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Rigidity of an Isometric SL(3, ℝ)-action

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Abstract. We describe the universal covering space of a finite volume connected analytic pseudo-Riemannian manifold with dimension at most 14 that admits a non-trivial isometric analytic action of the simple Lie group $\mathrm{SL}(3,\mathbb{R})$ with a dense orbit. If such manifold is also weakly irreducible we prove that $\widetilde{M}$ is isometric to, or a quotient space of a simple Lie group containing $\mathrm{SL}(3,\mathbb{R})$.
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1. Introduction

Let G be a connected non-compact simple Lie group acting on a connected analytic manifold M preserving a pseudo-Riemannian structure with finite volume. It has been conjectured that such actions are rigid in the sense that the manifold M belongs to a small family of possibilities. More precisely, assuming non-triviality conditions, the manifold has been conjectured to be basically an algebraic double coset $K\backslash H/\Gamma$, where H is a Lie group containing G and centralizing the compact subgroup K of H, and $\Gamma \subset H$ is a lattice. Furthermore, the G-action is given by a natural left action on $K\backslash H/\Gamma$.

For the above setup, in [5] it is assumed that $G = \widetilde{\mathrm{SO}}_0(p,q)$ and that M is a complete and weakly irreducible manifold with dimension $\leq \dim \widetilde{\mathrm{SO}}_0(p,q) + p + q$. It is also assumed that M and the action are both analytic. Then, it is proved that the manifold M is (up to a finite covering) the quotient of either $\widetilde{\mathrm{SO}}_0(p,q+1)$ or $\widetilde{\mathrm{SO}}_0(p+1,q)$ by a lattice. A similar result for actions of the reductive group $\mathrm{U}(p,q)$ is obtained in [6].

Following this line of research, in this work we consider an isometric action of $\mathrm{SL}(3,\mathbb{R})$ with a dense orbit on a complete pseudo-Riemannian manifold M such that $8 < \dim(M) \leq 14$. Our main result is the following. In the rest of this work we denote by $G_{2(2)}$ the simply connected (centerless) group whose Lie algebra is the non-compact simple real Lie algebra of type G_2.

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Theorem 1.1. *Let M be a connected analytic pseudo-Riemannian manifold. Suppose that M is complete, has finite volume and admits an analytic and isometric $\mathrm{SL}(3,\mathbb{R})$-action with a dense orbit. If $8 < \dim(M) \leq 14$, then there is an $\mathrm{SL}(3,\mathbb{R})$-invariant metric $\widehat{g}$ on M and an isometry φ from $\widetilde{M}$ onto one of the following*

1. $\widetilde{\mathrm{SL}}(3,\mathbb{R}) \times \widetilde{N}$*, where $\widetilde{N}$ is a complete pseudo-Riemannian manifold, with the product metric and for $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ carrying a bi-invariant metric.*
2. $G_{2(2)}$ *endowed with some bi-invariant metric.*
3. $\mathbb{R}\backslash\widetilde{\mathrm{SL}}(4,\mathbb{R})$*, where $\mathbb{R}$ is the pull-back of a subgroup of diagonal matrices in $\mathrm{SL}(4,\mathbb{R})$, and with a metric induced by a bi-invariant metric on $\widetilde{\mathrm{SL}}(4,R)$.*

For the next result, we recall that a connected pseudo-Riemannian manifold is weakly irreducible if there is no proper non-degenerate subspace of the tangent space at some (and hence any) point invariant under the restricted holonomy group at that point. Hence, the following is a particular case of Theorem 1.1

Theorem 1.2. *With the hypotheses of Theorem 1.1, if M is weakly irreducible manifold, then the only conclusions that can occur in Theorem 1.1 are 2. and 3.*

Finally, we have a positive answer to the above mentioned conjecture for the first case of Theorem 1.2.

Theorem 1.3. *Let M be as in the hypotheses of Theorem 1.2. If $\widetilde{M}$ is isometric to $G_{2(2)}$, then there exist*

- *a lattice $\Gamma \subset G_{2(2)}$, and*
- *an analytic finite covering map $\pi : G_{2(2)}/\Gamma \to M$*

such that π is a $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-equivariant map, where the $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-action on $G_{2(2)}/\Gamma$ is induced by some non-trivial homomorphism $\widetilde{\mathrm{SL}}(3,\mathbb{R}) \to G_{2(2)}$.

We observe that further information for the case 3. of Theorem 1.1 is provided by Proposition 6.2.

In [5] and [6] the study of the irreducible representations of the corresponding groups preserving a non-degenerate, symmetric, bilinear form is fundamental to analyze the properties of the manifold M. Similarly, we analyze these sort of representations of $\mathrm{SL}(3,R)$ and in particular the non-trivial representation of $\mathrm{SL}(3,\mathbb{R})$ with minimal dimension that admits an invariant non-degenerate, symmetric bilinear form. Hence, this work builds on the ideas from the previous references.

The proof of Theorems 1.1, 1.2 and 1.3 are based on the application of representation theory to the Killing vector fields that centralize the action of the group $\widetilde{\mathrm{SL}}(3,\mathbb{R})$. A fundamental result for this work is Proposition 3.2, which appears in [5] and [7] and whose detailed proof can be found in [8]. In particular, our techniques rely heavily on Gromov-Zimmer machinery.

The organization of this paper is the following. In Section 2 we present the lowest dimensional non-trivial representation W of $\mathfrak{sl}(3,\mathbb{R})$ that admits an invariant metric. We also describe the decomposition of $\mathfrak{so}(W)$ as a direct sum of irreducible $\mathfrak{sl}(3,\mathbb{R})$-modules. In Section 3, we show results that guarantee the existence of a large centralizer for the action on the manifold M. In Section 4 we study the properties of the centralizer as a $\mathfrak{sl}(3,\mathbb{R})$-module. In Section 5 we analyze the structure of the centralizer which permit us to restrict the possibilities of the manifold M. Finally, in Section 6 we use the previous result to prove the theorems stated in this introduction.

2. Some remarks on the representations of $\mathfrak{sl}(3,\mathbb{R})$

We recall that a non-trivial representation of $\mathfrak{sl}(3,\mathbb{R})$ is called auto-dual if it preserves a non-degenerated bilinear form.

Since $\mathfrak{sl}(3,\mathbb{R})$ is split we can apply Weyl's dimension formulas. In particular, it is easy to see that the non-trivial irreducible representations of $\mathfrak{sl}(3,\mathbb{R})$ with dimension ≤ 8 occur in dimensions 3, 6 and 8. There are precisely two with dimension 3, which are $\mathbb{R}^3$ and $\mathbb{R}^{3*}$ and correspond to the fundamental weights ω_1 and ω_2. For dimension 6 there are exactly two irreducible representations corresponding to the highest weights $2\omega_1$ and $2\omega_2$, which are dual of each other. Hence, for dimension ≤ 8, the only auto-dual non-trivial irreducible representation is the adjoint representation.

Lemma 2.1. *The lowest dimensional non-trivial real representation of* $\mathfrak{sl}(3,\mathbb{R})$ *preserving a non-degenerated bilinear symmetric form is* $\mathbb{R}^3\oplus\mathbb{R}^{3*}$. *More precisely, the bilinear form*

$$\langle\cdot,\cdot\rangle_0 : (\mathbb{R}^3\oplus\mathbb{R}^{3*})\times(\mathbb{R}^3\oplus\mathbb{R}^{3*})\to\mathbb{R}$$
$$\langle(p,q),(p',q')\rangle_0 = q(p') + q'(p),$$

is symmetric, $\mathfrak{sl}(3,\mathbb{R})$*-invariant and has signature* $(3,3)$. *Furthermore, any other* $\mathfrak{sl}(3,\mathbb{R})$*-invariant symmetric bilinear form on* $\mathbb{R}^3\oplus\mathbb{R}^{3*}$ *is a (real) constant multiple of* $\langle\cdot,\cdot\rangle_0$.

Proof. The symmetry, invariance and signature of $\langle\cdot,\cdot\rangle_0$ are easy to verify.

On the other hand, let V be a non-trivial representation with dimension ≤ 6. By the previous remarks on the low dimensional irreducible representations of $\mathfrak{sl}(3,\mathbb{R})$ it follows that V is isomorphic to one of the following

$$V^{2\omega_1},\ V^{2\omega_2},\ \mathbb{R}^3\oplus\mathbb{R}^{3*},\ \mathbb{R}^3\oplus k\mathbb{R},\ \mathbb{R}^{3*}\oplus k\mathbb{R},$$

where for $k=0,1,2,3$ the representation $k\mathbb{R}$ denotes the trivial k-dimensional representation of $\mathfrak{sl}(3,\mathbb{R})$. It follows again by the remarks above that $\mathbb{R}^3\oplus\mathbb{R}^{3*}$ is the only auto-dual representation in the list and so the first claim follows.

If $\langle \cdot, \cdot \rangle$ is some other $\mathfrak{sl}(3,\mathbb{R})$-invariant symmetric bilinear form on $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$, then there is a homomorphism

$$\varrho : \mathbb{R}^3 \oplus \mathbb{R}^{3*} \to \mathbb{R}^3 \oplus \mathbb{R}^{3*},$$

of $\mathfrak{sl}(3,\mathbb{R})$-modules such that

$$\langle \cdot, \cdot \rangle = \langle \varrho(\cdot), \cdot \rangle_0.$$

By Schur's Lemma and since there are no invariant complex structures for the irreducible spaces involved, there exists $c_1, c_2 \in \mathbb{R}$ such that

$$\varrho(p, q) = (c_1 p, c_2 q),$$

for all $(p, q) \in \mathbb{R}^3 \oplus \mathbb{R}^{3*}$. Hence the symmetry of $\langle \cdot, \cdot \rangle$ implies that

$$c_2 q(p') + c_1 q'(p) = c_1 q(p') + c_2 q'(p),$$

for all $(p, q), (p', q') \in \mathbb{R}^3 \oplus \mathbb{R}^{3*}$. This implies that $c_1 = c_2$ and so the result follows. ■

From now on, we will denote by $\langle \cdot, \cdot \rangle_0$ the $\mathfrak{sl}(3,\mathbb{R})$-invariant inner product on $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ defined in the proof of Lemma 2.1. It is easy to check that this bilinear form is also $\mathfrak{gl}(3,\mathbb{R})$-invariant. Hence we have naturally induced homomorphisms

$$\mathfrak{sl}(3,\mathbb{R}) \subset \mathfrak{gl}(3,\mathbb{R}) \hookrightarrow \mathfrak{so}(3,3),$$

that we will fix from now on. In particular, $\mathfrak{so}(3,3)$ can be seen as an $\mathfrak{sl}(3,\mathbb{R})$-module and we will describe its decomposition into irreducible submodules.

Lemma 2.2. *The decomposition of* $\mathfrak{so}(3,3)$ *as a sum of irreducible* $\mathfrak{sl}(3,\mathbb{R})$*-submodules is given by*

$$\mathfrak{so}(3,3) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \oplus \mathbb{R}.$$

Proof. First we note that $\mathfrak{gl}(3,\mathbb{R})$ is a $\mathfrak{sl}(3,\mathbb{R})$-submodule whose decomposition as a sum of irreducible submodules is $\mathfrak{gl}(3,\mathbb{R}) \cong \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}$.

We recall that by Table II in [1], and for the above embedding, the pair $(\mathfrak{so}(3,3), \mathfrak{gl}(3,\mathbb{R}))$ is a symmetric pair. Furthermore, from [1] we also have that

$$\mathfrak{so}(3,3) = \mathfrak{gl}(3,\mathbb{R}) \oplus V,$$

where V is a non-trivial 6-dimensional auto-dual $\mathfrak{sl}(3,\mathbb{R})$-module that admits an invariant non-degenerate symmetric bilinear form. Hence, the result follows from Lemma 2.1. ■

Using the fact that $(\mathfrak{so}(3,3), \mathfrak{gl}(3,\mathbb{R}))$ is a symmetric pair, a straightforward computation allows us to conclude the following result.

Corollary 2.3. *With respect to the decomposition of $\mathfrak{so}(3,3)$ given by Lemma 2.2, the following is a complete list of the subalgebras of $\mathfrak{so}(3,3)$ that are at the same time $\mathfrak{sl}(3,\mathbb{R})$-submodules*

$0,\ \mathfrak{so}(3,3),\ \mathfrak{sl}(3,\mathbb{R}),$
$\mathbb{R}^3,\ \mathbb{R}^{3*},\ \mathbb{R},\ \mathbb{R}^3 \oplus \mathbb{R},\ \mathbb{R}^{3*} \oplus \mathbb{R},$
$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3,\ \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^{3*},\ \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R},\ \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R},\ \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^{3*} \oplus \mathbb{R}.$

We now state and prove some useful results. The following is a consequence of the fact that $\mathfrak{sl}(3,\mathbb{R})$ is simple with simple complexification.

Lemma 2.4. *There is, up to a multiple by a real scalar, exactly one $\mathfrak{sl}(3,\mathbb{R})$-invariant non-degenerate bilinear form on $\mathfrak{sl}(3,\mathbb{R})$.*

The following is now an easy consequence. We recall that the decomposition of $\mathfrak{g}_{2(2)}$ into irreducible $\mathfrak{sl}(3,\mathbb{R})$-submodules is given by $\mathfrak{g}_{2(2)} = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$.

Lemma 2.5. *Let $\langle \cdot,\cdot \rangle_1$ and $\langle \cdot,\cdot \rangle_2$ be inner products on $\mathfrak{sl}(3,\mathbb{R})$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$, respectively. If $\langle \cdot,\cdot \rangle_1$ and $\langle \cdot,\cdot \rangle_2$ are $\mathfrak{sl}(3,\mathbb{R})$-invariant, then there exist $a_1, a_2 \in \mathbb{R}$ such that $a_1\langle \cdot,\cdot \rangle_1 + a_2\langle \cdot,\cdot \rangle_2$ is the Killing form of $\mathfrak{g}_{2(2)} = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$.*

Finally, we have the following results relating $\mathfrak{sl}(3,\mathbb{R})$ to $\mathfrak{g}_{2(2)}$.

Lemma 2.6. *Suppose that $\rho : \mathfrak{sl}(3,\mathbb{R}) \to \mathfrak{g}_{2(2)}$ is an injective Lie algebra homomorphism. Then, the centralizer of $\mathfrak{s} = \rho(\mathfrak{sl}(3,\mathbb{R}))$ in $\mathfrak{g}_{2(2)}$ is zero.*

Proof. As a consequence of Lemma 2.1, we necessarily have $\mathfrak{g}_{2(2)} \cong \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3^*}$ as $\mathfrak{sl}(3,\mathbb{R})$-modules with $\mathfrak{s}$ mapped onto $\mathfrak{sl}(3,\mathbb{R})$. Hence the result follows easily. ∎

Lemma 2.7. *Let $\rho : \widetilde{\mathrm{SL}}(3,\mathbb{R}) \to G_{2(2)}$ be a non trivial homomorphism of Lie groups. Then, the centralizer $Z(\rho(\widetilde{\mathrm{SL}}(3,\mathbb{R})))$ of $\rho(\mathrm{SL}(3,\mathbb{R}))$ in $G_{2(2)}$ is finite.*

Proof. By Lemma 2.6, it follows that $Z(\rho(\widetilde{\mathrm{SL}}(3,\mathbb{R})))$ is discrete. Then, the proof that $Z(\rho(\widetilde{\mathrm{SL}}(3,\mathbb{R})))$ is finite is a now consequence of Lemma 1.1.3.7 from [10] and the fact that $G_{2(2)}$ is centerless. ∎

3. Isometric actions of the simple Lie group SL(3, $\mathbb{R}$)

In this section we assume G is a connected non-compact simple Lie group with Lie algebra $\mathfrak{g}$ and M a connected and analytic finite-volume pseudo-Riemannian manifold where G acts analytically and isometrically with a dense orbit.

Since every isometric G-action on a manifold M with a dense orbit is locally free (see [9]), the orbits of the G-action define a foliation that we denote with $\mathcal{F}$. We will assume that the leaves of $\mathcal{F}$ are non-degenerate for the pseudo-Riemannian metric of M; this holds, for example, when $\dim(M) \leq 2\dim(G)$ (see [5]); note that this condition is satisfied under the assumptions of Theorem 1.1. Hence, we can decompose $TM = T\mathcal{F} \oplus T\mathcal{F}^\perp$.

On the other hand, the bundle $T\mathcal{F}$ is a trivial vector bundle isomorphic to $M \times \mathfrak{g}$ under the isomorphism $M \times \mathfrak{g} \to T\mathcal{F}$, given by $(x, X) \mapsto X^*_x$. In particular, this defines an isomorphism of the fibers $T_x\mathcal{F}$ with $\mathfrak{g}$. We recall that for an H-action on a manifold N and $X \in \mathfrak{h}$, we denote by X^* the vector field on N whose one-parameter group of diffeomorphism is given by $(\exp(tX))_t$ through the action of H on N.

The space of Killing fields of a geometric structure ω in a manifold M is denoted by $\mathrm{Kill}(M,\omega)$, and $\mathrm{Kill}_0(M,\omega,x)$ will denote the subspace of $\mathrm{Kill}(M,\omega)$ consisting of vector fields that vanish on x. Here, the geometric structure of the pseudo-Riemannian metric on M is denoted by σ. Unless otherwise indicated, in the rest of this paper we will omit the symbol that denotes the structure of pseudo-Riemannian metric, in particular, we write $\mathrm{Kill}(M) = \mathrm{Kill}(M,\sigma)$ and $\mathrm{Kill}_0(M,x) = \mathrm{Kill}_0(M,\sigma,x)$.

For a vector space V with a non-degenerate symmetric bilinear form, we denote by $\mathfrak{so}(V)$ the Lie algebra of linear maps on V that are skew-symmetric with respect to the form on V. An immediate consequence of the previous remarks and the Jacobi identity is the following result.

Lemma 3.1. *Let M be a pseudo-Riemannian manifold and $x \in M$. Then, the map $\lambda_x : \mathrm{Kill}_0(M,x) \to \mathfrak{so}(T_xM)$ given by $\lambda_x(Z)(w) = [Z,W]_x$, where W is any vector field such that $W_x = w$, is a well defined homomorphism of Lie algebras.*

The following result is a fundamental consequence of Gromov-Zimmer machinery.

Proposition 3.2 ([8, Proposition 2.3])**.** *Let G be a connected non-compact simple Lie group acting isometrically and with a dense orbit on a connected finite volume pseudo-Riemannian manifold M. Consider the $\widetilde{G}$-action on $\widetilde{M}$, lifted from the G-action on M. Assume that M and the G-action on M are both analytic. Then, there exists a conull subset $S \subset \widetilde{M}$ such that for every $x \in S$ the following properties are satisfied:*

1. *There is a homomorphism $\rho_x : \mathfrak{g} \to \mathrm{Kill}(\widetilde{M})$ which is an isomorphism onto its image $\rho_x(\mathfrak{g}) = \mathfrak{g}(x)$.*
2. *$\mathfrak{g}(x) \subset \mathrm{Kill}_0(\widetilde{M},x)$, i.e. every element of $\mathfrak{g}(x)$ vanishes at x.*
3. *For every $X, Y \in \mathfrak{g}$ we have*
$$[\rho_x(X), Y^*] = [X,Y]^* = -[X^*,Y^*].$$
In particular, the elements in $\mathfrak{g}(x)$ and their corresponding local flows preserve both $\mathcal{F}$ and $T\mathcal{F}^\perp$.
4. *The homomorphism of Lie algebras $\lambda_x \circ \rho_x : \mathfrak{g} \to \mathfrak{so}(T_x\widetilde{M})$ induces a $\mathfrak{g}$-module structure on $T_x\widetilde{M}$ for which subspaces $T_x\mathcal{F}$ and $T_x\mathcal{F}^\perp$ are $\mathfrak{g}$-submodules.*

We consider the $\mathfrak{g}$-valued 1-form θ on $\widetilde{M}$ which is defined, at every $x \in \widetilde{M}$, by the composition of the projection $T_x\widetilde{M} \to T_x\mathcal{F}$ and the isomorphism of $T_x\mathcal{F}$ with $\mathfrak{g}$. We also consider the $\mathfrak{g}$-valued 2-form given by $\Theta = d\theta|_{\wedge^2 T\mathcal{F}^\perp}$.

The following result, whose proof can be founded in [8], involves the forms defined in the previous paragraph.

Lemma 3.3. *Let G, M and S be as in Proposition 3.2. If we assume that the G-orbits are non-degenerate, then:*

1. *For every $x \in S$, the maps $\theta_x : T_x\widetilde{M} \to \mathfrak{g}$ and $\Theta_x : \wedge^2 T_x\mathcal{F}^\perp \to \mathfrak{g}$ are both homomorphism of $\mathfrak{g}$-modules, for the $\mathfrak{g}$-module structures from Proposition 3.2.*
2. *The normal bundle $T\mathcal{F}^\perp$ is integrable if and only if $\Theta = 0$.*

Under the assumptions of the preceding Lemma and by the analyticity of the elements involved we have two possible cases. Either $\Theta \equiv 0$ or for almost every $x \in \widetilde{M}$ we have $\Theta_x \neq 0$.

For the G-action as in Proposition 3.2, we consider $\widetilde{M}$ endowed with the $\widetilde{G}$-action obtained by lifting the G-action on M. Let us denote by $\mathcal{H}$ the Lie subalgebra of $\mathrm{Kill}(\widetilde{M})$ consisting of the fields that centralize the $\widetilde{G}$-action on $\widetilde{M}$. Our first result involving $\mathcal{H}$ is the embedding of the Lie algebra $\mathfrak{g}$ into $\mathcal{H}$. Such result allows us to apply representation theory to the study of $\mathcal{H}$.

Lemma 3.4 ([5, Lemma 1.7])**.** *Let S as in Proposition 3.2. Then, for every $x \in S$ and for ρ_x given as in Proposition 3.2, the map $\widehat{\rho}_x : \mathcal{H} \to \mathrm{Kill}(\widetilde{M})$ defined as:*

$$\widehat{\rho}_x(X) = \rho_x(X) + X^*,$$

is an injective homomorphism of Lie algebras whose image $\mathcal{G}(x)$ lies in $\mathcal{H}$. In particular, $\widehat{\rho}_x$ induces on $\mathcal{H}$ a $\mathfrak{g}$-module structure such that $\mathcal{G}(x)$ is a submodule isomorphic to $\mathfrak{g}$.

The following lemma relates the $\mathfrak{g}$-module structure of $\mathcal{H}$ to that of $T_x\widetilde{M}$.

Lemma 3.5 ([5, Lemma 1.8])**.** *Let S as in Proposition 3.2. Consider $T_x\widetilde{M}$ and $\mathcal{H}$ endowed with the $\mathfrak{g}$-module structure given by Proposition 3.2(4) and Lemma 3.4, respectively. Then, for almost every $x \in S$, the evaluation map:*

$$ev_x : \mathcal{H} \to T_x\widetilde{M},$$
$$Z \mapsto Z_x$$

is a homomorphism of $\mathfrak{g}$-modules that satisfies $ev_x(\mathcal{G}(x)) = T_x\mathcal{F}$. Furthermore, for almost every $x \in S$ we have $ev_x(\mathcal{H}) = T_x\widetilde{M}$.

Next, the following result provides the link between representations of Lie algebras on Killing vector fields and actions of groups for complete manifolds. Its proof can be found in [5].

Lemma 3.6. *Let N be a complete pseudo-Riemannian manifold and H a simply connected Lie group with Lie algebra $\mathfrak{h}$. If $\psi : \mathfrak{h} \to \mathrm{Kill}(N)$ is a homomorphism of Lie algebras, then there exists an isometric right H-action, $N \times H \to N$, such that $\psi(X) = X^*$, for every $X \in \mathfrak{h}$. Furthermore, if N is analytic, then the H-action is analytic as well.*

4. Analysis and properties of isometric $\mathbf{SL(3,\mathbb{R})}$-actions

From now on we assume that $G = \mathrm{SL}(3,\mathbb{R})$ and that the hypotheses of Theorem 1.1 are satisfied. In particular, the assumptions of Section 3 hold as well and so the results of this section are satisfied.

The following result can be found in [8].

Theorem 4.1. *With the hypotheses of Theorem 1.1, if we assume that our normal bundle $T\mathcal{F}^\perp$ is integrable, then $\widetilde{M} \cong \widetilde{\mathrm{SL}}(3,\mathbb{R}) \times \widetilde{N}$, where $\widetilde{N}$ is a complete pseudo-Riemannian manifold. Furthermore, the metric on M can be replaced by some other* $\mathrm{SL}(3,\mathbb{R})$*-invariant metric so that the diffeomorphism is in fact an isometry.*

The previous result corresponds to the case $\Theta \equiv 0$ discussed in Section 3. Hence, as remarked in this section, we can assume from now on that $T\mathcal{F}$ is not integrable and that $\Theta_x \neq 0$ for almost every x.

Lemma 4.2. *Let S as in Proposition 3.2 and for every $x \in S$ consider $T_x\mathcal{F}^\perp$ endowed with the $\mathfrak{sl}(3,\mathbb{R})$-module structure given by Proposition 3.2. Then, for almost every $x \in S$, the module $T_x\mathcal{F}^\perp$ is isomorphic to $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ and we have* $\dim(M) = 14$*. In particular, the algebra $\mathfrak{so}(T_x\mathcal{F}^\perp)$ is isomorphic to $\mathfrak{so}(3,3)$ as a Lie algebra and as a $\mathfrak{sl}(3,\mathbb{R})$-module.*

Proof. By the previous remarks, we have $\Theta_x \neq 0$ for almost every $x \in S$. Choose and fix such an element $x \in S$. By Lemma 3.3 we conclude that $T_x\mathcal{F}^\perp$ is a non-trivial $\mathfrak{sl}(3,\mathbb{R})$-module that admits a non-degenerate symmetric bilinear form. Since $1 < \dim(T_x\mathcal{F}^\perp) \leq 6$, then from Lemma 2.1 we conclude that $T_x\mathcal{F}^\perp \simeq \mathbb{R}^3 \oplus \mathbb{R}^{3*}$ as a $\mathfrak{sl}(3,\mathbb{R})$-module. Hence, $\dim(T_x\mathcal{F}^\perp) = 6$ and, therefore, $\dim(M) = 14$.

On the other hand, by Lemma 2.1, the representation of $\mathfrak{sl}(3,\mathbb{R})$ on $T_x\mathcal{F}^\perp$ defines a non-trivial homomorphism of Lie algebras $\mathfrak{so}(T_x\mathcal{F}^\perp) \to \mathfrak{so}(3,3)$, which is also an isomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules. Since $\mathfrak{so}(3,3)$ is a simple Lie algebra, this latter homomorphism is injective and so it is an isomorphism. ∎

The results of the previous lemma allow us to obtain a decomposition of the centralizer $\mathcal{H}$ of the $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-action into submodules related to the pseudo-Riemannian metric on $\widetilde{M}$. First, recall that Lemma 3.4 induces on $\mathcal{H}$ a structure of $\mathfrak{sl}(3,\mathbb{R})$-module.

Lemma 4.3. *Let S be as in Proposition 3.2. Then, for almost every $x \in S$ there is a decomposition of $\mathcal{H}$ into $\mathfrak{sl}(3,\mathbb{R})$-submodules, $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{W}(x)$,* *satisfying:*

1. $\mathcal{G}(x) = \widehat{\rho}_x(\mathfrak{sl}(3,\mathbb{R}))$ *is a Lie subalgebra of* $\mathcal{H}$ *isomorphic to* $\mathfrak{sl}(3,\mathbb{R})$ *and* $ev_x(\mathcal{G}(x)) = T_x\mathcal{F}$.

2. $\mathcal{H}_0(x) = \ker(ev_x)$ *is a Lie subalgebra and a* $\mathfrak{sl}(3,\mathbb{R})$*-module of* $\mathcal{H}$. *Furthermore, there is a linear embedding of* $\mathcal{H}_0(x)$ *into* $\mathfrak{so}(T_x\mathcal{F}^\perp)$ *which is a homomorphism of Lie algebras and of* $\mathfrak{sl}(3,\mathbb{R})$*-modules.*

3. $\mathcal{W}(x)$ *is isomorphic to* $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ *as* $\mathfrak{sl}(3,\mathbb{R})$*-module and* $ev_x(\mathcal{W}(x)) = T_x\mathcal{F}^\perp$.

In particular, the evaluation map ev_x *defines an isomorphism of* $\mathfrak{sl}(3,\mathbb{R})$*-modules* $\mathcal{G}(x) \oplus \mathcal{W}(x) \cong T_x\mathcal{F} \oplus T_x\mathcal{F}^\perp$*, preserving the summands in that order.*

Proof. Let us choose and fix an element $x \in S$ that satisfies Lemma 3.5 and Lemma 4.2. By Lemma 3.4 we conclude that $\mathcal{G}(x) = \widehat{\rho}_x(\mathfrak{sl}(3,\mathbb{R}))$ is a Lie algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R})$.

Define $\mathcal{H}_0(x) = \ker(ev_x)$. By Lemma 3.5, it follows that $\mathcal{H}_0(x)$ is an $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$. On the other hand, since $\mathcal{H}_0(x) = \mathcal{H} \cap \mathrm{Kill}_0(\widetilde{M}, x)$, it is clear that $\mathcal{H}_0(x)$ is a Lie subalgebra of $\mathcal{H}$.

Recall that the elements of $\mathcal{G}(x)$ are of the form $\rho_x(X) + X^*$, with $X \in \mathfrak{sl}(3,\mathbb{R})$. Hence, for any such element we have $ev_x(\rho_x(X) + X^*) = X^*_x$. That and the condition $ev_x(\rho_x(X) + X^*) = 0$ imply that $X = 0$. In other words, $\mathcal{G}(x) \cap \mathcal{H}_0(x) = 0$. Therefore, there exists an $\mathfrak{sl}(3,\mathbb{R})$-submodule complementary $\mathcal{W}'(x)$ to $\mathcal{G}(x) \oplus \mathcal{H}_0(x)$ in $\mathcal{H}$. Note that we have an isomorphism from $\mathcal{G}(x) \oplus \mathcal{W}'(x)$ onto $T_x\widetilde{M}$ via the evaluation map. We choose $\mathcal{W}(x)$ as the inverse image of $T_x\mathcal{F}^\perp$ under our previous isomorphism. Hence, we have our desired decomposition of $\mathcal{H}$ into $\mathfrak{sl}(3,\mathbb{R})$-submodules.

Let $\mathrm{Kill}_0(\widetilde{M}, x, \mathcal{F})$ be the Lie algebra of Killing vector fields on $\widetilde{M}$ which preserves the foliation $\mathcal{F}$ and vanish at x. Note that every vector field in $\mathrm{Kill}_0(\widetilde{M}, x, \mathcal{F})$ leaves invariant the normal bundle. On the other hand, the map λ_x restricted to $\mathrm{Kill}_0(\widetilde{M}, x, \mathcal{F})$ induces the following homomorphism of Lie algebras:

$$\begin{aligned} \lambda_x^\perp : \mathrm{Kill}_0(\widetilde{M}, x, \mathcal{F}) &\to \mathfrak{so}(T_x\mathcal{F}^\perp), \\ X &\mapsto \lambda_x(X)|_{T_x\mathcal{F}^\perp}. \end{aligned}$$

Observe that both $\rho_x(\mathfrak{sl}(3,\mathbb{R}))$ and $\mathcal{H}_0(x)$ lie inside of $\mathrm{Kill}_0(\widetilde{M}, x, \mathcal{F})$.

Claim 1: $\lambda_x^\perp$ *is injective when is restricted to* $\mathfrak{sl}(3,\mathbb{R})(x)$. By our choice of the element $x \in S$ and the results in Lemma 4.2, the map $\lambda_x^\perp \circ \rho_x : \mathfrak{sl}(3,\mathbb{R}) \to \mathfrak{so}(T_x\mathcal{F}^\perp)$ is a non-trivial homomorphism of Lie algebras. Since $\mathfrak{sl}(3,\mathbb{R})$ is a simple Lie algebra then the map $\lambda_x^\perp$ restricted to $\mathfrak{sl}(3,\mathbb{R})$ is injective.

Claim 2: $\lambda_x^\perp$ *restricted to* $\mathcal{H}_0(x)$ *is injective.* Recall that pseudo-Riemannian metric structures are 1-rigid structures (see [2]), therefore a Killing vector space is completely determined by its 1-jet at x. If $Z \in \mathcal{H}_0(x)$ is given, then $Z_x = ev_x(Z) = 0$; so it is determined by the values $[Z,V]_x$, for V vector field on a neighborhood of x. Since Z is in the centralizer of the $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-action, then $[Z, X^*]_x = 0$ for all $X \in \mathfrak{sl}(3,\mathbb{R})$, so $[Z,V]_x = 0$ when $V_x \in T_x\mathcal{F}$. Hence, if

$[Z,V]_x = 0$ when $V_x \in T_x\mathcal{F}^\perp$, this implies that $Z = 0$. Therefore, we have that $\lambda_x^\perp$ is injective when it is restricted to $\mathcal{H}_0(x)$.

On the other hand, if $X \in \mathfrak{sl}(3,\mathbb{R})$ and $Y \in \mathcal{H}_0(x)$ then

$$\begin{aligned}\lambda_x^\perp(X \cdot Y) &= \lambda_x^\perp([\widehat{\rho}_x(X), Y]) = \lambda_x^\perp([\rho_x(X) + X^*, Y]) \\ &= \lambda_x^\perp([\rho_x(X), Y]) = [\lambda_x^\perp(\rho_x(X)), \lambda_x^\perp(Y)] \\ &= X \cdot \lambda_x^\perp(Y).\end{aligned}$$

This shows that the map $\lambda_x^\perp$ restricted to $\mathcal{H}_0(x)$ is a homomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules. ∎

By the previous lemma we have that $\mathcal{H}_0(x)$ is a subalgebra, and an $\mathfrak{sl}(3,\mathbb{R})$-submodule, isomorphic to $\lambda_x^\perp\big(\mathcal{H}_0(x)\big) \subset \mathfrak{so}(T_x\mathcal{F}^\perp)$. Since $\mathfrak{so}(T_x\mathcal{F}^\perp)$ is isomorphic to $\mathfrak{so}(3,3)$ (and hence to $\mathfrak{sl}(4,\mathbb{R})$), then $\mathcal{H}_0(x)$ is isomorphic to one of the Lie subalgebras in Corollary 2.3.

On the other hand, by Lemma A.5 in [6], we have that $\wedge^2 T_x\mathcal{F}^\perp$ is isomorphic to $\mathfrak{so}(T_x\mathcal{F}^\perp)$ as $\mathfrak{so}(T_x\mathcal{F}^\perp)$-module. Thus, from the definition of the map Θ_x in Lemma 3.3 and Lemma A.5 in [6], we can consider Θ_x as a map from $\mathfrak{so}(T_x\mathcal{F}^\perp)$ to $\mathfrak{sl}(3,\mathbb{R})$.

Additional information on the subalgebra $\mathcal{H}_0(x)$ can be obtained considering the map Θ_x as in the previous paragraph. In particular, we have the following result which follows from Propositions 3.10 and 3.11 from [6].

Theorem 4.4. *For almost every $x \in \mathrm{S}$ as in Lemma 4.3. $\lambda_x^\perp(\mathcal{H}_0(x))$ is a $\lambda_x^\perp(\mathcal{G}(x))$-submodule and a Lie algebra of $\mathfrak{so}(T_x\mathcal{F}^\perp)$ that satisfies*

$$[\lambda_x^\perp(\mathcal{H}_0(x)), \mathfrak{so}(T_x\mathcal{F}^\perp)] \subset \ker(\Theta_x).$$

A straightforward computation using the previous result and the list from Corollary 2.3 provides the following.

Proposition 4.5. *For almost every $x \in S$ as in Lemma 4.3, either $\mathcal{H}_0(x) = 0$ or it is isomorphic to $\mathbb{R}$.*

5. Structure of the centralizer $\mathcal{H}$ and its consequences

Let us choose and fix an element $x \in S$ that satisfies the conclusions of both Lemma 4.3 and Proposition 4.5.

By Lemma 4.2 we can choose subspaces $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$ of $\mathcal{W}(x)$ such that $\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ with $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$ isomorphic as $\mathfrak{sl}(3,\mathbb{R})$-modules to $\mathbb{R}^3$ and $\mathbb{R}^{3*}$, respectively.

From the structure of $\mathcal{H}$ as an $\mathfrak{sl}(3,\mathbb{R})$-module and the properties of the evaluation map in Lemma 4.3, we have the following properties

$$\begin{aligned} [\mathcal{G}(x), \mathcal{H}_0(x)] &\subseteq \mathcal{H}_0(x), \\ [\mathcal{G}(x), \mathcal{W}(x)] &= \mathcal{W}(x), \\ [\mathcal{H}_0(x), \mathcal{W}(x)] &\subseteq \mathcal{H}_0(x) \oplus \mathcal{W}(x), \\ [\mathcal{H}_0(x), \mathcal{H}_0(x)] &\subseteq \mathcal{H}_0(x), \\ [\mathcal{V}(x), \mathcal{V}(x)] &\subset \mathcal{V}^*(x), \quad [\mathcal{V}^*(x), \mathcal{V}^*(x)] \subset \mathcal{V}(x). \end{aligned} \tag{5.1}$$

We have also used the fact that the Lie brackets defines a homomorphism of modules as a consequence of the Jacobi identity.

We describe $\mathcal{H}$ in terms of the cases provided by Proposition 4.5.

Lemma 5.1. *For $x \in S$ as above, if $\mathcal{H}_0(x) = 0$ then, one of the following holds*

1. *The radical of $\mathcal{H}$ is $\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.*
2. *$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is isomorphic to $\mathfrak{g}_{2(2)}$.*

Proof. In this case we have $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$, which is its decomposition into irreducible $\mathcal{G}(x)$-modules.

Since $\mathcal{G}(x)$ is a simple Lie algebra, we can choose a Levi factor $\mathfrak{s}$ of $\mathcal{H}$ which contains $\mathcal{G}(x)$. Since $\mathcal{G}(x) \subseteq \mathfrak{s}$ it follows that $\mathfrak{s}$ is also a $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$. Let W be an $\mathfrak{sl}(3,\mathbb{R})$-submodule of $\mathcal{H}$ such that $\mathfrak{s} = \mathcal{G}(x) \oplus W$. Hence, we obtain the following decomposition of $\mathcal{H}$ as a direct sum of $\mathfrak{sl}(3,\mathbb{R})$-submodules

$$\mathcal{H} = \mathfrak{s} \oplus \mathrm{rad}(\mathcal{H}) = \mathcal{G}(x) \oplus W \oplus \mathrm{rad}(\mathcal{H}).$$

Comparing with the above decomposition, we conclude that one of the following holds

(a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}^*(x)$,

(b) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}(x)$,

(c) $\mathfrak{s} = \mathcal{G}(x)$ and $\mathrm{rad}(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$,

(d) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

We proceed to consider each case.

If (a) is satisfied, then $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x) = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$, a direct product of simple ideals. Since all such ideals are invariant under $\mathcal{G}(x)$, then these ideals are $\mathfrak{sl}(3,\mathbb{R})$-submodules. In particular, k is either 1 or 2.

If $k = 1$, then $\mathfrak{s}$ is an 11-dimensional real simple Lie algebra, which is a contradiction (see [3]). If $k = 2$, then we necessarily have (after renumbering the ideals) $\mathfrak{h}_1 = \mathcal{G}(x)$ and $\mathfrak{h}_2 = \mathcal{V}(x)$. This implies that $[\mathcal{G}(x), \mathcal{V}(x)] = 0$, which is a contradiction since $\mathcal{V}(x)$ is a non-trivial $\mathcal{G}(x)$-module.

We observe that case (b) is dealt with similarly. We also note that case (c) yields the first stated conclusion of the Lemma.

If case (d) holds, then $\mathcal{H} = \mathfrak{h}_1 \times \dots \mathfrak{h}_k$ a direct product of simple ideals. As before, we necessarily have $k = 1, 2, 3$.

If $k = 1$, we conclude that $\mathcal{H}$ is a 14-dimensional real simple Lie algebra. Since $\mathcal{H}$ is non-compact (it contains $\mathcal{G}(x) \cong \mathfrak{sl}(3, \mathbb{R})$) it follows that $\mathcal{H} \cong \mathfrak{g}_{2(2)}$ (see [3]). This yields the second stated conclusion of the Lemma.

For $k = 2, 3$ we compare the decompositions $\mathcal{H} = \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_k = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ to conclude that each ideal $\mathfrak{h}_j$ is the sum of some of the subspaces $\mathcal{G}(x)$, $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$. As before, we cannot have $\mathcal{G}(x)$ lying in some ideal and either $\mathcal{V}(x)$ or $\mathcal{V}^*(x)$ lying in a different ideal, since this would contradict the non-triviality of $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$ as $\mathcal{G}(x)$-modules. Hence this possibility cannot occur. ■

Lemma 5.2. *For $x \in S$ as above, if $\mathcal{H}_0(x) \cong \mathbb{R}$, then one of the following holds*

1. *The radical of $\mathcal{H}$ is $\mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ and $\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a Lie subalgebra.*
2. *$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a simple Lie algebra isomorphic to $\mathfrak{sl}(4, \mathbb{R})$.*

Proof. As before, we let $\mathfrak{s}$ a Levi factor of $\mathcal{H}$ that contains $\mathcal{G}(x)$ and W an $\mathfrak{sl}(3, \mathbb{R})$-submodule of $\mathcal{H}$ such that $\mathfrak{s} = \mathcal{G}(x) \oplus W$. Then we compare the decompositions

$$\mathcal{H} = \mathcal{G}(x) \oplus W \oplus \operatorname{rad}(\mathcal{H}) = \mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x).$$

And we now have the following possibilities.

(a) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x)$ and $\operatorname{rad}(\mathcal{H}) = \mathcal{V}^*(x)$.

(b) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}^*(x)$ and $\operatorname{rad}(\mathcal{H}) = \mathcal{V}(x)$.

(c) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ and $\operatorname{rad}(\mathcal{H}) = \mathcal{H}_0(x)$.

(d) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{H}_0(x)$ and $\operatorname{rad}(\mathcal{H}) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(e) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ and $\operatorname{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}^*(x)$.

(f) $\mathfrak{s} = \mathcal{G}(x) \oplus \mathcal{V}^*(x)$ and $\operatorname{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x)$.

(g) $\mathfrak{s} = \mathcal{G}(x)$ and $\operatorname{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$.

(h) $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is semisimple.

For (a), and using the fact that Lie brackets define homomorphisms of $\mathcal{G}(x)$-modules, we conclude that $[\mathcal{V}(x), \mathcal{V}(x)]$ is either 0 or isomorphic to $\mathbb{R}^{3*}$. The latter cannot happen since $[\mathcal{V}(x), \mathcal{V}(x)] \subset \mathfrak{s}$. Hence, we have $[\mathcal{V}(x), \mathcal{V}(x)] = 0$. By the same argument $[\mathcal{H}_0(x), \mathcal{V}(x)]$ is either 0 or isomorphic to $\mathbb{R}^3$, and so it is contained in $\mathcal{V}(x)$. Since $\mathcal{V}(x)$ is invariant under $\mathcal{G}(x)$ we conclude that $\mathcal{V}(x)$ is an Abelian ideal of $\mathfrak{s}$, which is absurd. Similarly, (b) cannot occur either.

For the rest of the cases, we note that

$$[\mathcal{H}_0(x), \mathcal{W}(x)] = \mathcal{W}(x).$$

This is obtained by a straightforward computation using the the explicit realization of $\mathcal{H}_0(x)$ as a subalgebra of $\mathfrak{so}(T_x\mathcal{F}^\perp) \cong \mathfrak{so}(\mathcal{W}(x))$.

We note that the previous identity rules out (c), (e) and (f) since $\mathrm{rad}(\mathcal{H})$ is an ideal containing $\mathcal{H}_0(x)$ in these cases. Next we observe that (d) cannot occur since it would imply the existence of an Abelian ideal in $\mathfrak{s}$.

For case (g) it is enough to show that $\mathcal{W}(x) = \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is a Lie algebra. And by equations (5.1) it is enough to look at $[\mathcal{V}(x), \mathcal{V}^*(x)] \subset \mathcal{H}_0(x) \oplus \mathcal{W}(x)$. Using the fact that the brackets define a homomorphism of the $\mathcal{G}(x)$-module structure, it follows that $[\mathcal{V}(x), \mathcal{V}^*(x)]$ is a sum of modules isomorphic to 0, $\mathbb{R}$ and $\mathfrak{sl}(3,\mathbb{R})$. This implies that either $[\mathcal{V}(x), \mathcal{V}^*(x)] = 0$ or $\mathcal{H}_0(x)$. In the latter case and by the above displayed identity, it follows that $[\mathcal{H}_0(x) \oplus \mathcal{W}(x), \mathcal{H}_0(x) \oplus \mathcal{W}(x)] = \mathcal{H}_0(x) \oplus \mathcal{W}(x)$, which contradicts the solvability of $\mathrm{rad}(\mathcal{H}) = \mathcal{H}_0(x) \oplus \mathcal{W}(x)$. Hence, $\mathcal{V}(x) \oplus \mathcal{V}^*(x)$ is indeed a Lie subalgebra.

We now deal with case (h). As before, $\mathcal{H} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_k$ a direct product of simple ideals where $k = 1, 2, 3, 4$. We also have that every such ideal $\mathfrak{h}_j$ is a sum of the some of the spaces $\mathcal{G}(x)$, $\mathcal{H}_0(x)$, $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$. We recall that arguments used before and the non-triviality of $\mathcal{V}(x)$ and $\mathcal{V}^*(x)$ as $\mathcal{G}(x)$-modules imply that $\mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x)$ must lie in the same ideal. This implies that we necessarily have $k = 1$ and that $\mathcal{H}$ is simple. In particular, $\mathcal{H}^{\mathbb{C}}$ is complex simple Lie algebra with dimension 15 and so isomorphic to $\mathfrak{sl}(4,\mathbb{C})$ (see [3]). The noncompact real forms of the latter are $\mathfrak{su}(1,3)$, $\mathfrak{su}(2,2) \cong \mathfrak{so}(2,4)$, $\mathfrak{su}^*(4)$ and $\mathfrak{sl}(4,\mathbb{R})$.

On the other hand, $\mathrm{rank}_{\mathbb{R}}(\mathcal{H}) \geq \mathrm{rank}_{\mathbb{R}}(\mathfrak{sl}(3,\mathbb{R})) = 2$ and so $\mathcal{H}$ is not isomorphic to $\mathfrak{su}(1,3)$ or $\mathfrak{su}^*(4)$ since the latter have $\mathbb{R}$-rank 1. We also have $\mathcal{H} \not\cong \mathfrak{so}(2,4)$ since otherwise we would have a non-trivial representation of $\mathfrak{sl}(3,\mathbb{R})$ on $\mathbb{R}^6$ preserving an inner product with signature $(2,4)$ (see Lemma 2.1). This implies that $\mathcal{H} \cong \mathfrak{sl}(4,\mathbb{R})$, thus providing the second stated conclusion of the Lemma. ■

6. Proof of the main theorems

In the rest of this section we assume the hypotheses of Theorem 1.1. We study the structure of the manifold M through the analysis of the different possibilities of $\mathcal{H}$ obtained in Section 5. As in the previous section, we use the notation of Lemma 4.3.

We fix $x \in S$ as in Section 5, so that the conclusions of either Lemma 5.1 or 5.2 hold. In particular, we can assume that either $\mathcal{W}(x) \subset \mathrm{rad}(\mathcal{H})$ is a Lie subalgebra or that $\mathcal{H}$ is simple and isomorphic to $\mathfrak{g}_{2(2)}$ or to $\mathfrak{sl}(4,\mathbb{R})$. We also recall that we are assuming that $T\mathcal{F}^\perp$ is not integrable.

First, let us assume that $\mathcal{W}(x) \subset \mathrm{rad}(\mathcal{H})$ is a Lie subalgebra. We will see that this implies the integrability of $T\mathcal{F}^\perp$, and thus a contradiction.

Under such assumption, $\mathcal{G}(x) \oplus \mathcal{W}(x)$ is a Lie algebra given by a semidirect product of both summands. In particular, there is a simply connected Lie group

W so that $\widetilde{\mathrm{SL}}(3,\mathbb{R})\ltimes W$ has Lie algebra isomorphic to $\mathcal{G}(x)\oplus\mathcal{W}(x)$, for some representation of $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ into the automorphism group of W. Fix an isomorphism $\psi : \mathfrak{sl}(3,\mathbb{R})\ltimes\mathfrak{w}\to\mathcal{G}(x)\oplus\mathcal{W}(x)$ that preserves the summands. By Lemma 3.6, there exists an analytic isometric right action of $\widetilde{\mathrm{SL}}(3,\mathbb{R})\ltimes \mathrm{W}$ on $\widetilde{M}$ such that $\psi(X)=X^*$ for every $X\in\mathfrak{sl}(3,\mathbb{R})\ltimes\mathfrak{w}$.

Consider the orbit map

$$\begin{aligned} p : \widetilde{\mathrm{SL}}(3,\mathbb{R})\ltimes W &\to \widetilde{M} \\ h &\mapsto xh. \end{aligned}$$

This map is clearly $\widetilde{\mathrm{SL}}(3,\mathbb{R})\ltimes W$-equivariant for the right action on its domain. If e and 0 denote the identity element in the subgroups $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ and W, respectively, then

$$\begin{aligned} dp_{(e,0)} : \mathfrak{sl}(3,\mathbb{R})\ltimes\mathfrak{w} \to \mathcal{G}(x)\oplus\mathcal{W}(x) &\to T_x\widetilde{M} \\ X\mapsto X^* &\mapsto X^*_x. \end{aligned}$$

Since $\psi(X)=X^*$ for all $X\in\mathfrak{sl}(3,\mathbb{R})\ltimes\mathfrak{w}$, by Lemma 4.3, $dp_{(e,0)}$ maps $\mathfrak{sl}(3,\mathbb{R})$ onto $T_x\mathcal{F}$ and $\mathfrak{w}$ onto $T_x\mathcal{F}^\perp$. Therefore, p is a local diffeomorphism at $(e,0)$.

For every $w\in W$, let R_w denote the map on $\widetilde{\mathrm{SL}}(3,\mathbb{R})\ltimes W$ and on $\widetilde{M}$ given by the assignment $y\mapsto y\cdot(e,w)$. In particular, we have that $R_w(W)=W$. Let us consider $P=p(e\times W)$, which defines a submanifold of $\widetilde{M}$ in a neighborhood of $x=p(e,0)$. Here, by the previous remarks, we have that

$$T_{p(e,0)}P = dp_{(e,0)}(T_{(e,0)}(e\times W)) = T_{p(e,0)}\mathcal{F}^\perp,$$

which with the equivariance of p implies that

$$\begin{aligned} T_{p(e,w)}P &= dp_{(e,w)}(T_{(e,w)}(e\times W)) = dp_{(e,w)}(d(R_w)_{(e,0)}(T_{(e,0)}(e\times W))) \\ &= d(R_w\circ p)_{(e,0)}(T_{(e,0)}(e\times W)) = d(R_w)_{p(e,0)}(T_{p(e,0)}P) \\ &= d(R_w)_{p(e,0)}T_{p(e,0)}\mathcal{F}^\perp = T_{R_w(p(e,0))}\mathcal{F}^\perp = T_{p(e,w)}\mathcal{F}^\perp, \end{aligned}$$

where we have used that R_w preserves the bundle $T\mathcal{F}^\perp$. This proves that P is an integral submanifold of $T\mathcal{F}^\perp$ passing through the element $x=p(e,0)$.

From the left $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-action on $\widetilde{M}$ we obtain by restriction to P the following map

$$\begin{aligned} \phi : \widetilde{\mathrm{SL}}(3,\mathbb{R})\times P &\to \widetilde{M} \\ (g,y) &\mapsto g\cdot y, \end{aligned}$$

whose differential at (e,x) is given by

$$X+v\mapsto X^*_x+v,$$

for $X\in\mathfrak{sl}(3,\mathbb{R})$ and $v\in T_xP$. This shows that the differential at (e,x) is an isomorphism and therefore the map ϕ is a diffeomorphism from a neighborhood of (e,x) onto a neighborhood of x. Since the left $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-action preserves both $T\mathcal{F}$

and $T\mathcal{F}^\perp$, there is an integral submanifold of $T\mathcal{F}^\perp$ passing through every point in a neighborhood of x in $\widetilde{M}$. By analiticity, this implies that $T\mathcal{F}^\perp$ is integrable. This contradiction proves that this case cannot occur.

From the previous discussion, we conclude that $\mathcal{H}$ is a simple Lie algebra isomorphic to either $\mathfrak{g}_{2(2)}$ or $\mathfrak{sl}(4, \mathbb{R})$.

Let us first assume that $\mathcal{H} \cong \mathfrak{g}_{2(2)}$. We also know that in this case $\mathcal{H}_0(x) = 0$. Furthermore, from the proof of Lemma 5.1 there exists an isomorphism of Lie algebras

$$\psi : \mathfrak{g}_{2(2)} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \to \mathcal{G}(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^* = \mathcal{H}$$

that preserves the summands. By Lemma 3.6, there exists an analytic isometric right $G_{2(2)}$-action on $\widetilde{M}$ such that $\psi(X) = X^*$, for all $X \in \mathfrak{g}_{2(2)}$. Note that this action commutes with the $\widetilde{\mathrm{SL}}(3, \mathbb{R})$-action because the former comes from the centralizer of the latter.

We now consider the map

$$\begin{aligned} \varphi : G_{2(2)} &\to \widetilde{M} \\ g &\mapsto x \cdot g, \end{aligned}$$

This map is clearly $G_{2(2)}$-equivariant for the right action on its domain. Furthermore, its differential satisfies $d\varphi_e = ev_x \circ \psi$ and so maps $\mathfrak{sl}(3, \mathbb{R})$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ onto $T_x\mathcal{F}$ and $T_x\mathcal{F}^\perp$, respectively. In particular, φ is a local diffeomorphism. We also conclude that $d\varphi_e$ is a homomorphism of $\mathfrak{sl}(3, \mathbb{R})$-modules. Hence, by Lemma 2.5, we can rescale the metric along the bundles $T\mathcal{F}$ and $T\mathcal{F}^\perp$ in $\widetilde{M}$ such that the new metric, say $\widehat{g}$, on $\widetilde{M}$ satisfies that $K = (d\varphi_e)^*(\widehat{g}_x)$ is the Killing form on $\mathfrak{g}_{2(2)}$.

Since the elements of $\mathcal{H} \subset \mathrm{Kill}(\widetilde{M})$ preserve the direct sum decomposition, $T\widetilde{M} = T\mathcal{F} \oplus T\mathcal{F}^\perp$, then $\mathcal{H} \subset \mathrm{Kill}(\widetilde{M}, \widehat{g})$. Note that $\widehat{g}$ is invariant under both the left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$-action and the right $G_{2(2)}$-action on $\widetilde{M}$. We also observe that the metric $\widehat{g}$ can be obtained from the lift of a correspondingly rescaled metric on M.

Consider the bi-invariant metric on $G_{2(2)}$ induced by the Killing form K, which we denote with the same symbol. The previous remarks imply that the local diffeomorphism φ is a local isometry. Hence, the completeness of $(G_{2(2)}, K)$ and the simply connectedness of $\widetilde{M}$ imply, by Corollary 20 in [4, p. 202], that φ is an isometry. In particular, we can identify $(G_{2(2)}, K)$ with $(\widetilde{M}, \widehat{g})$, and so the former admits the $\widetilde{\mathrm{SL}}(3, \mathbb{R})$-action lifted from M. In what follows we will consider this action. This provides the second conclusion of Theorem 1.1.

By Proposition 4.5 of [5] we have that the isometry group of the pseudo-Riemannian manifold $(G_{2(2)}, K)$, which we denote by $\mathrm{Iso}(G_{2(2)})$, has only a finite number of connected components. Such proposition also shows that

$$\mathrm{Iso}(G_{2(2)})_0 = L(G_{2(2)})R(G_{2(2)}),$$

where $L(G_{2(2)})$ and $R(G_{2(2)})$ are the subgroups of left and right translations on $G_{2(2)}$, respectively.

Let $\varrho : \widetilde{\mathrm{SL}}(3, \mathbb{R}) \to \mathrm{Iso}(G_{2(2)})$ be the homomorphism induced by the isometric left $\widetilde{\mathrm{SL}}(3, \mathbb{R})$-action on $G_{2(2)}$. From the previous observations, the covering

$$G_{2(2)} \times G_{2(2)} \to L(G_{2(2)})R(G_{2(2)})$$

yields the existence of homomorphisms $\varrho_1, \varrho_2 : \widetilde{\mathrm{SL}}(3,\mathbb{R}) \to G_{2(2)}$ such that

$$\varrho(g) = L_{\varrho_1(g)} \circ R_{\varrho_2(g)^{-1}},$$

for all $g \in \widetilde{\mathrm{SL}}(3,\mathbb{R})$.

As we remarked before, the right $G_{2(2)}$-action and the lifted $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-action on $G_{2(2)}$ commute with each other. In other words, we have

$$\varrho(g) \circ R_h = R_h \circ \varrho(g),$$

for all $g \in \widetilde{\mathrm{SL}}(3,\mathbb{R})$ and $h \in G_{2(2)}$. This implies that $\varrho_2(\widetilde{\mathrm{SL}}(3,\mathbb{R}))$ is contained in the center of $G_{2(2)}$, which is trivial, and so we have

$$\varrho(g) = L_{\varrho_1(g)},$$

for all $g \in \widetilde{\mathrm{SL}}(3,\mathbb{R})$. Thus, the $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-action on $G_{2(2)}$ is induced by the homomorphism $\varrho_1 : \widetilde{\mathrm{SL}}(3,\mathbb{R}) \to G_{2(2)}$ and the left action of $G_{2(2)}$ on itself. In particular, the homomorphism ϱ_1 is non-trivial.

Through the identification of $\widetilde{M}$ with $G_{2(2)}$ we have $\pi_1(M) \subset \mathrm{Iso}(G_{2(2)})$. Hence, the previous remarks imply that $\Gamma_1 = \pi_1(M) \cap \mathrm{Iso}_0(G_{2(2)})$ is a finite index subgroup of $\pi_1(M)$. Therefore, for every $\gamma \in \Gamma_1$ there exist $h_1, h_2 \in G_{2(2)}$ such that $\gamma = L_{h_1} \circ R_{h_2}$. Since the left $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-action on $G_{2(2)}$ is the lift of an action on M, this left $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-action commutes with the Γ_1-action. Hence, we conclude that

$$\Gamma_1 \subset L(Z(\widetilde{\mathrm{SL}}(3,\mathbb{R})))R(G_{2(2)}),$$

where $Z(\widetilde{\mathrm{SL}}(3,\mathbb{R}))$ is the centralizer of $\varrho_1(\widetilde{\mathrm{SL}}(3,\mathbb{R}))$ in $G_{2(2)}$. By Lemma 2.7, the group $Z(\widetilde{\mathrm{SL}}(3,\mathbb{R}))$ is finite and we conclude that $R(G_{2(2)})$ has finite index in $L(Z(\widetilde{\mathrm{SL}}(3,\mathbb{R})))R(G_{2(2)})$. In particular, the group $\Gamma = \Gamma_1 \cap R(G_{2(2)})$ is a finite index subgroup of Γ_1 and of $\pi_1(M)$ as well.

The previous remarks prove that, for the natural identification of $R(G_{2(2)})$ with $G_{2(2)}$, the group Γ can be seen as a discrete subgroup of $G_{2(2)}$ such that $G_{2(2)}/\Gamma$ is a finite covering space of M. Let $\pi : G_{2(2)}/\Gamma \to M$ be the corresponding covering map. The constructions in the previous paragraphs show that the map π is an $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-equivariant isometric covering. To complete the proof of Theorem 1.3 it remains to show that Γ is a lattice in $G_{2(2)}$. This is a consequence of the following result.

Lemma 6.1. *Let* vol *and* $\mathrm{vol}_{\widehat{g}}$ *denote the volume elements on M for the original metric and the rescaled metric $\widehat{g}$, respectively. Then, there is some constant $C > 0$ such that* $\mathrm{vol}_{\widehat{g}} = C\mathrm{vol}$.

Proof. We consider $(x^1, x^2, \ldots, x^{14})$ some coordinates of M in a neighborhood U of a given point such that $(x^1, \ldots, x^8)$ defines a set of coordinates of the leaves of the foliation $\mathcal{F}$ in such neighborhood. For the original metric g on M, consider the orthogonal bundle $T\mathcal{F}^\perp$ and a set of 1-forms $\theta^1, \ldots, \theta^6$ that define a basis for its dual $(T\mathcal{F}^\perp)^*$ at every point in U. Thus, in U the metric g has an expression of the form:

$$g = \sum_{i,j=1}^{8} l_{ij} dx^i \otimes dx^j + \sum_{i,j=1}^{6} h_{ij} \theta^i \otimes \theta^j.$$

From this and the definition of the volume element as a 14-form, we have:

$$\mathrm{vol} = \sqrt{|\det(l_{ij}) \det(h_{ij})|} dx^1 \wedge \ldots \wedge dx^8 \wedge \theta^1 \wedge \cdots \wedge \theta^6.$$

On the other hand, since the metric $\widehat{g}$ is obtained by rescaling g along the bundles $T\mathcal{F}$ and $T\mathcal{F}^\perp$, then $\widehat{g}$ has an expression of the form:

$$\widehat{g} = \sum_{i,j=1}^{8} C_1 l_{ij} dx^i \otimes dx^j + \sum_{i,j=1}^{6} C_2 h_{ij} \theta^i \otimes \theta^j.$$

for some constants $C_1, C_2 \neq 0$. Therefore, the volume element of $\widehat{g}$ satisfies:

$$\begin{aligned} \mathrm{vol}_{\widehat{g}} &= \sqrt{|\det(C_1 l_{ij}) \det(C_2 h_{ij})|} dx^1 \wedge \ldots \wedge dx^8 \wedge \theta^1 \wedge \cdots \wedge \theta^6 \\ &= \sqrt{|C_1^8 C_2^6|} \mathrm{vol}. \end{aligned}$$

■

It remains to consider the case $\mathcal{H} \cong \mathfrak{sl}(4,\mathbb{R})$, which occurs for $\mathcal{H}_0(x) \cong \mathbb{R}$. As before we can consider an isomorphism of Lie algebras

$$\psi : \mathfrak{sl}(4,\mathbb{R}) = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*} \to \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \oplus \mathcal{V}^*(x) = \mathcal{H}$$

that preserves the summands in that order. In particular, ψ is also an isomorphism of $\mathfrak{sl}(3,\mathbb{R})$-modules. As before, applying Lemma 3.6 we obtain an isometric right $\widetilde{\mathrm{SL}}(4,\mathbb{R})$-action on $\widetilde{M}$ that commutes with the lifted $\widetilde{\mathrm{SL}}(3,\mathbb{R})$-action and that preserves the bundles $T\mathcal{F}$ and $T\mathcal{F}^\perp$.

If we let H be the analytic subgroup of $\widetilde{\mathrm{SL}}(4,\mathbb{R})$ corresponding to $\mathcal{H}_0(x)$, then H is closed (see the exercises in page 152 from [3]). Hence, the map

$$\begin{aligned} \varphi : H\backslash\widetilde{\mathrm{SL}}(4,\mathbb{R}) &\to \widetilde{M} \\ Hg &\mapsto x \cdot g, \end{aligned}$$

is a well-defined analytic $\widetilde{\mathrm{SL}}(4,\mathbb{R})$-equivariant map. Furthermore, $d\varphi_{He}$ is an isomorphism which maps $\mathfrak{sl}(3,\mathbb{R})$ onto $T_x\mathcal{F}$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ onto $T_x\mathcal{F}^\perp$. Note that we have a natural identification of the tangent space of $H\backslash\widetilde{\mathrm{SL}}(4,\mathbb{R})$ at He with $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$. With this identification, it follows that $d\varphi_{He} = ev_x \circ \psi_0$ where ψ_0 denotes the restriction of ψ to $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$.

Let us denote by K the Killing form of $\mathfrak{sl}(4,\mathbb{R})$. It is easily seen by a straightforward computation that the subspaces $\mathbb{R}$, $\mathfrak{sl}(3,\mathbb{R})$ and $\mathbb{R}^3 \oplus \mathbb{R}^{3*}$ of $\mathfrak{sl}(4,\mathbb{R})$ are mutually orthogonal with respect to K. In particular, the restrictions of K to each of these subspaces are non-degenerate. Hence, it follows from Lemmas 2.1 and 2.4 that we can rescale the metric on $\widetilde{M}$ along the bundles $T\mathcal{F}$ and $T\mathcal{F}^\perp$ to obtain a new metric $\widehat{g}$ so that $d\varphi_{He}^*(\widehat{g}_x)$ is the restriction of the Killing form K to $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*}$. By the $\widetilde{\mathrm{SL}}(4,\mathbb{R})$-equivariance of φ, it follows that the bi-invariant metric on $\widetilde{\mathrm{SL}}(4,\mathbb{R})$ induced by K descends to a pseudo-Riemannian metric on $H\backslash\widetilde{\mathrm{SL}}(4,\mathbb{R})$ so that φ is a local isometry into $(\widetilde{M},\widehat{g})$.

On the other hand, the decomposition

$$\mathfrak{sl}(4,\mathbb{R}) = \mathbb{R} \oplus (\mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}^{3*})$$

together with the usual adjoint invariance of the Killing form, implies that the pair $(\widetilde{\mathrm{SL}}(4,\mathbb{R}), H)$ defines a naturally reductive homogeneous space, as considered in Definition 23 from [4] in page 312. It follows from Proposition 25 in page 313 of the same reference that $H\backslash\widetilde{\mathrm{SL}}(4,\mathbb{R})$ is complete.

As before, the above remarks imply that

$$\varphi : (H\backslash\widetilde{\mathrm{SL}}(4,\mathbb{R}), K) \to (\widetilde{M},\widehat{g})$$

is an isometry. Thus, the completeness of $(H\backslash\widetilde{\mathrm{SL}}(4,\mathbb{R}), K)$ and the simply connectedness of $\widetilde{M}$ imply, by Corollary 29 in [4, p. 202], that φ is an isometry.

This completes the proof of Theorem 1.1. Further information obtained so far is stated in the following result.

Proposition 6.2. *Let us assume the hypotheses of Theorem 1.1 and that case 3. from its conclusions hold. Then, there exist an analytic isometry*

$$\varphi : H\backslash\widetilde{\mathrm{SL}}(4,\mathbb{R}) \to (\widetilde{M},\widehat{g}),$$

and an analytic isometric right $\widetilde{\mathrm{SL}}(4,\mathbb{R})$*-action on* $(\widetilde{M},\widehat{g})$ *such that the following hold.*

1. *The left* $\widetilde{\mathrm{SL}}(3,\mathbb{R})$*-action and the right* $\widetilde{\mathrm{SL}}(4,\mathbb{R})$*-action on* $\widetilde{M}$ *commute with each other.*
2. *The map* φ *is* $\widetilde{\mathrm{SL}}(4,\mathbb{R})$*-equivariant for the right* $\widetilde{\mathrm{SL}}(4,\mathbb{R})$*-action on its domain.*

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