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Varieties of Elementary Subalgebras of Submaximal Rank in Type A

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Abstract. Let G be a connected simple algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > 0$, and $\mathfrak{g} := \mathrm{Lie}(G)$. We additionally assume that G is standard and is of type A_n. Motivated by the investigation of the geometric properties of the varieties $\mathbb{E}(r, \mathfrak{g})$ of r-dimensional elementary subalgebras of a restricted Lie algebra $\mathfrak{g}$, we will show in this article the irreducible components of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathfrak{g})$ when $\mathsf{rk}_p(\mathfrak{g})$ is the maximal dimension of an elementary subalgebra of $\mathfrak{g}$.
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1. Introduction

Let $(\mathfrak{g}, [p])$ be a finite dimensional restricted Lie algebra over an algebraically closed field $\Bbbk$ of positive characteristic $p > 0$. The closed subset of p-restricted nilpotent elements

$$V(\mathfrak{g}) := \{x \in \mathfrak{g}\ ;\ x^{[p]} = 0\}$$

has been studied in the modular representation theory of $(\mathfrak{g}, [p])$, aiming to understand the cohomological support variety. A Lie subalgebra $\mathfrak{e} \subset \mathfrak{g}$ is said to be elementary if it is abelian and has trivial p-restriction. The subset

$$\mathbb{E}(r, \mathfrak{g}) := \{\mathfrak{e} \in \mathrm{Gr}_r(\mathfrak{g})\ ;\ [\mathfrak{e}, \mathfrak{e}] = 0,\ \mathfrak{e} \subset V(\mathfrak{g})\}$$

of the Grassmannian $\mathrm{Gr}_r(\mathfrak{g})$ of r-planes in $\mathfrak{g}$ which consists of r-dimensional elementary subalgebras of $\mathfrak{g}$ has been expounded in [4] by Carlson, Friedlander and Pevtsova. For instance, they show that $\mathbb{E}(r, \mathfrak{g})$ can be endowed with a projective variety structure and it affords geometric invariant for the representations of $\mathfrak{g}$.

When concerning the geometric properties of $\mathbb{E}(r, \mathfrak{g})$, interest has been shown in determining its irreducible components. A prototypical example arises from $\mathbb{E}(1, \mathfrak{g})$, which is the projectivization of the restricted nullcone $V(\mathfrak{g})$. When $\mathfrak{g}$ is the Lie algebra of a simple algebraic group, $V(\mathfrak{g})$ is irreducible regardless of the characteristic p, so is $\mathbb{E}(1, \mathfrak{g})$. When r equals 2, Premet in [10] shows the

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correspondence between the irreducible components of the nilpotent commuting variety $\mathcal{C}^{nil}(\mathfrak{g})$ and the distinguished nilpotent orbits of $\mathfrak{g}$ when $\mathfrak{g}$ is a reductive Lie algebra. It follows that $\mathcal{C}^{nil}(\mathfrak{g})$ is irreducible when $\mathfrak{g}$ is of type A_n, in which case the same is true of $\mathbb{E}(2,\mathfrak{g})$ if $p \geq n+1$. Let

$$\mathsf{rk}_p(\mathfrak{g}) := \max\{r \in \mathbb{N}_0 \ ; \ \mathbb{E}(r,\mathfrak{g}) \neq \emptyset\}$$

be the p-rank of $\mathfrak{g}$. This rank of the restricted Lie algebra of a simple algebraic group was determined earlier in [4] and in recent work by Pevtsova-Stark in [12]. Irreducible components of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}),\mathfrak{g})$ for these Lie algebras were calculated case by case and were shown in [12, Table 4].

It is the purpose of this article to give a description of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{g})$ when $\mathfrak{g}$ is the Lie algebra of a connected standard simple algebraic group G of type A. Under the standard assumption one can show that $\mathfrak{g}$ is a Lie algebra isomorphic to $\mathfrak{sl}_n(\Bbbk)$ such that p does not divide n. In view of [11, Lemma 2.2], the determination of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{g})$ can be reduced to the unipotent case $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{u})$ where $\mathfrak{u} = \mathrm{Lie}(R_u(B))$ and $R_u(B) \subset B \subset G$ is the unipotent radical of a fixed Borel subgroup B of G. Let Φ be an irreducible root system with positive roots Φ^+. Recall from [12] that two positive roots commute if their sum is not a root. We define the set

$$\mathrm{Max}_r(\Phi) := \left\{ \begin{array}{c} R \subset \Phi^+ \ ; \ \alpha+\beta \notin \Phi^+, \forall \alpha,\beta \in R, |R| = r \text{ and } R \not\subset R' \\ \text{where } R' \text{ is any subset of commuting positive roots} \end{array} \right\}.$$

When $r = \mathsf{rk}_p(\mathfrak{g})$, we write $\mathrm{Max}_r(\Phi)$ as $\mathrm{Max}(\Phi)$ simply. By considering the set

$$\mathrm{Com}_r(\Phi) := \{R \subset \Phi^+ \ ; \ \alpha+\beta \notin \Phi^+, \forall \alpha,\beta \in R, |R| = r\},$$

we find that the map $\mathrm{LT} : \mathbb{E}(\mathsf{rk}_p(\mathfrak{g}),\mathfrak{u}) \longrightarrow \mathrm{Max}(\Phi)$ in [12, (3.1.2)] can be defined in a generalized fashion $\mathrm{LT} : \mathbb{E}(r,\mathfrak{u}) \longrightarrow \mathrm{Com}_r(\Phi)$ since $\mathrm{Com}_r(\Phi) = \mathrm{Max}(\Phi)$ when $r = \mathsf{rk}_p(\mathfrak{g})$. The problem now is that for any given total ordering on positive roots and any element $\mathfrak{e}$ of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{u})$ it is possible to have $\mathrm{LT}(\mathfrak{e}) \notin \mathrm{Max}_{\mathsf{rk}_p(\mathfrak{g})-1}(\Phi)$. Let $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{u})_{\max}$ be the subset of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{u})$ consisting of maximal elementary subalgebras. This raises the question concerning the ordering on Φ^+, the one giving rise to the map

$$\mathrm{LT} : \mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{u})_{\max} \longrightarrow \mathrm{Max}_{\mathsf{rk}_p(\mathfrak{g})-1}(\Phi) \ .$$

We find that for type A_n the ordering exists for n sufficiently large.

Since the set $\mathrm{Max}_{\mathsf{rk}_p(\mathfrak{g})-1}(\Phi)$ is tractable, it will be determined within the initial step. The calculation of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{u})$ then proceeds via three steps. First, we determine $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{u})_{\max}$ as a set by using the map LT. Secondly, we prove that the elements of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{u})_{\max}$ are given by the combinatorics of the root system of G, which largely relies on Malcev's linear algebraic approach. Finally, we have to utilize the result on $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}),\mathfrak{u})$ to understand the elements of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{u})$ which are not in $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{u})_{\max}$. The main result of this paper is:

Theorem. *Let G be a standard simple algebraic $\Bbbk$-group with root system Φ of type A_n $(n \geq 5)$ and $\mathfrak{g} := \mathrm{Lie}(G)$. Then the irreducible components of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathfrak{g})$ can be characterized as follows:*

Type	Restrictions on rank	Irreducible components
A_{2m+1}	$m \geq 2$	$G.\mathrm{Lie}(\Phi_m^{\mathrm{rad}})$, $G.\mathrm{Lie}(\Phi_{m+2}^{\mathrm{rad}})$, $G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi_{m+1}^{\mathrm{rad}}))$
A_{2m}	$m \geq 3$	$G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi_m^{\mathrm{rad}}))$, $G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi_{m+1}^{\mathrm{rad}}))$

Table 1: Characterization

Here $\Phi_i^{\mathrm{rad}} := \Phi^+ \setminus \Phi^+_{\Delta\setminus\{\alpha_i\}}$ for $i \in \{m, m+1, m+2\}$ is interpreted explicitly in Sect. 2 and $\mathrm{Lie}(\Phi_i^{\mathrm{rad}}) := \mathrm{Span}_{\Bbbk}\{x_\alpha \ ; \ \alpha \in \Phi_i^{\mathrm{rad}}\}$ is an elementary subalgebra.

Remark 1.1. (1). In [12] the authors have shown that $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}), \mathfrak{g})$ is a finite disjoint union of partial flag varieties unless G is of type A_2, which differs from the above result.
(2). We list the results of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathfrak{g})$ for A_n $(n \leq 4)$ in the following. The reference we give is the paper [14] of Warner, in which the author discusses the irreducibility of $\mathbb{E}(r, \mathfrak{gl}_n)$ in Section 5.

Type	$\mathsf{rk}_p(\mathfrak{g})$	$\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathfrak{g})$
A_2	2	irreducible
A_3	4	irreducible
A_4	6	unknown

Table 2: small rank cases

2. Preliminaries

2.1. Parabolic system. We assume that G is a simple algebraic $\Bbbk$-group with irreducible root system Φ. The interested reader may consult [1], [2], [3], [13] for the theory of algebraic groups. Let U_α be the root subgroup corresponding to a root α, and $B = \langle U_\alpha, T \ ; \ \alpha \in \Phi^+\rangle$ be a Borel subgroup of G containing T. Initially, we study the Weyl group $\mathscr{W}$ in tandem with an irreducible root system Φ. Let $\Delta := \{\alpha_1, \ldots, \alpha_n\}$ be the set of positive simple roots, and I be a subset of Δ. We define

$$\Phi_I := \Phi \cap \sum_{\alpha \in I} \mathbb{Z}\alpha$$

to be the parabolic subsystem of roots, and

$$\mathscr{W}_I := \langle s_\alpha \ ; \ \alpha \in I\rangle$$

to be the standard parabolic subgroup of $\mathscr{W}$ (see [8] for details). Then subgroups of the form $P_I := B\mathscr{W}_I B = \langle T, U_\alpha \ ; \ \alpha \in \Phi^+ \cup \Phi_I\rangle$ are called standard parabolic subgroups of G. The Levi decomposition $P_I = L_I \ltimes R_u(P_I)$ decomposes P_I into a semi-direct product of its Levi factor L_I and the unipotent radical $R_u(P_I)$, with the latter being generated by root subgroups $\{U_\alpha \ ; \ \alpha \in \Phi^+ \setminus \Phi_I^+\}$. Influenced by this, we set $S := \Delta \setminus I$ and then define

$$\Phi_S^{\mathrm{rad}} = \Phi^+ \setminus \Phi_I^+$$

to be the set of positive roots that cannot be written as a linear combination of the simple roots not in S. If $S = \{\alpha_i\}$, then we simply write Φ_i^{rad} instead of $\Phi_{\{\alpha_i\}}^{\mathrm{rad}}$.

2.2. Maximal subsets for type A. Suppose that G is of type A_n. The roots of A_n are the integer vectors in $\mathbb{R}^{n+1}$ of length $\sqrt{2}$ for which the coordinates sum to 0. Let $\{\epsilon_i \ ; \ 1 \le i \le n+1\}$ be the standard basis of $\mathbb{R}^{n+1}$. We denote by

$$\Phi = \{\epsilon_i - \epsilon_j \ ; \ i \neq j, 1 \le i, j \le n+1\}$$

the corresponding set of roots, and by $\Delta = \{\alpha_1, \dots, \alpha_n\}$ the base of Φ where $\alpha_i = \epsilon_i - \epsilon_{i+1}$. There is a bijection ϕ from the set of non-trivial proper subsets of $\{1, \dots, n+1\}$ to the set of maximal subsets of commuting roots of Φ by sending J to $\phi(J) := \{\epsilon_i - \epsilon_j \ ; \ i \in J, j \notin J\}$, and the condition $J < \{1, \dots, n+1\} \setminus J$ on J gives rise to a maximal subset of commuting positive roots; see [12, A.1].

Definition 2.1. Type A_n:

$$\begin{aligned}
n &= 2m+1, & \Phi^{\mathrm{odd}}_{m+1,m+2} &:= \phi(J) \cap \Phi^+, \text{ for } J = \{1, \dots, m, m+2\}; \\
n &= 2m, & \Phi^{\mathrm{ev}}_{m+1,m+2} &:= \phi(J) \cap \Phi^+, \text{ for } J = \{1, \dots, m, m+2\}; \\
n &= 2m, & \Phi^{\mathrm{ev}}_{m,m+1} &:= \phi(J) \cap \Phi^+, \text{ for } J = \{1, \dots, m-1, m+1\}.
\end{aligned}$$

Theorem 2.2. *Keep the notations as above and set* $\mathfrak{g} := \mathrm{Lie}(G)$. *Then the elements of the set* $\mathrm{Max}_{\mathsf{rk}_p(\mathfrak{g})-1}(\Phi)$ *are given as follows:*

Type	Restrictions on rank	$\mathrm{Max}_{\mathsf{rk}_p(\mathfrak{g})-1}(\Phi)$
A_{2m+1}	$m \ge 0$	Φ_m^{rad}, $\Phi_{m+2}^{\mathrm{rad}}$, $\Phi^{\mathrm{odd}}_{m+1,m+2}$
A_{2m}	$m \ge 1$	$\Phi^{\mathrm{ev}}_{m+1,m+2}$, $\Phi^{\mathrm{ev}}_{m,m+1}$

Table 3: Maximal subset: order $\mathsf{rk}_p(\mathfrak{g}) - 1$

Proof. Let $M(A) \in \mathrm{Max}_{\mathsf{rk}_p(\mathfrak{g})-1}(\Phi)$. Notice that the maximal dimension $\mathsf{rk}_p(\mathfrak{g})$ is $(m+1)^2$ (resp. $m(m+1)$) when $n = 2m+1$ (resp. $n = 2m$). If $M(A)$ is still maximal in Φ, then $M(A) = \phi(J)$ for certain J. By letting $|M(A)| = |\phi(J)| = |J|(n+1-|J|)$ equal $(m+1)^2 - 1$ when $n = 2m+1$ and equal $m(m+1) - 1$ when $n = 2m$, we get $|J| = m, m+2$ for $n = 2m+1$, and there is no solution for $n = 2m$. Continuing the consideration, if $|J| = m$ or $m+2$ then $M(A)$ has to be Φ_m^{rad} or $\Phi_{m+2}^{\mathrm{rad}}$ respectively.

Alternatively, $M(A)$ is maximal in Φ^+ but not in Φ. Then $M(A) \subset \phi(J)$ for some J with

$$|\phi(J)| = \mathsf{rk}_p(\mathfrak{g}) \text{ and } |\phi(J) \cap \Phi^+| = \mathsf{rk}_p(\mathfrak{g}) - 1.$$

One gets $M(A)$ equals $\Phi^{\mathrm{odd}}_{m+1,m+2}$ when $n = 2m+1$, and equals $\Phi^{\mathrm{ev}}_{m+1,m+2}$ or $\Phi^{\mathrm{ev}}_{m,m+1}$ when $n = 2m$. ■

Remark 2.3. We recall the set $\mathrm{Max}(\Phi)$ for type A_n, which is calculated by Malcev in [7]:

Type	Restrictions on rank	$\mathrm{Max}(\Phi)$
A_{2m+1}	$m \geq 0$	$\Phi^{\mathrm{rad}}_{m+1}$
A_{2m}	$m \geq 1$	$\Phi^{\mathrm{rad}}_{m+1}$, Φ^{rad}_{m}

Table 4: Maximal subset: order $\mathsf{rk}_p(\mathfrak{g})$

3. Main result

Now we concentrate on G being a standard connected simple algebraic $\Bbbk$-group of type A_n with $\mathfrak{g} := \mathrm{Lie}(G)$. Let Φ be the root system of G with positive roots Φ^+. Since p is a good prime for G, we have $[x_\alpha, x_\beta] = 0$ if and only if $\alpha + \beta \notin \Phi$ for $\alpha, \beta \in \Phi$ and their associated root vectors x_α, x_β. Recall that $x_\alpha^{[p]} = 0$ for $\alpha \in \Phi$, one does have an elementary subalgebra $\mathrm{Lie}(R) := \mathrm{Span}_\Bbbk\{x_\alpha \ ; \ \alpha \in R\}$ when R is a subset of commuting roots. Let $\mathfrak{u}$ be the Lie algebra of the unipotent radical. We will show the map

$$\mathrm{Lie}: \ \mathrm{Max}_{\mathsf{rk}_p(\mathfrak{g})-1}(\Phi) \longrightarrow \mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathfrak{u})_{\max}, \quad R \mapsto \mathrm{Lie}(R)$$

is surjective up to conjugacy by G. This will be done by employing the map (cf. [12, (3.1.2)])

$$\mathrm{LT}: \mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathfrak{u})_{\max} \longrightarrow \mathrm{Max}_{\mathsf{rk}_p(\mathfrak{g})-1}(\Phi)$$

according to the chosen total ordering.

3.1. Total ordering for map LT**.** Suppose that G is of type A_{2m+1}. We fix the total ordering $\succeq$ by letting it be the reverse lexicographic ordering given by $\alpha_{m+1} \prec \alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{2m+1}$. We first show that the map LT is well-defined under such setting for A_{2m+1}.

Lemma 3.1. *Suppose G is of type A_{2m+1} $(m \geq 1)$. If $\mathfrak{e} \in \mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1, \mathfrak{u})_{\max}$, then $\mathrm{LT}(\mathfrak{e}) \in \mathrm{Max}_{\mathsf{rk}_p(\mathfrak{g})-1}(\Phi)$ with respect to $\succeq$.*

Proof. Assume that $\mathrm{LT}(\mathfrak{e}) \notin \mathrm{Max}_{\mathsf{rk}_p(\mathfrak{g})-1}(\Phi)$, then $\mathrm{LT}(\mathfrak{e}) \subsetneq \Phi^{\mathrm{rad}}_{m+1}$ by Table 2.3. Since $\Phi^+ \setminus \Phi^{\mathrm{rad}}_{m+1} \succ \Phi^{\mathrm{rad}}_{m+1}$, it implies that all terms of basis vectors correspond to the roots lying in $\Phi^{\mathrm{rad}}_{m+1}$. As a result, $\mathfrak{e}$ is contained in the elementary subalgebra $\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+1})$. Notice that $\dim \mathfrak{e} < \dim \mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+1})$, the containment is proper which contradicts maximality. ■

Now we consider the $\Bbbk$-group G which is of type A_{2m}. We choose the total ordering $\succeq$ to be the reverse lexicographic ordering given by $\alpha_{m+1} \prec \alpha_m \prec \alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{2m}$. According to this choice, one can easily check that

$$\Phi^+ \setminus (\Phi^{\mathrm{rad}}_m \cup \Phi^{\mathrm{rad}}_{m+1}) \succ \Phi^{\mathrm{rad}}_m \setminus \Phi^{\mathrm{rad}}_{m+1} \succ \Phi^{\mathrm{rad}}_{m+1} \setminus \Phi^{\mathrm{rad}}_m \succ \Phi^{\mathrm{rad}}_m \cap \Phi^{\mathrm{rad}}_{m+1}.$$

Lemma 3.2. *Suppose G is of type A_{2m} with $m \geq 3$. If $\mathfrak{e} \in \mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1, \mathfrak{u})_{\max}$, then $\mathrm{LT}(\mathfrak{e}) \in \mathrm{Max}_{\mathsf{rk}_p(\mathfrak{g})-1}(\Phi)$ with respect to $\succeq$.*

Proof. If $\mathrm{LT}(\mathfrak{e}) \notin \mathrm{Max}_{\mathsf{rk}_p(\mathfrak{g})-1}(\Phi)$, then either $\mathrm{LT}(\mathfrak{e}) \subsetneq \Phi^{\mathrm{rad}}_m$, or $\mathrm{LT}(\mathfrak{e}) \subsetneq \Phi^{\mathrm{rad}}_{m+1}$ according to Table 2.3.

• Case 1. $\mathrm{LT}(\mathfrak{e}) \subsetneq \Phi^{\mathrm{rad}}_{m+1}$. Then $\Phi^{\mathrm{rad}}_{m+1} \setminus \mathrm{LT}(\mathfrak{e}) = \{\epsilon_u - \epsilon_v\}$ for some (u, v). Notice that $\Phi^+ \setminus \Phi^{\mathrm{rad}}_{m+1} \succ \Phi^{\mathrm{rad}}_{m+1}$, thus the reduced echelon form basis of $\mathfrak{e}$ is as follows:

$$x_{\epsilon_i - \epsilon_j} + a_{ij} x_{\epsilon_u - \epsilon_v}, \quad a_{ij} = 0 \text{ if } i < u \text{ or } i = u, j > v,$$

for $1 \leq i \leq m+1, m+2 \leq j \leq 2m+1$ and $(i,j) \neq (u,v)$. Then it is readily seen that $\mathfrak{e} \subsetneq \mathfrak{e} \oplus \Bbbk x_{\epsilon_u - \epsilon_v}$, and the maximality of $\mathfrak{e}$ leads to a contradiction.

• Case 2. $\Phi^{\mathrm{rad}}_m \setminus \mathrm{LT}(\mathfrak{e}) = \{\epsilon_u - \epsilon_v\} \subset \Phi^{\mathrm{rad}}_m \cap \Phi^{\mathrm{rad}}_{m+1}$. Then the reduced basis of $\mathfrak{e}$ consists of the following elements for $1 \leq i \leq m, m+2 \leq j \leq 2m+1$ and $(i,j) \neq (u,v)$:

$$x_{ij} = x_{\epsilon_i - \epsilon_j} + a_{ij} x_{\epsilon_u - \epsilon_v}, \quad a_{ij} = 0 \text{ if } i < u \text{ or } i = u, j > v,$$
$$y_i = x_{\epsilon_i - \epsilon_{m+1}} + \sum_{s=m+2}^{2m+1} b_{is} x_{\epsilon_{m+1} - \epsilon_s} + d_i x_{\epsilon_u - \epsilon_v}.$$

Now we compute

$$[y_i, y_{i'}] = \sum_{s=m+2}^{2m+1} b_{i's} N_{\epsilon_i - \epsilon_{m+1}, \epsilon_{m+1} - \epsilon_s} x_{\epsilon_i - \epsilon_s} + \sum_{s=m+2}^{2m+1} b_{is} N_{\epsilon_{m+1} - \epsilon_s, \epsilon_{i'} - \epsilon_{m+1}} x_{\epsilon_{i'} - \epsilon_s}.$$

As $m \geq 3$, we may take $i \neq i'$, this gives $b_{is} = 0$ for all i and s. As a result, we will have $\mathfrak{e} \subsetneq \mathfrak{e} \oplus \Bbbk x_{\epsilon_u - \epsilon_v}$, a contradiction.

• Case 3. $\Phi_m^{\mathrm{rad}} \setminus \mathrm{LT}(\mathfrak{e}) = \{\epsilon_u - \epsilon_{m+1}\} \subseteq \Phi_m^{\mathrm{rad}} \setminus \Phi_{m+1}^{\mathrm{rad}}$. Then the reduced echelon form basis of $\mathfrak{e}$ is $x_{\epsilon_i - \epsilon_j}$ for $1 \leq i \leq m$ and $m+2 \leq j \leq 2m+1$ together with

$$y_i = x_{\epsilon_i - \epsilon_{m+1}} + q_i x_{\epsilon_u - \epsilon_{m+1}} + \sum_{s=m+2}^{2m+1} d_{is} x_{\epsilon_{m+1} - \epsilon_s}, \quad q_i = 0 \text{ if } i < u,$$

for $1 \leq i \leq m$ and $i \neq u$. If $i, i^{'}$ are distinct and different from u (which is possible as $m \geq 3$), then the coefficient of $x_{\epsilon_{i^{'}} - \epsilon_s}$ in $[y_i, y_{i^{'}}]$ is $N_{\epsilon_{m+1} - \epsilon_s, \epsilon_{i^{'}} - \epsilon_{m+1}} d_{is}$, so $d_{is} = 0$ for all i and s. Thus $\mathfrak{e} \subsetneq \mathfrak{e} \oplus \Bbbk x_{\epsilon_u - \epsilon_{m+1}}$, a contradiction, and we finish the proof. ■

3.2. Surjectivity for map Lie.

Theorem 3.3. *Suppose G is of type A_{2m+1} with $m \geq 2$. If $\mathfrak{e} \in \mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathfrak{u})$ satisfies* $\mathrm{LT}(\mathfrak{e}) = \Phi_m^{\mathrm{rad}}, \Phi_{m+2}^{\mathrm{rad}}$, *or* $\Phi_{m+1,m+2}^{\mathrm{odd}}$, *then* $\mathfrak{e} = \mathrm{Lie}(\Phi_m^{\mathrm{rad}}), \mathrm{Lie}(\Phi_{m+2}^{\mathrm{rad}})$, *or* $\mathrm{Lie}(\Phi_{m+1,m+2}^{\mathrm{odd}})^{\exp(\mathrm{ad}(a x_{\alpha_{m+1}}))}$ *for some a, respectively.*

Proof. • Case 1. $\mathrm{LT}(\mathfrak{e}) = \Phi_m^{\mathrm{rad}}$. We write the reduced echelon form basis for $\mathfrak{e}$:

$$x_{ij} = x_{\epsilon_i - \epsilon_j}, \quad 1 \leq i \leq m \text{ and } m+2 \leq j \leq 2m+2,$$
$$y_i = x_{\epsilon_i - \epsilon_{m+1}} + \sum_{s=1}^{i-1} \sum_{t=s+1}^{m} a_{ist} x_{\epsilon_s - \epsilon_t} + \sum_{r=m+2}^{2m+2} b_{ir} x_{\epsilon_{m+1} - \epsilon_r}, \quad 1 \leq i \leq m.$$

Let $1 \leq i \leq m$ and $2 \leq t \leq m$, the coefficient of $x_{\epsilon_s - \epsilon_j}$ in $[y_i, x_{tj}]$ is $a_{ist} N_{\epsilon_s - \epsilon_t, \epsilon_t - \epsilon_j}$, this gives $a_{ist} = 0$ for all i, s and t. If $i, j \leq m$ are distinct, then the coefficient of $x_{\epsilon_j - \epsilon_r}$ in $[y_i, y_j]$ is $b_{ir} N_{\epsilon_{m+1} - \epsilon_r, \epsilon_j - \epsilon_{m+1}}$. As $m \geq 2$, this gives all $b_{ir} = 0$. Therefore, we have $\mathfrak{e} = \mathrm{Lie}(\Phi_m^{\mathrm{rad}})$.

• Case 2. $\mathrm{LT}(\mathfrak{e}) = \Phi_{m+2}^{\mathrm{rad}}$. The reduced echelon form basis of $\mathfrak{e}$ is of the form

$$x_{ij} = x_{\epsilon_i - \epsilon_j} + \sum_{s=1}^{i-1} a_{ijs} x_{\epsilon_s - \epsilon_{m+2}}, \quad 1 \leq i \leq m+1 \text{ and } m+3 \leq j \leq 2m+2,$$
$$y_j = x_{\epsilon_{m+2} - \epsilon_j} + \sum_{s=1}^{m+1} \sum_{t=s+1}^{m+2} b_{jst} x_{\epsilon_s - \epsilon_t}, \quad m+3 \leq j \leq 2m+2.$$

Let $m+3 \leq j, j^{'} \leq 2m+2$ and $2 \leq t \leq m+1$. If j and $j^{'}$ are distinct, then the coefficient of $x_{\epsilon_s - \epsilon_{j^{'}}}$ in $[y_j, x_{tj^{'}}]$ is $b_{jst} N_{\epsilon_s - \epsilon_t, \epsilon_t - \epsilon_{j^{'}}}$, it gives $b_{jst} = 0$ for all j, s and $t < m+2$ as $m \geq 2$. Then the coefficient of $x_{\epsilon_s - \epsilon_j}$ in $[y_j, x_{ij^{'}}]$ is $a_{ij^{'}s} N_{\epsilon_{m+2} - \epsilon_j, \epsilon_s - \epsilon_{m+2}}$, this implies $a_{ijs} = 0$ for all i, j, s. It remains to consider $b_{js(m+2)}$. If $m+3 \leq i, j \leq 2m+2$ are distinct, then the coefficient of $x_{\epsilon_s - \epsilon_i}$ in $[y_i, y_j]$ is $b_{js(m+2)} N_{\epsilon_{m+2} - \epsilon_i, \epsilon_s - \epsilon_{m+2}}$. According to this together with $m \geq 2$, we get $b_{js(m+2)} = 0$ for all j and s. Therefore, we have $\mathfrak{e} = \mathrm{Lie}(\Phi_{m+2}^{\mathrm{rad}})$.

• Case 3. $\mathrm{LT}(\mathfrak{e}) = \Phi^{\mathrm{odd}}_{m+1,m+2}$. The reduced echelon form basis of $\mathfrak{e}$ consists of

$$
\begin{aligned}
x_{ij} &= x_{\epsilon_i-\epsilon_j} + \sum_{s=1}^{i-1} a_{ijs} x_{\epsilon_s-\epsilon_{m+2}},\\
y_i &= x_{\epsilon_i-\epsilon_{m+1}} + \sum_{s=1}^{i-1}\sum_{t=s+1}^{m} b_{ist} x_{\epsilon_s-\epsilon_t} + \sum_{r=m+2}^{2m+2} d_{ir} x_{\epsilon_{m+1}-\epsilon_r} + \sum_{r=1}^{m} k_{ir} x_{\epsilon_r-\epsilon_{m+2}},\\
z_j &= x_{\epsilon_{m+2}-\epsilon_j} + \sum_{s=1}^{m-1}\sum_{t=s+1}^{m} h_{jst} x_{\epsilon_s-\epsilon_t} + \sum_{r=m+2}^{2m+2} \ell_{jr} x_{\epsilon_{m+1}-\epsilon_r} + \sum_{r=1}^{m} \xi_{jr} x_{\epsilon_r-\epsilon_{m+2}},
\end{aligned}
$$

where $1 \le i \le m$ and $m+3 \le j \le 2m+2$. By the same argument as before we deduce that $b_{ist} = h_{jst} = 0$ for all s and t. If $i, j \le m$ are distinct, then the coefficient of $x_{\epsilon_j-\epsilon_r}$ in $[y_i, y_j]$ is $d_{ir} N_{\epsilon_{m+1}-\epsilon_r,\epsilon_j-\epsilon_{m+1}}$. As $m \ge 2$, this gives $d_{ir} = 0$ and the argument can also be applied to z_j which ensures that $\xi_{jr} = 0$. Let $\lambda = -k_{11} N_{\epsilon_{m+1}-\epsilon_{m+2},\epsilon_1-\epsilon_{m+1}}$. Conjugation by $\exp(\mathrm{ad}(\lambda x_{\alpha_{m+1}}))$ to $\mathfrak{e}$ ensures that the image of y_1 has no term $x_{\epsilon_1-\epsilon_{m+2}}$, so we may assume $k_{11} = 0$. We compute the coefficient of $x_{\epsilon_1-\epsilon_r}$ in $[y_1, z_j]$ which is $\ell_{jr} N_{\epsilon_1-\epsilon_{m+1},\epsilon_{m+1}-\epsilon_r}$, giving $\ell_{jr} = 0$ for all j and r. Then the coefficient of $x_{\epsilon_r-\epsilon_j}$ in $[y_i, z_j]$ is $k_{ir} N_{\epsilon_r-\epsilon_{m+2},\epsilon_{m+2}-\epsilon_j}$, this gives $k_{ir} = 0$ for all i and r, and this also applies to $[x_{ij}, z_{j'}]$ from which we can get $a_{ijs} = 0$. As a result, we get $\mathfrak{e} = \mathrm{Lie}(\Phi^{\mathrm{odd}}_{m+1,m+2})^{\exp(\mathrm{ad}(a x_{\alpha_{m+1}}))}$ where $a = -\lambda$. ■

Theorem 3.4. *Suppose G is of type A_{2m} with $m \ge 3$. If $\mathfrak{e} \in \mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathfrak{u})$ satisfies $\mathrm{LT}(\mathfrak{e}) = \Phi^{\mathrm{ev}}_{m,m+1}$ or $\Phi^{\mathrm{ev}}_{m+1,m+2}$ then there exists some a such that $\mathfrak{e} = \mathrm{Lie}(\Phi^{\mathrm{ev}}_{m,m+1})^{\exp(\mathrm{ad}(a x_{\alpha_m}))}$ or $\mathrm{Lie}(\Phi^{\mathrm{ev}}_{m+1,m+2})^{\exp(\mathrm{ad}(a x_{\alpha_{m+1}}))}$ respectively.*

Proof. • Case 1. $\mathrm{LT}(\mathfrak{e}) = \Phi^{\mathrm{ev}}_{m,m+1}$. Then the reduced echelon form basis of $\mathfrak{e}$ is $x_{\epsilon_i-\epsilon_j}$ for $1 \le i \le m-1$ and $m+2 \le j \le 2m+1$, and

$$
\begin{aligned}
y_j &= x_{\epsilon_{m+1}-\epsilon_j} + \sum_{s=m+2}^{2m+1} a_{js} x_{\epsilon_m-\epsilon_s},\\
z_i &= x_{\epsilon_i-\epsilon_m} + \sum_{u=1}^{i-1}\sum_{v=u+1}^{m-1} b_{iuv} x_{\epsilon_u-\epsilon_v} + \sum_{s=1}^{m} c_{is} x_{\epsilon_s-\epsilon_{m+1}} + \sum_{s=m+2}^{2m+1} d_{is} x_{\epsilon_m-\epsilon_s},
\end{aligned}
$$

where $m+2 \le j \le 2m+1$ for y_j and $1 \le i < m$ for z_i. Let $\lambda = -a_{(m+2)(m+2)} N_{\epsilon_m-\epsilon_{m+1},\epsilon_{m+1}-\epsilon_{m+2}}$. Using conjugation given by $\exp(\mathrm{ad}(\lambda x_{\epsilon_m-\epsilon_{m+1}}))$ to $\mathfrak{e}$, we have explicitly

$$
\exp(\mathrm{ad}(\lambda x_{\epsilon_m-\epsilon_{m+1}}))(y_{m+2}) = x_{\epsilon_{m+1}-\epsilon_{m+2}} + \sum_{s=m+3}^{2m+1} a_{(m+2)s} x_{\epsilon_m-\epsilon_s},
$$

which allows us to assume $a_{(m+2)(m+2)} = 0$. Then we compute for $1 \le i < m$

$$
[y_{m+2}, z_i] = \sum_{s=1}^{m} c_{is} N_{\epsilon_{m+1}-\epsilon_{m+2},\epsilon_s-\epsilon_{m+1}} x_{\epsilon_s-\epsilon_{m+2}} + \sum_{s=m+3}^{2m+1} a_{(m+2)s} N_{\epsilon_m-\epsilon_s,\epsilon_i-\epsilon_m} x_{\epsilon_i-\epsilon_s}.
$$

Notice that these items $x_{\epsilon_s-\epsilon_{m+2}}$ and $x_{\epsilon_i-\epsilon_s}$ are different, so $c_{is}=0$ for all i and s and $a_{(m+2)s}=0$ for all s. Further we can get $a_{js}=0$ for all j and s by seeing $[y_j,z_1]=0$. When $1<v<m$, we compute the coefficient of $x_{\epsilon_u-\epsilon_{m+2}}$ in $[x_{\epsilon_v-\epsilon_{m+2}},z_i]$, that is $N_{\epsilon_v-\epsilon_{m+2},\epsilon_u-\epsilon_v}b_{iuv}$. This gives $b_{iuv}=0$ for all i,u and v. If $1\le i,i^{'}<m$ are distinct, then the coefficient of $x_{\epsilon_i-\epsilon_s}$ in $[z_i,z_{i'}]$ is $N_{\epsilon_i-\epsilon_m,\epsilon_m-\epsilon_s}d_{i's}$. As $m\ge 3$, this gives $d_{is}=0$ for all i,s, and consequently $\mathfrak{e}=\mathrm{Lie}(\Phi^{\mathrm{ev}}_{m,m+1})^{\exp(\mathrm{ad}(ax_{\alpha_m}))}$ for $a=-\lambda$.

• Case 2. $\mathrm{LT}(\mathfrak{e})=\Phi^{\mathrm{ev}}_{m+1,m+2}$. We write the reduced basis with $1\le i\le m$ and $m+3\le j\le 2m+1$:

$$x_{ij}=x_{\epsilon_i-\epsilon_j}+\sum_{t=1}^{i-1}a_{ijt}x_{\epsilon_t-\epsilon_{m+2}},$$

$$y_i=x_{\epsilon_i-\epsilon_{m+1}}+\sum_{s=m+2}^{2m+1}b_{is}x_{\epsilon_{m+1}-\epsilon_s}+\sum_{s=1}^{m}c_{is}x_{\epsilon_s-\epsilon_{m+2}},$$

$$z_j=x_{\epsilon_{m+2}-\epsilon_j}+\sum_{u=1}^{m-1}\sum_{v=u+1}^{m}d_{juv}x_{\epsilon_u-\epsilon_v}+\sum_{s=m+2}^{2m+1}f_{js}x_{\epsilon_{m+1}-\epsilon_s}+\sum_{s=1}^{m}k_{js}x_{\epsilon_s-\epsilon_{m+2}}.$$

If $m+3\le j,j^{'}\le 2m+1$ and $j\ne j^{'}$, choose $1<v<m+1$, then we compute

$$[x_{vj},z_{j'}]=\sum_{u=1}^{v-1}d_{j'uv}N_{\epsilon_v-\epsilon_j,\epsilon_u-\epsilon_v}x_{\epsilon_u-\epsilon_j}+\sum_{t=1}^{i-1}a_{vjt}N_{\epsilon_t-\epsilon_{m+2},\epsilon_{m+2}-\epsilon_{j'}}x_{\epsilon_t-\epsilon_{j'}}.$$

As $m\ge 3$, this gives $d_{juv}=0$ and consequently $a_{ijt}=0$ by seeing the coefficient of $x_{\epsilon_t-\epsilon_j}$ in $[x_{ij},z_j]$. If $1\le i,i^{'}\le m$ and $i\ne i^{'}$, then the coefficient of $x_{\epsilon_i-\epsilon_s}$ in $[y_i,y_{i'}]$ is $N_{\epsilon_i-\epsilon_{m+1},\epsilon_{m+1}-\epsilon_s}b_{i's}$, so $b_{is}=0$ for all i and s. Now let $\xi=-c_{11}N_{\epsilon_{m+1}-\epsilon_{m+2},\epsilon_1-\epsilon_{m+1}}$, conjugation given by $\exp(\mathrm{ad}(\xi x_{\epsilon_{m+1}-\epsilon_{m+2}}))$ lets us assume that $c_{11}=0$. Then we compute

$$[y_1,z_j]=\sum_{s=m+2}^{2m+1}f_{js}N_{\epsilon_1-\epsilon_{m+1},\epsilon_{m+1}-\epsilon_s}x_{\epsilon_1-\epsilon_s}+\sum_{s=2}^{m}c_{1s}N_{\epsilon_s-\epsilon_{m+2},\epsilon_{m+2}-\epsilon_j}x_{\epsilon_s-\epsilon_j}.$$

It follows that c_{1s} and f_{js} are zero. Further $c_{is}=0$ for all i and s by computing $[y_i,z_1]$. Finally, the coefficient of $x_{\epsilon_s-\epsilon_i}$ in $[z_i,z_j]$ for $i\ne j$ is $N_{\epsilon_{m+2}-\epsilon_i,\epsilon_s-\epsilon_{m+2}}k_{js}$, so $k_{js}=0$ for all j and s. Now we have $\mathfrak{e}=\mathrm{Lie}(\Phi^{\mathrm{ev}}_{m+1,m+2})^{\exp(\mathrm{ad}(ax_{\alpha_{m+1}}))}$ for $a=-\xi$ and complete the proof. ■

3.3. Irreducible components.

Definition 3.5. ([12, Definition 2.10]) We say $R\subset\Phi^+$ is an *ideal* if $\alpha+\beta\in R$ whenever $\alpha\in R,\beta\in\Phi^+$ and $\alpha+\beta\in\Phi^+$.

Lemma 3.6. *Suppose that G is of type A_n with $n\ge 5$. Then*

$$\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{u})_{\max}\subseteq\bigcup_{R\text{ an ideal}}G.\mathrm{Lie}(R),$$

and the ideals occurring here for each type are listed in the third column of the following Table

Type	Restrictions on rank	Ideal R
A_{2m+1}	$m \geq 2$	$\Phi_m^{\text{rad}}, \Phi_{m+2}^{\text{rad}}, \Phi_{m+1}^{\text{rad}} \setminus \{\alpha_{m+1}\}$
A_{2m}	$m \geq 3$	$\Phi_m^{\text{rad}} \setminus \{\alpha_m\}, \Phi_{m+1}^{\text{rad}} \setminus \{\alpha_{m+1}\}$

Table 5: Ideals for Lemma 3.6

Proof. For type A_{2m+1}, Φ_m^{rad} and Φ_{m+2}^{rad} both are ideals, and $\Phi_{m+1,m+2}^{\text{odd}}$ can be conjugated to $\Phi_{m+1}^{\text{rad}} \setminus \{\alpha_{m+1}\}$ by a simple reflection s_{m+1}. For type A_{2m}, $\Phi_{m,m+1}^{\text{ev}}$ is conjugate to $\Phi_m^{\text{rad}} \setminus \{\alpha_m\}$ by s_m, and $\Phi_{m+1,m+2}^{\text{ev}}$ is conjugate to $\Phi_{m+1}^{\text{rad}} \setminus \{\alpha_{m+1}\}$ by s_{m+1}. Then it is a summarization of Theorem 3.3 and Theorem 3.4. ■

Corollary 3.7. *Let G be a standard simple algebraic $\Bbbk$-group with root system A_n $(n \geq 5)$. Then*

$$\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{g}) = \bigcup_{R \text{ an ideal}} G.\text{Lie}(R) \cup \bigcup_{I \text{ an ideal}} G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1, \text{Lie}(I)) \qquad (*)$$

is the union of irreducible closed subsets, where R is taken from Table 5 and I is given as follows:

Type	Restrictions on rank	Ideal I
A_{2m+1}	$m \geq 2$	Φ_{m+1}^{rad}
A_{2m}	$m \geq 3$	$\Phi_m^{\text{rad}}, \Phi_{m+1}^{\text{rad}}$

Table 6: Ideals for Corollary 3.7

Proof. Let R be an ideal in Table 4 and I be an ideal in Table 5. We define $X_1 := \text{Lie}(R), X_2 := \mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1, \text{Lie}(I))$ and $Y := \mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{g})$. Since X_2 is a projective variety, it is complete, implying that X_2 is closed in Y. Since R and I are ideals, it follows that X_i is stabilized by a parabolic subgroup of G for $i \in \{1,2\}$ respectively. By [12, Theorem 4.9] for X_1 and [6, Proposition 0.15] for X_2, we have

$$G.X_i \text{ is closed in } Y, \text{ where } i \in \{1,2\}.$$

Since Y is a G-variety, $G.X_1$ is irreducible as a G-orbit. Since Lie(I) is an elementary subalgebra of $\mathfrak{g}$, it follows that $X_2 = \text{Gr}_{\mathsf{rk}_p(\mathfrak{g})-1}(\text{Lie}(I))$ is the Grassmannian which is irreducible. Then $G.X_2$ as the image of X_2 under G is irreducible. As a result, the right hand of $(*)$ is the union of irreducible closed subsets.

By utilizing Lemma 3.6 along with [12, Sect. 3.2/3.4], we have

$$\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{u}) \subset \bigcup_{R \text{ an ideal}} G.\mathrm{Lie}(R) \cup \bigcup_{I \text{ an ideal}} G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathrm{Lie}(I)).$$

Therefore, we arrive at the equality of $(*)$ according to [11, Lemma 2.2]. ∎

Lemma 3.8. *Let $\mathfrak{e}$ be an element of $\mathbb{E}(r,\mathfrak{g})$ and R be an ideal of commuting roots with $|R|=r$. Assume that there is $g\in G$, satisfying*

$$g.\mathfrak{e} = \mathrm{Lie}(R).$$

Then $\mathrm{LT}(\mathfrak{e})$ and R are conjugate by an element of $\mathscr{W}$.

Proof. By Bruhat decomposition of G, there exist elements $b,b^{'}\in B$ and $w\in\mathscr{W}$ such that $g=b\dot{w}b^{'}$ where $\dot{w}$ is an element of $N_G(T)$ whose image in the Weyl group $\mathscr{W}$ is w. Since $g.\mathfrak{e}=\mathrm{Lie}(R)$, we have $\dot{w}b^{'}.\mathfrak{e}=b^{-1}.\mathrm{Lie}(R)$. Notice that R is an ideal, implying $B\subset \mathrm{Stab}_G(\mathrm{Lie}(R))$. Thus $\dot{w}b^{'}.\mathfrak{e}=\mathrm{Lie}(R)$ and

$$b^{'}.\mathfrak{e} = \dot{w}^{-1}.\mathrm{Lie}(R) = \mathrm{Lie}(w^{-1}.R). \qquad (**)$$

Observe that the action of U_α on $\mathfrak{e}$ is lower triangular with respect to $\succeq$ for $\alpha\in\Phi^+$. Then the equality $(**)$ gives $\mathrm{LT}(\mathfrak{e})=w^{-1}.R$, as desired. ∎

Theorem 3.9. *Let G be a standard simple algebraic $\mathbb{k}$-group with root system Φ of type A_n $(n\geq 5)$. Then the irreducible components of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathfrak{g})$ can be characterised; see Table 7.*

Type	Restrictions on rank	Irreducible components
A_{2m+1}	$m\geq 2$	$G.\mathrm{Lie}(\Phi^{\mathrm{rad}}_m)$, $G.\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+2})$, $G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+1}))$
A_{2m}	$m\geq 3$	$G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathrm{Lie}(\Phi^{\mathrm{rad}}_m))$, $G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+1}))$

Table 7: Irreducible components for Theorem 3.9

Proof. By Corollary 3.7, it suffices to check the maximality of each irreducible closed subset. Let R_v be an ideal of commuting roots of order $\mathsf{rk}_p(\mathfrak{g})-1$ for $v\in J:=\{1,2\}$. Let $I_v\in\mathrm{Max}(\Phi)$ be an ideal for $v\in J$. We will apply Lemma 3.8 to the following three cases for $\{u,v\}=J$:

(1) $G.\mathrm{Lie}(R_v)\subseteq G.\mathrm{Lie}(R_u)$;

(2) $G.\mathrm{Lie}(R_v)\subseteq G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathrm{Lie}(I_u))$;

(3) $G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathrm{Lie}(I_v))\subseteq G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g})-1,\mathrm{Lie}(I_u))$.

We conclude that R_u and R_v are $\mathscr{W}$-conjugate in (1). In (2), we have $\mathrm{Lie}(R_v) = g.\mathfrak{e}$ for some $g \in G$ and $\mathfrak{e} \in \mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(I_u))$. Therefore R_v and $\mathrm{LT}(\mathfrak{e})$ are conjugate by an element of $\mathscr{W}$. In (3), let γ be the unique positive simple root in I_v and $\mathfrak{e}$ be an element of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(I_u))$ such that $\mathrm{Lie}(I_v \setminus \{\gamma\}) = g.\mathfrak{e}$ for some $g \in G$. Then we have $I_v \setminus \{\gamma\}$ and $\mathrm{LT}(\mathfrak{e})$ are $\mathscr{W}$-conjugate.

Now we are in the position to classify the irreducible components for A_n $(n \geq 5)$:

- Type A_{2m+1}.

  (a) $G.\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+1} \setminus \{\alpha_{m+1}\})$ is not maximal because $\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+1} \setminus \{\alpha_{m+1}\})$ is an element of $\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+1}))$.

  (b) If $G.\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m}) \subseteq G.\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+2})$ or $G.\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m}) \subseteq G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+1}))$ then Φ^{rad}_{m} is $\mathscr{W}$-conjugate to $\Phi^{\mathrm{rad}}_{m+2}$ or conjugate to $\mathrm{LT}(\mathfrak{e}(\Phi^{\mathrm{rad}}_{m}))$. Both cases are impossible when we look at [12, Lemma 2.6], this gives $G.\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m})$ is maximal.

  (c) $G.\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+2})$ and $G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+1}))$ are maximal by the same argument of (b).

- Type A_{2m}.

  (a) $G.\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m} \setminus \{\alpha_m\})$ and $G.\mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+1} \setminus \{\alpha_{m+1}\})$ are not maximal since they are contained in $G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(R'))$ for $R' = \Phi^{\mathrm{rad}}_{m}, \Phi^{\mathrm{rad}}_{m+1}$ respectively.

  (b) We claim $G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi^{\mathrm{rad}}_{m}))$ and $G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+1}))$ are maximal. Without loss of generality, we may assume that

  $$G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi^{\mathrm{rad}}_{m})) \subseteq G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi^{\mathrm{rad}}_{m+1})).$$

  Then $\Phi^{\mathrm{rad}}_{m} \setminus \{\alpha_m\}$ is $\mathscr{W}$-conjugate to $\Phi^{\mathrm{rad}}_{m+1} \setminus \{\gamma\}$ where $\gamma = \alpha_1 + \cdots + \alpha_{2m}$ is the highest root, and consequently $\Phi^{\mathrm{rad}}_{m} \setminus \{\gamma\}$ and $\Phi^{\mathrm{rad}}_{m+1} \setminus \{\gamma\}$ are conjugate. Notice that the Weyl group of A_{2m} is the permutation group $\mathfrak{S}_{2m+1}$. Let $w.\Phi^{\mathrm{rad}}_{m} \setminus \{\gamma\} = \Phi^{\mathrm{rad}}_{m+1} \setminus \{\gamma\}$ for some $w \in \mathscr{W}$ and $m + 1 \leq j_0 < 2m + 1$. We denote by $w(j_0)$ the corresponding action for j_0 when w acts on $\{\epsilon_i - \epsilon_{j_0}\}_{1 \leq i < m+1} \subseteq \Phi^{\mathrm{rad}}_{m} \setminus \{\gamma\}$. Then

  $$w.\{\epsilon_i - \epsilon_{j_0}\}_{1 \leq i < m+1} \subseteq \left\{\epsilon_i - \epsilon_{w(j_0)}\right\}_{1 \leq i < m+2} \subseteq \Phi^{\mathrm{rad}}_{m+1} \setminus \{\gamma\}$$

  and $w.\{\epsilon_i - \epsilon_r\}_{1 \leq i < m+1} \nsubseteq \left\{\epsilon_i - \epsilon_{w(j_0)}\right\}_{1 \leq i < m+2}$ for $m + 1 \leq r < 2m + 1$ with $r \neq j_0$. As $m \geq 3$, there exists j_0 such that $w(j_0) \neq 2m + 1$. Then the equality $|\{\epsilon_i - \epsilon_{j_0}\}_{1 \leq i < m+1}| < |\left\{\epsilon_i - \epsilon_{w(j_0)}\right\}_{1 \leq i < m+2}|$ shows the impossibility. ∎

Remark 3.10. We would like to refer the reader to the CAU-thesis [9] for other classical types:

Type	Restrictions on rank	Irreducible components
B_n	$n \geq 5$	$G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(S_1))$
C_n	$n \geq 3$	$G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi_n^{\mathrm{rad}}))$
D_n	$n \geq 6$	$G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi_{n-1}^{\mathrm{rad}}))$, $G.\mathbb{E}(\mathsf{rk}_p(\mathfrak{g}) - 1, \mathrm{Lie}(\Phi_n^{\mathrm{rad}}))$

Table 8: Irreducible components for other classical types

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