Homogeneous Distributions
on Finite Dimensional Vector Spaces

Huajian Xue

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Abstract.   Let V be a finite dimensional vector space over a local field F. Let $\chi\colon F^\times \to \mathbb{C}^\times$ be an arbitrary character of $F^\times$. We determine the structure of the natural representation of $\mathrm{GL}(V)$ on the space $\mathcal{S}^*(V)^\chi$ of χ-invariant distributions on V.
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1.   Introduction

Let V be a vector space over a local field F, of finite dimension $n \geq 1$. For every $g \in \mathrm{GL}(V)$ and every distribution η on V, define

$$g.\eta := \text{the push-forward of } \eta \text{ through the map } g : V \to V. \tag{1}$$

For each character $\chi : F^\times \to \mathbb{C}^\times$, a distribution η on V is said to be χ-invariant if

$$a.\eta = \chi(a)\,\eta, \quad \text{for all } a \in F^\times.$$

Here and as usual, $F^\times$ is identified with the center of $\mathrm{GL}(V)$. By [2, Theorem 4.0.2], we know that every χ-invariant distribution on V is tempered when F is archimedean. By convention, every distribution on V is defined to be tempered when F is non-archimedean. The goal of this paper is to understand the space

$$\mathcal{S}^*(V)^\chi := \{\eta \in \mathcal{S}^*(V) \mid a.\eta = \chi(a)\,\eta\} \tag{2}$$

of χ-invariant tempered distributions on V, as a representation of $\mathrm{GL}(V)$ under the action (1). Here and as usual, $\mathcal{S}(V)$ denotes the space of Schwartz or Schwartz-Bruhat functions on V, when F is respectively archimedean or non-archimedean; and $\mathcal{S}^*(V)$ denotes the space of all (continuous in the archimedean case) linear functionals on $\mathcal{S}(V)$. It is a fundamental fact in Tate's thesis that the space (2) is one dimensional when $n = 1$ (see [17, Section 1]). Thus we will focus on the case when $n \geq 2$.

Dualizing the action (1), we have a representation of $\mathrm{GL}(V)$ on $\mathcal{S}(V)$ by

$$(g.f)(x) := f(g^{-1}x), \quad \text{for all } g \in \mathrm{GL}(V),\ f \in \mathcal{S}(V),\ x \in V.$$

Define the normalized $(F^\times, \chi)$-coinvariant space

$$\mathcal{S}_\chi(V) := \left(\mathcal{S}(V) \otimes |\det|_F^{-\frac{1}{2}} \right)_{F^\times, \chi}, \tag{3}$$

where $|\det|_F$ denotes the positive character

$$\mathrm{GL}(V) \to \mathbb{C}^\times, \quad g \mapsto |\det(g)|_F,$$

$|\cdot|_F$ denotes the normalized absolute value on F, and for a smooth representation U of $\mathrm{GL}(V)$, $U_{F^\times, \chi}$ denotes the maximal (Hausdorff in the archimedean case) quotient of U on which $F^\times$ acts through the character χ. Here and as usual, we do not distinguish a one dimensional representation with its corresponding character. Then we have

$$\mathcal{S}^*(V)^\chi \cong \left(\mathcal{S}_{\chi^{-1} \cdot |\cdot|_F^{-\frac{n}{2}}}(V) \right)^* \otimes |\det|_F^{-\frac{1}{2}}, \tag{4}$$

as representations of $\mathrm{GL}(V)$. Here and henceforth, we use a superscript * to indicate the dual space in various contexts. Thus we only need to study the representation (3).

As before, write $\mathcal{S}(V \setminus \{0\})$ for the space of Schwartz or Schwartz-Bruhat functions on $V \setminus \{0\}$, when F is respectively archimedean or non-archimedean (see [1] for the definition of Schwartz functions in the archimedean case). Similar to (3), define

$$\mathcal{S}_\chi(V \setminus \{0\}) := \left(\mathcal{S}(V \setminus \{0\}) \otimes |\det|_F^{-\frac{1}{2}} \right)_{F^\times, \chi}. \tag{5}$$

Then the embedding $\mathcal{S}(V \setminus \{0\}) \hookrightarrow \mathcal{S}(V)$ induces a homomorphism

$$\mathrm{j}_\chi : \mathcal{S}_\chi(V \setminus \{0\}) \to \mathcal{S}_\chi(V) \tag{6}$$

of representations of $\mathrm{GL}(V)$.

It is easy to see that the representation $\mathcal{S}_\chi(V \setminus \{0\})$ is isomorphic to a degenerate principal series. More precisely, fix an arbitrary nonzero vector $v_0 \in V$, and write $\mathrm{P}(v_0) = F^\times \times \mathrm{P}^\circ(v_0)$ for the maximal parabolic subgroup of $\mathrm{GL}(V)$ stabilizing Fv_0, where $\mathrm{P}^\circ(v_0)$ denotes the stabilizer of v_0 in $\mathrm{GL}(V)$. Then the linear map

$$\begin{aligned} \mathcal{S}(V \setminus \{0\}) \otimes |\det|_F^{-\frac{1}{2}} &\to C^\infty(\mathrm{GL}(V)), \\ \phi \otimes 1 &\mapsto \left(g \mapsto \int_{F^\times} \phi((ga)^{-1}v_0) \cdot |\det(ga)|_F^{-\frac{1}{2}} \cdot \chi^{-1}(a)\ \mathrm{d}^\times a \right) \end{aligned}$$

induces a $\mathrm{GL}(V)$-intertwining isomorphism

$$\mathcal{S}_\chi(V \setminus \{0\}) \cong \mathrm{Ind}_{\mathrm{P}(v_0)}^{\mathrm{GL}(V)}(\chi \boxtimes 1). \tag{7}$$

Here $\mathrm{d}^{\times}a$ is a Haar measure on the multiplicative group $F^{\times}$. And "Ind" indicates the normalized smooth induction, on which $\mathrm{GL}(V)$ acts by right translation. While "1" stands for the trivial character of $\mathrm{P}^{\circ}(v_0)$ and "$\boxtimes$" indicates the external tensor product. The structure of this degenerate principal series is well-known (see Lemma 4.1).

Denote by C_F the submonoid of $\mathrm{Hom}(F^{\times}, \mathbb{C}^{\times})$ generated by characters of the form $\iota|_{F^{\times}} : F^{\times} \to \mathbb{C}^{\times}$, where $\iota : F \hookrightarrow \mathbb{C}$ is a continuous field embedding. Explicitly, denote by $\mathbb{N}$ the set of non-negative integers. If $F = \mathbb{R}$, let ι be the natural imbedding of $\mathbb{R}$ into $\mathbb{C}$. If $F = \mathbb{C}$, let ι_1, ι_2 be the identity map and complex conjugate respectively. Then

$$
\mathrm{C}_F = \begin{cases} \{(\iota|_{\mathbb{R}^{\times}})^r | r \in \mathbb{N}\}, & \text{if } F = \mathbb{R}; \\ \{(\iota_1|_{\mathbb{C}^{\times}})^r (\iota_2|_{\mathbb{C}^{\times}})^s | r, s \in \mathbb{N}\}, & \text{if } F = \mathbb{C}; \\ \{1\}, & \text{if } F \text{ is non-archimedean.} \end{cases}
$$

In Section 2, we will define an irreducible finite dimensional representation $\sigma_{V,\chi}$ of $\mathrm{GL}(V)$ for each $\chi \in \mathrm{C}_F$. Note that

$$
\mathrm{C}^+(n) := |\cdot|_F^{\frac{n}{2}} \, \mathrm{C}_F \quad \text{and} \quad \mathrm{C}^-(n) := \{\chi^{-1} \mid \chi \in \mathrm{C}^+(n)\}
$$

are disjoint subsets of $\mathrm{Hom}(F^{\times}, \mathbb{C}^{\times})$.

Now the main result of this paper is formulated as follows.

Theorem 1.1. *Let V be an n-dimensional ($n \geq 2$) vector space over a local field F, and let χ be a character of $F^{\times}$. Define the homomorphism $\mathrm{j}_{\chi} : \mathcal{S}_{\chi}(V \setminus \{0\}) \to \mathcal{S}_{\chi}(V)$ as (6). Then we have the followings:*

(a) *If $\chi \notin \mathrm{C}^+(n) \cup \mathrm{C}^-(n)$, then j_{χ} is an isomorphism of irreducible representations.*

(b) *If $\chi \in \mathrm{C}^+(n)$, then j_{χ} is an isomorphism and $\mathcal{S}_{\chi}(V)$ has a unique irreducible subrepresentation, and the corresponding quotient representation is isomorphic to $\sigma_{V,\chi \cdot |\cdot|_F^{-\frac{n}{2}}} \otimes |\det|_F^{\frac{1}{2}}$.*

(c) *If $\chi \in \mathrm{C}^-(n)$, then both $\mathcal{S}_{\chi}(V)$ and $\mathcal{S}_{\chi}(V \setminus \{0\})$ have length 2 and have a unique irreducible subrepresentation. The unique irreducible subrepresentation of $\mathcal{S}_{\chi}(V)$ is isomorphic to*

$$
\ker(\mathrm{j}_{\chi}) \cong \mathrm{coker}(\mathrm{j}_{\chi}) \cong \left(\sigma_{V,\chi^{-1} \cdot |\cdot|_F^{-\frac{n}{2}}} \otimes |\det|_F^{\frac{1}{2}} \right)^*.
$$

The space $\mathcal{S}(V)$ carries an action of $\mathrm{GL}_1(F) \times \mathrm{GL}(V)$ which is defined by

$$
((g_1, g_2).f)(x) := f(g_2^{-1} x g_1), \quad (g_1, g_2) \in \mathrm{GL}_1(F) \times \mathrm{GL}(V), f \in \mathcal{S}(V), x \in V. \quad (8)
$$

The Weil representation ω of $\mathrm{GL}_1(F) \times \mathrm{GL}(V)$ is defined to be

$$
\omega := \mathcal{S}(V) \otimes \left(|\cdot|_F^{\frac{n}{2}} \boxtimes |\det|_F^{-\frac{1}{2}} \right).
$$

In the theory of local theta correspondence, one is interested in the decomposition of ω into irreducible representations of $\mathrm{GL}_1(F) \times \mathrm{GL}(V)$. Let χ be a character of $\mathrm{GL}_1(F)$. View $\mathrm{GL}_1(F)$ (resp. $\mathrm{GL}(V)$) as the subgroup $\mathrm{GL}_1(F) \times \{1\}$ (resp. $\{1\} \times \mathrm{GL}(V)$) of $\mathrm{GL}_1(F) \times \mathrm{GL}(V)$, and view χ as the character $\chi \boxtimes 1$ of $\mathrm{GL}_1(F) \times \{1\}$. Denote by $(\omega)_{\mathrm{GL}_1(F),\chi}$ the maximal (Hausdorff in the archimedean case) quotient of ω on which $\mathrm{GL}_1(F)$ acts through the character χ. By [12, Chapter 2, Lemma III.4], there is a unique (up to isomorphism) smooth representation $\Theta(\chi)$, called the full theta lift of χ, of $\mathrm{GL}(V)$ such that

$$(\omega)_{\mathrm{GL}_1(F),\chi} \cong \chi \boxtimes \Theta(\chi)$$

as representations of $\mathrm{GL}_1(F) \times \mathrm{GL}(V)$. Note that in (8), $\mathrm{GL}_1(F)$ acts on $\mathcal{S}(V)$ by right translation. While in (3), $\mathrm{GL}_1(F)$ is viewed as a subgroup of $\mathrm{GL}(V)$ and acts on $\mathcal{S}(V)$ by left translation. It follows that

$$\Theta(\chi) \cong \mathcal{S}_{\chi^{-1}}(V)$$

in our setting. It is a fundamental fact that the full theta lift always has a unique irreducible quotient whenever it is nonzero (see [7] [16] [11] [5] [4]). Howe also expects that in many cases, the full theta lift also has a unique irreducible subrepresentation, and the irreducible subrepresentation is "large" and the irreducible quotient representation is "small". Theorem 1.1 provides some evidences for these expectations.

2.   Distributions supported at 0

We continue with the notation of the Introduction. For every $\chi \in \mathrm{C}_F$, we shall define a representation $\sigma_{V,\chi}$ of $\mathrm{GL}(V)$ in what follows. If F is non-archimedean, we define $\sigma_{V,1}$ to be the one dimensional trivial representation for the unique member $1 \in \mathrm{C}_F$. If $F \cong \mathbb{R}$, let $\chi = (\iota|_{F^\times})^r \in \mathrm{C}_F, (r \in \mathbb{N})$ (as is defined in the Introduction), we define

$$\sigma_{V,\chi} := \mathrm{Sym}^r(V \otimes_{F,\iota} \mathbb{C}).$$

Here and henceforth, Sym^r indicates the r-th symmetric power. If $F \cong \mathbb{C}$, let $\chi = (\iota_1|_{F^\times})^r \cdot (\iota_2|_{F^\times})^s \in \mathrm{C}_F, (r, s \in \mathbb{N})$, we define

$$\sigma_{V,\chi} := \mathrm{Sym}^r(V \otimes_{F,\iota_1} \mathbb{C}) \otimes \mathrm{Sym}^s(V \otimes_{F,\iota_2} \mathbb{C}).$$

In all cases, $\sigma_{V,\chi}$ is an irreducible finite dimensional representation of $\mathrm{GL}(V)$ of central character χ.

Let $\mathcal{S}^*(V, \{0\})$ denotes the space of tempered distributions on V whose support is contained in $\{0\}$. For each character $\chi : F^\times \to \mathbb{C}^\times$, put

$$\mathcal{S}^*(V, \{0\})^\chi := \mathcal{S}^*(V, \{0\}) \cap \mathcal{S}^*(V)^\chi,$$

which is a representation of $\mathrm{GL}(V)$.

Denote by δ_0 the Dirac distribution. Set $V = F^n$. If $F = \mathbb{R}$, for $I = (k_1, k_2, \cdots, k_n) \in \mathbb{N}^n$, let ∂^I be the differential operator which takes k_i-th derivative for the i-th variable for all $1 \leq i \leq n$. If $F = \mathbb{C}$, define $\partial^{I_1} \bar{\partial}^{I_2}$ for $I_1, I_2 \in \mathbb{N}^n$ similarly. By [13, Theorem 1.7], we have

$$\mathcal{S}^*(V, \{0\}) = \begin{cases} \mathrm{span}\{\partial^I \delta_0 | I \in \mathbb{N}^n\}, & \text{if } F = \mathbb{R}; \\ \mathrm{span}\{\partial^{I_1} \bar{\partial}^{I_2} \delta_0 | I_1, I_2 \in \mathbb{N}^n\}, & \text{if } F = \mathbb{C}; \\ \mathrm{span}\{\delta_0\}, & \text{if } F \text{ is non-archimedean.} \end{cases}$$

Then it is easy to deduce the following

Lemma 2.1. *If $\chi \in \mathrm{C}_F$, then $\mathcal{S}^*(V, \{0\})^\chi \cong \sigma_{V, \chi}$; otherwise it is zero. Moreover,*

$$\mathcal{S}^*(V, \{0\}) = \bigoplus_{\chi \in \mathrm{C}_F} \mathcal{S}^*(V, \{0\})^\chi.$$

3. Fourier transform

Denote by V^* the dual space of V. The Fourier transform yields a linear isomorphism

$$\mathcal{F} : \mathcal{S}(V) \otimes |\det|_F^{-\frac{1}{2}} \to \mathcal{S}(V^*) \otimes |\det|_F^{-\frac{1}{2}}, \quad \phi \otimes 1 \mapsto \widehat{\phi} \otimes 1,$$

where

$$\widehat{\phi}(\lambda) := \int_V \phi(x) \psi(\lambda(x)) \, \mathrm{d}x.$$

Here $\mathrm{d}x$ is a fixed Haar measure on V, and ψ is a fixed non-trivial unitary character ψ on F. It is routine to check that

$$\mathcal{F}(g.\eta) = g^{-t}.(\mathcal{F}(\eta)), \quad \text{for all } g \in \mathrm{GL}(V), \eta \in \mathcal{S}(V) \otimes |\det|_F^{-\frac{1}{2}}.$$

Here $g^{-t} \in \mathrm{GL}(V^*)$ denotes the inverse transpose of g. Consequently, $\mathcal{F}$ induces a linear isomorphism

$$\mathcal{S}_\chi(V) \cong \mathcal{S}_{\chi^{-1}}(V^*) \tag{9}$$

which is intertwining with respect to the isomorphism $\mathrm{GL}(V) \to \mathrm{GL}(V^*), g \mapsto g^{-t}$.

Lemma 3.1. *If $n \geq 2$, then the space $\mathcal{S}_\chi(V)$ is infinite dimensional.*

Proof. Since $\mathrm{C}^+(n)$ and $\mathrm{C}^-(n)$ are disjoint, by using the isomorphism (9), it suffices to prove the lemma in the case when $\chi \notin \mathrm{C}^-(n)$.

As a reformulation of (4), we have

$$\mathcal{S}^*(V)^{\chi^{-1} \cdot |\cdot|_F^{-\frac{n}{2}}} \cong (\mathcal{S}_\chi(V))^* \otimes |\det|_F^{-\frac{1}{2}}, \tag{10}$$

Note that $\chi \notin \mathrm{C}^-(n)$ if and only if $\chi^{-1} \cdot |\cdot|_F^{-\frac{n}{2}} \notin \mathrm{C}_F$. Thus we only need to show that $\mathcal{S}^*(V)^\chi$ is infinite dimensional when $\chi \notin \mathrm{C}_F$.

For each one dimensional subspace L of V, let η_L denote a nonzero distribution in the one dimensional space $\mathcal{S}^*(L)^\chi$. Assuming $\chi \notin \mathrm{C}_F$, we know from Lemma 2.1 that the support of η_L equals L. Then the infinite family

$$\{\text{the push-forward of } \eta_L \text{ through the embedding } L \hookrightarrow V \},$$

where L runs over all one dimensional subspace of V, is linearly independent in $\mathcal{S}^*(V)^\chi$. This shows that the space $\mathcal{S}^*(V)^\chi$ is infinite dimensional. ∎

4. Proof of theorem 1.1

In this section, assume that $n \geq 2$. Recall the infinite dimensional representation $\mathcal{S}_\chi(V \setminus \{0\}) \cong \mathrm{Ind}_{\mathrm{P}(v_0)}^{\mathrm{GL}(V)}(\chi \boxtimes 1)$ from the Introduction. Combining [8, Section 2.4 and 3.4] and [6, Theorem 1.1], we get the following lemma.

Lemma 4.1. (a) *If* $\chi \notin \mathrm{C}^+(n) \cup \mathrm{C}^-(n)$, *then the representation* $\mathrm{Ind}_{\mathrm{P}(v_0)}^{\mathrm{GL}(V)}(\chi \boxtimes 1)$ *is irreducible; otherwise, it has length* 2 *and has a unique irreducible subrepresentation.*

(b) *If* $\chi \in \mathrm{C}^+(n)$, *then the irreducible quotient representation of* $\mathrm{Ind}_{\mathrm{P}(v_0)}^{\mathrm{GL}(V)}(\chi \boxtimes 1)$ *is isomorphic to* $\sigma_{V, \chi \cdot |\cdot|_F^{-\frac{n}{2}}} \otimes |\det|_F^{\frac{1}{2}}$.

(c) *If* $\chi \in \mathrm{C}^-(n)$, *then the irreducible subrepresentation of* $\mathrm{Ind}_{\mathrm{P}(v_0)}^{\mathrm{GL}(V)}(\chi \boxtimes 1)$ *is isomorphic to* $\left(\sigma_{V, \chi^{-1} \cdot |\cdot|_F^{-\frac{n}{2}}} \otimes |\det|_F^{\frac{1}{2}} \right)^*$.

Proof. Let $F = \mathbb{R}$. Set $G = \mathrm{GL}(V)$ and $P = \mathrm{P}(v_0)$. We may assume that $v_0 = (1, 0, \cdots, 0)^t$, thus P has the form $P = \begin{pmatrix} \mathbb{R}^\times & * \\ 0 & \mathrm{GL}_{n-1}(\mathbb{R}) \end{pmatrix}$, and $P^\circ = \mathrm{P}^\circ(v_0)$ is the subgroup of P with the first column being $(1, 0, \cdots, 0)^t$. The modular character Δ_P of P satisfies

$$\Delta_P(p) = |\det(\mathrm{Ad}_p)| = |a|^{n-1} |\det g|^{-1} = |a|^n |\det p|^{-1},$$

where $p = \begin{pmatrix} a & x \\ 0 & g \end{pmatrix} \in P$. Denote by ${}^{\mathrm{u}}\mathrm{Ind}_P^G(\chi \otimes 1)$ the non-normalized induction and observe that

$${}^{\mathrm{u}}\mathrm{Ind}_P^G((\chi \boxtimes 1) \otimes \Delta_P^{\frac{1}{2}}) = \mathrm{Ind}_P^G(\chi \boxtimes 1).$$

For any representation π of P and any character χ' of G, there is a natural isomorphism

$${}^{\mathrm{u}}\mathrm{Ind}_P^G(\pi \otimes \chi'|_P) \cong \chi' \otimes {}^{\mathrm{u}}\mathrm{Ind}_P^G \pi. \tag{11}$$

It follows that

$$\mathrm{Ind}_P^G(\chi \boxtimes 1) \cong |\det|^{-\frac{1}{2}} \otimes {}^{\mathrm{u}}\mathrm{Ind}_P^G(\chi \cdot |\cdot|^{\frac{n}{2}} \boxtimes 1) \tag{12}$$

by setting $\pi = \chi \boxtimes 1$ and $\chi' = |\det|^{-\frac{1}{2}}$.

Let $Q = \begin{pmatrix} \mathrm{GL}_{n-1}(\mathbb{R}) & * \\ 0 & \mathbb{R}^\times \end{pmatrix}$ be another parabolic subgroup of G, and let Q° be the subgroup of Q with the last row being $(0, \cdots, 0, 1)$. Then $Q = Q^\circ \times \mathbb{R}^\times$. Set $w = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. We have an isomorphism

$$\begin{array}{rcl} \tau: \ {}^{\mathrm{u}}\mathrm{Ind}_P^G(\chi \boxtimes 1) & \xrightarrow{\sim} & {}^{\mathrm{u}}\mathrm{Ind}_Q^G(1 \boxtimes \chi^{-1}), \\ f & \mapsto & (g \mapsto f(wg^{-t}w)), \end{array} \tag{13}$$

which is intertwining with respect to the isomorphism $G \to G, g \mapsto wg^{-t}w$.

By ([8], Theorem 3.41, 3.42, 3.43 and 3.44, applied with $k = 1$), the representation ${}^{\mathrm{u}}\mathrm{Ind}_Q^G(1 \boxtimes \chi)$ is irreducible if $\chi \notin |\cdot|^{-\frac{n}{2}} \mathrm{C}^+(n) \cup |\cdot|^{-\frac{n}{2}} \mathrm{C}^-(n)$; otherwise, it has length 2 and has a unique irreducible subrepresentation. Hence the assertion (a) follows from the isomorphisms (12) and (13).

If $\chi = (\iota|_{\mathbb{R}^\times})^r (r \in \mathbb{N})$. For $I = (k_1, k_2, \cdots, k_n) \in \mathbb{N}^n$ which satisfies $|I| := k_1 + k_2 + \cdots + k_n = r$, we define

$$
f_I \begin{pmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} := a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}.
$$

It is easy to check that $f_I \in {}^{\mathrm{u}}\mathrm{Ind}_Q^G(1 \boxtimes \chi)$, and $W := \mathrm{span}\{f_I | I \in \mathbb{N}^n, |I| = r\}$ is a (finite dimensional) subrepresentation of ${}^{\mathrm{u}}\mathrm{Ind}_Q^G(1 \boxtimes \chi)$ which is isomorphic to $\sigma_{V,\chi}$. Applying the isomorphisms (12) and (13), we get part (c).

Note that the representations $\mathrm{Ind}_P^G(\chi \boxtimes 1)$ and $\mathrm{Ind}_P^G(\chi^{-1} \boxtimes 1)$ are contragredient ([15, Lemma 5.2.4]). Thus (c) implies (b), and we complete the proof for $F = \mathbb{R}$.

The proof for the case that $F = \mathbb{C}$ is analogous.

Now suppose that F is non-archimedean, then part (a) of the lemma follows from Theorem 1.1 (i) and Lemma 4.2 of [6]. And parts (b) and (c) follow from Corollary 3.5 of [6].   ∎

The short exact sequence

$$
0 \to \mathcal{S}(V \setminus \{0\}) \to \mathcal{S}(V) \to (\mathcal{S}^*(V, \{0\}))^* \to 0
$$

induces an exact sequence

$$
\mathcal{S}_\chi(V \setminus \{0\}) \xrightarrow{\mathrm{j}_\chi} \mathcal{S}_\chi(V) \to \left(\mathcal{S}^*(V, \{0\})^{\chi^{-1} \cdot |\cdot|_F^{-\frac{n}{2}}} \right)^* \otimes |\det|_F^{-\frac{1}{2}} \to 0. \quad (14)
$$

Here we have used the natural isomorphism

$$
\left((\mathcal{S}^*(V, \{0\}))^* \otimes |\det|_F^{-\frac{1}{2}} \right)_{F^\times, \chi} \cong \left(\mathcal{S}^*(V, \{0\})^{\chi^{-1} \cdot |\cdot|_F^{-\frac{n}{2}}} \right)^* \otimes |\det|_F^{-\frac{1}{2}}
$$

of representations of $\mathrm{GL}(V)$.

If $\chi \notin \mathrm{C}^-(n)$, then Lemma 2.1 and the exactness of (14) imply that j_χ is surjective. We argue according to the three cases of Theorem 1.1.

Case a: $\chi \notin \mathrm{C}^+(n) \cup \mathrm{C}^-(n)$.

In this case, $\mathcal{S}_\chi(V \setminus \{0\})$ is irreducible by Part (a) of Lemma 4.1. By Lemma 3.1, $\mathcal{S}_\chi(V \setminus \{0\})$ is nonzero. Therefore the surjective homomorphism j_χ is an isomorphism. This proves Part (a) of Theorem 1.1.

Case b: $\chi \in \mathrm{C}^+(n)$.

In this case, by Lemma 3.1, $\ker(\mathrm{j}_\chi)$ is a subrepresentation of $\mathcal{S}_\chi(V \setminus \{0\})$ of infinite codimension. Then Parts (a) and (b) of Lemma 4.1 imply that $\ker(\mathrm{j}_\chi) = \{0\}$, which further implies Part (b) of Theorem 1.1.

Case c: $\chi \in C^-(n)$.

In this case, applying Part (b) of Theorem 1.1 to V^*, and using the isomorphism (9), we know that $\mathcal{S}_\chi(V)$ has length 2 and has a unique irreducible subrepresentation. The exact sequence (14) and Lemma 2.1 imply that

$$\mathrm{coker}(j_\chi) \cong \left(\sigma_{V, \chi^{-1} \cdot |\cdot|_F^{-\frac{n}{2}}} \otimes |\det|_F^{\frac{1}{2}} \right)^*.$$

Thus the image of j_χ is irreducible, and hence $\ker(j_\chi)$ is irreducible as $\mathcal{S}_\chi(V \setminus \{0\})$ has length 2. Applying Part (c) of Lemma 4.1, this proves Part (c) of Theorem 1.1.

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Huajian Xue
Academy of Mathematics
and Systems Science
Chinese Academy of Sciences
Beijing, 100190, China
xuehuajian@126.com