}}$ as indicated above: $[Q\varphi]\mapsto[Q\varphi]+[p_\nu]$.

The $\mathfrak{h}$-part of the Jacobi identity can be written for some $p_\nu\in(\Lambda^2\mathfrak{m}^*\otimes\mathfrak{m})^{\mathfrak{h}}$ (the freedom in a choice of $\theta_{\mathfrak{m}}$) as

$$Q\varphi+p_\nu=d\theta_{\mathfrak{h}}\quad\Leftrightarrow\quad[Q\varphi]\equiv 0 \bmod \Pi_\varphi.$$

This implies (2). ■

Now the reconstruction algorithm from the data $(\mathfrak{h},\mathfrak{m},\rho)$ is the following:

- Compute $H^1(\mathfrak{h},\mathbb{V})$ for $\mathbb{V}=\mathfrak{m}^*\otimes\mathfrak{h},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{m},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h}$.
- Constrain the Lie bracket parameters by (1) and (2) in the theorem.
- Constrain them further by the quadratic relations $\mathrm{Jac}_{\mathfrak{m}}:\Lambda^3\mathfrak{m}\to\mathfrak{h}\oplus\mathfrak{m}$ so:

$$\mathfrak{S}\left[\varphi(\theta_{\mathfrak{h}}(u_1,u_2),u_3)+\theta_{\mathfrak{h}}(\theta_{\mathfrak{m}}(u_1,u_2),u_3)\right]=0,$$
$$\mathfrak{S}\left[\theta_{\mathfrak{h}}(u_1,u_2)\cdot u_3+\theta_{\mathfrak{m}}(\theta_{\mathfrak{m}}(u_1,u_2),u_3)\right]=0,$$

where $\mathfrak{S}$ is the cyclic summation by indices $1,2,3$.

2. Some specifications

In this section we discuss computation of the cohomology involved in the constraints of the theorem in several cases.

1: Reductive isotropy. Let us first note that if $\mathfrak{h}$ is a semi-simple Lie algebra then $H^1(\mathfrak{h},\mathbb{V})=0$ for any $\mathfrak{h}$-module $\mathbb{V}$ (Whitehead's lemma [13]). The same holds true if $\mathfrak{h}$ is reductive with the center action on $\mathbb{V}$ being semi-simple and with no linear invariants. Then choosing $\varphi=0$ the conditions of the theorem equivalently mean that

$$\theta_{\mathfrak{h}}:\Lambda^2\mathfrak{m}\to\mathfrak{h},\ \theta_{\mathfrak{m}}:\Lambda^2\mathfrak{m}\to\mathfrak{m}$$

are $\mathfrak{h}$-equivariant maps. The parameters of these maps are constrained by the quadratic conditions $\mathrm{Jac}_{\mathfrak{m}}=0$. Examples of reconstruction of $\mathfrak{g}$ in the case of $\mathfrak{h}=\mathfrak{sl}(2),\mathfrak{su}(2)$ and $\dim\mathfrak{m}=6$ can be found in [1].

2: Internal gradings. Notice that if $\mathfrak{h}$ is graded $\mathfrak{h}=\oplus_\alpha\mathfrak{h}_\alpha$ with $\mathfrak{h}_0$ Abelian and $\mathbb{V}$ is a graded $\mathfrak{h}$-module $\mathbb{V}=\oplus\mathbb{V}_\alpha$, then [4] the cohomology $H^\bullet(\mathfrak{h},\mathbb{V})$ of the complex $\Lambda^\bullet\mathfrak{h}^*\otimes\mathbb{V}$ coincides with that of the subcomplex $(\Lambda^\bullet\mathfrak{h}^*\otimes\mathbb{V})_0$ of total degree zero cochains, and the same is true if we use multi-grading given by $\mathfrak{h}$ (i.e. α takes values in a multi-dimensional lattice).

In particular, if $\mathfrak{h}$ is Abelian, and $\mathbb{V}$ completely reducible, then $H^1(\mathfrak{h},\mathbb{V})=\mathfrak{h}^*\otimes\mathbb{V}_0$, where $\mathbb{V}_0=\mathbb{V}^{\mathfrak{h}}$ is the trivial component. Indeed, for any reductive Lie algebra $\mathfrak{h}$ with the center acting semi-simply on $\mathbb{V}$ we have $H^\bullet(\mathfrak{h},\mathbb{V}) = H^\bullet(\mathfrak{h})\otimes\mathbb{V}^{\mathfrak{h}}$ [4] and if $\mathfrak{h}$ is Abelian, then the differential of its cohomology complex vanishes.

3: One-dimensional space of splittings. By this we mean the case $\dim H^1(\mathfrak{h},\mathfrak{m}^*\otimes\mathfrak{h}) = 1$ and we can normalize $[\varphi] = 0 \vee 1$. In the case $[\varphi] = 0$ an $\mathfrak{h}$-invariant splitting $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$ exists, and the solutions $(\theta_{\mathfrak{m}},\theta_{\mathfrak{h}})$ of the theorem can be found as $\mathfrak{h}$-equivariant elements $(\Lambda^2\mathfrak{m}^*\otimes\mathfrak{g})^{\mathfrak{h}}$. If $[\varphi] = 1$, then the solutions space to the constraints of Theorem 1.5 is an affine space modelled on $(\Lambda^2\mathfrak{m}^*\otimes\mathfrak{g})^{\mathfrak{h}}$. Indeed, the system of equations $d(\theta_{\mathfrak{m}},\theta_{\mathfrak{h}}) = (\delta\varphi, Q\varphi)$ is linear inhomogeneous in the unknowns $(\theta_{\mathfrak{m}},\theta_{\mathfrak{h}})$.

With one solution (often) a-priori known it is easy to parametrize all solutions (to the constraints of Theorem 1.5; these parameters are yet subject to the Jacobi constraints with 3 arguments from $\mathfrak{m}$).

4: Cartan subalgebra. Let $\mathfrak{h}$ be a Cartan subalgebra of a semi-simple Lie algebra $\mathfrak{g}$, and $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$. Since $\mathfrak{m}_0 = \mathfrak{m}^{\mathfrak{h}} = 0$ we have $H^\bullet(\mathfrak{h},\mathfrak{m}^*\otimes\mathfrak{h}) = 0$. In particular, there is a reductive complement. For the split real form $\mathfrak{g}$, we can take the sum of root spaces $\mathfrak{m} = \sum_\alpha \mathbb{R}\cdot e_\alpha$ as such.

The other cohomologies involved in Theorem 1.5 are non-trivial (this will have an implication in the next section): $H^\bullet(\mathfrak{h},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{m}) = \Lambda^\bullet\mathfrak{h}^*\otimes(\Lambda^2\mathfrak{m}^*\otimes\mathfrak{m})_0$ (with 0 referring to the multi-grading induced by a set of simple roots; if the real form $\mathfrak{g}$ is not split, the complexification can be used at this step), and similarly $H^\bullet(\mathfrak{h},\Lambda^2\mathfrak{m}^*\otimes\mathfrak{h}) = \Lambda^\bullet\mathfrak{h}^*\otimes(\Lambda^2\mathfrak{m}^*)_0\otimes\mathfrak{h}$.

Note that the submodule $(\Lambda^2\mathfrak{m}^*)_0$ is generated by the elements $e_\alpha\otimes\theta^\alpha$ ($\equiv \theta^{-\alpha}\wedge\theta^\alpha$ via the Killing form), where θ^α is the co-basis dual to the basis e_α of $\mathfrak{m}$, and similarly the submodule $(\Lambda^2\mathfrak{m}^*\otimes\mathfrak{m})_0$ is generated by the elements $e_{\alpha+\beta}\otimes\theta^\alpha\wedge\theta^\beta$. Thus both submodules and hence the indicateed cohomology groups are non-trivial.

5: Nilradical of a parabolic. Let $\mathfrak{g} = \oplus\mathfrak{g}_i$ be a semi-simple Lie algebra with the grading induced by the parabolic subalgebra $\mathfrak{p} = \oplus_{i\geq 0}\mathfrak{g}_i$. We choose as subalgebra the nilradical of the opposite parabolic: $\mathfrak{h} = \oplus_{i<0}\mathfrak{g}_i$, and $\mathfrak{m} = \mathfrak{g}/\mathfrak{h} = \mathfrak{p}$.

The cohomology $H^\bullet(\mathfrak{h},\mathbb{V})$ for $\mathfrak{g}$-modules $\mathbb{V}$ (restricted to $\mathfrak{h}\subset\mathfrak{g}$) is given (as a $\mathfrak{g}_0$-module) by Kostant's version of the Bott-Borel-Weyl theorem [7], however in the case $\mathfrak{m} = \mathfrak{p}$ it is not a $\mathfrak{g}$-module and the computations are more involved.

We compute the cohomology $H^1(\mathfrak{h},\mathfrak{m}^*\otimes\mathfrak{h})$ in the case of $|1|$-gradings[1]: $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{g}_0\oplus\mathfrak{h}^*$ (where $\mathfrak{g}_1$ is identified with $\mathfrak{h}^* = \mathfrak{g}_{-1}^*$ via the Killing form, and similarly $\mathfrak{g}_0^* = \mathfrak{g}_0$). We have: $\mathfrak{m} = \mathfrak{g}_0\oplus\mathfrak{h}^*$, $\mathfrak{m}^* = \mathfrak{g}_0\oplus\mathfrak{h}$ (this decomposition of modules is not $\mathfrak{h}$-invariant; to get an invariant computation one should use the spectral sequences based on the $\mathfrak{h}$-invariant filtration, but we skip doing so).

The cochain complex for the cohomology is

$$0\to(\mathfrak{g}_0\oplus\mathfrak{h})\otimes\mathfrak{h}\xrightarrow{d_0}\mathfrak{h}^*\otimes(\mathfrak{g}_0\oplus\mathfrak{h})\otimes\mathfrak{h}\xrightarrow{d_1}\Lambda^2\mathfrak{h}^*\otimes(\mathfrak{g}_0\oplus\mathfrak{h})\otimes\mathfrak{h}\to\dots$$

where d_0 projects the first term to $\mathfrak{g}_0\otimes\mathfrak{h}\subset\mathfrak{h}^*\otimes\mathfrak{h}\otimes\mathfrak{h}$ and d_1 projects the second term to $\mathfrak{h}^*\otimes\mathfrak{g}_0\otimes\mathfrak{h}\xrightarrow{\delta\otimes\mathrm{id}_{\mathfrak{h}}}\Lambda^2\mathfrak{h}^*\otimes\mathfrak{h}\otimes\mathfrak{h}$, with $\delta:\mathfrak{h}^*\otimes\mathfrak{g}_0\subset\mathfrak{h}^*\otimes\mathfrak{h}^*\otimes\mathfrak{h}\to\Lambda^2\mathfrak{h}^*\otimes\mathfrak{h}$

[1]The $|1|$-gradings of simple complex Lie algebras are: A_ℓ/P_k, B_ℓ/P_1, C_ℓ/P_ℓ, D_ℓ/P_1, D_ℓ/P_ℓ, E_6/P_6, E_7/P_7 in Bourbaki's ordering of the nodes on the Dynkin diagram. Which of those extend to the real versions is decided by the Satake diagram.

being the Spencer operator (skew-symmetrization). Thus $B^1 = \mathfrak{g}_0 \otimes \mathfrak{h} \subset \mathfrak{h}^* \otimes \mathfrak{h} \otimes \mathfrak{h}$, $Z^1 = \mathfrak{g}_0^{(1)} \otimes \mathfrak{h} \oplus \mathfrak{h}^* \otimes \mathfrak{h} \otimes \mathfrak{h}$, where $\mathfrak{g}_0^{(1)} = \mathrm{Ker}(\delta) = \mathfrak{g}_0 \otimes \mathfrak{h} \cap S^2\mathfrak{h}^* \otimes \mathfrak{h}$ is the Sternberg-Spencer prolongation of $\mathfrak{g}_0$, whence $H^1 = Z^1/B^1 = [\mathfrak{g}_0^{(1)} + \mathfrak{h}^* \otimes \mathfrak{h}/\mathfrak{g}_0] \otimes \mathfrak{h}$.

By Yamaguchi's prolongation theorem [14] we have $\mathfrak{g}^{(1)} = \mathfrak{g}_1 = \mathfrak{h}^*$ in all $|1|$-graded cases except A_ℓ/P_1 (and dually A_ℓ/P_ℓ). In the latter case $\mathfrak{g}_0 = \mathfrak{gl}(\mathfrak{h})$, so $\mathfrak{g}^{(1)} = S^2\mathfrak{h}^* \otimes \mathfrak{h}$. Thus we conclude

$$H^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h}) = \begin{cases} S^2\mathfrak{h}^* \otimes \mathfrak{h} \otimes \mathfrak{h} & \text{when the grading is } A_\ell/P_1, \\ \left[\mathfrak{h}^* + \dfrac{\mathfrak{h}^* \otimes \mathfrak{h}}{\mathfrak{g}_0}\right] \otimes \mathfrak{h} & \text{otherwise.} \end{cases}$$

3. Applications

In this section we consider some simple applications, illustrating the developed technique. More substantial outcomes can be extracted from [10, 11] (the first reference is in retrospective, while the second essentially uses the results of this paper). Some applications to reconstruction from the deformation technique can be found in [5, 9].

We start with a toy example: one-dimensional subalgebra $\mathfrak{h} \subset \mathfrak{sl}_2$. Let $\mathfrak{m} = \mathfrak{sl}_2/\mathfrak{h}$ and ρ be the isotropy representation. If $\mathfrak{h}$ is a (non-compact) Cartan subalgebra, then the triple $(\mathfrak{h}, \mathfrak{m}, \rho)$ recovers either the original Lie algebra $\mathfrak{sl}_2$ or the algebra $\mathbb{R} \ltimes \mathbb{R}^2$, where the action of $\mathfrak{h} = \mathbb{R}$ on the Abelian piece $\mathbb{R}^2$ is by the matrix $\mathrm{diag}(-1, 1)$. If, on the other hand, $\mathfrak{h}$ is nilpotent, then the triple $(\mathfrak{h}, \mathfrak{m}, \rho)$ recovers either the original semi-simple Lie algebra or a solvable algebra $\mathfrak{g}$ with $[\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{h}$.

Now if $\mathfrak{h}$ is two-dimensional (Borel), then $(\mathfrak{h}, \mathfrak{m}, \rho)$ recovers either the original $\mathfrak{g} = \mathfrak{sl}_2$ or the above $\mathbb{R} \ltimes \mathbb{R}^2 = \langle e_1, e_2, e_3 : [e_1, e_2] = e_2, [e_1, e_3] = -e_3] \rangle$, $\mathfrak{h} = \langle e_1, e_2 \rangle$. Note that for the corresponding homogeneous space G/H the isotropy has a kernel: in the Lie subalgebra $\mathfrak{h}$ the element e_2 generates an ideal and thus acts non-effectively. As a result we obtain an effective homogeneous representation $G/H = G'/H'$ with Lie algebras corresponding to the groups: $\mathfrak{g}' = \langle e_1, e_3 \rangle$, $\mathfrak{h} = \langle e_1 \rangle$.

This motivates the following

Definition 3.1. A pair $(\mathfrak{h}, \mathfrak{g})$ reconstruction rigid, if $\mathfrak{g}$ is the unique non-flat algebra with the isotropy data $(\mathfrak{h}, \mathfrak{m}, \rho)$ and no nontrivial $\mathfrak{g}$-ideals supported in $\mathfrak{h}$.

Recall that the flat algebra is $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$, where $\mathfrak{m}$ is Abelian and the bracket $\mathfrak{h} \wedge \mathfrak{m} \to \mathfrak{m}$ is ρ. Now we consider three examples.

1: Borel subalgebra in $\mathfrak{sl}_3$. Let $\mathfrak{h} = \mathfrak{b}$ be a minimal parabolic (maximal solvable) subalgebra in $\mathfrak{sl}_3$ (realized as the set of upper-triangular matrices) and $\mathfrak{m} = \mathfrak{sl}_3/\mathfrak{h}$. The cohomology group $H^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h}) = \mathbb{R}$, so there exists a unique (one-dimensional) obstruction to finding a reductive complement to $\mathfrak{h}$. If the obstruction vanishes, then the Lie algebra structure on $\mathfrak{h} \oplus \mathfrak{m}$ is encoded by $(\Lambda^2\mathfrak{m}^* \otimes (\mathfrak{h} \oplus \mathfrak{m}))^{\mathfrak{h}} = 0$. So this case is flat: the only possible Lie algebra structure is $\mathfrak{g} = \mathfrak{b} \ltimes \mathbb{R}^3$.

If the obstruction $[\varphi] \in \mathbb{R}$ is non-zero, it can be normalized to 1. In this case we compute the other cohomologies[2]: $H^1(\mathfrak{h}, \Lambda^2\mathfrak{m}^*\otimes\mathfrak{m}) = \mathbb{R}^3$, $H^1(\mathfrak{h}, \Lambda^2\mathfrak{m}^*\otimes\mathfrak{h}) = 0$. The first constraint of our theorem, $\delta\varphi = d\theta_{\mathfrak{m}}$, is non-trivial and locks all the structure constants to be equal to that of $\mathfrak{sl}_3$. Thus $(\mathfrak{b}, \mathfrak{sl}_3)$ is reconstruction rigid. This shows the uniqueness of (effective) homogeneous representation with 5D isotropy of the flag variety $F(1,2,3) = \mathrm{SL}_3(\mathbb{R})/B$ (this Klein geometry is non-reductive unless we reduce the symmetry group: $F(1,2,3) = SO(3)/\mathbb{Z}_2^2$).

Note that the isotropy data of the homogeneous space $M^3 = G/H$ include the invariant contact 2-distribution (it is integrable in the flat case) that is split as a sum of two line fields $\Pi^2 = L_1^1 \oplus L_2^1 \subset TM$. This geometric structure corresponds to a second order ODE with respect to point transformations. Our computation shows that there exists a unique ODE with symmetry of the maximal dimension 8 (this is indeed $y''(x) = 0$ viewed as a pair of line fields – vertical and the total derivative – on $J^1(\mathbb{R}, \mathbb{R}) = M^3(x, y, y')$). In fact, it is known that the other homogeneous geometries encoding second order ODEs can exist on 3-dimensional Lie groups only (trivial isotropy). Our approach allows us to independently verify this.

2: Cartan subalgebra in $\mathfrak{sl}_3$. Let $\mathfrak{h} = \mathfrak{c}$ be a Cartan subalgebra in $\mathfrak{sl}_3$ (realized as the set of diagonal matrices) and $\mathfrak{m} = \mathfrak{sl}_3/\mathfrak{h}$. Since we know from the previous section that $H^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h}) = 0$ and $\mathfrak{m} \subset \mathfrak{g}$ can be $\mathfrak{h}$-invariant, the reconstruction is reduced to classifying $\mathfrak{h}$-equivariant maps $\Lambda^2\mathfrak{m} \to \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. A simple computation shows that the results of reconstruction is as follows:

1. $\mathfrak{g} = \mathfrak{sl}_3$, $\mathfrak{h} = \mathfrak{c}$;
2. $\mathfrak{g} = \mathfrak{gl}_2 \ltimes W$, $\mathfrak{h} \subset \mathfrak{gl}_2$ is the diagonal subalgebra, W is Abelian and is $\mathbb{R}^2 \oplus \mathbb{R}^{2*}$ as $\mathfrak{gl}_2$-representation;
3. $\mathfrak{g} = \mathbb{R}^2 \ltimes W$, where $W = \mathbb{R}^3 + \Lambda^2\mathbb{R}^3$ is the 2-step nilpotent algebra with the derived series of dimensions $(6, 3, 0)$;
4. $\mathfrak{g} = \mathbb{R}^2 \ltimes W$, where $W = ((\mathbb{R} \oplus \mathbb{R}^2) + \mathbb{R} \otimes \mathbb{R}^2) \oplus \mathbb{R}$ and the first summand is the 2-step nilpotent algebra with the derived series of dimensions $(5, 2, 0)$;
5. $\mathfrak{g} = \mathbb{R}^2 \ltimes W$, where $W = \mathfrak{heis}_3 \oplus \mathfrak{heis}_3$ is the sum of two Heisenberg algebras;
6. $\mathfrak{g} = \mathbb{R}^2 \ltimes W$, where $W = \mathfrak{heis}_3 \oplus \mathbb{R}^3$;
7. $\mathfrak{g} = \mathbb{R}^2 \ltimes W$, where $W = \mathbb{R}^6$ is the Abelian algebra (this is the flat case).

In cases (3)-(7) the action of $\mathfrak{h} = \mathbb{R}^2$ on W is chosen to match the isotropy data.

Thus $(\mathfrak{c}, \mathfrak{sl}_3)$ is not reconstruction rigid, and the real version $\mathrm{SL}_3(\mathbb{R})/(\mathbb{R}^\times)^2$ of the complex flag variety $F^{\mathbb{C}}(1,2,3) = \mathrm{SL}_3(\mathbb{C})/B^{\mathbb{C}} = U(3)/T^3 = SU(3)/T^2$ is not (uniquely) recoverable from its isotropy data.

3: Homogeneous 4D almost complex spaces with Sol_2 isotropy. Let us investigate 4-dimensional almost complex homogeneous spaces with 2-dimensional solvable (non-abelian) isotropy, i.e. $\mathfrak{h}$ preserves a complex structure on

[2]In this and subsequent computations the *DifferentialGeometry* package of MAPLE was used.

$\mathfrak{m}$. We assume the isotropy representation ρ to be effective. If the almost complex structure J is non-integrable, then the isotropy representation $\rho : \mathfrak{sol}_2 \to \mathfrak{gl}_2(\mathbb{C})$ also leaves invariant the Nijenhuis tensor $0 \neq N_J \in \Lambda^2_{\mathbb{C}}\mathbb{C}^{2*} \otimes_{\bar{\mathbb{C}}} \mathbb{C}^2$: $\rho(v) \cdot N_J = 0$ $\forall v \in \mathfrak{sol}_2$. This uniquely normalizes the representation in some complex basis of $\mathbb{C}^2$ so:

$$e_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad [e_1, e_2] = e_2.$$

Now we compute the cohomology:

$$H^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h}) = \mathbb{R}^2,\ H^1(\mathfrak{h}, \Lambda^2\mathfrak{m}^* \otimes \mathfrak{m}) = \mathbb{R}^8,\ H^1(\mathfrak{h}, \Lambda^2\mathfrak{m}^* \otimes \mathfrak{h}) = \mathbb{R}^4.$$

Thus a-priori we get a non-reductive decomposition and the bracket $\mathfrak{h} \wedge \mathfrak{m} \to \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ takes values in both summands, being parametrized by 2 real numbers. These two parameters are however forced to vanish by the further homological constraints of Theorem 1.5 (to be precise, by $[\delta\varphi] = 0 \in H^1(\mathfrak{h}, \Lambda^2\mathfrak{m}^* \otimes \mathfrak{m})$ forming 8 scalar linear equations). Thus a-posteriori the $\mathfrak{h}$-module $\mathfrak{g}$ splits, and the remaining brackets can be chosen as elements of the module $(\Lambda^2\mathfrak{m}^* \otimes \mathfrak{g})^{\mathfrak{h}}$ satisfying the Jacobi identity.

These are computable, however another look at the Jacobi identity shows that the complement $\mathfrak{m} \subset \mathfrak{g}$ can be chosen in such a way that both $\mathfrak{h}$ and $\mathfrak{m}$ are subalgebras, though the bracket $[\mathfrak{h}, \mathfrak{m}]$ takes values in the whole $\mathfrak{g}$. This still allows to conclude that the almost complex structure J on the homogeneous space $M = G/H$ coincides with the left-invariant structure on some Lie group M^4. Such groups are classified, but we do not include this rather long list.

Finally notice that by [8] a symmetry of a (non-integrable) almost complex structure J on a 4-manifold M is at most 4D (in this case J is the left-invariant structure on a Lie group) unless the 2-distribution $\mathrm{Im}(N_J) \subset TM$ is integrable and J, N_J are projectible along its leaves. We may also observe this from the obtained structure equations of $\mathfrak{h} \oplus \mathfrak{m}$ in our case.

A. Lie algebra cohomology

For completeness recall the formula for the Lie algebra differential in the complex $\Lambda^\bullet\mathfrak{h}^* \otimes \mathbb{V}$ for $H^\bullet(\mathfrak{h}, \mathbb{V})$. If $\varphi \in \Lambda^k\mathfrak{h}^* \otimes \mathbb{V}$ and $h_i \in \mathfrak{h}$, then

$$d\varphi(h_0, \dots, h_k) = \sum (-1)^i h_i \cdot \varphi(h_0, \dots, \check{h}_i, \dots, h_k) \\ +(-1)^{i+j}\varphi([h_i, h_j], h_0, \dots, \check{h}_i, \dots, \check{h}_j, \dots, h_k).$$

Note that if $d\varphi = 0$, then $h \cdot \varphi = d\, i_h\varphi$, where $i_h\varphi = \varphi(h, \cdot) \in \Lambda^{k-1}\mathfrak{h}^* \otimes \mathbb{V}$ is the result of substitution of h as the first argument. Thus $H^\bullet(\mathfrak{h}, \mathbb{V}) = H^\bullet(\mathfrak{h}, \mathbb{V})^{\mathfrak{h}}$ and the equivariancy (of e.g. $[\varphi]$) is not a constraint on the cohomology classes.

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