

## Cartan Pairs and Shared Orbit Pairs

Boris Širola \*

Communicated by T. Kobayashi

**Abstract.** We study a class of pairs of Lie algebras  $(\mathfrak{g}, \mathfrak{g}_1)$  that we call Cartan pairs; here  $\mathfrak{g}$  is semisimple and  $\mathfrak{g}_1$  is a reductive in  $\mathfrak{g}$  subalgebra. For these pairs, which generalize symmetric ones, we have standardly defined Cartan subspaces, and consequently the set of restricted roots  $\Sigma(\mathfrak{g}, \mathfrak{a})$ . We prove that there are infinitely many interesting nonsymmetric Cartan pairs. Next we prove that every pair of the well known Brylinski-Kostant list of shared orbit pairs is a Cartan pair. As a continuation of the previous research we obtained some further useful and clarifying results and examples related to Cartan pairs and Cartan subspaces.

*Mathematics Subject Classification 2010:* Primary 17B20; Secondary 17B05, 17B22.

*Key Words and Phrases:* Semisimple Lie algebra, Cartan subalgebra, nonsymmetric pair, Kostant pair, Cartan subspace, Cartan pair, restricted root, set of restricted roots, shared orbit pair.

### Introduction

Throughout this paper by the letter  $\mathbb{K}$  we always denote a field of characteristic zero. Every Lie algebra we consider is finite dimensional over the ground field. Given a  $\mathbb{K}$ -Lie algebra  $\mathfrak{g}$  by  $B_{\mathfrak{g}}$  we denote its Killing form. Also we define  $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \bar{\mathbb{K}}$ , where  $\bar{\mathbb{K}}$  is a fixed algebraic closure of  $\mathbb{K}$ . For any  $\mathfrak{g}$  and its proper subalgebra  $\mathfrak{g}_1$  which is not an ideal, let us formulate the following condition:

(C) *The restriction of  $B_{\mathfrak{g}}$  to  $\mathfrak{g}_1$  is nondegenerate.*

We refer to [BK, Ks2, Ks3] in order to see why this condition is worth of consideration. Now assume that  $\mathfrak{g}$  is a semisimple Lie algebra,  $\mathfrak{g}_1$  is a proper subalgebra reductive in  $\mathfrak{g}$  and the condition (C) is fulfilled. For such a pair  $(\mathfrak{g}, \mathfrak{g}_1)$  it is worth to know if the following stronger condition hold.

(Q1) *For any Cartan subalgebra  $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$  there is a unique Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$  containing  $\mathfrak{h}_1$ .*

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\* The author was supported in part by the Croatian Science Foundation Grant No. 2634, and by the Croatian Scientific Center of Excellence QuantixLie

It is well known that a number of interesting pairs  $(\mathfrak{g}, \mathfrak{g}_1)$  satisfy the above condition. For example, every symmetric pair does. More generally, assume  $\mathfrak{g}$  is (complex) semisimple and  $\sigma$  is its finite-order automorphism; see [K] and [He, Ch. X, Sect. 5] for the Kac's classification of all such automorphisms. Then the pair  $(\mathfrak{g}, \mathfrak{g}_1)$ , where  $\mathfrak{g}_1$  is the fixed point algebra  $\mathfrak{g}^\sigma$ , satisfies **(Q1)**. As an example for the latter we can take the pair of type  $(D_4, G_2)$ , where now  $\sigma$  is of order 3; see Section 3 below. Besides, there are nonsymmetric pairs satisfying **(Q1)** that are not of the above form. Perhaps as the most famous example of that kind we have the pair of type  $(B_3, G_2)$ ; see again Section 3. Further, suppose for the moment that  $\mathfrak{g}$  is a classical simple Lie algebra defined over an algebraically closed ground field, and  $\mathfrak{s}$  is its principal  $\mathfrak{sl}_2$ -subalgebra. Then the (nonsymmetric) pair  $(\mathfrak{g}, \mathfrak{s})$  satisfies our condition. For more details concerning all the mentioned facts we refer the reader to [Š1, Š2, Š4, Š5].

Now suppose that  $G$  is a noncompact and connected semisimple Lie group. Let  $\mathfrak{g}$  be its Lie algebra. Assume that we have an involution  $\theta$  on  $G$  such that the  $\theta$ -fixed point subgroup  $K$  is compact. By the same letter we denote the differential of  $\theta$ , an involution of  $\mathfrak{g}$ . Define  $\mathfrak{k}$  and  $\mathfrak{p}$  as the 1 and  $-1$  eigenspace for  $\theta$ , respectively. Thus we get a symmetric pair of Lie algebras  $(\mathfrak{g}, \mathfrak{k})$ , and the corresponding Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ ; the latter sum is Killing-orthogonal. Having such a setting we define a Cartan subspace as a maximal commutative subalgebra  $\mathfrak{a} \subseteq \mathfrak{p}$ . The notion of Cartan subspace for real and complex semisimple symmetric Lie algebras is important and standard; see, e.g., [Bu, Ch. 29]. Dixmier generalized the same notion for semisimple symmetric Lie algebras over any ground field [D, Ch. I, §1.13]. A further generalization for not necessarily semisimple Lie algebras is due to Lepowsky and McCollum [LM]. In [Š4] we showed that the standard definition of a Cartan subspace is meaningful for an interesting class of semisimple (nonsymmetric) pairs of Lie algebras. More precisely we consider a subclass of pairs that we call *Cartan pairs*, either of *type I* or *type II*, within the class of pairs  $(\mathfrak{g}, \mathfrak{g}_1)$  satisfying **(Q1)**; see Definition 2.4 below. For Cartan pairs and their Cartan subspaces we proved several results that generalize the classical counterparts for symmetric pairs.

For later needs recall the following. Suppose we have a symmetric semisimple  $\mathbb{K}$ -Lie algebra  $(\mathfrak{g}, \theta)$ , and let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be the corresponding Killing-orthogonal decomposition. Let  $\mathfrak{a} \subseteq \mathfrak{p}$  be a Cartan subspace of  $(\mathfrak{g}, \theta)$ . Define  $\mathfrak{m}$  to be the centralizer of  $\mathfrak{a}$  in  $\mathfrak{k}$ , and let  $\mathfrak{l}$  be its Cartan subalgebra. Then  $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{a}$  is a Cartan subalgebra of  $\mathfrak{g}$ . Define  $\Delta'$  and  $\Delta''$  as the set of roots  $\phi \in \Delta(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$  satisfying  $\phi|_{\bar{\mathfrak{a}}} = 0$  and  $\phi|_{\bar{\mathfrak{a}}} \neq 0$ , respectively. Then  $\{\phi|_{\bar{\mathfrak{l}}} \mid \phi \in \Delta'\}$  is equal to  $\Delta(\bar{\mathfrak{m}}, \bar{\mathfrak{l}})$ , the root system of  $\bar{\mathfrak{m}}$  relative to  $\bar{\mathfrak{l}}$ . More generally, supposing that  $\mathfrak{a}$  is a splitting Cartan subspace of a symmetric semisimple Lie algebra  $(\mathfrak{g}, \theta)$ , for each  $\phi \in \mathfrak{a}^*$  we define  $\mathfrak{g}_\phi$  as the root subspace consisting of all  $x \in \mathfrak{g}$  satisfying  $[A, x] = \phi(A)x$ , for all  $A \in \mathfrak{a}$ . And then  $\Sigma(\mathfrak{g}, \mathfrak{a})$  as the set of all nontrivial  $\phi$  such that  $\mathfrak{g}_\phi \neq 0$ . As we already noted,  $\Sigma(\mathfrak{g}, \mathfrak{a})$  is an abstract root system; in general not necessarily reduced. It is called the restricted root system of the pair  $(\mathfrak{g}, \theta)$ .

The purpose of the present paper is to extend the previous research in a nontrivial way. We obtained some further interesting and clarifying results, examples and remarks for Cartan pairs. Besides, we offer further examples of Kostant

pairs; which complements our previous research in [Š6]. Let us emphasize that our primary interest here are nonsymmetric pairs of Lie algebras, as the symmetric ones are necessarily Cartan pairs; see Lemma 2.9. The following theorem is our first main result; see Theorem 2.14 for a bit more stronger formulation. It shows that there are infinitely many nonsymmetric Cartan pairs; and implicitly suggests that there should be more ones.

**Theorem 0.1.** *Suppose that  $\mathfrak{g}$  is a classical simple Lie algebra defined over an algebraically closed ground field, and  $\mathfrak{s}$  is its principal  $\mathfrak{sl}_2$ -subalgebra. Then  $(\mathfrak{g}, \mathfrak{s})$  is a Cartan pair of type I.*

Having a Kostant pair  $(\mathfrak{g}, \mathfrak{g}_1)$ , see Subsection 1, and the usual  $B_{\mathfrak{g}}$ -orthogonal decomposition  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}$ , we have the notion of a Cartan subspace  $\mathfrak{a} \subseteq \mathfrak{p}$ . As in the standard semisimple symmetric setting, for Cartan pairs  $(\mathfrak{g}, \mathfrak{g}_1)$  and their Cartan subspaces  $\mathfrak{a}$  we consider the set of restricted roots  $\Sigma(\mathfrak{g}, \mathfrak{a})$ . Although the latter set will not always be a root system in the standard sense, see (1) below, it will be useful having quite similar meaning. For example, by [Š4, Prop. 2.8] we know that  $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\Sigma(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_{\lambda}$ . Related to the previous theorem, in Example 2.16 we computed  $\Sigma(\mathfrak{g}, \mathfrak{a})$  for some small rank classical Lie algebras  $\mathfrak{g}$ . In particular for  $\mathfrak{g} = \mathfrak{sp}(6, \mathbb{K})$  we showed that  $\Sigma(\mathfrak{g}, \mathfrak{a})$  may be understood as a (reducible) root system of type  $A_2 \times A_2 \times A_2$ ; see (3) of Examples 2.16 for the details.

In particular our goal in this research was to compare Cartan pairs and the shared orbit pairs of Brylinski and Kostant; see Theorem 1.4. The following result gives a definite explanation. Let us say that its proof is technical and rather tedious; see Lemma 3.1, Lemma 3.5 and Proposition 3.11. The pair of type  $(B_3, G_2)$  was treated in [Š4].

**Theorem 0.2.** *Every pair of the Brylinski-Kostant list is a Cartan pair. More precisely, the pairs of type  $(B_3, G_2)$  and  $(D_4, G_2)$  are nonsymmetric Cartan pairs of type I, while  $(G_2, A_2)$  and  $(F_4, D_4)$  are nonsymmetric pairs of type II. All the other pairs of the list are symmetric; and therefore Cartan pairs as well.*

As a consequence of the latter theorem we have that the subclass of Cartan pairs of type II is nonempty. Here it must be said that we are not aware of any other such nonsymmetric Cartan pair. On the other hand it is important to observe that within the class of all nonsymmetric Cartan pairs in fact only the subclass of the pairs of type I is really interesting for research.

For better understanding of the previous theorem it might be instructive to state here the following basic facts; see Corollaries 3.2 and 3.6.

**Scholium.** Consider the standard inclusions of Lie algebras

$$\mathfrak{g}_1 = \mathfrak{G}_2(\mathbb{C}) \subseteq \mathfrak{g} = \mathfrak{so}(7, \mathbb{C}) \subseteq \mathfrak{g}_0 = \mathfrak{so}(8, \mathbb{C});$$

here  $\mathfrak{G}_2(\mathbb{C})$  is the exceptional complex Lie algebra of type  $G_2$ .

- (i) For the corresponding Killing-orthogonal decompositions  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}$  and  $\mathfrak{g}_0 = \mathfrak{g}_1 \oplus \mathfrak{p}_0$  we choose Cartan subspaces  $\mathfrak{a} \subseteq \mathfrak{p}$ , of dimension 1, and

$\mathfrak{a}_0 \subseteq \mathfrak{p}_0$ , of dimension 2. Then  $(\mathfrak{g}, \mathfrak{g}_1)$  is a nonsymmetric Cartan pair (of type  $I$ ) and the set of restricted roots  $\Sigma(\mathfrak{g}, \mathfrak{a})$  is a (nonreduced) root system of type  $A_1$ . Also  $(\mathfrak{g}_0, \mathfrak{g}_1)$  is a nonsymmetric Cartan pair (of type  $I$ ) and the set of restricted roots  $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$  is a (reduced) root system of type  $A_2$ .

- (ii) If we denote by  $\mathfrak{s}$  the centralizer  $C_{\mathfrak{g}_1}(\mathfrak{a})$ , then  $(\mathfrak{g}_1, \mathfrak{s})$  is a pair of Lie algebras of type  $(G_2, A_2)$ , and it is a nonsymmetric Cartan pair (of type  $II$ ). Further, for a convenient Cartan subspace  $\mathfrak{a}_1$  and certain  $\lambda \in \mathfrak{a}_1^*$  we have

$$\Sigma(\mathfrak{g}_1, \mathfrak{a}_1) = \{\pm\lambda, \pm 2\lambda, \pm 3\lambda\}. \quad (1)$$

Let us say a few words about our motivation for the presented research. There are many instances where Cartan pairs  $(\mathfrak{g}, \mathfrak{g}_1)$  arise; of course, primarily symmetric ones. For example in branching problems, if we want to decompose certain restrictions  $\pi|_{\mathfrak{g}_1}$  of some  $\mathfrak{g}$ -representations  $\pi$ . As a particular “singular” example we think about the already mentioned nonsymmetric pair  $(\mathfrak{g}, \mathfrak{g}_1)$  of type  $(B_3, G_2)$ . Influenced by some Vogan’s ideas [V1, V2], Levasseur and Smith [LS] studied thoroughly that pair. As a surprising fact, they showed that there is a certain infinite-dimensional simple highest weight  $\mathfrak{g}$ -module which is as a  $\mathfrak{g}_1$ -module simple as well; see [V3] for a consequence about the unitary representations. Note that the paper [BK] was influenced by the mentioned research due to Levasseur, Smith and Vogan. As the second source of inspiration we would like to mention an important research due to Kobayashi on the side of Lie groups and unitary representations; see [Ko1, Ko3, Ko4, Ko5, Ko6], and also [KO]. In particular [Ko6] includes some of his unpublished results around 1990 about restrictions of representations. More precisely, he considered triples  $(G, G', \pi)$ , where  $G' \subseteq G$  are reductive groups and  $\pi$  is a representation of  $G$ ; with a special emphasis on the case when  $(G, G')$  is a symmetric pair. One of the main goals was to see when the restriction of  $\pi$  to  $G'$  is almost irreducible; i.e.,  $\pi$  is as a  $G'$ -representation a finite direct sum of irreducible representations. He treated a general case where  $\pi$  is not necessarily a highest weight module (cf. [LS]) and  $(G, G')$  is not necessarily a symmetric pair; see Theorem 3.5 of the cited paper, which has been previously presented in [Ko2]. Besides Kobayashi introduced an important notion of discrete decomposability that works very nice for symmetric pairs. As the example of  $(B_3, G_2)$  shows, we believe that a part of the mentioned research can be performed for some nonsymmetric pairs of Lie algebras (and Lie groups) as well. Finally observe that the pairs considered in our Theorem 0.1 had the central role within the seminal Kostant’s paper [Ks1]; see also [Š2].

The paper is organized as follows. Preliminary Section 1 has two subsections. The first one recalls some basic facts about pairs of Lie algebras we consider. In the second one we recall the necessary facts about the Brylinski-Kostant classification of shared orbit pairs. In Section 2 we prove Theorem 0.1. Section 3, which has two subsections, starts with some preliminary structural results about the embeddings of Lie algebras  $\mathfrak{g}_1 \subseteq \mathfrak{g} \subseteq \mathfrak{g}_0$ , mentioned in Scholium. We also introduce a notation that will be needed later. Then in the first subsection we treat the pairs of type  $(D_4, G_2)$  and  $(G_2, A_2)$ . In the second one we treat the pair  $(F_4, D_4)$ . Section 4 contains some interesting general observations about Cartan subspaces and Cartan pairs; given as Propositions 4.1, 4.2 and 4.5.

## 1. Preliminaries on certain pairs of Lie algebras

Let us fix our notation that will be used throughout the paper. If  $\mathfrak{g}$  is a Lie algebra and  $\mathfrak{r}$  is its subalgebra, then  $C_{\mathfrak{g}}(\mathfrak{r})$  and  $N_{\mathfrak{g}}(\mathfrak{r})$  are the centralizer and normalizer, of  $\mathfrak{r}$  in  $\mathfrak{g}$ , respectively. Given reductive  $\mathfrak{g}$  and a split Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ , by  $\Delta(\mathfrak{g}, \mathfrak{h})$  we denote the root system of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ . By  $\Pi(\mathfrak{g}, \mathfrak{h})$  we denote a choice of simple roots. For a root  $\phi$ ,  $X_{\phi}$  denotes a nonzero root vector from the root subspace  $\mathfrak{g}_{\phi}$ . By  $\Delta^{\pm} = \Delta^{\pm}(\mathfrak{g}, \mathfrak{h})$  we denote the positive/negative roots. Define a nilpotent radical  $\mathfrak{n} = \bigoplus_{\phi \in \Delta^+} \mathfrak{g}_{\phi}$ , and analogously  $\mathfrak{n}^-$ , which gives the standard triangular decomposition  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ . Recall that a subset of nonzero elements  $\{\mathfrak{f}, \mathfrak{h}, \mathfrak{e}\} \subseteq \mathfrak{g}$  is called a *standard triple* if we have  $[\mathfrak{h}, \mathfrak{e}] = 2\mathfrak{e}$ ,  $[\mathfrak{h}, \mathfrak{f}] = -2\mathfrak{f}$  and  $[\mathfrak{e}, \mathfrak{f}] = \mathfrak{h}$ .

### 1.1. Kostant pairs.

For later use here we recall in short some definitions, facts and notation concerning certain pairs of Lie algebras.

Suppose that a pair  $(\mathfrak{g}, \mathfrak{g}_1)$  satisfies the condition **(C)**. Let  $r: \mathfrak{g}^* \rightarrow \mathfrak{g}_1^*$  be the restriction map and  $K: \mathfrak{g} \rightarrow \mathfrak{g}^*$  the Killing homomorphism. By  $K_1: \mathfrak{g}_1 \rightarrow \mathfrak{g}_1^*$  we denote an isomorphism  $K_1(x) = B_{\mathfrak{g}}(x, \cdot)$ . Then define the associated homomorphism  $\pi = K_1^{-1} \circ r \circ K: \mathfrak{g} \rightarrow \mathfrak{g}_1$ . Now we have the following lemma; see [Š1, Sect. 3] or [Š3, Sect. 2] for more details.

**Lemma 1.1.** *Let  $(\mathfrak{g}, \mathfrak{g}_1)$  be a pair of Lie algebras as above. Define a subspace  $\mathfrak{p} = \ker \pi$  of  $\mathfrak{g}$ . Then we have a Killing-orthogonal direct sum decomposition*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p},$$

where  $[\mathfrak{g}_1, \mathfrak{p}] \subseteq \mathfrak{p}$ . Further, the pair  $(\mathfrak{g}, \mathfrak{g}_1)$  is symmetric if and only if  $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{g}_1$ .

Now assume that  $\mathfrak{g}$  is a semisimple Lie algebra and  $\mathfrak{g}_1$  is reductive in  $\mathfrak{g}$ . We say that  $(\mathfrak{g}, \mathfrak{g}_1)$  is a *Kostant pair* if the condition **(C)** holds. For the two facts of the following auxiliary lemma see [Š6, Thm. 0.1]

**Lemma 1.2.** *Suppose  $(\mathfrak{g}, \mathfrak{g}_1)$  is a Kostant pair of Lie algebras. Then we have a semidirect product of Lie algebras*

$$N_{\mathfrak{g}}(\mathfrak{g}_1) = [\mathfrak{g}_1, \mathfrak{g}_1] \oplus C_{\mathfrak{g}}(\mathfrak{g}_1),$$

and the pair  $(\mathfrak{g}, N_{\mathfrak{g}}(\mathfrak{g}_1))$  is a Kostant pair as well.

Given a (reductive) Lie algebra  $\mathfrak{g}$  it is often interesting to have precisely defined some of its semisimple, and nilpotent, elements. For later use recall the following useful observation; see [Š4, Sect. 1].

**Lemma 1.3.** *Let  $\mathfrak{g}$  be a semisimple Lie algebra, and suppose that  $\{\mathfrak{h}, \mathfrak{e}, \mathfrak{f}\}$  is a standard triple. Then for arbitrary nonzero  $a, b \in \mathbb{K}$  the element  $a\mathfrak{e} + b\mathfrak{f}$  is semisimple in  $\mathfrak{g}$ . In particular, suppose that  $\mathfrak{h}$  is a Cartan subalgebra of  $\mathfrak{g}$ . Then for an arbitrary root  $\phi \in \Delta(\mathfrak{g}, \mathfrak{h})$ , and root vectors  $X_{\pm\phi} \in \mathfrak{g}_{\pm\phi}$ , the elements  $X_{\phi} \pm X_{-\phi}$  are semisimple in  $\mathfrak{g}$ .*

## 1.2. Shared orbit pairs.

Levasseur and Smith in [LS] considered a pair of complex Lie algebras  $(\mathfrak{so}(7), \mathfrak{G}_2)$ , where  $\mathfrak{G}_2$  is the simple Lie algebra of type  $G_2$  standardly embedded in  $\mathfrak{so}(7)$ , which is of type  $B_3$ . Motivated by questions posed by Vogan [V1], the authors obtained several interesting geometric and algebraic results, about that pair. Then Brylinski and Kostant, influenced by the above cited two papers, in [BK] obtained some important results about the geometry of nilpotent orbits and the related symmetry groups for a class of interesting pairs of complex Lie groups, i.e., Lie algebras. To be more precise assume that  $\mathbf{G}$  is a simply connected simple complex Lie group. Let  $\mathfrak{g}$  be its Lie algebra, and  $\mathbf{G}_1 \subseteq \mathbf{G}$  a simple subgroup with Lie algebra  $\mathfrak{g}_1 \subseteq \mathfrak{g}$ . Suppose there is a nilpotent  $\mathbf{G}_1$ -orbit  $\mathcal{O}_1$  in  $\mathfrak{g}_1$  which (after adding a boundary of codimension  $\geq 2$ ) can be recognized as a certain nilpotent  $\mathbf{G}$ -orbit  $\mathcal{O}$  in  $\mathfrak{g}$ . Then we say that  $(\mathcal{O}, \mathcal{O}_1)$  is a *shared orbit pair*. Being slightly incorrect in this paper we will also say that  $(\mathfrak{g}, \mathfrak{g}_1)$ , or  $(\mathbf{G}, \mathbf{G}_1)$ , is a shared orbit pair if for it we have a shared orbit pair of nilpotent orbits. The main result of [BK] is the determination of all shared orbit pairs of complex simple Lie algebras (and of course the corresponding pairs of shared nilpotent orbits); see their Corollary 5.4 and Theorem 5.9. For the convenience of the reader we recall the list, in a slightly changed notation.

**Theorem 1.4.** *The Brylinski-Kostant list of shared orbit pairs of (complex) Lie algebras  $(\mathfrak{g}, \mathfrak{g}_1)$  is as follows.*

- (i) *For the equal rank case  $\text{rk } \mathfrak{g}_1 = \text{rk } \mathfrak{g}$  we have the following possible types:  $(G_2, A_2)$ ,  $(F_4, D_4)$ ,  $(F_4, B_4)$  and  $(B_n, D_n)$  for  $n \geq 3$ .*
- (ii) *For the non-equal rank case  $\text{rk } \mathfrak{g}_1 < \text{rk } \mathfrak{g}$  we have the following possible types:  $(B_3, G_2)$ ,  $(E_6, F_4)$ ,  $(D_4, G_2)$  and  $(D_{n+1}, B_n)$ ,  $(A_{2n-1}, C_n)$  for  $n \geq 2$ .*

**Remark 1.5.** As we will see below all the pairs of the above list are symmetric pairs, except the following four small rank pairs:

$$(G_2, A_2), \quad (B_3, G_2), \quad (D_4, G_2), \quad (F_4, D_4).$$

## 2. Cartan pairs

In what follows for the convenience of the reader and later use, where it is necessary we recall some definitions and facts from [Š4, Sect. 2].

Now suppose that  $(\mathfrak{g}, \mathfrak{g}_1)$  is a Kostant pair of Lie algebras. And let then  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}$  be the usual decomposition; see Lemma 1.1. As in [BK, Eq.(5-5)] for any Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  define the corresponding “vector part” by

$$\mathfrak{a}_{\mathfrak{h}} = \mathfrak{h} \cap \mathfrak{p}. \tag{2}$$

Supposing the above setting recall the following [Š1, Lemma 3.9].

**Lemma 2.1.** *Let  $\mathfrak{h}_1$  be a splitting Cartan subalgebra of  $\mathfrak{g}_1$  and  $\mathfrak{h}$  be any splitting Cartan subalgebra of  $\mathfrak{g}$  containing  $\mathfrak{h}_1$ .*

(i) We have a Killing-orthogonal direct sum

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{a}_{\mathfrak{h}}.$$

(ii) If we define  $\Omega(\mathfrak{h}) = \{\phi \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \phi|_{\mathfrak{h}_1} = 0\}$ , then

$$C_{\mathfrak{g}}(\mathfrak{h}_1) = \mathfrak{h} \oplus \bigoplus_{\Omega(\mathfrak{h})} \mathfrak{g}_{\phi}.$$

**Corollary 2.2.** *Let  $(\mathfrak{g}, \mathfrak{g}_1)$  be a Kostant pair and  $\mathfrak{h}_1$  any splitting Cartan subalgebra of  $\mathfrak{g}_1$ . Then both  $(\mathfrak{g}, \mathfrak{h}_1)$  and  $(\mathfrak{g}, C_{\mathfrak{g}}(\mathfrak{h}_1))$  are Kostant pairs as well.*

**Proof.** First observe that  $\mathfrak{h}_1$  is reductive in  $\mathfrak{g}$ . Next, by the part (i) of the latter lemma it easily follows that the restriction of  $B_{\mathfrak{g}}$  to  $\mathfrak{h}_1$  is nondegenerate. Thus  $(\mathfrak{g}, \mathfrak{h}_1)$  is a Kostant pair.

By Lemma 1.2 we first have the equality  $N_{\mathfrak{g}}(\mathfrak{h}_1) = C_{\mathfrak{g}}(\mathfrak{h}_1)$ , and therefore it follows that  $(\mathfrak{g}, C_{\mathfrak{g}}(\mathfrak{h}_1))$  is a Kostant pair.  $\blacksquare$

As we already said in the Introduction, the notion of Cartan subspace is standard both for real and complex semisimple symmetric Lie algebras; where for a semisimple symmetric pair  $(\mathfrak{g}, \mathfrak{g}_1)$  we define a Cartan subspace  $\mathfrak{a}$  of  $\mathfrak{g}$  as a maximal commutative subspace  $\mathfrak{a} \subseteq \mathfrak{p}$ . Recall that Dixmier generalized the mentioned notion for semisimple symmetric Lie algebras over an arbitrary field of characteristic zero. A further generalization, for not necessarily semisimple Lie algebras, have been obtained in [LM]. Concerning that notion, we have the following one for Kostant pairs  $(\mathfrak{g}, \mathfrak{g}_1)$ .

**Definition 2.3.** *A Cartan subspace of  $\mathfrak{g}$  is a commutative subalgebra  $\mathfrak{a} \subseteq \mathfrak{p}$  which is reductive in  $\mathfrak{g}$  and satisfies  $C_{\mathfrak{p}}(\mathfrak{a}) = \mathfrak{a}$ .*

Now we can recall a definition of a Cartan pair which has been introduced in [Š4].

**Definition 2.4.** *A Kostant pair  $(\mathfrak{g}, \mathfrak{g}_1)$  satisfying (Q1) will be called a Cartan pair if the following two conditions hold:*

(Cp1) There exist a Cartan subspace  $\mathfrak{a}$  and Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  such that

$$(\mathfrak{h} \cap \mathfrak{g}_1) \oplus \mathfrak{a} = \mathfrak{h};$$

(Cp2) If we put

$$\Delta' = \{\phi \in \Delta(\overline{\mathfrak{g}}, \overline{\mathfrak{h}}) \mid \phi|_{\overline{\mathfrak{a}}} = 0\},$$

then  $\phi \in \Delta'$  implies that  $X_{\phi} \in \overline{\mathfrak{g}}_1$ .

**Remark 2.5.** *Observe that in the latter definition in order to get a Cartan pair we had to choose a particular Cartan subspace  $\mathfrak{a}$ , and a Cartan subalgebra  $\mathfrak{h}$ . Later we will see that having such a particular choice of  $\mathfrak{a}$ , and  $\mathfrak{h}$ , we will in principle*

be able to obtain a number of other Cartan subspaces, and Cartan subalgebras, for which the two conditions of our definition will hold as well. Further, for the examples  $(\mathfrak{g}, \mathfrak{g}_1)$  that we consider below it turns out that every Cartan subspace will do.

The following simple lemma is a part of [Š4, Prop. 2.7].

**Lemma 2.6.** *Suppose that  $(\mathfrak{g}, \mathfrak{g}_1)$  is a Cartan pair, and  $\mathfrak{a}$  is a Cartan subspace as in **(Cp1)**. Then*

$$C_{\mathfrak{g}}(\mathfrak{a}) = C_{\mathfrak{g}_1}(\mathfrak{a}) \oplus \mathfrak{a}, \quad (3)$$

*the restrictions of the Killing form  $B_{\mathfrak{g}}$  to  $\mathfrak{a}$  and  $C_{\mathfrak{g}_1}(\mathfrak{a})$  are nondegenerate, and  $C_{\mathfrak{g}_1}(\mathfrak{a})$  is reductive in  $\mathfrak{g}$ .*

**Remark 2.7.** Suppose that  $(\mathfrak{g}, \mathfrak{g}_1)$  is a nonsymmetric Cartan pair; i.e., by Lemma 1.1, that  $[\mathfrak{p}, \mathfrak{p}] \not\subseteq \mathfrak{g}_1$ . Then, concerning the latter fact and the above stated nonobvious equality (3), observe that in general we can have that moreover  $[\mathfrak{a}, \mathfrak{p}] \not\subseteq \mathfrak{g}_1$ . For example, the latter will happen for the pair  $(\mathfrak{g}_1, \mathfrak{m})$  considered in Lemma 3.5 below.

**Corollary 2.8.** *Let  $(\mathfrak{g}, \mathfrak{g}_1)$  and  $\mathfrak{a}$  be as above. Then  $(\mathfrak{g}, \mathfrak{a})$ ,  $(\mathfrak{g}, C_{\mathfrak{g}_1}(\mathfrak{a}))$  and  $(\mathfrak{g}, C_{\mathfrak{g}}(\mathfrak{a}))$  are Kostant pairs.*

**Proof.** By the latter lemma we know that  $(\mathfrak{g}, \mathfrak{a})$  and  $(\mathfrak{g}, C_{\mathfrak{g}_1}(\mathfrak{a}))$  are Kostant pairs. Now as in the previous corollary we conclude that  $N_{\mathfrak{g}}(\mathfrak{a}) = C_{\mathfrak{g}}(\mathfrak{a})$ , and therefore  $(\mathfrak{g}, C_{\mathfrak{g}}(\mathfrak{a}))$  is a Kostant pair as well. ■

Before we proceed let us note the following basic observation [Š4, Prop. 2.13]; of course, it comes as no surprise.

**Lemma 2.9.** *A semisimple symmetric pair  $(\mathfrak{g}, \mathfrak{g}_1)$  is a Cartan pair as well.*

The next lemma introduces a subclass of Cartan pairs that is of the primary interest; see [Š4, Lemma 2.10]. That will enable us to refine the above definition of Cartan pair.

**Lemma 2.10.** *Let  $(\mathfrak{g}, \mathfrak{g}_1)$  be a Kostant pair satisfying the inequality  $\text{rk } \mathfrak{g}_1 < \text{rk } \mathfrak{g}$  on ranks. Suppose there is a pair of the corresponding Cartan subalgebras  $\mathfrak{h}_1 \subseteq \mathfrak{h}$ , and let then  $\mathfrak{a} = \mathfrak{a}_{\mathfrak{h}}$  be as in (2). Suppose also that the condition **(Cp2)** holds. Then  $\mathfrak{a}$  is a Cartan subspace of  $\mathfrak{g}$ , and  $\mathfrak{h} \cap \mathfrak{g}_1 = \mathfrak{h}_1$ . As a consequence we have that  $(\mathfrak{g}, \mathfrak{g}_1)$  is a Cartan pair if it satisfies **(Q1)**.*

**Definition 2.11.** A Cartan pair  $(\mathfrak{g}, \mathfrak{g}_1)$  is called a *Cartan pair of type I* if we have  $\text{rk } \mathfrak{g}_1 < \text{rk } \mathfrak{g}$ , and for  $\mathfrak{a}$  and  $\mathfrak{h}$  as in **(Cp1)** there exists a Cartan subalgebra  $\mathfrak{h}_1 \subseteq \mathfrak{h}$  of  $\mathfrak{g}_1$  so that  $\mathfrak{a}$  is given as in (2). If  $(\mathfrak{g}, \mathfrak{g}_1)$  is not of type I, we say that it is a *Cartan pair of type II*.

The following corollary is an immediate consequence of the previous definition and Lemma 2.1(i).

**Corollary 2.12.** *Suppose that  $(\mathfrak{g}, \mathfrak{g}_1)$  is a Cartan pair of type I, and let a Cartan subspace  $\mathfrak{a}$  be as in (Cp1). Then*

$$\dim \mathfrak{a} = \operatorname{rk} \mathfrak{g} - \operatorname{rk} \mathfrak{g}_1.$$

For a Cartain pair  $(\mathfrak{g}, \mathfrak{g}_1)$  let  $\mathfrak{h}$  and  $\mathfrak{a}$  be as in (Cp1). For  $\lambda \in \mathfrak{a}^*$  define

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [A, x] = \lambda(A)x, \forall A \in \mathfrak{a}\},$$

and then

$$\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\lambda \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}_\lambda \neq 0\}.$$

Loosely speaking, it turns out that for nonsymmetric Cartan pairs the set  $\Sigma(\mathfrak{g}, \mathfrak{a})$  takes the role of the restricted root system of the classical symmetric setting; see, e.g., [Kn1, Ch. V] or [Kn2, Ch. VI, §§4,5]. Thus it makes sense to call any  $\lambda \in \Sigma(\mathfrak{g}, \mathfrak{a})$  a *restricted root* of  $(\mathfrak{g}, \mathfrak{a})$ , and any corresponding nonzero element of  $\mathfrak{g}_\lambda$  a *restricted root vector*. Observe that we in particular have a direct sum decomposition

$$\mathfrak{g} = C_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\Sigma(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\lambda;$$

to avoid ambiguity in the sequel, here we prefer to write  $C_{\mathfrak{g}}(\mathfrak{a})$  instead of  $\mathfrak{g}_0$ .

Keeping the above setting, an element  $A \in \mathfrak{a}$  is called *regular* if  $\lambda(A) \neq 0$  for every  $\lambda \in \Sigma(\mathfrak{g}, \mathfrak{a})$ . For later use let us define

$$\mathfrak{a}_{reg} = \text{set of all regular elements of } \mathfrak{a};$$

clearly,  $\mathfrak{a}_{reg}$  is a nonempty open subset of  $\mathfrak{a}$ . The following simple observation, which in a sense improve (3), is an analog of a well-known standard fact, and has the same proof; see, e.g., [Kn2, Lemma 6.50].

**Lemma 2.13.** *Suppose that  $A \in \mathfrak{a}$  is a regular element. Then*

$$C_{\mathfrak{g}}(A) = C_{\mathfrak{g}_1}(\mathfrak{a}) \oplus \mathfrak{a},$$

and hence

$$C_{\mathfrak{p}}(A) = \mathfrak{a}.$$

The following result is the first one showing us that there are infinitely many nonsymmetric Cartan pairs. Besides it suggests to look at some analogues settings, for example when we have semisimple Lie algebras  $\mathfrak{s} \subseteq \mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{K})$  so that there is no faithful representation of  $\mathfrak{s}$  of dimension less than  $n$ . Further, as will be seen below (Examples 2.16), this theorem enables one more useful insight. More precisely, although our pairs are in a sense on the “edge” of the whole spectrum, we will see what one can obtain for the set  $\Sigma(\mathfrak{g}, \mathfrak{a})$ , of restricted roots.

**Theorem 2.14.** *Let  $\mathbb{K}$  be an algebraically closed field. Suppose that  $\mathfrak{g}$  is a classical simple  $\mathbb{K}$ -Lie algebra and  $\mathfrak{s}$  is its principal  $\mathfrak{sl}_2$ -subalgebra; or  $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{K})$ , with  $n > 1$  odd, and  $\mathfrak{s}$  is a subregular  $\mathfrak{sl}_2$ -subalgebra. Then  $(\mathfrak{g}, \mathfrak{s})$  is a Cartan pair of type I. Moreover, suppose that  $\{\mathfrak{f}, \mathfrak{h}, \mathfrak{e}\}$  is a standard triple which is a basis of  $\mathfrak{s}$ , and  $\mathfrak{h}$  is a Cartan subalgebra of  $\mathfrak{g}$ . Then the Killing-orthogonal of  $\mathfrak{h}$  in  $\mathfrak{h}$  is a Cartan subspace for the pair  $(\mathfrak{g}, \mathfrak{s})$ .*

**Proof.** First observe that by [Š2, Sect. 4] and [Š5, Lemma 2.7] we know that every such  $(\mathfrak{g}, \mathfrak{s})$  satisfies **(Q1)**. Next suppose for the moment that  $(\mathfrak{g}, \mathfrak{h})$  is any split semisimple Lie algebra, and  $\Delta^+$  is a choice of positive roots in  $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ . Then recall that every element  $e = \sum_{\Delta^+} c_\phi X_\phi$ , with  $c_\phi \neq 0$  for every  $\phi$  of the basis  $\Pi(\mathfrak{g}, \mathfrak{h}) \subseteq \Delta^+$ , is a (positive) principal nilpotent element; see [Bo2, Ch. VIII, §11].

Suppose first that  $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{K})$ ,  $n > 1$ . Define  $H_i = E_{ii} - E_{i+1, i+1}$ ; then  $\mathfrak{h} = \text{span}\{H_1, \dots, H_n\}$  is a Cartan subalgebra of  $\mathfrak{g}$ . Define  $(\epsilon_i)_1^{n+1}$  to be the dual basis of  $(E_{ii})_1^{n+1}$ ; i.e.,  $\epsilon_i(E_{jj}) = \delta_{ij}$ . Define  $\beta_i = \epsilon_i - \epsilon_{i+1}$ ; then  $\Pi = \{\beta_1, \dots, \beta_n\}$  is a root system basis, and  $\Delta^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n+1\}$  are the corresponding positive roots. Let  $\langle \cdot | \cdot \rangle$  be the standard trace-form. Now consider a principal nilpotent element  $e$ , and an attached  $\mathfrak{h} \in \mathfrak{h}$  satisfying  $[\mathfrak{h}, e] = 2e$ ; we can take

$$e = \sum_{i=1}^n i(n+1-i)X_{\beta_i} \quad \text{and} \quad \mathfrak{h} = \sum_{i=1}^{n+1} (n+2-2i)E_{ii}.$$

Define

$$\mathfrak{a} = \mathfrak{a}(n) = \{A \in \mathfrak{h} \mid \langle A, \mathfrak{h} \rangle = 0\}; \quad (4)$$

i.e.,  $\mathfrak{a}$  is the Killing-orthogonal of  $\mathfrak{h}$  in  $\mathfrak{h}$ . It is easy to see that every  $A$  has a form  $\sum_{i=1}^{n+1} a_i E_{ii}$ , where

$$\sum_{i=1}^n (n+1-i)a_i = 0 = \sum_{i=1}^{n+1} a_i.$$

We may further assume that  $n > 2$ ; as the pair  $(\mathfrak{sl}(3, \mathbb{K}), \mathfrak{s})$  is symmetric, and therefore a Cartan pair. Now for every  $1 \leq i < j \leq n$  consider any  $S_{ij} = \sum_{t=1}^{n+1} a_t E_{tt}$  satisfying  $a_i = 1$  and  $a_j = 0$ . Then  $(\epsilon_i - \epsilon_j)(S_{ij}) = 1$ . Similarly we can find some  $S_i$  such that  $(\epsilon_i - \epsilon_{n+1})(S_i) \neq 0$  for all  $i \leq n$ . This shows that the set of roots  $\Delta'$ , given in **(Cp2)**, is the empty set. It remains to note that for  $\mathfrak{p}$ , the Killing-orthogonal of  $\mathfrak{s}$  in  $\mathfrak{g}$ , we have that it is a direct sum  $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}'$ , for a certain vector subspace  $\mathfrak{p}' \leq \mathfrak{n}^- \oplus \mathfrak{n}$ . Here  $\mathfrak{n}$  and  $\mathfrak{n}^-$  have the standard meaning; see the beginning of Section 1. Thus by Lemma 2.10 we get that  $(\mathfrak{g}, \mathfrak{s})$  is a Cartan pair, and  $\mathfrak{a}$  is its Cartan subspace.

Let again  $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{K})$ , but now  $n \geq 2$  even; and consider a subregular nilpotent element  $e = X_{\beta_1} + \dots + X_{\beta_{n-1}}$ . Here we can take for the corresponding  $\mathfrak{h}$  the element  $\sum_{i=1}^n (n+1-2i)E_{ii}$ ; see Lemma 1.2 in [Š5], and the paragraph following it. Let  $\mathfrak{a}$  be again defined by (4). Then every  $A \in \mathfrak{a}$  has a form  $\sum_{i=1}^{n+1} a_i E_{ii}$ , where  $\sum_{i=1}^n (n+1-2i)a_i = 0$ . If  $n = 2$ , then  $\mathfrak{a} = \mathbb{K}A$ , where  $A = E_{11} + E_{22} - 2E_{33}$ . And therefore  $\Delta' = \{\beta_1\}$ , which shows that **(Cp2)** holds. For  $n \geq 4$  we can check that again  $\Delta'$  is the empty set. Thus we have the theorem for this case as well.

Now suppose that  $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{K}) \leq \mathfrak{sl}(2n, \mathbb{K})$ ; here we consider a realization as in [Š5, Subsubsection 1.2.1]. Define  $H_i = E_{ii} - E_{2n+1-i, 2n+1-i}$ ; then  $\mathfrak{h} = \text{span}\{H_1, \dots, H_n\}$  is a Cartan subalgebra of  $\mathfrak{g}$ . This time  $(\epsilon_i)_1^n$  is dual to the basis  $(H_i)_1^n$ . Define  $\beta_i = \epsilon_i - \epsilon_{i+1}$  for  $1 \leq i < n$  and  $\beta_n = 2\epsilon_n$ ; then  $\Pi = \{\beta_1, \dots, \beta_n\}$  is a root system basis, and

$$\Delta^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\epsilon_1, \dots, 2\epsilon_n\}$$

are the corresponding positive roots. Consider a principal nilpotent element  $\mathfrak{e} = \sum_{i=1}^n X_{\beta_i}$ . An easy computation gives that now for the corresponding  $\mathfrak{h}$  we can put

$$\mathfrak{h} = \sum_{i=1}^n (2n - 2i + 1)H_i,$$

and its Killing-orthogonal in  $\mathfrak{h}$  is equal to

$$\mathfrak{a} = \left\{ \sum_{i=1}^n a_i H_i \mid \sum_{i=1}^n (2n - 2i + 1)a_i = 0 \right\}.$$

In particular for  $n = 2$  we have  $\mathfrak{a} = \mathbb{K}A$ , where  $A = H_1 - 3H_2$ . Hence it is clear that  $\Delta' = \emptyset$ . For  $n > 2$  we have the same. Thus we conclude that the theorem holds for the pair  $(\mathfrak{g}, \mathfrak{s})$ , where  $\mathfrak{s}$  is the corresponding principal  $\mathfrak{sl}_2$ -subalgebra.

The cases when  $\mathfrak{g}$  is either of type  $B_n$  or type  $D_n$  are left to the reader. ■

**Remark 2.15.** (1) Consider again  $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{K})$ , but this time a slightly different realization than the one of the above proof; see [Š5, Subsubsection 1.2.3]. Here we also have  $\Pi = \{\beta_1, \dots, \beta_n\}$ , and then consider a subregular standard triple  $\mathcal{T} = \{\mathfrak{f}, \mathfrak{h}, \mathfrak{e}\} \subseteq \mathfrak{g}$  as in [Š6, Lemma 4.4]; see also [CM, Ch. 5]. It is easy to see that the pair  $(\mathfrak{g}, \mathfrak{s})$ , where  $\mathfrak{s} = \text{span } \mathcal{T}$ , does not satisfy **(Q1)**. We conclude that we cannot expect that the previous theorem can be further generalized for the setting when  $\mathfrak{s}$  is some other  $\mathfrak{sl}_2$ -subalgebra, than the ones we considered.

(2) Given a splittable semisimple Lie algebra  $\mathfrak{g}$ , with a splitting Cartan subalgebra  $\mathfrak{h}$ , let  $\Delta^+$  be a choice of positive roots in  $\Delta(\mathfrak{g}, \mathfrak{h})$ . Consider any standard triple  $\mathcal{T} = (\mathfrak{f}, \mathfrak{h}, \mathfrak{e})$  such that  $\mathfrak{e}$  is a principal nilpotent element belonging to the nilpotent radical  $\mathfrak{n}$  determined by  $\Delta^+$ . Let  $\mathfrak{s}$  be a subalgebra spanned by  $\mathcal{T}$ ; so,  $\mathfrak{s}$  is a principal  $\mathfrak{sl}_2$ -subalgebra. Now take  $\mathfrak{g}$  to be a classical splittable simple Lie algebra. By adapting the given proof of the above theorem, one can proceed in order to prove that  $(\mathfrak{g}, \mathfrak{s})$  is again a Cartan pair of type *I*. (We believe that the same result holds for exceptional splittable simple Lie algebras as well; but did not check that in details.)

For what follows it will be useful to have the following, more or less standard, notation. Suppose that we have a Cartan pair  $(\mathfrak{g}, \mathfrak{s})$  with a Cartan subspace  $\mathfrak{a}$ , of dimension greater than one. Fix a basis  $(S_1, \dots, S_k)$  of  $\mathfrak{a}$ . Then for any  $\lambda \in \mathfrak{a}^*$  satisfying  $\lambda(S_i) = r_i$  we write  $\lambda = (r_1, \dots, r_k)$ .

**Examples 2.16.** (1) Consider first  $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{K})$  and let  $\mathfrak{s}$  be a subregular  $\mathfrak{sl}_2$ -subalgebra attached to  $\mathfrak{e} = X_{\beta_1}$  and  $\mathfrak{h} = H_1$ , and also  $\mathfrak{a} = \mathbb{K}A$  for  $A = H_1 + 2H_2$ ;

as in the proof of the above theorem. It is easy to check the following. If we define  $\lambda \in \mathfrak{a}^*$  by  $\lambda(A) = 3$ , then  $\mathfrak{g}_{\pm\lambda} = \text{span}\{X_{\pm\beta_2}, X_{\pm(\beta_1+\beta_2)}\}$ ; and therefore  $\Sigma(\mathfrak{g}, \mathfrak{a})$  is a root system of type  $A_1$ , while  $C_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{s} \oplus \mathfrak{a}$ .

(2) Next take  $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{K})$ , and let  $\mathfrak{s}$ ,  $\mathfrak{e}$ ,  $A$  and  $\mathfrak{a}$  be as in the theorem. Then consider  $\lambda \in \mathfrak{a}^*$  defined by  $\lambda(A) = 2$ . It is easy to check that  $\mathfrak{g}_{\pm\lambda} = \text{span}\{X_{\pm(2\beta_1+\beta_2)}, X_{\mp(\beta_1+\beta_2)}\}$ ,  $\mathfrak{g}_{\pm 2\lambda} = \mathbb{K}X_{\pm\beta_1}$  and  $\mathfrak{g}_{\pm 3\lambda} = \mathbb{K}X_{\mp\beta_2}$ . Thus it follows that

$$C_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{h} \quad \text{and} \quad \Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\lambda, \pm 2\lambda, \pm 3\lambda\}.$$

(3) Now consider  $\mathfrak{g} = \mathfrak{sp}(6, \mathbb{K})$ . Here we have  $\mathfrak{a} = \text{span}\{S_1, S_2\}$ , where  $S_1 = H_1 - 5H_3$  and  $S_2 = H_2 - 3H_3$ . If we write down the values  $\phi(S_i)$ , for every  $\phi \in \Delta^+$  and  $i = 1, 2$ , it follows the following observation. For the basis  $\{S_1, S_2\}$  of  $\mathfrak{a}$  define  $\lambda_i, \mu_i, \nu_i \in \mathfrak{a}^*$  as follows:  $\lambda_1 = (5, 2)$  and  $\lambda_2 = (5, 4)$ ,  $\mu_1 = (2, 0)$  and  $\mu_2 = (4, 3)$ ,  $\nu_1 = (1, -1)$  and  $\nu_2 = (0, 2)$ . It is easy to check that here we have the set of positive roots

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{\lambda_1, \lambda_2, \lambda_1 + \lambda_2, \mu_1, \mu_2, \mu_1 + \mu_2, \nu_1, \nu_2, \nu_1 + \nu_2\},$$

and thus  $\Sigma(\mathfrak{g}, \mathfrak{a}) = (-\Sigma^+(\mathfrak{g}, \mathfrak{a})) \cup \Sigma^+(\mathfrak{g}, \mathfrak{a})$ .

Here it might be interesting to note the following. We can take  $\mathfrak{a}_\lambda = \mathfrak{a}_\mu = \mathfrak{a}_\nu = \mathfrak{a}$ , and then  $\mathfrak{a} = \mathfrak{a}_\lambda \oplus \mathfrak{a}_\mu \oplus \mathfrak{a}_\nu$ . Consider  $\lambda_i \in \mathfrak{a}^* = \mathfrak{a}_\lambda^*$  and identify it with its trivial extension to  $\lambda_i \in \mathfrak{a}^*$ ; and analogously for  $\mu_i, \nu_i$ . Then put

$$\Sigma_\lambda^+ = \Sigma_\lambda^+(\mathfrak{g}, \mathfrak{a}_\lambda) = \{\lambda_1, \lambda_2, \lambda_1 + \lambda_2\},$$

and also

$$\Sigma_\lambda = \Sigma_\lambda(\mathfrak{g}, \mathfrak{a}_\lambda) = (-\Sigma_\lambda^+) \cup \Sigma_\lambda^+.$$

Analogously we define  $\Sigma_\mu$  and  $\Sigma_\nu$ . Thus we have a disjoint union

$$\Sigma(\mathfrak{g}, \mathfrak{a}) = \Sigma_\lambda \cup \Sigma_\mu \cup \Sigma_\nu,$$

where each of the three sets of roots is of type  $A_2$ . Hence, somewhat loosely speaking,  $\Sigma(\mathfrak{g}, \mathfrak{a})$  is a reducible root system of type  $A_2 \times A_2 \times A_2$ .

### 3. Nonsymmetric pairs of the Brylinski-Kostant list

In this rather technical section we treat the four small rank pairs of the Brylinski-Kostant list that were pointed out at Remark 1.5. Before starting to perform our computations, for the convenience of the reader we recall the necessarily notation and facts; cf. [Š5, Sect. 3.1], where the presentation and notation are slightly different.

Given  $n \in \mathbb{N}$ , by  $M_n(\mathbb{K})$  we denote the space of  $n \times n$  matrices and by  $E_{ij} \in M_n(\mathbb{K})$  the standard matrix with 1 on the place  $(i, j)$  and 0 elsewhere. For  $k \in \mathbb{N}$ , by  $\mathfrak{s}_k \in M_k(\mathbb{K})$  we denote the matrix with 1 on the skew diagonal and 0 elsewhere. For  $A \in M_k(\mathbb{K})$  we define the *skew transpose*  $A^\tau = \mathfrak{s}_k A^t \mathfrak{s}_k$ , where  $A^t$  is the ordinary transpose.

We will precisely describe the inclusions

$$\mathfrak{g}_1 = \mathfrak{G}_2(\mathbb{K}) \subseteq \mathfrak{g} = \mathfrak{so}(7, \mathbb{K}) \subseteq \mathfrak{g}_0 = \mathfrak{so}(8, \mathbb{K}); \quad (5)$$

$\mathfrak{G}_2(\mathbb{K})$  is the exceptional Lie algebra of type  $G_2$ . First consider a map  $A \mapsto A^\dagger$  on block-matrices  $A \in M_{2n}(\mathbb{K})$ , where each block is of size  $n \times n$ , defined by

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = A \longmapsto A^\dagger = \begin{pmatrix} T^\tau & Y^\tau \\ Z^\tau & X^\tau \end{pmatrix}.$$

Then define a map  $\Theta(A) = -A^\dagger$ . It turns out that  $(\mathfrak{sl}(2n, \mathbb{K}), \Theta)$  is a symmetric Lie algebra with  $\mathfrak{g}_0 = \mathfrak{so}(2n, \mathbb{K})$  as its corresponding fixed point algebra. For each  $1 \leq i \leq n$  we define  $H_i = E_{ii} - E_{2n+1-i, 2n+1-i}$ , and then a Cartan subalgebra  $\mathfrak{h}_0 = \text{span}\{H_1, \dots, H_n\}$  of  $\mathfrak{g}_0$ . Let  $(\epsilon_i)_1^n$ ,  $\epsilon_i \in \mathfrak{h}_0^*$ , be the dual basis of  $(H_i)_1^n$ . Define  $\gamma_i = \epsilon_i - \epsilon_{i+1}$ , for  $1 \leq i < n$ , and  $\gamma_n = \epsilon_{n-1} + \epsilon_n$ . Then  $\Pi_0 = \Pi(\mathfrak{g}_0, \mathfrak{h}_0) = \{\gamma_1, \dots, \gamma_n\}$  is a choice of simple roots in  $\Delta_0 = \Delta(\mathfrak{g}_0, \mathfrak{h}_0)$ . The set of positive roots relative to  $\Pi_0$  is

$$\Delta_0^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\}.$$

Choose the root vectors

$$X_{\epsilon_i - \epsilon_j} = E_{ij} - E_{2n+1-j, 2n+1-i}, \quad X_{\epsilon_i + \epsilon_j} = E_{i, 2n+1-j} - E_{j, 2n+1-i};$$

and note that in particular

$$X_{\gamma_i} = \begin{cases} E_{i, i+1} - E_{2n-i, 2n-i+1} & \text{for } 1 \leq i < n, \\ E_{n-1, n+1} - E_{n, n+2} & \text{for } i = n. \end{cases}$$

Now we specialize to  $n = 4$ ; i.e., we consider  $\mathfrak{g}_0 = \mathfrak{so}(8, \mathbb{K}) \subseteq \mathfrak{sl}(8, \mathbb{K})$ . So here we in particular have  $\mathfrak{h}_0 = \text{span}\{H_1, \dots, H_4\}$ , where  $H_i = E_{ii} - E_{9-i, 9-i}$ ; and  $\Pi_0 = \{\gamma_1, \dots, \gamma_4\}$ , where  $\gamma_1 = \epsilon_1 - \epsilon_2$ ,  $\gamma_2 = \epsilon_2 - \epsilon_3$ ,  $\gamma_3 = \epsilon_3 - \epsilon_4$  and  $\gamma_4 = \epsilon_3 + \epsilon_4$ . A (nonzero) root vector corresponding to a root  $\phi \in \Delta_0$  will be denoted by  $X_\phi^0$ . Further for easier notation a root  $\phi = \sum_{i=1}^4 c_i \gamma_i$  satisfying  $c_1 + \dots + c_4 > 2$  will be written as  $\gamma(c_1 \dots c_4)$ .

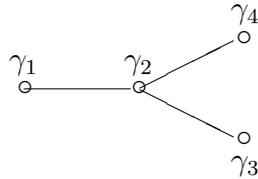
Next define  $\mathfrak{h} = \text{span}\{H_1, H_2, H_3\}$ , and let  $\eta_i$  be the restrictions of  $\epsilon_i$  to  $\mathfrak{h}$ . Also put  $\beta_1 = \eta_1 - \eta_2$ ,  $\beta_2 = \eta_2 - \eta_3$  and  $\beta_3 = \eta_3$ . It turns out that  $\tilde{\Pi} = \{\beta_1, \beta_2, \beta_3\}$  is a basis of the root system  $\tilde{\Delta}$ , where

$$\tilde{\Delta}^+ = \{\eta_1, \eta_2, \eta_3\} \cup \{\eta_i \pm \eta_j \mid 1 \leq i < j \leq 3\},$$

is the corresponding set of positive roots. For a convenient choice of root vectors  $\tilde{X}_\phi \in M_8(\mathbb{K})$ ,  $\phi \in \tilde{\Delta}$ ,

$$\tilde{\mathfrak{g}} = \mathfrak{h} \oplus \bigoplus_{\phi \in \tilde{\Delta}} \mathbb{K} \tilde{X}_\phi$$

is a  $B_3$ -type simple Lie algebra  $\mathfrak{so}(7, \mathbb{K})$ . (Observe that  $\tilde{\mathfrak{g}}$  is not a subalgebra of  $\mathfrak{g}_0$ .) Now consider the Dynkin diagram of type  $D_4$  labeled as



Let  $\theta$  be its order 2 automorphism given by  $\theta(\gamma_3) = \gamma_4$  and  $\theta(\gamma_i) = \gamma_i$  for  $i = 1, 2$ . By the same letter we denote the corresponding lift to an involution of  $\mathfrak{g}_0$ . Then define the fixed point algebra

$$\mathfrak{g} = \mathfrak{g}_0^\theta.$$

It turns out that  $\mathfrak{g}$  is a Lie algebra isomorphic to  $\tilde{\mathfrak{g}}$ ,  $\mathfrak{h}$  is its Cartan subalgebra and the root system  $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) = \tilde{\Delta}$ . In particular we have the following table, see [Š5], for root vectors  $X_\phi$  where  $\phi \in \Delta^+ = \tilde{\Delta}^+$ :

$$\begin{array}{ll} X_{\beta_1} = X_{\gamma_1}^0 & X_{\beta_2} = X_{\gamma_2}^0 \\ X_{\beta_3} = X_{\gamma_3}^0 + X_{\gamma_4}^0 & X_{\beta_1+\beta_2} = X_{\gamma_1+\gamma_2}^0 \\ X_{\beta_2+\beta_3} = X_{\gamma_2+\gamma_3}^0 + X_{\gamma_2+\gamma_4}^0 & X_{\beta_1+\beta_2+\beta_3} = X_{\gamma(1110)}^0 + X_{\gamma(1101)}^0 \\ X_{\beta_2+2\beta_3} = -X_{\gamma(0111)}^0 & X_{\beta_1+\beta_2+2\beta_3} = -X_{\gamma(1111)}^0 \\ X_{\beta_1+2\beta_2+2\beta_3} = -X_{\gamma(1211)}^0 & \end{array}$$

Note that also

$$X_{-\phi} = -(X_\phi)^t. \quad (6)$$

In order to describe the inclusion  $\mathfrak{g}_1 \subseteq \mathfrak{g}$ , let  $\sigma$  be an order 3 automorphism of the above Dynkin diagram of type  $D_4$ , given by  $\sigma(\gamma_4) = \gamma_3$ . By the same letter we denote the lift to an order 3 automorphism of  $\mathfrak{g}$ . Next define the fixed point algebra

$$\mathfrak{g}_1 = \mathfrak{g}_0^\sigma;$$

and also  $\mathfrak{h}_1 = \text{span}\{C_1, C_2\}$ , where

$$C_1 = H_1 - H_2 + 2H_3, \quad C_2 = H_2 - H_3. \quad (7)$$

Then  $\mathfrak{g}_1$  is a simple Lie algebra of type  $G_2$ , and  $\mathfrak{h}_1$  is its Cartan subalgebra. Let  $\Pi_1 = \{\alpha_1, \alpha_2\}$  be a basis of the root system  $\Delta_1 = \Delta(\mathfrak{g}_1, \mathfrak{h}_1)$  so that  $\alpha_1$  is the short root and  $\alpha_2$  is the long one. Recall the table, see [Š5], giving root vectors  $X_\phi^1 \in \mathfrak{g}_1$  corresponding to  $\phi \in \Delta_1$  in terms of the root vectors  $X_\phi^0 \in \mathfrak{g}_0$ ,  $\phi \in \Delta_0$ :

$$\begin{array}{ll} X_{\alpha_1}^1 = X_{\gamma_1}^0 + X_{\gamma_3}^0 + X_{\gamma_4}^0 & X_{-\alpha_1}^1 = X_{-\gamma_1}^0 + X_{-\gamma_3}^0 + X_{-\gamma_4}^0 \\ X_{\alpha_2}^1 = X_{\gamma_2}^0 & X_{-\alpha_2}^1 = X_{-\gamma_2}^0 \\ X_{\alpha_1+\alpha_2}^1 = -X_{\gamma_1+\gamma_2}^0 + X_{\gamma_2+\gamma_3}^0 & X_{-(\alpha_1+\alpha_2)}^1 = -X_{-(\gamma_1+\gamma_2)}^0 + X_{-(\gamma_2+\gamma_3)}^0 \\ & \quad + X_{-(\gamma_2+\gamma_4)}^0 \\ X_{2\alpha_1+\alpha_2}^1 = X_{\gamma(1110)}^0 + X_{\gamma(1101)}^0 & X_{-(2\alpha_1+\alpha_2)}^1 = X_{-\gamma(1110)}^0 + X_{-\gamma(1101)}^0 \\ & \quad + X_{-\gamma(0111)}^0 \\ X_{3\alpha_1+\alpha_2}^1 = X_{\gamma(1111)}^0 & X_{-(3\alpha_1+\alpha_2)}^1 = -X_{-\gamma(1111)}^0 \\ X_{3\alpha_1+2\alpha_2}^1 = X_{\gamma(1211)}^0 & X_{-(3\alpha_1+2\alpha_2)}^1 = -X_{-\gamma(1211)}^0 \end{array}$$

### 3.1. Pairs of type $(B_3, G_2)$ , $(D_4, G_2)$ and $(G_2, A_2)$ .

In this subsection we treat the (nonsymmetric) pairs of type  $(B_3, G_2)$ ,  $(D_4, G_2)$  and  $(G_2, A_2)$ ; i.e., we show that they are Cartan pairs.

Now let  $\mathfrak{g}_1 \subseteq \mathfrak{g} \subseteq \mathfrak{g}_0$  be exactly the inclusions (5). Below we freely use the above introduced notation. Let  $\mathfrak{g}_0 = \mathfrak{g}_1 \oplus \mathfrak{p}_0$  be the usual Killing-orthogonal decomposition. In particular by  $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$  we denote the Killing-orthogonal of  $\mathfrak{h}_1$  in  $\mathfrak{h}_0$ . It is immediate that

$$\mathfrak{a}_0 = \text{span}\{\mathfrak{a}, H_4\}, \quad (8)$$

where  $\mathfrak{a} = -H_1 + H_2 + H_3$ . Clearly,  $\mathfrak{a}_0$  is commutative and reductive in  $\mathfrak{g}_0$ . Let us compute the centralizer  $C_{\mathfrak{g}_0}(\mathfrak{a}_0)$ . For that purpose suppose that  $w^+ = \sum_{\phi \in \Delta_0^+} t_\phi X_\phi^0$  satisfies  $[w^+, \mathfrak{a}_0] = 0$ . It is easy to see that

$$w^+ \in \text{span}\{X_{\gamma_2}^0, X_{\gamma(1111)}^0, X_{\gamma(1211)}^0\} \subseteq \mathfrak{g}_1.$$

Hence we conclude at once that  $C_{\mathfrak{p}_0}(\mathfrak{a}_0) = \mathfrak{a}_0$ . In other words we have proved the part (i) of the following lemma.

**Lemma 3.1.** (i) *The set  $\mathfrak{a}_0$  is a Cartan subspace of  $\mathfrak{g}_0$ .*

(ii) *The pair  $(\mathfrak{g}_0, \mathfrak{g}_1)$  is a Cartan pair of type I.*

**Proof.** (ii) We only have to check that the condition (**Cp2**) of Definition 2.4 holds. For that purpose we have to see that the set of roots

$$\Delta'_0 = \{\phi \in \Delta_0 \mid \phi|_{\mathfrak{a}_0} = 0\}$$

is equal to  $\{\pm\gamma_2, \pm\gamma(1111), \pm\gamma(1211)\}$ . Hence it follows that  $X_\phi^0 \in \mathfrak{g}_1$ , for every  $\phi \in \Delta'_0$ , as we had to see.  $\blacksquare$

Now define  $\lambda_1, \lambda_2 \in \mathfrak{a}_0^*$  by

$$\begin{aligned} \lambda_1(\mathfrak{a}) &= 1, & \lambda_2(\mathfrak{a}) &= 1, \\ \lambda_1(H_4) &= -1, & \lambda_2(H_4) &= 1. \end{aligned}$$

As an easy exercise we check that

$$\Sigma_0 = \Sigma(\mathfrak{g}_0, \mathfrak{a}_0) = \Sigma_0^+ \cup (-\Sigma_0^+),$$

where

$$\Sigma_0^+ = \Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0) = \{\lambda_1, \lambda_2, \lambda_1 + \lambda_2\},$$

and then

$$\begin{aligned} (\mathfrak{g}_0)_{\lambda_1} &= \text{span}\{X_{\gamma_3}^0, X_{\gamma_2+\gamma_3}^0, X_{-\gamma(1101)}^0\}, \\ (\mathfrak{g}_0)_{\lambda_2} &= \text{span}\{X_{\gamma_4}^0, X_{\gamma_2+\gamma_4}^0, X_{-\gamma(1110)}^0\}, \\ (\mathfrak{g}_0)_{\lambda_1+\lambda_2} &= \text{span}\{X_{\gamma(0111)}^0, X_{-\gamma_1}^0, X_{-(\gamma_1+\gamma_2)}^0\} \end{aligned}$$

Similarly we compute the subspaces  $(\mathfrak{g}_0)_{-\phi}$ , for  $\phi \in \Sigma_0^+$ . The above computations can be summarized as the part (ii) of the following corollary. For (i) see [Š4, Ex. 2.14] where actually we considered another realization  $\tilde{\mathfrak{g}}_1 \subseteq \tilde{\mathfrak{g}}$  of the Lie algebra of type  $G_2$  and the corresponding pair  $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_1)$ ; see again [Š5, Sect. 3.1].

**Corollary 3.2.** *Let  $\mathfrak{g}_1 \subseteq \mathfrak{g} \subseteq \mathfrak{g}_0$  be the inclusions given by (5); i.e.,  $\mathfrak{g}_1$  is of type  $G_2$ ,  $\mathfrak{g}$  is of type  $B_3$  and  $\mathfrak{g}_0$  is of type  $D_4$ . Let  $\mathfrak{a}$  and  $\mathfrak{a}_0$  be as above.*

- (i) *Let  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}$  be the usual  $B_{\mathfrak{g}}$ -orthogonal decomposition, and define  $\mathfrak{a} \subseteq \mathfrak{p}$  to be the one-dimensional subspace spanned by  $\mathfrak{a}$ . Then the pair  $(\mathfrak{g}, \mathfrak{g}_1)$  is a nonsymmetric Cartan pair of type I, and  $\mathfrak{a}$  is its Cartan subspace. If we define  $\lambda \in \mathfrak{a}^*$  by  $\lambda(\mathfrak{a}) = 1$ , then*

$$\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\lambda, \pm 2\lambda\};$$

*i.e.,  $\Sigma(\mathfrak{g}, \mathfrak{a})$  is a (nonreduced) root system of type  $A_1$ .*

- (ii) *The pair  $(\mathfrak{g}_0, \mathfrak{g}_1)$  is a nonsymmetric Cartan pair of type I, and  $\mathfrak{a}_0$  is its Cartan subspace. Further,  $\Sigma(\mathfrak{g}_0, \mathfrak{a}_0)$  is a (reduced) root system of type  $A_2$ .*

**Remark 3.3.** Define  $\mathfrak{d}^{\pm} = \text{span}\{X_{\pm\gamma_2}^0, X_{\pm\gamma(1111)}^0, X_{\pm\gamma(1211)}^0\}$ . Then  $\mathfrak{m}_0 = C_{\mathfrak{g}_1}(\mathfrak{a}_0)$  is equal to  $\mathfrak{h}_1 \oplus \mathfrak{d}^- \oplus \mathfrak{d}^+$ ; note that  $\mathfrak{m}_0 = \mathfrak{s}$ , where  $\mathfrak{s}$  is defined below by (9). Next define

$$\mathfrak{n}_0 = (\mathfrak{g}_0)_{\lambda_1} \oplus (\mathfrak{g}_0)_{\lambda_2} \oplus (\mathfrak{g}_0)_{\lambda_1 + \lambda_2},$$

which is a two-step nilpotent subalgebra of  $\mathfrak{g}_0$ . And then  $\mathfrak{u}_0 = \mathfrak{d}^+ \oplus \mathfrak{n}_0$ ; by [Š4, Prop. 2.8] it is clear that  $\mathfrak{u}_0$  is a subalgebra of  $\mathfrak{g}_0$ . An easy exercise gives that for the lower central series we have  $\mathcal{C}^6\mathfrak{u}_0 = 0$ , while  $\mathcal{C}^5\mathfrak{u}_0 = \mathbb{K}X_{\gamma(0111)}^0$ . In other words,  $\mathfrak{u}_0$  is a six-step nilpotent Lie algebra. In fact,  $\mathfrak{b}_0 = \mathfrak{h}_0 \oplus \mathfrak{u}_0$  is a Borel subalgebra of  $\mathfrak{g}_0$ , while  $\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$  is a parabolic subalgebra containing  $\mathfrak{b}_0$ . Note that here we do not have an analogue of the Iwasawa decomposition. But related to that it is worth to note the fact that  $\mathfrak{g}_1$  is not a maximal semisimple subalgebra of  $\mathfrak{g}_0$ ; cf. Remark 3.7 below.

Having Definition 2.11 and the above corollary, a natural further task is to show that (nonsymmetric) Cartan pairs of type II do exist. We will show that the pairs of type  $(G_2, A_2)$  and  $(F_4, D_4)$  are such. In order to treat the first one define  $\mathfrak{s} = C_{\mathfrak{g}_1}(\mathfrak{a})$ . In [Š5] we observed that

$$\mathfrak{s} = \mathfrak{h}_1 \oplus \bigoplus_{\phi \in \Lambda} \mathbb{K}X_{\phi}^1, \quad (9)$$

where  $\Lambda^+ = \{\alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\} \subseteq \Delta_1^+$  and  $\Lambda = \Lambda^+ \cup (-\Lambda^+)$ , is isomorphic to  $\mathfrak{sl}(3, \mathbb{K})$ ; there we wrote  $\mathfrak{m}$  instead of  $\mathfrak{s}$ . So we have the pair  $(\mathfrak{g}_1, \mathfrak{s})$ , of type  $(G_2, A_2)$ . Let us denote by  $\mathfrak{p}_1$  the  $B_{\mathfrak{g}_1}$ -orthogonal of  $\mathfrak{s}$  in  $\mathfrak{g}_1$ . Thus we have a direct sum decomposition

$$\mathfrak{g}_1 = \mathfrak{s} \oplus \mathfrak{p}_1.$$

A straightforward computation gives that

$$\mathfrak{p}_1 = \text{span}\{X_{\pm\alpha_1}^1, X_{\pm(\alpha_1 + \alpha_2)}^1, X_{\pm(2\alpha_1 + \alpha_2)}^1\}.$$

Next define

$$\begin{aligned} \mathfrak{a}_1 &= X_{-\alpha_1}^1 + X_{\alpha_1}^1, \\ \mathfrak{a}_2 &= X_{-(\alpha_1 + \alpha_2)}^1 + X_{\alpha_1 + \alpha_2}^1, \\ \mathfrak{a}_3 &= X_{-(2\alpha_1 + \alpha_2)}^1 + X_{2\alpha_1 + \alpha_2}^1; \end{aligned}$$

and observe that by Lemma 1.3 these elements are semisimple in  $\mathfrak{g}_1$ . Also define the one-dimensional subspaces  $\mathfrak{a}_i \subseteq \mathfrak{p}_1$  by  $\mathfrak{a}_i = \mathbb{K}\mathfrak{a}_i$ .

**Lemma 3.4.** *The sets  $\mathfrak{a}_i$  are Cartan subspaces of  $\mathfrak{g}_1$ .*

**Proof.** We only have to check that  $C_{\mathfrak{p}_1}(\mathfrak{a}_i) = \mathfrak{a}_i$ . Let us take  $i = 1$ ; the other two cases are left to the reader. Let then an element  $E \in \mathfrak{p}_1$  be such that  $[E, \mathfrak{a}_1] = 0$ . Decompose  $E = F + G$ , where

$$\begin{aligned} F &= t_{-\alpha_1}X_{-\alpha_1}^1 + t_{\alpha_1}X_{\alpha_1}^1, \\ G &= t_{-(\alpha_1+\alpha_2)}X_{-(\alpha_1+\alpha_2)}^1 + t_{\alpha_1+\alpha_2}X_{\alpha_1+\alpha_2}^1 \\ &\quad + t_{-(2\alpha_1+\alpha_2)}X_{-(2\alpha_1+\alpha_2)}^1 + t_{2\alpha_1+\alpha_2}X_{2\alpha_1+\alpha_2}^1. \end{aligned}$$

It is easy to obtain that then necessarily  $G = 0$  and  $t_{\alpha_1} = t_{-\alpha_1}$ . In other words,  $E = F \in \mathfrak{a}_1$ , as we had to see.  $\blacksquare$

**Lemma 3.5.** *The pair  $(\mathfrak{g}_1, \mathfrak{s})$  is a nonsymmetric Cartan pair of type II.*

**Proof.** First observe that this pair is indeed nonsymmetric; see Remark 2.7.

Define the elements  $\mathfrak{b}_i \in \mathfrak{s}$  by

$$\begin{aligned} \mathfrak{b}_1 &= X_{-(3\alpha_1+2\alpha_2)}^1 + X_{3\alpha_1+2\alpha_2}^1, \\ \mathfrak{b}_2 &= X_{-(3\alpha_1+\alpha_2)}^1 + X_{3\alpha_1+\alpha_2}^1, \\ \mathfrak{b}_3 &= X_{-\alpha_2}^1 + X_{\alpha_2}^1; \end{aligned}$$

and observe that these elements are semisimple in  $\mathfrak{g}_1$  as well. We obviously have  $[\mathfrak{a}_i, \mathfrak{b}_i] = 0$ , and thus  $\mathfrak{c}_i = \text{span}\{\mathfrak{a}_i, \mathfrak{b}_i\}$  are Cartan subalgebras of  $\mathfrak{g}_1$ . Next observe that  $\mathfrak{c}_i \cap \mathfrak{s} = \mathbb{K}\mathfrak{b}_i$ , and then  $(\mathfrak{c}_i \cap \mathfrak{s}) \oplus \mathbb{K}\mathfrak{a}_i = \mathfrak{c}_i$ . In other words, we have the condition **(Cp1)** of Definition 2.4 fulfilled. Now we have to check that the condition **(Cp2)** holds as well. For that purpose consider the root systems  $\Delta(i) = \Delta(\mathfrak{g}_1, \mathfrak{c}_i)$ , and for the corresponding Cartan subspaces  $\mathfrak{a}_i$  the subsets of roots

$$\Delta(i)' = \{\phi \in \Delta(i) \mid \phi|_{\mathfrak{a}_i} = 0\}.$$

We have to show the following implication: If  $\phi \in \Delta(i)'$ , then  $X_\phi \in \mathfrak{s}$ . As in Lemma 3.4 we will again consider only the case  $i = 1$ ; i.e., we will compute  $\Delta(1)'$ . For that purpose assume that  $\phi \in \Delta(1)'$ , which means that  $\phi(\mathfrak{a}_1) = 0$  and

$$(\mathfrak{g}_1)_\phi = \{x \in \mathfrak{g}_1 \mid [U, x] = \phi(U)x, \forall U \in \mathfrak{c}_1\}$$

is nonzero. This means that the above  $x$  satisfies both  $[\mathfrak{a}_1, x] = 0$  and  $[\mathfrak{b}_1, x] = \phi(\mathfrak{b}_1)x$ . So let us first see what kind of an element  $x$  belong to the centralizer  $C_{\mathfrak{g}_1}(\mathfrak{a}_1)$ . But to do that we just have to take into account the equality  $C_{\mathfrak{g}_1}(\mathfrak{a}_1) = C_{\mathfrak{s}}(\mathfrak{a}_1) \oplus \mathfrak{a}_1$ , which is an analog of (3) for the present situation, and then determine  $C_{\mathfrak{s}}(\mathfrak{a}_1)$ . An easy computation gives that

$$C_{\mathfrak{s}}(\mathfrak{a}_1) = \text{span}\{H_1 + H_2, X_{\pm(3\alpha_1+2\alpha_2)}^1\}.$$

In other words, we can write  $x$  as

$$x = s(H_1 + H_2) + t\mathbf{a}_1 + t_-X_{-(3\alpha_1+2\alpha_2)}^1 + t_+X_{3\alpha_1+2\alpha_2}^1,$$

for some  $s, t, t_-, t_+ \in \mathbb{K}$ .

Now we compute

$$\begin{aligned} [\mathbf{b}_1, x] &= [\mathbf{b}_1, s(H_1 + H_2)] + (t_- - t_+)[X_{3\alpha_1+2\alpha_2}^1, X_{-(3\alpha_1+2\alpha_2)}^1] \\ &= 2s(X_{-(3\alpha_1+2\alpha_2)}^1 - X_{3\alpha_1+2\alpha_2}^1) + (t_- - t_+)(H_1 + H_2), \end{aligned}$$

and ask whether there is some nonzero  $\nu \in \mathbb{K}$  so that  $[\mathbf{b}_1, x] = \nu x$ . It is straightforward to see that  $\nu \in \{\pm 2\}$  works. As the final conclusion we have that  $\Delta(1)' = \{\pm\phi\}$ , where  $\phi(\mathbf{a}_1) = 0$  and  $\phi(\mathbf{b}_1) = 2$ ; and then  $(\mathfrak{g}_1)_\phi = \mathbb{K}X_\phi$ , where

$$X_\phi = X_{3\alpha_1+2\alpha_2}^1 - X_{-(3\alpha_1+2\alpha_2)}^1 - (H_1 + H_2).$$

But clearly  $X_\phi \in \mathfrak{s}$ , which finishes the proof of our lemma.  $\blacksquare$

To the end of this subsection we assume that the ground field  $\mathbb{K}$  is algebraically closed. In what follows by  $\iota$  we denote an ‘‘imaginary unit’’; i.e., a fixed element of  $\mathbb{K}$  satisfying  $\iota^2 = -1$ . Let us now compute the set of restricted roots  $\Sigma(\mathfrak{g}_1, \mathbf{a}_1)$ . Assume that  $\varepsilon$  is any such root and  $x \in (\mathfrak{g}_1)_\varepsilon$  is a restricted root vector. Decompose

$$x = s_1C_1 + s_2C_2 + \sum_{\psi \in \Delta_1} t_\psi X_\psi^1,$$

where  $s_1, s_2, t_\psi \in \mathbb{K}$ ;  $C_1$  and  $C_2$  are the same as in (7). Let us identify  $\varepsilon$  with its (nonzero) value  $\varepsilon(\mathbf{a}_1)$ . We are searching for  $\varepsilon$  and  $x$  such that

$$[\mathbf{a}_1, x] = \varepsilon x. \tag{10}$$

A straightforward computation gives that

$$\begin{aligned} [\mathbf{a}_1, x] &= (t_{\alpha_1} - t_{-\alpha_1})C_1 + (2s_1 - s_2)X_{-\alpha_1}^1 + (s_2 - 2s_1)X_{\alpha_1}^1 \\ &\quad + 3t_{-(\alpha_1+\alpha_2)}X_{-\alpha_2}^1 + 3t_{\alpha_1+\alpha_2}X_{\alpha_2}^1 \\ &\quad + (-t_{-\alpha_2} - 2t_{-(2\alpha_1+\alpha_2)})X_{-(\alpha_1+\alpha_2)}^1 + (-t_{\alpha_2} - 2t_{2\alpha_1+\alpha_2})X_{\alpha_1+\alpha_2}^1 \\ &\quad + (2t_{-(\alpha_1+\alpha_2)} + t_{-(3\alpha_1+\alpha_2)})X_{-(2\alpha_1+\alpha_2)}^1 + (2t_{\alpha_1+\alpha_2} - t_{3\alpha_1+\alpha_2})X_{2\alpha_1+\alpha_2}^1 \\ &\quad - 3t_{-(2\alpha_1+\alpha_2)}X_{-(3\alpha_1+\alpha_2)}^1 + 3t_{2\alpha_1+\alpha_2}X_{3\alpha_1+\alpha_2}^1. \end{aligned}$$

Hence we immediately have that  $t_{\pm(3\alpha_1+2\alpha_2)} = s_2 = 0$ . Now first consider the case  $s_1 \neq 0$  and  $t_{\pm\alpha_2} = 0$ . Then it easily follows that necessarily  $\varepsilon \in \{\pm 2\iota\}$ . For later use define

$$\begin{aligned} Z(2\iota) &= C_1 - \iota X_{-\alpha_1}^1 + \iota X_{\alpha_1}^1, \\ Z(-2\iota) &= C_1 + \iota X_{-\alpha_1}^1 - \iota X_{\alpha_1}^1. \end{aligned}$$

As the second case we take  $t_{\pm\alpha_1} = s_1 = 0$ ; and so  $t_{\pm\alpha_2}$  cannot be both equal to zero. By the equality (10), where its left side is written as above, we deduce that

$$\begin{aligned} t_{\pm(\alpha_1+\alpha_2)} &= \frac{\varepsilon}{3} t_{\pm\alpha_2}, \\ t_{\pm(2\alpha_1+\alpha_2)} &= -\frac{3+\varepsilon^2}{6} t_{\pm\alpha_2}, \\ t_{\pm(3\alpha_1+\alpha_2)} &= \pm\varepsilon \frac{7+\varepsilon^2}{6} t_{\pm\alpha_2}, \end{aligned}$$

and also

$$\pm 3t_{\pm(2\alpha_1+\alpha_2)} = \varepsilon t_{\pm(3\alpha_1+\alpha_2)}.$$

Hence it follows at once that  $\varepsilon$  necessarily satisfies the biquadratic equation  $\varepsilon^4 + 10\varepsilon^2 + 9 = 0$ ; i.e., that  $\varepsilon \in \{\pm i, \pm 3i\}$ . In order to formulate the corollary below let us now define the following elements of  $\mathfrak{g}_1$ :

$$\begin{aligned} Z^-(i) &= X_{-\alpha_2}^1 + \frac{i}{3} X_{-(\alpha_1+\alpha_2)}^1 - \frac{1}{3} X_{-(2\alpha_1+\alpha_2)}^1 - i X_{-(3\alpha_1+\alpha_2)}^1, \\ Z^+(i) &= X_{\alpha_2}^1 + \frac{i}{3} X_{\alpha_1+\alpha_2}^1 - \frac{1}{3} X_{2\alpha_1+\alpha_2}^1 + i X_{3\alpha_1+\alpha_2}^1, \\ Z^-(-i) &= X_{-\alpha_2}^1 - \frac{i}{3} X_{-(\alpha_1+\alpha_2)}^1 - \frac{1}{3} X_{-(2\alpha_1+\alpha_2)}^1 + i X_{-(3\alpha_1+\alpha_2)}^1, \\ Z^+(-i) &= X_{\alpha_2}^1 - \frac{i}{3} X_{\alpha_1+\alpha_2}^1 - \frac{1}{3} X_{2\alpha_1+\alpha_2}^1 - i X_{3\alpha_1+\alpha_2}^1, \end{aligned}$$

and also

$$\begin{aligned} Z^-(3i) &= X_{-\alpha_2}^1 + i X_{-(\alpha_1+\alpha_2)}^1 + X_{-(2\alpha_1+\alpha_2)}^1 + i X_{-(3\alpha_1+\alpha_2)}^1, \\ Z^+(3i) &= X_{\alpha_2}^1 + i X_{\alpha_1+\alpha_2}^1 + X_{2\alpha_1+\alpha_2}^1 - i X_{3\alpha_1+\alpha_2}^1, \\ Z^-(-3i) &= X_{-\alpha_2}^1 - i X_{-(\alpha_1+\alpha_2)}^1 + X_{-(2\alpha_1+\alpha_2)}^1 - i X_{-(3\alpha_1+\alpha_2)}^1, \\ Z^+(-3i) &= X_{\alpha_2}^1 - i X_{\alpha_1+\alpha_2}^1 + X_{2\alpha_1+\alpha_2}^1 + i X_{3\alpha_1+\alpha_2}^1. \end{aligned}$$

As a conclusion of our calculations we have the following corollary.

**Corollary 3.6.** *Let the Cartan pair  $(\mathfrak{g}_1, \mathfrak{s})$ , the element  $\mathfrak{a}_1$  and the Cartan subspace  $\mathfrak{a}_1$  be as above. Define  $\lambda \in \mathfrak{a}_1^*$  by  $\lambda(\mathfrak{a}_1) = i$ . Then the set of restricted roots of  $(\mathfrak{g}_1, \mathfrak{a}_1)$  equals*

$$\Sigma(\mathfrak{g}_1, \mathfrak{a}_1) = \{\pm\lambda, \pm 2\lambda, \pm 3\lambda\}.$$

Further, keeping the introduced notation, for the restricted root subspaces we have:

$$\begin{aligned} (\mathfrak{g}_1)_{\pm\lambda} &= \text{span}\{Z^-(\pm i), Z^+(\pm i)\}, \\ (\mathfrak{g}_1)_{\pm 2\lambda} &= \mathbb{K} Z(\pm 2i), \\ (\mathfrak{g}_1)_{\pm 3\lambda} &= \text{span}\{Z^-(\pm 3i), Z^+(\pm 3i)\}. \end{aligned}$$

Analogous of the previous corollary for other two Cartan subspaces,  $\mathfrak{a}_2$  and  $\mathfrak{a}_3$ , are again left to the reader.

**Remark 3.7.** Define  $\mathfrak{n}_1 = (\mathfrak{g}_1)_\lambda \oplus (\mathfrak{g}_1)_{2\lambda} \oplus (\mathfrak{g}_1)_{3\lambda}$ . Then it is immediate that

$$\mathfrak{g}_1 = \mathfrak{s} \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1;$$

i.e., here we have an analogue of the classical Iwasawa decomposition.

### 3.2. Pair of type $(F_4, D_4)$ .

In this subsection we treat the only pair from the Brylinski-Kostant list that remained; i.e., the pair of type  $(F_4, D_4)$ .

For later use recall a realization of Lie algebra  $\mathfrak{G}_1$  of type  $D_n$  as a subalgebra of the simple Lie algebra  $\mathfrak{G}$  of type  $B_n$ . First suppose we have  $\mathfrak{G} \subseteq \mathfrak{sl}(2n+1, \mathbb{K})$  as it was done in [Š5, Subsection 1.2.4]; notice a slightly different notation. Let  $H_i = E_{ii} - E_{2n+1-i, 2n+1-i}$ , for  $1 \leq i \leq n$ ; then  $\mathfrak{H} = \text{span}\{H_1, \dots, H_n\}$  is a Cartan subalgebra of  $\mathfrak{G}$ . Let  $(\epsilon_i)_1^n$  be the dual basis of  $(H_i)_1^n$ . Define  $\beta_i = \epsilon_i - \epsilon_{i+1}$  for  $1 \leq i < n$ , and  $\beta_n = \epsilon_n$ . Then  $\Pi = \{\beta_1, \dots, \beta_n\}$  is a basis of the root system  $\Delta = \Delta(\mathfrak{G}, \mathfrak{H})$ ; this is the same as in [Bo1, Plate II, p. 267]. Put  $\Delta_1^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\}$ , and  $\Delta_1 = (-\Delta_1^+) \cup \Delta_1^+$ . Then

$$\mathfrak{G}_1 = \mathfrak{H} \oplus \bigoplus_{\phi \in \Delta_1} \mathfrak{G}_\phi \quad (11)$$

is a simple Lie algebra of type  $D_n$ , with  $\mathfrak{H}$  as its Cartan subalgebra and  $\Delta(\mathfrak{G}_1, \mathfrak{H}) = \Delta_1$  as its root system. Observe that  $\Delta^+$ , the set of positive roots in  $\Delta$  relative to  $\Pi$ , is equal to  $\Delta_1^+ \cup \{\epsilon_1, \dots, \epsilon_n\}$ . Hence it is clear that for the Killing-orthogonal  $\mathfrak{P}$ , of  $\mathfrak{G}_1$  in  $\mathfrak{G}$ , we have  $\mathfrak{P} = \text{span}\{X_{\pm\epsilon_i} \mid 1 \leq i \leq n\}$ . Thus we in particular see that  $[\mathfrak{P}, \mathfrak{P}] \subseteq \mathfrak{G}_1$ , and so by Lemma 1.1 we obtain a well known fact that the pair  $(\mathfrak{G}, \mathfrak{G}_1)$ , of type  $(B_n, D_n)$ , is symmetric.

Although the above pairs of type  $(B_n, D_n)$  are symmetric, it will be instructive to note the following simple fact; an easy proof is omitted.

**Lemma 3.8.** *For each  $1 \leq i \leq n$  define  $U_i = X_{\epsilon_i} + X_{-\epsilon_i}$ , and then  $\mathfrak{a}_i$  to be the one-dimensional space spanned by  $U_i$ . Then every  $\mathfrak{a}_i$  is a Cartan subspace of  $\mathfrak{G}$ .*

Now we have to describe the inclusions

$$\mathfrak{g}_3 = \mathfrak{g}(D_4) \subseteq \mathfrak{g}_2 = \mathfrak{g}(B_4) \subseteq \mathfrak{g}_1 = \mathfrak{g}(F_4); \quad (12)$$

more details about some sketchy recalled facts can be found in [Š5, Subsection 3.3]. We start with a realization of the simple exceptional Lie algebra  $\mathfrak{e}_6$  of type  $E_6$ ; for more about realizations of exceptional Lie algebras see [Ad], [KS, §1], and [Ba, Sect. 4]. Consider an obvious automorphism  $\theta$  of the Dynkin diagram of type  $E_6$ , and lift it to an automorphism of  $\mathfrak{e}_6$ . Define  $\mathfrak{g}_1$  to be the fix point algebra  $\mathfrak{e}_6^\theta$ . For certain elements  $\mathcal{H}_1, \dots, \mathcal{H}_4$ , chosen within a convenient Cartan subalgebra of  $\mathfrak{e}_6$ , we get a Cartan subalgebra

$$\mathfrak{h}_1 = \text{span}\{\mathcal{H}_1, \dots, \mathcal{H}_4\}$$

of  $\mathfrak{g}_1$ . For the root system  $\Delta_1 = \Delta(\mathfrak{g}_1, \mathfrak{h}_1)$  we choose a basis  $\Pi_1 = \{\delta_1, \dots, \delta_4\}$  so that the corresponding labeled Dynkin diagram has a form



In order to list the (positive) roots in  $\Delta_1$ , let us introduce the following notation. By  $\Delta_1(t)$  we denote the set of positive roots of level  $t$ . Also for  $\phi \in \Delta_1(t)$ , when  $t \geq 3$ , we write  $\phi = \delta(c_1 \cdots c_4)$  instead of  $\sum_{i=1}^4 c_i \delta_i$ ; analogously as we did for  $\mathfrak{g} = \mathfrak{so}(8, \mathbb{K})$ . Then we have

$$\begin{aligned} \Delta_1^+ = \{ & \delta_1, \delta_2, \delta_3, \delta_4, \delta_1 + \delta_2, \delta_2 + \delta_3, \delta_3 + \delta_4, \\ & \delta(1110), \delta(0111), \delta(0120), \delta(1111), \delta(1120), \delta(0121), \\ & \delta(1121), \delta(1220), \delta(0122), \delta(1122), \delta(1221), \delta(1222), \delta(1231), \\ & \delta(1232), \delta(1242), \delta(1342), \delta(2342)\}. \end{aligned}$$

For  $\phi \in \Delta_1$  by  $X_\phi \in (\mathfrak{g}_1)_\phi$  we denote the corresponding root vectors. In particular for positive roots  $\phi$  these root vectors will be the same as in [Š5, Lemma 3.12]; note that there we write  $X_\phi^1$ . (Let us use this opportunity to correct two minor mistakes in the proof of the mentioned lemma. Namely, contrary to what we claimed, for there considered scalars  $N_{\phi, \psi}$  we take them to be  $1/2$ , while computing the commutators  $[\mathcal{X}_3, \mathcal{Z}_2]$  and  $[\mathcal{X}_4, \mathcal{Z}_6]$ .)

Next we need a realization of  $\mathfrak{g}_2$ , the simple Lie algebra of type  $B_4$ , as a subalgebra of  $\mathfrak{g}_1$ . For that purpose define

$$\kappa_1 = \delta_2 + 2\delta_3 + 2\delta_4, \quad \kappa_2 = \delta_1, \quad \kappa_3 = \delta_2, \quad \kappa_4 = \delta_3.$$

Using the same notational convention as for the roots in  $\Delta_1^+$ , here we have  $\Delta_2(t)$  for  $1 \leq t \leq 7$ , and then

$$\begin{aligned} \Delta_2^+ = \{ & \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_1 + \kappa_2, \kappa_2 + \kappa_3, \kappa_3 + \kappa_4, \\ & \kappa(1110), \kappa(0111), \kappa(0012), \kappa(1111), \kappa(0112), \\ & \kappa(1112), \kappa(0122), \kappa(1122), \kappa(1222)\}; \end{aligned}$$

see [Š5, Lemma 3.13]. By putting  $\Delta_2 = (-\Delta_2^+) \cup \Delta_2^+$ , as a conclusion we get that

$$\mathfrak{g}_2 = \mathfrak{h}_1 \oplus \bigoplus_{\phi \in \Delta_2} (\mathfrak{g}_1)_\phi$$

is a simple Lie algebra of type  $B_4$ ; of course,  $\mathfrak{h}_1$  is its Cartan subalgebra as well. Observe that the root system  $\Delta(\mathfrak{g}_2, \mathfrak{h}_1) = \Delta_2$ , and  $\Pi_2$  is a basis with the corresponding labeled Dynkin diagram



Finally we give a realization of  $\mathfrak{g}_3$ , the simple Lie algebra of type  $D_4$ , as a subalgebra of  $\mathfrak{g}_2$ ; this is a special case of the above considered realization of pairs  $(B_n, D_n)$ . More precisely, let

$$\Delta_3^+ = \Delta_2^+ \setminus \{\kappa_4, \kappa_3 + \kappa_4, \kappa(0111), \kappa(1111)\}, \tag{13}$$

and then  $\Delta_3 = (-\Delta_3^+) \cup \Delta_3^+$ . Thus as a special case of (11) we get

$$\mathfrak{g}_3 = \mathfrak{h}_1 \oplus \bigoplus_{\phi \in \Delta_3} (\mathfrak{g}_1)_\phi,$$

a simple Lie algebra of type  $D_4$ ; again we have  $\mathfrak{h}_1$  as a Cartan subalgebra.

Thus we have the inclusions (12). As we already know, the pair  $(\mathfrak{g}_2, \mathfrak{g}_3)$  is symmetric. As we could not find it in the literature, we include the following simple, but non-obvious, fact.

**Lemma 3.9.** *The pair  $(\mathfrak{g}_1, \mathfrak{g}_2)$ , of type  $(F_4, B_4)$ , is symmetric as well.*

**Proof.** Let  $\mathfrak{g}_1 = \mathfrak{g}_2 \oplus \mathfrak{f}$  be the usual Killing-orthogonal decomposition of  $\mathfrak{g}_1$ . Using [Š5, Lemma 3.12], a careful checking shows that we have a direct sum  $\mathfrak{f} = \mathfrak{f}^- \oplus \mathfrak{f}^+$ , where

$$\begin{aligned} \mathfrak{f}^\pm = \text{span}\{ & X_{\pm\delta_4}, X_{\pm(\delta_3+\delta_4)}, X_{\pm\delta(0111)}, X_{\pm\delta(1111)}, \\ & X_{\pm\delta(0121)}, X_{\pm\delta(1121)}, X_{\pm\delta(1221)}, X_{\pm\delta(1231)}\}. \end{aligned} \quad (14)$$

Now a direct computation, using the above listed basis vectors of  $\mathfrak{f}^\pm$ , gives that  $[\mathfrak{f}^\pm, \mathfrak{f}^\pm] \subseteq \mathfrak{g}_2$ . Analogously we check that  $[\mathfrak{f}^+, \mathfrak{f}^-]$  is also a subspace of  $\mathfrak{g}_2$ . Let us emphasize that during this computations we have to look all the time at the listed roots for  $\Delta_1^+$  and  $\Delta_2^+$ . By Lemma 1.1 we conclude that  $(\mathfrak{g}_1, \mathfrak{g}_2)$  is a symmetric pair. (More precisely, by decomposing any  $x \in \mathfrak{g}_1$  as  $y + z$ , according to the sum  $\mathfrak{g}_2 = \mathfrak{g}_1 \oplus \mathfrak{f}$ , we see that  $\theta(x) = y - z$  defines an involutive automorphism of  $\mathfrak{g}_1$  satisfying  $\mathfrak{g}_1^\theta = \mathfrak{g}_2$ .) ■

**Remark 3.10.** The referee of this paper informed the author that some related results, on the side of (split) algebraic groups of type  $F_4$ , have been obtained in the Hutchens' paper [Hu].

So in fact the only one interesting pair, among the three, is  $(\mathfrak{g}_1, \mathfrak{g}_3)$ ; i.e., we have the following result.

**Proposition 3.11.** *The pair  $(\mathfrak{g}_1, \mathfrak{g}_3)$ , of type  $(F_4, D_4)$ , is a nonsymmetric Cartan pair.*

**Proof.** Denote by  $\mathfrak{p}$  the Killing-orthogonal of  $\mathfrak{g}_3$  in  $\mathfrak{g}_1$ . Then define

$$\mathfrak{j}^\pm = \text{span}\{X_{\pm\delta_3}, X_{\pm(\delta_2+\delta_3)}, X_{\pm\delta(1110)}, X_{\pm\delta(1232)}\};$$

cf. (13). If we put  $\mathfrak{p}^\pm = \mathfrak{f}^\pm \oplus \mathfrak{j}^\pm$ , with  $\mathfrak{f}^\pm$  as in (14), it is clear that we have a direct sum  $\mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{p}^+$ .

First observe that for example the commutator  $[X_{\delta_3}, X_{\delta_4}]$  is a nonzero element of  $\mathfrak{f}^+$ ; and therefore the considered pair is nonsymmetric. Next we want to show that it is a Cartan pair. For that purpose define

$$A = X_{-\delta_3} + X_{\delta_3},$$

and then  $\mathfrak{a}$  to be the one-dimensional subspace of  $\mathfrak{p}$  spanned by  $A$ . Note that  $\mathfrak{a}$  is a (commutative) subalgebra, reductive in  $\mathfrak{g}$ . It remains to see that it is self-centralizing in  $\mathfrak{p}$ . In order to do that denote by  $\Theta^+$  the set of twelve roots  $\phi \in \Delta_1^+$  so that  $\mathfrak{p}^+ = \{X_\phi \mid \phi \in \Theta^+\}$ . Let then  $x \in C_{\mathfrak{p}}(\mathfrak{a})$  be written as  $x = x^- + x^+$ , where

$$x^\pm = \sum_{\phi \in \Theta^+} c_\phi^\pm X_{\pm\phi}.$$

We have

$$\begin{aligned} [X_{\delta_3}, x^+] &= [X_{\delta_3}, c_{\delta_4}^+ X_{\delta_4} + c_{\delta_2+\delta_3}^+ X_{\delta_2+\delta_3} + c_{\delta(1110)}^+ X_{\delta(1110)} + c_{\delta(0111)}^+ X_{\delta(0111)} \\ &\quad + c_{\delta(1111)}^+ X_{\delta(1111)} + c_{\delta(1221)}^+ X_{\delta(1221)} + c_{\delta(1232)}^+ X_{\delta(1232)}], \end{aligned}$$

and also

$$\begin{aligned} [X_{-\delta_3}, x^+] &= [X_{-\delta_3}, c_{\delta_3}^+ X_{\delta_3} + c_{\delta_2+\delta_3}^+ X_{\delta_2+\delta_3} + c_{\delta_3+\delta_4}^+ X_{\delta_3+\delta_4} \\ &\quad + c_{\delta(1110)}^+ X_{\delta(1110)} + c_{\delta(0121)}^+ X_{\delta(0121)} + c_{\delta(1121)}^+ X_{\delta(1121)} \\ &\quad + c_{\delta(1231)}^+ X_{\delta(1231)} + c_{\delta(1232)}^+ X_{\delta(1232)}]. \end{aligned}$$

Similarly we write  $[X_{\pm\delta_3}, x^-]$ . Using the equality  $[A, x] = 0$ , by careful inspection during the descent on the level roots (from the top to the bottom), we conclude that  $c_{\delta(1232)}^+ = 0$ ,  $c_{\delta(1221)}^+ = 0$ ,  $c_{\delta(1231)}^+ = 0$ , etc. We obtain that necessarily  $x^\pm = c_{\delta_3}^\pm X_{\pm\delta_3}$ . Hence it is immediate that moreover  $c_{\delta_3}^+ = c_{\delta_3}^-$ ; i.e.,  $x \in \mathfrak{a}$  as we had to see.

Now we have to check that  $\mathfrak{a}$  satisfies the two conditions of Definition 2.4. For that first observe that

$$\mathbf{B} = \{\mathcal{H}_1, \mathcal{H}_2 + \mathcal{H}_3, \mathcal{H}_3 + 2\mathcal{H}_4\}$$

is a basis of  $\ker \delta_3$ . Let then  $\mathfrak{c}$  be a subspace spanned by  $\mathbf{B} \cup \{A\}$ . Clearly  $\mathfrak{c}$  is a Cartan subalgebra of  $\mathfrak{g}_1$ . Further, we have  $(\mathfrak{c} \cap \mathfrak{g}_3) \oplus \mathfrak{a} = \mathfrak{c}$ ; i.e., the first condition (**Cp1**) holds.

In order to see that (**Cp2**) holds as well, we proceed as follows; cf. the proof of Lemma 3.5. We may assume that the ground field  $\mathbb{K}$  is algebraically closed. Let then  $\psi \in \Delta(\mathfrak{g}_1, \mathfrak{c})$  be such that  $\psi|_{\mathfrak{a}} = 0$ , and then take any  $w \in (\mathfrak{g}_1)_\psi$ . We in particular have  $[A, w] = 0$ ; i.e.,  $w \in C_{\mathfrak{g}_1}(A)$ . By (3) we can write  $w = v + cA$ , for some  $v \in C_{\mathfrak{g}_3}(A)$  and scalar  $c$ . Let then  $U$  be any of the three basis vectors in  $\mathbf{B}$  satisfying  $\psi(U) \neq 0$ . Observe that  $[U, w] = [U, v]$ . On the other hand,  $[U, w] = \psi(U)w$ . Thus it follows that

$$[U, v] - \psi(U)v = c\psi(U)A.$$

But the left-hand side is an element of  $\mathfrak{g}_3$ , while the right-hand side is an element of  $\mathfrak{p}$ . We conclude that  $c = 0$ ; i.e.,  $w = v$ . Thus we have our proposition proved. ■

#### 4. Some remarks about Cartan subspaces

In this section we conclude our present research by a few further facts and remarks about Cartan subspaces. Unless otherwise said, in order to avoid some

technical complications we assume that throughout the section the ground field is algebraically closed.

Assume that  $(\mathfrak{g}, \mathfrak{g}_1)$  is a Cartan pair,  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}$  is the usual decomposition, and  $\mathfrak{a}$  is a Cartan subspace as in **(Cp1)** of Definition 2.4; choose also a Cartan subalgebra  $\mathfrak{h}$  as there. Assume that  $G$  is a connected (linear) algebraic group with  $\mathfrak{g}$  as its Lie algebra. Let  $G_1 \subseteq G$  be a connected closed subgroup with Lie algebra  $\mathfrak{g}_1$ . The following basic observations are again analogs of well-known standard claims in the symmetric setting. For the sake of completeness we give a proof.

**Proposition 4.1.** *Suppose  $(\mathfrak{g}, \mathfrak{g}_1)$ ,  $\mathfrak{p}$ ,  $\mathfrak{a}$  and the groups  $G_1 \subseteq G$  are as above.*

(i) *The group  $G_1$  acts on  $\mathfrak{p}$  via the adjoint representation, and so in particular we have  $g_1 \cdot \mathfrak{p} = \mathfrak{p}$  for every  $g_1 \in G_1$ .*

(ii) *The set  $g_1 \cdot \mathfrak{a}$  is a Cartan subspace of  $\mathfrak{g}$  for every  $g_1 \in G_1$ .*

**Proof.** The claim (i) is an immediate consequence of Lemma 1.1 and the invariance of the Killing form  $B_{\mathfrak{g}}$  for  $\text{Aut } \mathfrak{g}$ , the group of automorphisms of  $\mathfrak{g}$ .

(ii) Fix some  $g_1 \in G_1$  and define  $\tilde{\mathfrak{a}} = g_1 \cdot \mathfrak{a}$ . Clearly, using (i),  $\tilde{\mathfrak{a}}$  is a commutative subalgebra of  $\mathfrak{p}$ . Next, let  $\tilde{z} \in C_{\mathfrak{p}}(\tilde{\mathfrak{a}})$ . Then  $z = g_1^{-1} \cdot \tilde{z}$  is from  $C_{\mathfrak{p}}(\mathfrak{a})$ ; i.e.,  $z \in \mathfrak{a}$ , and hence  $\tilde{z} \in \tilde{\mathfrak{a}}$ . Thus we have  $C_{\mathfrak{p}}(\tilde{\mathfrak{a}}) \subseteq \tilde{\mathfrak{a}}$ . But  $\tilde{\mathfrak{a}}$  is commutative which gives that moreover  $C_{\mathfrak{p}}(\tilde{\mathfrak{a}}) = \tilde{\mathfrak{a}}$ . It remains to see that every  $\tilde{A} \in \tilde{\mathfrak{a}}$  is a semisimple element of  $\mathfrak{g}$ . But this follows at once by the fact that  $A = g_1^{-1} \cdot \tilde{A} \in \mathfrak{a}$  is semisimple.  $\blacksquare$

Now fix again some  $g_1 \in G_1$  and define the Cartan subspace  $\tilde{\mathfrak{a}}$  as in the latter proof and  $\tilde{\mathfrak{h}} = g_1 \cdot \mathfrak{h}$ ; of course,  $\tilde{\mathfrak{h}}$  is another Cartan subalgebra of  $\mathfrak{g}$ . The content of the next proposition was implicitly announced in Remark 2.5.

**Proposition 4.2.** *Suppose that a Cartan pair  $(\mathfrak{g}, \mathfrak{g}_1)$ ,  $\mathfrak{h}$  and  $\mathfrak{a}$  are as in Definition 2.4. Then the above defined  $\tilde{\mathfrak{a}}$  and  $\tilde{\mathfrak{h}}$ , for every  $g_1 \in G_1$ , satisfy the two conditions of the mentioned definition as well.*

**Proof.** First we easily see that the condition **(Cp1)** holds; i.e., that we have the equality

$$(\tilde{\mathfrak{h}} \cap \mathfrak{g}_1) \oplus \tilde{\mathfrak{a}} = \tilde{\mathfrak{h}}.$$

In order to treat **(Cp2)** we proceed as follows. Consider a map  $\gamma : \mathfrak{h} \rightarrow \tilde{\mathfrak{h}}$ ,  $\gamma(H) = g_1 \cdot H$ , and then the corresponding map on duals  $\gamma^* : \tilde{\mathfrak{h}}^* \rightarrow \mathfrak{h}^*$  given by  $\gamma^*(\tilde{\lambda}) = \tilde{\lambda} \circ \gamma$ . Then  $\gamma^* : \Delta(\mathfrak{g}, \tilde{\mathfrak{h}}) \rightarrow \Delta(\mathfrak{g}, \mathfrak{h})$  is an isomorphism of root systems. At the same time we consider the restriction of  $\gamma$  to  $\mathfrak{a}$ ; i.e.,  $\gamma : \mathfrak{a} \rightarrow \tilde{\mathfrak{a}}$ , and then the corresponding map  $\gamma^* : \tilde{\mathfrak{a}}^* \rightarrow \mathfrak{a}^*$ . It is easy to check that then

$$\gamma^* : \Sigma(\mathfrak{g}, \tilde{\mathfrak{a}}) \rightarrow \Sigma(\mathfrak{g}, \mathfrak{a})$$

is a bijection; note that in general here we cannot speak of a root system isomorphism understood in the standard sense. Next, let  $\Delta'$  be as in Definition 2.4, and define

$$\tilde{\Delta}' = \{\tilde{\phi} \in \Delta(\mathfrak{g}, \tilde{\mathfrak{h}}) \mid \tilde{\phi}|_{\tilde{\mathfrak{a}}} = 0\}.$$

Now pick any  $\tilde{\phi} \in \tilde{\Delta}'$  and a root vector  $X_{\tilde{\phi}} \in \mathfrak{g}_{\tilde{\phi}}$ . Define  $\phi = \gamma^*(\tilde{\phi})$ , which is a root from  $\Delta(\mathfrak{g}, \mathfrak{h})$ . We have the root subspace  $\mathfrak{g}_{\phi} = \mathbb{K}X_{\phi}$ , where

$$X_{\tilde{\phi}} = g_1 \cdot X_{\phi}. \quad (15)$$

Further, for any  $A \in \mathfrak{a}$  and  $\tilde{A} = g_1 \cdot A \in \tilde{\mathfrak{a}}$  we have the equality  $\tilde{\phi}(\tilde{A}) = \phi(A)$ . Thus we deduce at once that the restriction

$$\gamma^* : \tilde{\Delta}' \rightarrow \Delta'$$

is again a bijection. To finish the proof it remains to see that  $X_{\tilde{\phi}} \in \mathfrak{g}_1$ . But this is an immediate consequence of the assumption that  $X_{\phi} \in \mathfrak{g}_1$ , for every  $\phi \in \Delta'$ , and the equality (15). ■

Let us also formulate the following obvious lemma.

**Lemma 4.3.** *Let  $\mathfrak{a}$  be a Cartan subspace, and for arbitrary  $g_1 \in G_1$  define another Cartan subspace  $\tilde{\mathfrak{a}} = g_1 \cdot \mathfrak{a}$  as before. If  $A \in \mathfrak{a}$  is a regular element, then  $\tilde{A} = g_1 \cdot A \in \tilde{\mathfrak{a}}$  is regular as well.*

**Remark 4.4.** Suppose  $V$  is a finite dimensional complex vector space,  $X$  is its subset and  $G$  is a closed subgroup of  $\mathrm{GL}(V)$ . Suppose that moreover  $G$  is connected and reductive. Having such a setting for certain sets  $X$  it is interesting to understand the topology of the orbit space  $G \cdot X$ . In particular we would like to know when  $G \cdot X$  is a closed subset of  $V$ . But in general such questions are quite involved. As an illustration for what we said consider the following example. Let  $V = \mathbb{C}^2$  and  $G$  be the diagonal group

$$G = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C}^* \right\}.$$

Take a vector  $w = (1, 1) \in V$ . An easy exercise shows that  $G \cdot w$  is closed in  $V$ . But on the other hand if we put  $W$  to be the one-dimensional subspace spanned by  $w$ , then  $G \cdot W$  is *not* closed in  $V$ . Nevertheless the following elementary fact must be well known; we leave details of a proof to the reader.

**Claim.** Let  $W$  be any subspace of a finite dimensional complex vector space  $V$ . Let  $G$  be any compact subgroup of  $\mathrm{GL}(V)$ . Then the orbit space  $G \cdot W$  is closed in  $W$ .

Assume for the moment that  $G$  is a linear connected semisimple group, real or complex. Let  $\mathfrak{g}$  be its Lie algebra and  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  a standard Cartan decomposition for  $\mathfrak{g}$ . Then we have a maximal compact subgroup  $K$  of  $G$ , which is connected having  $\mathfrak{k}$  as its Lie algebra. As we already mentioned now we define a Cartan subspace of  $\mathfrak{g}$  as a maximal commutative subspace of  $\mathfrak{p}$ . Keeping the latter setting and notation recall the following well known basic facts; see, e.g., [Kn1, Thm. 5.13]. For any Cartan subspace  $\mathfrak{a} \subseteq \mathfrak{p}$  we have

$$K \cdot \mathfrak{a} = \mathfrak{p}. \quad (16)$$

More precisely, if  $\mathfrak{a}$  and  $\mathfrak{a}'$  are two Cartan subspaces of  $\mathfrak{g}$ , then

$$k \cdot \mathfrak{a} = \mathfrak{a}', \quad \text{for some } k \in K. \quad (17)$$

Let again  $(\mathfrak{g}, \mathfrak{g}_1)$  be a Cartan pair,  $G_1 \subseteq G$  the corresponding pair of groups and the decomposition  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}$  as before. The claim (i) of the following proposition might be understood as a weak analog of (16) and [D, Prop. 1.13.13], while (ii) is just the first step toward an analog of the above conjugating result (17).

**Proposition 4.5.** *Let  $(\mathfrak{g}, \mathfrak{g}_1)$  be a Cartan pair and  $\mathfrak{a}$  be as in Definition 2.4.*

(i) *Suppose there exist  $a \in \mathfrak{a}_{reg}$  such that*

$$[a, \mathfrak{g}_1] + \mathfrak{a} = \mathfrak{p}. \quad (18)$$

*Then  $G_1 \cdot \mathfrak{a}$  is dense in  $\mathfrak{p}$ .*

(ii) *Suppose that  $(\mathfrak{g}, \mathfrak{g}_1)$  is of type I. Let  $\mathfrak{a}'$  be another Cartan subspace as in Definition 2.4, and assume that the intersection  $\gamma_1 \cdot \mathfrak{a}_{reg} \cap \mathfrak{a}'$  is nonempty, for certain  $\gamma_1 \in G_1$ . Then  $\gamma_1 \cdot \mathfrak{a} = \mathfrak{a}'$ .*

**Proof.** (i) The technical assumption (18) ensures the claim; here the same argument as in [D] applies.

(ii) Let  $A'$  be an element of the mentioned intersection, and  $A \in \mathfrak{a}_{reg}$  so that  $\gamma_1 \cdot A = A'$ . Observe, by Lemma 4.3, that then  $A' \in \mathfrak{a}'_{reg}$ . Now take an arbitrary element  $Z \in \mathfrak{a}'$ . As  $\mathfrak{a}'$  is commutative we clearly have  $[Z, A'] = 0$ , and hence it immediately follows that  $\gamma_1^{-1} \cdot Z \in C_{\mathfrak{p}}(A)$ ; here we use Proposition 4.1(i). Therefore, by Lemma 2.13, it follows at once that  $Z \in \gamma_1 \cdot \mathfrak{a}$ . Thus we have proved the inclusion  $\mathfrak{a}' \subseteq \gamma_1 \cdot \mathfrak{a}$ . To finish the proof we just have to take into account Corollary 2.12.  $\blacksquare$

Concerning the above proposition the following two examples will be helpful. Let us emphasize that again we use below the notation from the beginning of Section 3.

**Example 4.6.** Let  $(\mathfrak{g}, \mathfrak{g}_1)$  be the pair of type  $(B_3, G_2)$  considered before. Recall the decomposition  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{p}$ , and also that  $\mathfrak{a} = -H_1 + H_2 + H_3$  and  $\mathfrak{a} = \mathbb{K}\mathfrak{a}$ .

**Claim.** We have a direct sum decomposition

$$[\mathfrak{a}, \mathfrak{g}_1] + \mathfrak{a} = \mathfrak{p}.$$

**Proof.** First we compute the following:

$$\gamma_1(\mathfrak{a}) = -2, \quad \gamma_2(\mathfrak{a}) = 0, \quad \gamma_3(\mathfrak{a}) = \gamma_4(\mathfrak{a}) = 1. \quad (19)$$

Hence it is an easy exercise to deduce the following table:

$$\begin{aligned}
[\mathbf{a}, X_{\alpha_1}^1] &= -2X_{\gamma_1}^0 + X_{\gamma_3}^0 + X_{\gamma_4}^0 \\
[\mathbf{a}, X_{\alpha_2}^1] &= 0 \\
[\mathbf{a}, X_{\alpha_1+\alpha_2}^1] &= 2X_{\gamma_1+\gamma_2}^0 + X_{\gamma_2+\gamma_3}^0 + X_{\gamma_2+\gamma_4}^0 \\
[\mathbf{a}, X_{2\alpha_1+\alpha_2}^1] &= -X_{\gamma_1+\gamma_2+\gamma_3}^0 - X_{\gamma_1+\gamma_2+\gamma_4}^0 + 2X_{\gamma_2+\gamma_3+\gamma_4}^0 \\
[\mathbf{a}, X_{3\alpha_1+\alpha_2}^1] &= [\mathbf{a}, X_{3\alpha_1+2\alpha_2}^1] = 0
\end{aligned}$$

Using the first table at the preliminary part of Section 3 the latter nonzero commutators can be written as follows:

$$\begin{aligned}
[\mathbf{a}, X_{\alpha_1}^1] &= -2X_{\beta_1} + X_{\beta_3} \\
[\mathbf{a}, X_{\alpha_1+\alpha_2}^1] &= 2X_{\beta_1+\beta_2} + X_{\beta_2+\beta_3} \\
[\mathbf{a}, X_{2\alpha_1+\alpha_2}^1] &= -X_{\beta_1+\beta_2+\beta_3} - 2X_{\beta_2+2\beta_3}
\end{aligned}$$

Further, by (6), we also have:

$$\begin{aligned}
[\mathbf{a}, X_{-\alpha_1}^1] &= 2X_{-\beta_1} - X_{-\beta_3} \\
[\mathbf{a}, X_{-(\alpha_1+\alpha_2)}^1] &= -2X_{-(\beta_1+\beta_2)} - X_{-(\beta_2+\beta_3)} \\
[\mathbf{a}, X_{-(2\alpha_1+\alpha_2)}^1] &= X_{-(\beta_1+\beta_2+\beta_3)} + 2X_{-(\beta_2+2\beta_3)}
\end{aligned}$$

Now, as we have  $\dim \mathfrak{p} = 7$ , the above computations show that we have the stated claim.  $\blacksquare$

**Example 4.7.** Let  $(\mathfrak{g}_1, \mathfrak{s})$  be the pair of type  $(G_2, A_2)$  considered before; see the paragraph after Remark 3.3. Recall the decomposition  $\mathfrak{g}_1 = \mathfrak{s} \oplus \mathfrak{p}_1$ , and let  $\mathfrak{p}_1$ ,  $\mathbf{a}_i$  and  $\alpha_i$  be as there.

**Claim.** We have direct sum decompositions

$$[\mathbf{a}_i, \mathfrak{s}] + \alpha_i = \mathfrak{p}_1.$$

**Proof.** Again we consider only the case  $i = 1$ . Recall the decomposition (9), where the Cartan subalgebra  $\mathfrak{h}_1 = \text{span}\{C_1, C_2\}$  and  $\Lambda, \Lambda^+$  are as there. It is well known that  $\alpha_1(C_1) = 2 = \alpha_2(C_2)$ ,  $\alpha_1(C_2) = -1$  and  $\alpha_2(C_1) = -3$ . Hence we easily deduce that

$$[\mathfrak{h}_1, \mathbf{a}_1] = \mathbb{K}(X_{\alpha_1}^1 - X_{-\alpha_1}^1),$$

and also:

$$\begin{aligned}
[\mathbf{a}_1, X_{\alpha_2}^1] &= -X_{(\alpha_1+\alpha_2)}^1 \\
[\mathbf{a}_1, X_{-\alpha_2}^1] &= -X_{-(\alpha_1+\alpha_2)}^1 \\
[\mathbf{a}_1, X_{3\alpha_1+\alpha_2}^1] &= -X_{2\alpha_1+\alpha_2}^1 \\
[\mathbf{a}_1, X_{-(3\alpha_1+\alpha_2)}^1] &= X_{-(2\alpha_1+\alpha_2)}^1 \\
[\mathbf{a}_1, X_{\pm(3\alpha_1+2\alpha_2)}^1] &= 0
\end{aligned}$$

As we have  $\dim \mathfrak{p}_1 = 6$ , the above computations give at once the proposed equality.  $\blacksquare$

**Remark 4.8.** Compared to the previous examples, let us point out at one more fact which shows that we have to be careful while working with (nonsymmetric) Cartan pairs. That is, observe that the equality (18) will not hold for every such pair. To see this take the pair  $(\mathfrak{g}_0, \mathfrak{g}_1)$  of type  $(D_4, G_2)$  considered before. (We will see once more that this pair is in a sense “pathological”.) Then we have the decomposition  $\mathfrak{g}_0 = \mathfrak{g}_1 \oplus \mathfrak{p}_0$  and  $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$  given by (8). Let also  $\lambda_1, \lambda_2 \in \mathfrak{a}_0^*$  be as in the proof of Lemma 3.1. It immediately follows that

$$(\mathfrak{a}_0)_{reg} = \{sa + tH_4 \mid 0 \neq s \neq \pm t\}.$$

**Claim.** Let  $A_0 \in (\mathfrak{a}_0)_{reg}$  be arbitrary, written as above. Then

$$[A_0, \mathfrak{g}_1] + \mathfrak{a}_0 \neq \mathfrak{p}_0.$$

**Proof.** We have  $\gamma_1(H_4) = 0 = \gamma_2(H_4)$ ,  $\gamma_3(H_4) = -1$  and  $\gamma_4(H_4) = 1$ . Using (19) it follows that

$$\gamma_1(A_0) = -2s, \quad \gamma_2(A_0) = 0, \quad \gamma_3(A_0) = s - t, \quad \gamma_4(A_0) = s + t.$$

Hence we deduce at once that in particular

$$[A_0, X_{\pm\alpha_2}^1] = [A_0, X_{\pm(3\alpha_1+\alpha_2)}^1] = [A_0, X_{\pm(3\alpha_1+2\alpha_2)}^1] = 0,$$

and as a consequence that  $\dim([A_0, \mathfrak{g}_1] + \mathfrak{a}_0) = 8$ . It remains to take into account that  $\dim \mathfrak{p}_0 = 14$ . ■

**Acknowledgments.** The present paper has been finished during the author’s stay (October 2016 till February 2017) at the Department of Mathematics, University of Utah. The author is very grateful to Dragan Miličić who enabled this very stimulating and enjoyable visit; and for all his help as well. He is also grateful to the referee for a careful reading of the manuscript, and some valuable suggestions and/or remarks.

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Boris Širola  
Department of Mathematics  
University of Zagreb  
Bijenička 30  
10000 Zagreb, Croatia  
sirola@math.hr

Received February 2, 2017  
and in final form June 4, 2017