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On Integrable Modules for the Twisted Full Toroidal Lie Algebra

Punita Batra and Senapathi Eswara Rao

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Abstract. The paper is to classify irreducible integrable modules for twisted full toroidal Lie algebras with some technical conditions. Twisted full toroidal Lie algebras are extensions of multiloop algebras twisted by several finite order automorphisms. This result genaralizes a result by Fu Jiayuan and Cuipo Jiang (Integrable representations for the twisted full toroidal Lie algebra, Journal of Algebra **307** (2007), 769–794) where they consider only one automorphism, and the result in Eswara Rao, S., and C. Jiang, Classifications of irreducible integrable representations for the full toroidal Lie algebra, Journal of Pure and Applied Algebra **200** (2005), 71–85.
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Introduction

The main purpose of this paper is to classify irreducible integrable modules for twisted full toroidal Lie algebra under certain technical conditions. Twisted full toroidal Lie algebra in several variables is defined using several commuting finite order automorphisms of the underlying finite dimensional simple Lie algebra $\mathfrak{g}$. The result of this paper generalizes the main theorem of [16], where they consider only one automorphism, and the result in [14].

Twisted full toroidal Lie algebra is a natural genaralization of the classical twisted affine Lie algebra. The classical procedure of realizing twisted affine Lie algebras using loop algebra in one variable proceeds in two steps [21]. In the first step, the derived algebra modulo its center of the affine Lie algebra is constructed as the algebra of a diagram automorphism of a finite dimensional simple Lie algebra. In the second step the affine Lie algebra is built from the graded loop algebra by forming universal central extension (one dimensional center) and adding graded algebra derivations (one dimensional).

The replacement of derived Lie algebra is multiloop algebra (Sec. 1.5) in several variables twisted by finitely many automorphisms. Then we consider the

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Universal central extension of the multiloop algebra (infinite dimensional) and add graded algebra of derivations (infinite dimensional) which we call twisted full toroidal Lie algebra and denoted by τ.

The classification of irreducible integrable modules for affine algebras with finite dimensional weight spaces is done in [7]. The twisted case has been done in [8] and [9]. Sevaral modules for full toroidal Lie algebras are constructed in [3] and [4]. The twisted case is done in [5]. The classification of irreducible integrable modules for full toroidal Lie algebras with finite dimensional weight spaces is done in [14].

In this paper we classify irreducible integrable modules for twisted full toroidal Lie algebra τ with finite dimensional weight spaces with non-zero central action and with some technical conditions that are satisfied for the well known Lie algebras called Lie Torus. See [1].

The contents of the paper are the following. We fix an irreducible integrable module for τ with non-zero central action.

The central operators act as scalars. Upto choice of co-ordinates (in other words upto an automorphism) we can assume that K_0 acts as $C_0 > 0$ and K_i acts trivially for $i \neq 0$. In the first main Theorem 5.2 we prove that V is a highest weight module with respect to some natural triangular decomposition

$$\tau = \tau^- \oplus \tau^0 \oplus \tau^+.$$

Let T be the highest weight space which is an irreducible τ_0-module(Proposition 6.1) and naturally $\mathbb{Z}^n$- graded. In Sections 7, 8 and 9, we describe T as τ_0-module. We noticed that some parts of τ_0 acts as scalars and hence we only consider a subalgebra L (See 7.3) for which T is irreducible. Here T and L are $\mathbb{Z}^n$-graded and T is a graded modules for L. We will now consider a subalgebra $\widetilde{L}$ of L (see 8.1) which is not graded. We further consider a finite dimensional quotient $\widetilde{V}$ of T and it is an $\widetilde{L}$-module. As $\widetilde{V}$ is a finite dimensional it is possible to describe $\widetilde{V}$ as $\widetilde{L}$-module. We then lift the module for L. Then we have a decent description of T as an L-module. This is our classification result (Theorem 9.6).

In the process of the proof we define an L-module $L(\widetilde{V})$ (See 8.3) and prove $L(\widetilde{V})$ naturally decomposes into L-modules (See notes after Proposition 8.4). Also see Lemma 8.5, 8.10 and Proposition 8.6. We prove one of the component is isomorphic to T. We also prove in Theorem 8.13 that $\widetilde{V}$ is irreducible $\widetilde{L}$-module if and only if $L(\widetilde{V})$ decomposes into mutually non-isomorphic irreducible L-modules. In this case T can be identified in a natural way as a component of $L(\widetilde{V})$. In the case $\widetilde{V}$ is reducible, we prove that $\widetilde{V}$ is completely reducible as $\widetilde{L}$ -module and all components are isomorphic (See Theorem 8.3). In this case also T is a submodule of $L(\widetilde{V})$ but the inclusion is twisted. In the rest of the paper we describe each component of $\widetilde{V}$. It turns out to be an irreducible module for $gl_n \oplus \overset{0}{\mathfrak{g}}$ (Theorem 9.4). See 7.5 for the Definition of $\overset{0}{\mathfrak{g}}$. We will indicate in 9.2, given an irreducible module for $gl_n \oplus \overset{0}{\mathfrak{g}}$, how to obtain a module for L and thereby for τ_0. In the Theorem 9.6 we will state our final result that the original module V is an irreducible quotient of an induced module of T.

1. Notation and Preliminaries

Throughout this paper we will use the following notation

(1.1) All vector spaces, algebras and tensor products are over complex numbers $\mathbb{C}$. Let $\mathbb{Z}, \mathbb{N}$ and $\mathbb{Z}_+$ denote integers, non-negative integers and positive integers.

(1.2) Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and let $(,)$ be a non-degenerate symmetric bilinear form on $\mathfrak{g}$. We fix a positive integer n. Let $\sigma_0, \sigma_1, \cdots, \sigma_n$ be commuting finite order automorphisms of $\mathfrak{g}$ of order $m_0, m_1, \cdots, m_n$ respectively. Let $m = (m_1, \cdots, m_n) \in \mathbb{Z}^n$. Let $k = (k_1, \cdots, k_n)$ and $l = (l_1, \cdots, l_n)$ denote vectors in $\mathbb{Z}^n$.

(1.3) Let $\Gamma = m_1\mathbb{Z} \oplus \cdots \oplus m_n\mathbb{Z}$ and $\Gamma_0 = m_0\mathbb{Z}$. Let $\Lambda = \mathbb{Z}^n/\Gamma$ and $\Lambda_0 = \mathbb{Z}/\Gamma_0$. Let $\overline{k}$ and $\bar{l}$ denote the images in Λ. For any integers k_0 and l_0, let $\overline{k}_0$ and $\bar{l}_0$ denote the images in Λ_0.
Let

$$\begin{aligned} A &= \mathbb{C}[t_0^{\pm 1}, \cdots, t_n^{\pm 1}], \\ A_n &= \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}], \\ A(m) &= \mathbb{C}[t_1^{\pm m_1}, \cdots, t_n^{\pm m_n}], \\ A(m_0, m) &= \mathbb{C}[t_0^{\pm m_0}, t_1^{\pm m_1}, \cdots, t_n^{\pm m_n}]. \end{aligned}$$

(1.4) For $k \in \mathbb{Z}^n$, let $t^k = t_1^{k_1} \cdots t_n^{k_n} \in A_n$. Let Ω_A be the vector space spanned by symbols $t_0^{k_0} t^k K_i, 0 \leq i \leq n, k_0 \in \mathbb{Z}, k \in \mathbb{Z}^n$. Let dA be the subspace spanned by $\sum_{i=0}^{n} k_i t_0^{k_0} t^k K_i$.

Let $L(\mathfrak{g}) = \mathfrak{g} \otimes A$ and notice that it has a natural structure of a Lie algebra. We will now define toroidal Lie algebra

$$\widetilde{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \Omega_A/dA.$$

Let $X(k_0, k) = X \otimes t_0^{k_0} t^k$ and $Y = Y \otimes t_0^{l_0} t^l$ for $X, Y \in \mathfrak{g}, k_0, l_0 \in \mathbb{Z}$ and $k, l \in \mathbb{Z}^n$.

1. $[X(k_0, k), Y(l_0, l)] = [X, Y](l_0 + k_0, l + k) + (X, Y) \sum k_i t_0^{l_0 + k_0} t^{l+k} K_i$.

2. Ω_A/dA is central in $\widetilde{L}(\mathfrak{g})$.

   It is well known that $\widetilde{L}(\mathfrak{g})$ is the universal central extension of $L(\mathfrak{g})$. See [15] and [22].

(1.5) We will now define multiloop algebra as a subalgebra of $L(\mathfrak{g})$. See [1] for more details. For $0 \le i \le n$, let ξ_i be a m_ith primitive root of unity.

Let
$$\mathfrak{g}(\overline{k}_0, \overline{k}) = \{x \in \mathfrak{g} | \sigma_i x = \xi_i^{k_i} x, 0 \le i \le n\}.$$

Then define
$$L(\mathfrak{g}, \sigma) = \bigoplus_{(k_o,k)\in\mathbb{Z}^{n+1}} \mathfrak{g}(\overline{k}_0, \overline{k}) \otimes t_0^{k_0} t^k,$$
which is called a multiloop algebra.

(1.6) The finite dimensional irreducible modules for $L(\mathfrak{g}, \sigma)$ are classified by Michael Lau [24].

(1.7) Suppose $\mathfrak{h}_1$ is a finite dimensional ad-diagonalizable subalgebra of a Lie algebra $\mathfrak{g}_1$. We set for $\alpha \in \mathfrak{h}_1^*$
$$\mathfrak{g}_{1,\alpha} = \{x \in \mathfrak{g}_1 | [h.x] = \alpha(h)x, h \in \mathfrak{h}_1\}.$$

Then we have
$$\mathfrak{g}_1 = \bigoplus_{\alpha \in \mathfrak{h}_1^*} \mathfrak{g}_{1,\alpha}.$$

Let $\Delta(\mathfrak{g}_1, \mathfrak{h}_1) = \{\alpha \in \mathfrak{h}_1^* | \mathfrak{g}_{1,\alpha} \neq 0\}$, which includes zero.

Let $\Delta^\times(\mathfrak{g}_1, \mathfrak{h}_1) = \Delta(\mathfrak{g}_1, \mathfrak{h}_1) \backslash \{0\}$.

(1.8) We will now define the universal central extension of $L(\mathfrak{g}, \sigma)$. Define $\Omega_A(m_0, m)$ and $dA(m_0, m)$ similar to the definition of Ω_A and dA by replacing A by $A(m_0, m)$. Denote $Z(m_0, m) = \Omega_A(m_0, m)/dA(m_0, m)$ and note that $Z(m_0, m) \subseteq \Omega_A/dA$.

Define
$$\widetilde{L}(\mathfrak{g}, \sigma) = L(\mathfrak{g}, \sigma) \oplus Z(m_0, m).$$

Let $X \in \mathfrak{g}(\overline{k}_0, \overline{k})$ and $Y \in \mathfrak{g}(\overline{l}_0, \overline{l})$ and let $X(k_0, k) = X \otimes t_0^{k_0} t^k$ and $Y(l_0, l) = Y \otimes t_0^{l_0} t^l$.

Define

1. $[X(k_0, k), Y(l_0, l)] = [X, Y](k_0 + l_0, k + l) + (X, Y) \sum k_i t_0^{l_0+k_0} t^{l+k} K_i$.
2. $Z(m_0, m)$ is central.

Notice that $(X, Y) \neq 0 \Rightarrow k + l \in \Gamma$ and $k_0 + l_0 \in \Gamma_0$. This follows from the standard fact that $(,)$ is invariant under σ_i for $0 \le i \le n$. This proves that the above Lie bracket is well defined. This Lie bracket is nothing but the restriction defined in (1.4).

Proposition 1.1. *$\widetilde{L}(\mathfrak{g}, \sigma)$ is the universal central extension of $L(\mathfrak{g}, \sigma)$.*

See Corollary (3.27) of [20].

2. Derivation algebra of $A(m_0, m)$ and its extension to $Z(m_0, m)$.

(2.1) Let $D(m_0, m)$ be the derivation algebra of $A(m_0, m)$. ¿From now onwards we let s and r to be in Γ and s_0 and r_0 to be in Γ_0.

For $0 \leq i \leq n$, consider $t_0^{s_0} t^s t_i \frac{d}{dt_i}$ which acts on $A(m_0, m)$ as derivations. It is well known that $D(m_0, m)$ has the following basis

$$\left\{ t_0^{s_0} t^s t_i \frac{d}{dt_i} | 0 \leq i \leq n, s_o \in \Gamma_0, s \in \Gamma \right\}.$$

Let $d_i = t_i \frac{d}{dt_i}$ and it is easy to see that

$$[t_0^{s_0} t^s d_a, t_0^{r_0} t^r d_b] = r_a t_0^{r_0+s_0} t^{r+s} d_b - s_b t_0^{r_0+s_0} t^{r+s} d_a.$$

(2.2) $D(m_0, m)$ acts on $Z(m_0, m)$ in the following way

$$t_0^{s_0} t^s d_a.(t_0^{r_0} t^r K_b) = r_a t_0^{r_0+s_0} t^{r+s} K_b + \delta_{ab} \sum_{p=0}^{n} s_p t_0^{r_0+s_0} t^{r+s} K_p.$$

(2.3) We also consider two non-trivial 2-cocycles of $D(m_0, m)$ with values in $Z(m_0, m)$. See $[BB]$ for details

$$\begin{aligned} \varphi_1(t_0^{r_0} t^r d_a, t_0^{s_0} t^s d_b) &= -s_a r_b \sum_{p=0}^{n} r_p t_0^{r_0+s_0} t^{r+s} K_p, \\ \varphi_2(t_0^{r_0} t^r d_a, t_0^{s_0} t^s d_b) &= r_a s_b \sum_{p=0}^{n} r_p t_0^{r_0+s_0} t^{r+s} K_p. \end{aligned}$$

(2.4) Let φ be arbitrary linear combinations of φ_1 and φ_2. Then there is a corresponding Lie algebra

$$\tau = L(\mathfrak{g}, \sigma) \oplus Z(m_0, m) \oplus D(m_0, m)$$

The Lie brackets are defined in the following way in addition to 1.8(1) and 1.8(2).

(a) $[t_0^{r_0} t^r d_a, X(k_0, k)] = k_a X(k_0 + r_0, k + r)$,

(b) $[t_0^{r_0} t^r d_a, t^{s_0} t^s K_b] = s_a t_0^{r_0+s_0} t^{r+s} K_b + \delta_{ab} \sum_{p=0}^{n} r_p t_0^{r_0+s_0} t^{r+s} K_p$,

(c) $[t_0^{r_0} t^r d_a, t_0^{s_0} t^s d_b] = s_a t_0^{r_0+s_0} t^{r+s} d_b - r_b t_0^{r_0+s_0} t^{r+s} d_a + \varphi(t_0^{r_0} t^r d_a, t^{s_0} t^s d_b)$, where $r, s \in \Gamma, r_0, s_0 \in \Gamma_0, X \in \mathfrak{g}(\overline{k}_0, \overline{k})$.

3. Assumptions and automorphisms.

In this section we will make some assumptions on $L(\mathfrak{g}, \sigma)$ which will hold throughout this paper. We will also define a class of automorphisms on τ.

(3.1) Assumptions

(a) $\mathfrak{g}(\overline{0},\overline{0})$ is simple Lie algebra.

(b) We can choose Cartan subalgebra $\mathfrak{h}(0)$ and $\mathfrak{h}$ for $\mathfrak{g}(\overline{0},\overline{0})$ and $\mathfrak{g}$ such that $\mathfrak{h}(0)\subseteq\mathfrak{h}$. This will follow from Lemma A.1 in [6].

(c) It is known that $\Delta_0^\times=\Delta(\mathfrak{g}(\overline{0},\overline{0}),\mathfrak{h}(0))\backslash\{0\}$ is an irreducible reduced finite root system and has atmost two root lengths. Let $\Delta_{0,sh}^\times$ be the set of non-zero short roots. Define

$$\Delta_{0,en}^\times=\begin{cases}\Delta_0^\times\cup 2\Delta_{0,sh}^\times \text{ if } \Delta_0^\times \text{ is of type } B_l\\ \Delta_0^\times \text{ otherwise}\end{cases}$$

$\Delta_{0,en}=\Delta_{0,en}^\times\cup\{0\}$. We assume that $\Delta(\mathfrak{g},\mathfrak{h}(0))=\Delta_{0,en}$.

Remark 3.1. The assumptions are very strong and will not be true in genaral. For example $\mathfrak{g}(\overline{0},\overline{0})$ could be zero. But these assumptions are true for Lie tori. See Proposition 3.2.5 of [1]. It should be mentioned that Lie tori is very important class of Lie algebras and give rise to almost all Extended Affine Lie algebras [EALA]. EALA's are extensively studied. See [23], [1] and references there in.

(3.2) Change of co-ordinates

In this subsection we will define a class of automorphisms for the Lie algebra

$$\tau(1,\mathbf{1})=\mathfrak{g}\otimes A\oplus\Omega_A/d_A\oplus D(1,\mathbf{1}).$$

It is standard fact that $GL(n+1,\mathbb{Z})$ acts on $\mathbb{Z}^{n+1}$ and we denote the action by dot. Let $B=(b_{ij})\in GL(n+1,\mathbb{Z})$, then define automorphism, again denote by B on $\tau(1,\mathbf{1})$.

Let $t_0^{k_0}t^k=t(k_0,k)$, then define

$$\begin{aligned}B\cdot x\otimes t(k_0,k) &= x\otimes t^{B\cdot(k_0,k)},\\ B\cdot t(k_0,k)K_j &= \sum_{p=0}^{n} b_{pj}t^{B\cdot(k_0,k)}K_p,\\ B\cdot t(k_0,k)d_j &= \sum_{p=0}^{n} c_{pj}t^{B\cdot(k_0,k)}d_p,\end{aligned}$$

where $B^{-1}=(c_{pj})$.

The action of B on $\tau(1,1)$ is nothing but change of co-ordinates. Recall from (2.4) that the twisted full toroidal Lie algebra $\tau\subseteq\tau(1,1)$, we have $B\tau\cong\tau$. We will call this isomorphism "change of co-ordinates".

4. Root space decomposition and integrable modules for τ.

(4.1) First note the center of τ is spanned by $K_0, K_1, \cdots, K_n$.

Let $H = \mathfrak{h}(0) \oplus \sum \mathbb{C}K_i \oplus \sum \mathbb{C}d_i$ which is an abelian Lie algebra of τ and plays the role of Cartan subalgebra.

Define $\delta_i, w_i \in H^*(0 \leq i \leq n)$ be such that

$$\begin{aligned} w_i(\mathfrak{h}(0)) = 0, w_i(K_j) = \delta_{ij}, w_i(d_j) = 0, \\ \delta_i(\mathfrak{h}(0)) = 0, \delta_i(K_j) = 0, \delta_i(d_j) = \delta_{ij}. \end{aligned} \tag{4.1.1}$$

Let $\delta_k = \sum_{i=1}^{n} k_i \delta_i \ for \ k \in \mathbb{Z}^n$.

(4.1.2) Let $\mathfrak{g}(\overline{k}_0, \overline{k}, \alpha) = \{x \in \mathfrak{g}(\overline{k}_0, \overline{k}) | [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}(0)\}$ then τ has a root space decomposition.

(4.1.3) $\tau = \bigoplus_{\beta \in \Delta} \tau_\beta$ where $\Delta \subseteq \{\alpha + k_0\delta_0 + \delta_k, \alpha \in \Delta_{0,e_n}, k_0 \in \mathbb{Z}, k \in \mathbb{Z}^n\}$.

$$\begin{aligned} \tau_{\alpha+k_0\delta_0+\delta_k} &= \mathfrak{g}(\overline{k}_0, \overline{k}, \alpha) \otimes t_0^{k_0} t^k \quad \text{for} \quad \alpha \neq 0, \\ \tau_{k_0\delta_0+\delta_k} &= \mathfrak{g}(\overline{k}_0, \overline{k}, 0) \otimes t_0^{k_0} t^k \oplus \bigoplus_{i=0}^{n} \mathbb{C} t_0^{k_0} t^k K_i \oplus \bigoplus_{i=0}^{n} \mathbb{C} t_0^{k_0} t^k d_i. \end{aligned}$$

Notice that $\tau_0 = H$.

(4.1.4) Now we will define a non-degenerate bilinear form on H^*. For $\alpha \in \mathfrak{h}(0)^*$ extended α to H by $\alpha(K_i) = \alpha(d_i) = 0, 0 \leq i \leq n$.

Let $(\mathfrak{h}(0), K_i) = 0 = (\mathfrak{h}(0), d_i)$, $(\delta_k + \delta_{k_0}, \delta_l + \delta_{l_0}) = 0 = (w_i, w_j), (\delta_k, \delta_l) = 0$ $(\delta_i, w_j) = \delta_{ij}$. The form on $\mathfrak{h}(0)$ is the restriction of the form $(,)$ on $\mathfrak{g}$.

(4.1.5) For $\gamma = \alpha + k_0\delta_0 + \delta_k$ is called real root if $\alpha \neq 0$ which is equivalent to $(\gamma, \gamma) \neq 0$. Denote Δ^{re} be the set of real roots. For $\alpha \in \Delta_{0,en}$, denote $\alpha^\vee$ the co-root of α.

Define

$$\gamma^\vee = \alpha^\vee + \frac{2}{(\alpha, \alpha)} \sum_{i=0}^{n} k_i K_i$$

for γ real. Then $\gamma(\gamma^\vee) = \alpha(\alpha^\vee) = 2$. For γ real root, define reflection on H^* by

$$r_\gamma(\lambda) = \lambda - \lambda(\gamma^\vee)\gamma, \lambda \in H^*.$$

Let W be the Weyl group genarated by $r_\gamma, \gamma \in \Delta^{re}$.

Definition 4.1. A module V of τ is called integrable if

(a) $V = \bigoplus_{\lambda \in H^*} V_\lambda, V_\lambda = \{v \in V | hv = \lambda(h)v, h \in H\}, \dim V_\lambda < \infty$,

(b) $\mathfrak{g}(\overline{k}_0, \overline{k}, \alpha) \otimes t_0^{k_0} t^k$ acts locally nilpotently on V for $\alpha \neq 0$.

Let $P(V) = \{\gamma \in H^* | V_\gamma \neq 0\}$.

The following Lemma is very standard.

Lemma 4.2. *Suppose V is an irreducible integrable module for τ. Then*

(a) $P(V)$ is W - invariant.

(b) $\dim V_\gamma = \dim V_{w\gamma}$ *for all $w \in W$.*

(c) For $\alpha \in \Delta^{re}, \lambda \in P(V)$ we have $\lambda(\alpha^\vee) \in \mathbb{Z}$.

(d) For $\alpha \in \Delta^{re}, \lambda \in P(V)$. If $\lambda(\alpha^\vee) > 0$ then $\lambda - \alpha \in P(V)$.

(e) For $\lambda \in P(V), \lambda(K_i)$ is an integer independent of λ .

The purpose of this paper is to classify irreducible integrable modules for τ with non-zero central action.

For an irreducible integrable module with non zero central charge, we can assume that K_0 acts as $C_0 > 0$ and $K_i (i \neq 0)$ acts trivially upto a choice of co-ordinates. See (3.2).

5. Existence of highest weight

Throughout the rest of the paper we fix an irreducible integrable module for τ with K_0 acting as $C_0 > 0$ and $K_i (i \neq 0)$ acts trivially. Notice that for any $\lambda \in P(V), \lambda(K_i) = C_i = 0$ for $1 \leq i \leq n$ and $\lambda(K_0) = C_0$. Given a $\lambda \in H^*$ let λ' denote the restriction to $\mathfrak{h}(0)$. Given a λ' in $\mathfrak{h}^*(0)$, extend to H by $\lambda'(K_i) = \lambda(d_i) = 0$. Then λ can be uniquely written as

$$\lambda = \lambda' + \sum_{i=0}^{n} \lambda(d_i)\delta_i + \sum_{i=0}^{n} \lambda(K_i) w_i. \tag{5.1}$$

For $\lambda \in P(V)$

$$\lambda = \lambda' + \lambda(d_0)\delta_0 + \sum_{i=1}^{n} \lambda(d_i)\delta_i + \lambda(K_0) w_0$$

and note that $(\lambda, \delta_0) = \lambda(K_0) \neq 0$.

Put $\overline{\lambda} = \lambda' + \lambda(d_0)\delta_0 + \lambda(K_0)w_0$, so that $\lambda = \overline{\lambda} + \sum_{i=1}^{n} \lambda(d_i)\delta_i$. Let $\alpha_0 = -\beta_0 + \delta_0$ where β_0 is maximal root in $\Delta_{0,en}$. Note that α_0 may not be root of τ.

Let $\alpha_1, \alpha_2, \cdots, \alpha_p$ be a set of simple roots for $\Delta(\mathfrak{g}(\overline{0}, \overline{0}), \mathfrak{h}(0))$ and let $Q^+ = \bigoplus_{i=0}^{p} \mathbb{N}\alpha_i$. Define an ordering on $H^*, \lambda \leq \mu$ for $\lambda, \mu \in H^*$, if $\mu - \lambda \in Q^+$. Notice that in this case $\mu(d_i) = \lambda(d_i)$ for $1 \leq i \leq n$.

Proposition 5.1. *Assumptions as above. Given a* $\lambda \in P(V)$ *there exists* $\mu \in P(V)$ *such that* $\lambda \leq \mu$ *and* $\mu + \alpha \notin P(V)$ *for all* $\alpha > 0$ *where* μ *is dominant i.e.* $(\lambda, \alpha^\vee) \geq 0$ *for all* $\alpha > 0$*).*

Proof. The proof of the proposition follows from the results of section 6 of [13]. See also [12]. We will briefly sketch the proof. Suppose the proposition is false. Then the conclusion of Proposition 6.6 of [13] holds and it will lead to contradictions as explained in the proof of Theorem 6.1 of [13]. Also note that the proof in [13] is worked only in the case $n = 0$ but holds for any n. Here we need to observe in the construction of λ_i's in proposition 6.6 of [13], the $\delta_k, k \in \mathbb{Z}^n$ does not appear. In Proposition 5.1, μ is clearly dominant. ∎

Theorem 5.2. *Let* V *be an irreducible integrable module for* τ *with* K_0 *acting as* $C_0 > 0$ *and* $K_i (i \neq 0)$ *acting trivially. Then there exists* $\lambda \in P(V)$ *such that* $\lambda + \beta + \delta_k \notin P(V)$ *for any root* $\beta + \delta_k$ *with* $\beta > 0, k \in \mathbb{Z}^n$*.*

Proof. Suppose the Theorem is false. First some word about the notation. The $\eta(k), \eta'(k)$ and $\eta(g)$ that occur below are always linear combination of $\delta_1, \delta_2 \ldots \delta_n$. Fix $\lambda_1 \in P(V)$ and let $\eta(g) = \sum_{i=1}^{n} \lambda_1(d_i)\delta_i$. Note that $(\lambda_1, \delta_i) = 0$ for $1 \leq i \leq n$ which follows from the unique expression (5.1). Now by the Proposition 5.1 there exists $\mu_1 \in P(V)$ such that $\lambda_1 \leq \mu_1$ and $\mu_1 + \alpha \notin P(V)$ for $\alpha > 0$. By our assumption there exists a root $\beta_1 + \eta(1)$ with $\beta_1 > 0$ and $\eta(1) \neq 0$ such that

$$\lambda_2 = \mu_1 + \beta_1 + \eta(1) \in P(V)$$

Notice that $\lambda_2 = \overline{\lambda}_2 + \eta(g) + \eta(1)$ and $\overline{\lambda}_1 \leq \overline{\mu}_1 < \overline{\lambda}_2$. By repeating the argument infinitely many times we get dominant integral weights $\mu_b \in P(V), b \in \mathbb{Z}_+$ and roots $\beta_b + \eta(b)$ with $\beta_b > 0$ and $\eta(b) \neq 0$ such that

(a) $\lambda_{b+1} = \mu_b + \beta_b + \eta(b) \in P(V)$

(b) $\mu_b + \beta \notin P(V)$ for all $\beta > 0$

(c) $\lambda_b \leq \mu_b$

(d) $\overline{\lambda}_b \leq \overline{\mu}_b < \overline{\lambda}_{b+1} \leq \overline{\mu}_{b+1}$. It is easy to see that

(e) $\mu_b = \overline{\mu}_b + \eta(g) + \eta'(b)$ where $\eta'(b) = \sum_{j=1}^{b} \eta(j)$

$\overline{\mu}_{b_1} < \overline{\mu}_{b_2}$ for $b_1 < b_2$. ∎

Claim 1. $(\mu_b + \beta_b, \beta_b) > 0$. Since μ_b is dominant integral we have $(\mu_b, \beta_b) \geq 0$. Also we have $(\beta_b, \beta_b) \geq 0$. Suppose $(\beta_b, \beta_b) > 0$ we are done. So suppose $(\beta_b, \beta_b) = 0$. Then it is easy checking that $\beta_b = P\delta_0$ for some $P \in \mathbb{Z}_+$. Thus $(\mu_b, \beta_b) = P\mu_b(K_0) > 0$ see (5.1). Thus claim 1 follows.

Claim 2. $\mu_{b_1} < \mu_{b_2} - \eta'(b_2) + \eta'(b_1)$ for $b_1 < b_2$. To see the claim 2, note that $\mu_{b_1} - \eta(g) - \eta'(b_1) < \mu_{b_2} - \eta(g) - \eta'(b_2)$ which follows from (e). The inequality still holds if we add $\eta(g) + \eta'(b_1)$ both sides and hence claim 2 follows.

Claim 3. There exists $b_1 < b_2$ such that $\eta'(b_2) - \eta'(b_1) \in \Gamma$. To see the claim consider the infinite set $\{\eta'(b), b \in \mathbb{Z}\}$ and its image in the finite set $\mathbb{Z}^n/\Gamma$. Clearly there exists b_1, b_2 such that $\eta'(b_2) - \eta'(b_1) = 0$ in $\mathbb{Z}^r/\Gamma$. Now the claim is obvious and can assume $b_1 < b_2$.

We know that $\beta_{b_2} + \eta(b_2) \in \Delta$ and from definitions it follows that

$$\beta_{b_2} + \eta(b_2) + \eta'(b_2) - \eta'(b_1) \in \Delta$$

Now from claim 1 and (5.1) it follows that

$$(\mu_{b_2} + \beta_{b_2} + \eta(b_2), \beta_{b_2} + \eta(b_2) + \eta'(b_2) - \eta'(b_1) > 0$$

Now from Lemma 4.2(b) it follows that

$$\mu' = \mu_{b_2} + \eta'(b_1) - \eta'(b_2) \in P(V)$$

But by claim 2 we have $\mu_{b_1} < \mu'$ which means $\mu' = \mu_{b_1} + \beta, \beta > 0$ a contradiction to (b). This completes the proof of Theorem 5.2.

6. Triangle decomposition

(6.1) We will now define triangular decomposition for τ. Let $Z = \Omega A/dA$. Let

$$\begin{aligned}
L^+(\mathfrak{g}, \sigma) &= \bigoplus_{\alpha + k_0\delta_0 > 0, k \in \mathbb{Z}^n} \mathfrak{g}(\overline{k}_0, \overline{k}, \alpha) \otimes t_0^{k_0} t^k; \\
L^-(\mathfrak{g}, \sigma) &= \bigoplus_{\alpha + k_0\delta_0 < 0, k \in \mathbb{Z}^n} \mathfrak{g}(\overline{k}_0, \overline{k}, \alpha) \otimes t_0^{k_0} t^k; \\
L^0(\mathfrak{g}, \sigma) &= \bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(\overline{0}, \overline{k}, 0) t^k; \\
D^+(m_0, m) &= \bigoplus_{\substack{0 \le i \le n \\ s_0 > 0, s \in \Gamma}} \mathbb{C} t_0^{s_0} t^s d_i; \\
D^-(m_0, m) &= \bigoplus_{\substack{0 \le i \le n \\ s_0 < 0, s \in \Gamma}} \mathbb{C} t_0^{s_0} t^s d_i; \\
D^0(m_0, m) &= \bigoplus_{0 \le i \le n, s \in \Gamma} \mathbb{C} t^s d_i;
\end{aligned}$$

$$
\begin{aligned}
Z^+ &= \bigoplus_{\substack{0\le i\le n\\ s_0>0, s\in\Gamma}} \mathbb{C}t_0^{s_0}t^sK_i;\\
Z^- &= \bigoplus_{\substack{0\le i\le n\\ s_0<0, s\in\Gamma}} \mathbb{C}t_0^{s_0}t^sK_i;\\
Z^0 &= \bigoplus_{0\le i\le n, s\in\Gamma} \mathbb{C}t^sK_i;\\
\tau^+ &= L^+(\mathfrak{g},\sigma)\oplus Z^+\oplus D^+(m_0,m);\\
\tau^- &= L^-(\mathfrak{g},\sigma)\oplus Z^-\oplus D^-(m_0,m);\\
\tau^0 &= L^0(\mathfrak{g},\sigma)\oplus Z^0\oplus D^0(m_0,m).
\end{aligned}
$$

Then clearly $\tau = \tau^-\oplus\tau^0\oplus\tau^+$ is a triangular decomposition. Let $T = \{v \in V|\tau^+v = 0\} \neq 0$ by Theorem 5.2.

Proposition 6.1. *T is a τ^0- module and in fact irreducible as τ^0- module. Further $V = U(\tau^-)T$.*

Proof. It is easy to check that $[\tau^0,\tau^+]\subset\tau^+$. ¿From this it follows that T is a τ^0- module. Now from PBW theorem, we have $U(\tau) = U(\tau^-)U(\tau^0)U(\tau^+)$. (Here U denotes the universal enveloping algebra.) Using this and the fact that V is τ-irreducible, it follows that T is τ^0-irreducible and $V = U(\tau^-)T$.

Recall that $\{d_1,\cdots,d_n\}\subseteq D^0(m_0,m)$ and hence T is $\mathbb{Z}^n$- graded.

Let

$$T_k = \{v\in T|d_iv = (\lambda(d_i)+k_i)v, 1\le i\le n\}$$

where λ is a fixed weight in $P(V)$ coming from Theorem 5.2. We will now record some result on T which can be proved similarly as in [19], [14] and [16]. ■

Lemma 6.2.

(a) *For $v\in T\backslash\{0\}, t^sK_0v\neq 0$ for all $s\in\Gamma$.*

(b) $\dim T_k = \dim T_{k+s} = p_k\ \forall\ s\in\Gamma, k\in\mathbb{Z}^n, p_k\in\mathbb{N}$.

(c) *Let $v_1(k), v_2(k),\cdots,v_{p_k}(k)$ be a basis of T_k for $k\in\mathbb{Z}^n, 0\le k_i<m_i$. Let $v_i(s+k) = \frac{1}{C_0}t^sK_0v_i(k)$, then $\{v_1(s+k),\cdots,v_{p_k}(s+k)\}$ is a basis of T_{k+s}.*

(d) $\frac{1}{C_0}t^sK_0(v_1(k+r),\cdots,v_{p_k}(k+r)) = (v_1(k+r+s),\cdots,v_{p_k}(k+r+s))$ *for all $r,s\in\Gamma$.*

(e) $h\otimes t^s(v_1(k+r),\cdots,v_{p_k}(k+r)) = \lambda(h)(v_1(k+r+s),\cdots,v_{p_k}(k+r+s))$ *for $h\in\mathfrak{h}(0)$ and all $r,s\in\Gamma$, where λ is a fixed weight of $P(V)$.*

(f) $t^sd_0(v_1(k+r),\cdots,v_{p_k}(k+r)) = \lambda(d_0)(v_1(k+r+s),\cdots,v_{p_k}(k+r+s))$ *for all $s,r\in\Gamma$ where λ is fixed.*

(g) $t^rK_p \cdot T = 0 \quad 1 \leq p \leq n, r \in \Gamma$.

(h) $t^rK_0 \cdot t^sK_0 \cdot v = C_0t^{r+s}K_0v$ *for all* $v \in T$ *and* $r, s \in \Gamma$.

7. More notation and co-finite ideals

(7.1) Let Der$A(m)$ be the derivation algebra of $A(m)$. Let $e_1, \cdots, e_n$ be the standard basis of $\mathbb{C}^n$ and let $u = \sum u_ie_i \in \mathbb{C}^n$. Let $D(u,r) = \sum_{i=1}^{n} u_it^rd_i, r \in \Gamma$.

(7.2) From the earlier section T can be identified with $V^1 \otimes A(m)$ where V^1 can be taken as

$$\bigoplus_{\substack{0 \leq k_i < m_i \\ 1 \leq i \leq n}} \mathbf{T}_k$$

(7.3) Now $D^0(m_0, m)$ is spanned by $t^rd_i, r \in \Gamma, 0 \leq i \leq n$. Thus $D^0(m_0, m)$ can be identified with $DerA(m) \oplus \sum_{r \in \Gamma} \mathbb{C}t^rd_0$. Z^0 can be identified with $\sum_{r \in \Gamma} \mathbb{C}t^rK_0$ as the rest of the space acts trivially on T. Thus $V^1 \otimes A(m)$ is an irreducible module for

$$L = L^0(\mathfrak{g}, \sigma) \oplus \mathrm{Der}A(m) \oplus \sum_{r \in \Gamma} \mathbb{C}t^rd_0 \oplus A(m),$$

where we identify $\sum_{r \in \Gamma} \mathbb{C}t^rK_0$ by $A(m)$ and $\frac{1}{C_0}t^rK_0$ goes to t^r which is well defined by Lemma 6.2(h).

(7.4) We note the following

$$\begin{aligned} t^r \cdot v \otimes t^s &= v \otimes t^{r+s}, \\ t^rd_0 \cdot v \otimes t^s &= \lambda(d_0)v \otimes t^{r+s} \text{ for } r, s \in \Gamma, v \in V^1. \end{aligned}$$

(7.5) Let $\overset{0}{\mathfrak{g}} = \{X \in \mathfrak{g} | \sigma_0X = X, [h, X] = 0, h \in \mathfrak{h}(0)\}$ and the following is easily checked.

(7.5.1) $\sigma_i(\overset{0}{\mathfrak{g}}) \subseteq \overset{0}{\mathfrak{g}}$ for $1 \leq i \leq n$.

(7.5.2) $\overset{0}{\mathfrak{g}} = \bigoplus_{\overline{k} \in \Lambda} \overset{0}{\mathfrak{g}}_{\overline{k}}$ is a natural Λ- grading where

$$\overset{0}{\mathfrak{g}}_{\overline{k}} = \{X \in \overset{0}{\mathfrak{g}} | \sigma_iX = \xi_i^{k_i}X, 1 \leq i \leq n\}.$$

The corresponding multiloop algebra is denoted by

$$L(\overset{0}{\mathfrak{g}}, \sigma) = \bigoplus_{k \in \Lambda} \overset{0}{\mathfrak{g}}_{\overline{k}} \otimes t^k.$$

It is clear that $L^0(\mathfrak{g},\sigma) = L(\overset{0}{\mathfrak{g}},\sigma)$. When we say $X(k) = X \otimes t^k \in L(\overset{0}{\mathfrak{g}},\sigma)$ we always mean $X \in \overset{0}{\mathfrak{g}}_{\overline{k}}$.

Thus $L \cong L(\overset{0}{\mathfrak{g}},\sigma) \oplus Der A(m) \oplus A(m) \oplus \sum_{r\in\Gamma} \mathbb{C}t^r d_0$.

(7.6) The brackets in L are given as follows :

(a) $[X(k), Y(l)] = [X,Y](k+l)$,

(b) $[D(u,r), D(v,s)] = D(w, r+s)$ where $w = (u,s)v - (v,r)u$,

(c) $[D(u,r), t^s] = (u,s)t^{r+s}$,

(d) $[D(u,r), X(k)] = (u,k)X(k+r)$,

(e) $[D(u,r), t^s d_0] = (u,s)t^{r+s}d_0$.

Now we would like to understand the irreducible L- module $V^1 \otimes A(m)$. We need some preparation for that.

(7.7) For $k \in \mathbb{Z}^n, X \in \overset{0}{\mathfrak{g}}_{\overline{k}}$ and $r_1, r_2, \cdots, r_d \in \Gamma$, define $X(k, r_1, \cdots, r_d) =$

$$X(k) - \sum_i X(k+r_i) + \sum_{i<j} X(k+r_i+r_j) \tag{7.1}$$

$$- \sum_{i<j<p} (k+r_i+r_j+r_p) \cdots (-1)^d X(k+r_1+r_2+\cdots+r_d). \tag{7.2}$$

(7.3)

Let F_d be the subspace of $L(\overset{0}{\mathfrak{g}},\sigma)$ spanned by $X(k, r_1, \cdots, r_d)$. It is easily checked that F_d is an ideal in $L(\overset{0}{\mathfrak{g}},\sigma)$

Lemma 7.1. *a)* $F_d \subseteq F_{d-1}$, *b)* $[F_d, F_d] \subseteq F_{d+1}$, *c)* $L(\overset{0}{\mathfrak{g}},\sigma)/F_1 \cong \overset{0}{\mathfrak{g}}$.

Proof. Note that $X(k, r_1, \cdots, r_d) = X(k, r_1, \cdots, r_{d-1}) - X(k+r_d, r_1, \cdots, r_{d-1})$ which proves (a).

Now consider for $l, k \in \mathbb{Z}^n, r_1, \cdots, r_d, s \in \Gamma, X \in \overset{0}{\mathfrak{g}}_{\overline{k}}, Y \in \overset{0}{\mathfrak{g}}_{\overline{l}}$.

$$\begin{aligned}[X(k, r_1, \cdots, r_d), Y(l) - Y(l+s)] &= [X,Y](k+l, r_1, \cdots, r_d) \\ &\quad - [X,Y](k+l+s, r_1, \cdots, r_d) \\ &= [X,Y](k+l, r_1, r_2, \cdots, r_d, s) \in F_{d+1}.\end{aligned}$$

By the above note we see that F_d is spanned by vectors of the form $X(k) - X(k+s)$. Thus from above (b) follows. To see (c), first notice it for non-twisted case and then deduce it for the twisted case using commutative algebra. See also [24]. ∎

(7.8) In this subsection we recall some facts from [11] on $DerA(m)$. Let $I(u,r) = D(u,r) - D(u,0), u \in \mathbb{C}^n, r \in \Gamma$. It is easy to check,

(7.8.1) $[I(u,r), I(v,s)] = (v,r)I(u,r) - (u,s)I(v,s) + I(w, s+r)$, where $w = (u,s)v - (v,r)u$.

Let I be the space spanned by $I(u,r), u \in \mathbb{C}^n, r \in \Gamma$ which can be seen as subalgebra of $DerA(m)$.

(7.9) For $d \geq 1, u \in \mathbb{C}^n, s_1, \cdots, s_d, r \in \Gamma$, let

$$I_d(u,r,s_1,s_2,\cdots,s_d) = I(u,r) - \sum_i I(u, r+s_i) + \sum_{i<j} I(u, r+s_i+s_j) + \cdots + (-1)^d I(u, r+s_1+s_2+\cdots+s_d)$$

Let I_d be the space spanned by $I_d(u,r,s_1,s_2,\cdots,s_d)$ for $u \in \mathbb{C}^n, r, s_1, \cdots s_d \in \Gamma$.

The following is proved in [11].

Lemma 7.2.

(a) I_d *is a co-finite ideal in* I.

(b) Any co-finite ideal of I *contains* I_d *for large* d.

(c) $I_1 = I$ *and* $I/I_2 \cong gl_n$.

8. Finite dimensional modules

(8.1) Let W be the subspace of $V^1 \otimes A(m)$ spanned by vectors of the form $t^r.v(s) - v(s)$ for $r,s \in \Gamma$ and $v \in V^1$.

Let $\widetilde{L} = I \ltimes L(\overset{0}{\mathfrak{g}}, \sigma)$

Lemma 8.1. *W is an* $\widetilde{L} \oplus A(m) \oplus \sum_{r\in\Gamma} \mathbb{C}t^r d_0$ *module.*

Proof. It is easy to check using the following

$$\begin{aligned} [D(u,r)] - D(u,0), t^s] &= (u,s)(t^{r+s} - t^s), \\ [L(\overset{0}{\mathfrak{g}}, \sigma), A(m)] &= 0, \\ [t^r d_0, A(m)] &= 0. \end{aligned}$$

Let $\widetilde{V} = (V^1 \otimes A(m))/W$ which is an $\widetilde{L}$- module. Notice that $A(m) \oplus \sum_{r\in\Gamma} \mathbb{C}t^r d_0$ acts as scalars on $\widetilde{V}$ and hence we ignore them. We would like to prove that $\widetilde{V}$ is completely reducible $\widetilde{L}$-module. ■

(8.2) Recall that the Lie brackets in $\widetilde{L}$ are given by

(a) $[I(v,s), I(u,r)] = (u,s)I(v,s) - (v,r)I(u,r) + I(w, r+s)$, where $I(u,r) = D(u,r) - D(u,0)$ and $w = (v,r)u - (u,s)v$,

(b) $[I(v,s), X(k)] = (v,k)(X(s+k) - X(k))$,

(c) $[X(k), Y(l)] = [X,Y](k+l)$,

where $X \in \mathfrak{g}_{\overline{k}}, Y \in \mathfrak{g}_{\overline{l}}, k,l \in \mathbb{Z}^n, r,s \in \Gamma$ and $u,v \in \mathbb{C}^n$.

(8.3) Recall that we fixed $\lambda \in P(V)$. Let $\alpha_i = \lambda(d_i)$. Let $\alpha = \sum \alpha_i e_i \in \mathbb{C}^n$ and let V_1 be any $\widetilde{L}$- module. Then we will define L- module structure on $L(V_1) = V_1 \otimes A_n$.

$$\begin{aligned} X(k) \cdot v_1 \otimes t^l &= (X(k)v_1) \otimes t^{l+k}, \\ D(u,r) \cdot v_1 \otimes t^l &= (I(u,r)v_1) \otimes t^{l+r} + (u, l+\alpha) v_1 \otimes t^{l+r}, \\ t^s v_1 \otimes t^l &= v_1 \otimes t^{s+l}, \\ t^r d_0 \cdot v_1 \otimes t^l &= \lambda(d_0) \cdot v_1 \otimes t^{l+r}, \end{aligned}$$

where $v_1 \in V_1, l,k \in \mathbb{Z}^n, r,s, \in \Gamma$.

We need to check the brackets in (7.6). We will first check 7.6(d). Consider

$$\begin{aligned} D(u,r)X(k)(v_1 \otimes t^l) &= D(u,r)((X(k)v_1 \otimes t^{l+k}) \\ &= I(u,r) \cdot X(k)v_1 \otimes t^{l+k+r} + (u, l+k+\alpha)X(k)v_1 \otimes t^{l+k+r} \end{aligned}$$

Consider

$$\begin{aligned} X(k)D(u,r)(v_1 \otimes t^l) &= X(k)(I(u,r)v_1 \otimes t^{l+r} + (u,l+\alpha)v_1 \otimes t^{l+r}) \\ &= X(k)I(u,r)v_1 \otimes t^{l+k+r} + (u,l+\alpha)X(k)v_1 \otimes t^{l+k+r} \end{aligned}$$

Now we will use the fact $[I(u,r), X(k)] = (u,k)(X(k+r) - X(k))$.

So

$$\begin{aligned} (D(u,r)X(k) - X(k)D(u,r)) \cdot (v_1 \otimes t^l) &= (u,k)(X(k+r) - X(k))v_1 \otimes t^{l+k+r} \\ &\quad + (u,k)X(k)v_1 \otimes t^{l+k+r} \\ &= (u,k)X(k+r)v_1 \otimes t^{l+k+r} \\ &= (u,k)X(k+r)(v_1 \otimes t^l). \end{aligned}$$

This proves 7.6(d). We will now check 7.6(b). Consider $D(v,s)D(u,r)(v_1 \otimes t^l)$

$$\begin{aligned} &= D(v,s)(I(u,r)v_1 \otimes t^{l+r} + (u,l+\alpha)v_1 \otimes t^{l+r}) \\ &= I(v,s)I(u,r)v_1 \otimes t^{l+r+s} + (v, l+r+\alpha)I(u,r)v_1 \otimes t^{l+r+s} \\ &\quad + (u,l+\alpha)I(v,s)v_1 \otimes t^{l+r+s} + (u,l+\alpha)(v,l+r+\alpha)v_1 \otimes t^{l+r+s}. \end{aligned}$$

Similarly we have $D(u,r)D(v,s)(v_1\otimes t^l)$

$$\begin{aligned}&= I(u,r)I(v,s)v_1\otimes t^{l+r+s}\\ &\quad + (u,l+s+\alpha)I(v,s)v_1\otimes t^{l+r+s} + (v,l+\alpha)I(u,r)v_1\otimes t^{l+r+s}\\ &\quad + (v,l+\alpha)(u,l+s+\alpha)v_1\otimes t^{l+r+s}.\end{aligned}$$

Now we will use 8.2(a). So $(D(v,s)D(u,r) - D(u,r)D(v,s))v_1\otimes t^l$

$$\begin{aligned}&= ((u,s)I(v,s) - (v,r)I(u,r)\\ &\quad + I(w,r+s))v_1\otimes t^{l+r+s} + (v,r)I(u,r)v_1\otimes t^{l+r+s}\\ &\quad - (u,s)I(v,s)v_1\otimes t^{l+r+s} + (w,l+\alpha)v_1\otimes t^{l+r+s}\\ &= D(w,r+s)(v_1\otimes t^l)\end{aligned}$$

This proves 7.6(b). The remaining brackets 7.6(a), 7.6(c), 7.6(e) are trivial to verify.

(8.4) Recall

(8.4.1) T is an irreducible L-module from Proposition 6.1.

(8.4.2) $\widetilde{V}$ is an $\widetilde{L}$-module from Lemma 8.1.

(8.4.3) $L(\widetilde{V})$ is an L-module from (8.3).

We will now establish that T is contained in $L(\widetilde{V})$ as L-modules.

For $v_k\in T_k$, let $\overline{v}_k$ be the image in $\widetilde{V}\cong T/W$. Let $\widetilde{\varphi}\colon T\to L(\widetilde{V})$, $\widetilde{\varphi}(v_k) = \overline{v}_k\otimes t^k, k\in\mathbb{Z}^n$.

Lemma 8.2. *$\widetilde{\varphi}$ is an L-module map.*

Proof. Consider $\widetilde{\varphi}(D(u,r)v_k) = \overline{D(u,r)v_k}\otimes t^{k+r}$. Now

$$\begin{aligned}D(u,r)(\overline{v}_k\otimes t^k) &= \overline{(D(u,r)-D(u,0))v_k}\otimes t^{k+r}\\ &+ (u,k+\alpha)\overline{v}_k\otimes t^{k+r}\\ &= \overline{D(u,r)v_k}\otimes t^{k+r}.\end{aligned}$$

Thus we have verified $\widetilde{\varphi}(D(u,r)v_k) = D(u,r)\widetilde{\varphi}(v_k)$.

The rest of the relations are easy to verify. Cleary $\widetilde{\varphi}$ is a non-zero map and since T is an irreducible L-module we have $T\subseteq L(\widetilde{V})$ as L-module. ∎

Theorem 8.3. *$\widetilde{V}$ is completely reducible as $\widetilde{L}$- module and all components are isomorphic.*

We will prove some results before proving Theorem 8.3. The following is very standard.

Proposition 8.4. *Let $\mathfrak{g}$ be any Lie algebra. Let $V_1, V_2, \cdots, V_n$ be mutually non-isomorphic irreducible $\mathfrak{g}$-modules. Suppose W is a non-zero $\mathfrak{g}$-submodule of $\bigoplus_{i=1}^{n} V_i$. Then there exists $S \subset \{1, 2, \cdots, n\}$ such that*

$$W = \bigoplus_{i \in S} V_i.$$

We will now prove a decomposition theorem for $L(\widetilde{V})$ as L-module. First we give some notation. Recall that

$$\widetilde{V} = \bigoplus_{\overline{p} \in \Lambda} \widetilde{V}_{\overline{p}}.$$

Let $p \in \mathbb{Z}^n$ and $\overline{p} \in \Lambda$.
Define

$$L(\widetilde{V})(\overline{p}) = \{\overline{v}_k \otimes t^{k+r+p}, \overline{v}_k \in \widetilde{V}_{\overline{k}}, r \in \Gamma, k \in \mathbb{Z}^n\}.$$

Clearly $L(\widetilde{V})(\overline{p})$ is closed under $A(m)$ and $\sum_{r \in \Gamma} \mathbb{C} t^r d_0$.

Consider for $X \in \overset{0}{\mathfrak{g}}_l$,

$$\begin{aligned} X(l).(\overline{v}_k \otimes t^{k+r+p}) &= \overline{X(l)v_k} \otimes t^{k+l+r+p} \in L(\widetilde{V})(\overline{p}). \\ D(u,s)(\overline{v}_k \otimes t^{k+r+p}) &= \overline{I(u,s)v_k} \otimes t^{k+r+p+s} + (u, k+r+p+\alpha)\overline{v}_k \otimes t^{k+r+p+s}, \end{aligned}$$

$s \in \Gamma$. We see that the above vector belongs to $L(\widetilde{V})(\overline{p})$. Thus $L(\widetilde{V})(\overline{p})$ is an L-module. Clearly

$$L(\widetilde{V}) = \bigoplus_{\overline{p} \in \Lambda} L(\widetilde{V})(\overline{p})$$

which is a finite sum of L-modules.

Lemma 8.5. *$\widetilde{V}$ is graded irreducible $\widetilde{L}$-module if and only if $L(\widetilde{V})(0)$ is an irreducible L-module.*

Proof. Recall that $\widetilde{V}$ and $\widetilde{L}$ are Λ-graded. Further using the map $\widetilde{\varphi}$, $T \cong \widetilde{\varphi}(T)$ as L-modules. We will first note that the L-module generated by $\overline{v}_k \otimes t^k, k \in \mathbb{Z}^n$ is equal to the $\widetilde{L} \oplus A(m)$-module generated by $\overline{v}_k \otimes t^k, k \in \mathbb{Z}^n$. This follows from the fact

$$D(u,r)(\overline{v}_k \otimes t^k) = \overline{I(u,r)v_k} \otimes t^{k+r} + (u, k+\alpha)\overline{v}_k \otimes t^{k+r}$$

and

$$t^r.\overline{v}_k \otimes t^k = \overline{v}_k \otimes t^{k+r}.$$

Also note that t^r acts trivially on $\widetilde{V}$. From this it is easy to see that both the statements of the Lemma are true. ∎

(8.5) It is easy to see that $T \cong L(\tilde{V})(\overline{0})$ as L-modules (See Lemma 8.2) and in particular $L(\tilde{V})(\overline{0})$ is an irreducible L-module. Thus by above Lemma $\tilde{V}$ is irreducible graded $\tilde{L}$-module.

Proposition 8.6. *Each $L(\tilde{V})(\overline{p})$ is an irreducible L-module.*

Proof. Consider the map for a fixed $p \in \mathbb{Z}^n$ such that $\overline{p} \neq 0$,

$$\pi(\overline{p}) : L(\tilde{V})(\overline{p}) \to L(\tilde{V})(\overline{0})$$

$$\pi(\overline{p})(\overline{v}_k \otimes t^{k+r+p}) = \overline{v}_k \otimes t^{k+r}.$$

It is easy to see that $\pi(\overline{p})$ is a vector space isomorphism and not a L-module map. For example

$$(u, k+r+p+\alpha)\overline{v}_k \otimes t^{k+r+p} = d_i(\overline{v}_k \otimes t^{k+r+p}) \neq d_i(\overline{v}_k \otimes t^{k+r}) = (u, k+r+\alpha)\overline{v}_k \otimes t^{k+r}.$$

Now suppose W is a non-zero proper submodule of $L(\tilde{V})(\overline{p})$. Clearly $\pi(\overline{p})(W)$ is a non-zero proper subspace of $L(\tilde{V})(\overline{0})$. To prove that $L(\tilde{V})(\overline{p})$ is irreducible, it is sufficient to prove that $\pi(\overline{p})(W)$ is an L-module. Since $\pi(\overline{p})$ commutes with $L(\overset{0}{\mathfrak{g}}, \sigma) \oplus A(m) \oplus \sum_{r \in \Gamma} \mathbb{C} t^r d_0$, it follows that $\pi(\overline{p})(W)$ is a module for the above space. Since W is a weight module, it is easy to check that $\pi(\overline{p})(W)$ is also a weight module. Let $w = \overline{v}_k \otimes t^{k+r} \in \pi(\overline{p})(W)$ be a weight vector, then

$$D(u, s)(\overline{v}_k \otimes t^{k+r+p}) = \overline{I(u, s)v_k} \otimes t^{k+r+s+p} + (u, k+r+p+\alpha)\overline{v}_k \otimes t^{k+r+s+p} \in W.$$

Also

$$t^s(\overline{v}_k \otimes t^{k+r+p}) = \overline{v}_k \otimes t^{k+r+s+p} \in W.$$

This proves $\overline{I(u, s)v_k} \otimes t^{k+r+s+p} + (u, k+r+\alpha)\overline{v}_k \otimes t^{k+r+s+p} \in W$. This means $\pi^{-1}(\overline{p})(D(u, s)(\overline{v}_k \otimes t^{k+r})) \in W$. So $D(u, s)(\overline{v}_k \otimes t^{k+r}) \in \pi(\overline{p})(W)$. Thus $\pi(\overline{p})(W)$ is an L-module. This proves that $L(\tilde{V})(\overline{p})$ is irreducible L-module.

It is possible that some of the modules in $L(\tilde{V})$ are isomorphic. We need to develop the notion of graded automorphisms of $\tilde{V}$. ∎

Definition 8.7. An $\tilde{L}$-module automorphism θ of $\tilde{V}$ is called $\overline{p}$-graded if $\theta(\tilde{V}_{\overline{k}}) = \tilde{V}_{\overline{k}-\overline{p}}$ for all $\overline{k} \in \Lambda$.

1. In this case dim $\tilde{V}_{\overline{k}} =$ dim $\tilde{V}_{\overline{k}-\overline{p}}$.

2. Suppose θ is a $\overline{p}$-graded automorphism of $\tilde{V}$. Choose minimal integer $N_p > 0$ such that $N_p.\overline{p} = 0$ in Λ.Then clearly $\theta^{N_p}\ \tilde{V}_{\overline{k}} = \tilde{V}_{\overline{k}}$. Thus there exists a vector v in $\tilde{V}_{\overline{k}}$ such that $\theta^{N_p} v = \lambda v$ for some non-zero scalar λ. Consider the space

$$W = \{v \in \tilde{V} \mid \theta^{N_p} v = \lambda v\},$$

which can be seen as a graded submodule of $\widetilde{V}$. Since $\widetilde{V}$ is graded irreducible by 8.5, we see that $W = \widetilde{V}$. Thus $\theta^{N_p} = \lambda$ on $\widetilde{V}$. Now by suitably multiplying θ by a scalar we can assume $\theta^{N_p} = 1$.

So here after we will work only with graded automorphisms of finite order. Recall we have fixed $\alpha \in \mathbb{C}^n$ from 8.3. For any L-module V, we define

$$V_k = \{v \in V | D(u,0)v = (u, k+\alpha)v\},$$

for $k \in \mathbb{Z}^n$. Thus we have

$$L(\widetilde{V})_k = \bigoplus_{\overline{p} \in \Lambda} L(\widetilde{V})(\overline{p})_k.$$

Lemma 8.8. $\dim \widetilde{V}_{\overline{k}} = dim\ L(\widetilde{V})(\overline{0})_k = \dim L(\widetilde{V})(\overline{p})_{k+p}$.

Proof. Proof follows from the definitions of $L(\widetilde{V})(\overline{p})$. ∎

(8.14) Suppose θ is a $\overline{p}$-graded automorphism of $\widetilde{V}$ and N_p be the order. Then θ defines an L -module isomorphism

$$\theta' : L(\widetilde{V})(\overline{0}) \to L(\widetilde{V})(\overline{p})$$
$$\theta'(\overline{v}_k \otimes t^{k+r}) = \theta(\overline{v}_k) \otimes t^{k+r}.$$

It is easy to check that θ' is an isomorphism of L-modules. Suppose there exists an L-module isomorphism θ' from $L(\widetilde{V})(\overline{0})$ to $L(\widetilde{V})(\overline{p})$, then k-weight vectors go to k-weight vectors under this isomorphism. Thus

$$\theta'(\overline{v}_k \otimes t^k) = \overline{w}_{k-p} \otimes t^k.$$

We now define $\theta(\overline{v}_k) = \overline{w}_{k-p}$. This can be checked to be $\overline{p}$-graded automorphism of $\widetilde{V}$ and can be assumed to be of finite order.

Proposition 8.9. *There is a one-one correspondence between $\overline{p}$-graded $\widetilde{L}$-module automorphism of $\widetilde{V}$ and isomorphism between $L(\widetilde{V})(\overline{0})$ and $L(\widetilde{V})(\overline{p})$.*

Proof. Proof follows from above discussion. ∎

Lemma 8.10. *Suppose such an automorphism exists, then*

1. $\dim \widetilde{V}_{\overline{k}} = \dim \widetilde{V}_{\overline{k}-\overline{p}}$,
2. $\dim L(\widetilde{V})(\overline{0})_k = \dim L(\widetilde{V})(\overline{0})_{k-p}$.

Corollary 8.11.

1. $\dim \widetilde{V}_{\overline{k}} = \dim \widetilde{V}_{\overline{k+jp}}, j \in \mathbb{Z}$.

2. $\dim L(\widetilde{V})(j\overline{p})_{\overline{k+ip}} = \dim \widetilde{V}_{\overline{k}}$ *for any* $i, j \in \mathbb{Z}$.

Proof. Repeated application of θ. Note that θ^j is also an automorphism. We will first prove the following Proposition which is also independent interest. ∎

Proposition 8.12. *Let* $\mathfrak{g}$ *be any* Λ*-graded Lie algebra. Let* V *be finite dimensional* Λ*-graded* $\mathfrak{g}$ *module. Let* $p \in \Lambda$ *and let* θ *be* p*-graded* $\mathfrak{g}$ *module automorphism of* V *of order* $N_p < \infty$.

1. *Then each eigenspace of* θ *is a* $\mathfrak{g}$*-module and all eigenspaces are isomorphic as* $\mathfrak{g}$ *modules.*

2. *Let* $\mathbb{Z}_p$ *be the finite subgroup of* Λ *genarated by* p. *Let* $\Lambda_p = \Lambda/\mathbb{Z}_p$. *Suppose* V *is* Λ*-graded* $\mathfrak{g}$ *irreducible then each eigen space is* Λ_p*-graded* $\mathfrak{g}$*-irreducible.*

Proof. Let $V = \bigoplus_{q\in\Lambda} V_q$ be Λ-gradation and let $\dim V_q = n_q$. Since θ is Λ-graded automorphism we have $n_{q+p} = n_q \ \forall q \in \Lambda$. We have $|\mathbb{Z}_p| = N_p$. Let $|\Lambda_p| = m_p$ so that $N_p m_p = |\Lambda|$. Let $q_1, q_2 \cdots, q_{m_p}$ be distinct coset representations of Λ_p. Let $S_p = \sum_{j=1}^{m_p} n_{q_j}$. Then in view of above we have $N_p S_p = \dim V$. Let ζ be the N_pth primitive root of unity. Let for $0 \le i < N_p$.

$$V^i = \{v \in V | \theta v = \zeta^{-i} v\}$$

Then

$$\bigoplus_{i=0}^{N_p-1} V^i = V.$$

For $v \in V$, let $v(i) = v + \zeta^i\theta(v) + \cdots + \zeta^{i(N_p-1)}\theta^{N_p-1}v$.
Then clearly $v(i) \in V^i$. Note that if $v \in V_q$ and $w = \theta^j v \in V_{q+jp}$. We have $w(i) = \zeta^{ij} v(i)$. It is easy to check that there is a injective map

$$\bigoplus_{j=1}^{m_p} V_{q_j} \longmapsto V^i, v \longmapsto v(i)$$

(a) Thus $S_p \le \dim V^i$ and note that L.H.S. is independent of i. Now summing over i we see that $\dim V = N_p S_p \le \sum_{i=0}^{N_p-1} \dim V^i = \dim V$. Thus equality holds at (a). This proves that $\dim V^i = \dim V^j = S_p$ for all $0 \le i, j < N_p$. Also, as equality holds at (a), we see that V^i is spanned by $v(i), v \in \bigoplus_{j=1}^{m_p} V_{q_j}$. Now consider the $\mathfrak{g}$-module map from

$$V^i \longmapsto V^j$$

$$v(i) \longmapsto v(j), v \in \bigoplus_{j=1}^{m_p} V_{q_j}$$

which is clearly injective and surjective. Thus proof of (1) is complete.

To see (2), note that any Λ-graded vector space is also Λ_p-graded. To prove V^i is Λ_p-graded $\mathfrak{g}$-irreducible it is sufficient to connect $v(i)$ and $w(i)$ for $v \in V_q, w \in V_{q^1}$ under $\mathfrak{g}$. Since we are assuming V is Λ-graded irreducible we know that v and w are $\mathfrak{g}$-connected. Then clearly $v(i)$ and $w(i)$ are $\mathfrak{g}$-connected. The proof of (2) is now complete. Note that each V^i is no longer graded by Λ but by Λ_p. ■

Theorem 8.13. *$\widetilde{V}$ is an $\widetilde{L}$ irreducible module if and only if $L(\widetilde{V})(\overline{p}), \overline{p} \in \Lambda$ are mutually non-isomorphic as L-modules.*

Proof. We can suppose $L(\widetilde{V})(\overline{0}) \cong L(\widetilde{V})(\overline{p})$ for $0 \neq \overline{p} \in \Lambda$. Then there exists a $\overline{p}$-graded automorphism of $\widetilde{V}$. Let N_p be the order. For $\overline{v}_k \in \widetilde{V}_{\overline{k}}$, define

$$\overline{v}_k(0) = \overline{v}_k + \theta(\overline{v}_k) + \cdots + \theta^{N_p - 1}(\overline{v}_k).$$

It is easy to check that $\theta(\overline{v}_k(0)) = \overline{v}_k(0)$.
Let $\widetilde{M}_0 = \{\overline{v}_k(0), \overline{v}_k \in \widetilde{V}_{\overline{k}}, \overline{k} \in \Lambda\}$, where $\{\}$ means the space spanned by the vectors inside $\widetilde{V}$. Clearly $\widetilde{M}_0$ is a non-zero proper submodule of $\widetilde{V}$. This proves one side of the theorem. Now suppose $L(\widetilde{V})(\overline{p}), \overline{p} \in \Lambda$ be mutually non-isomorphic modules. Suppose W is a $\widetilde{L}$ submodule of $\widetilde{V}$. Then clearly

$$L(W) \subset L(\widetilde{V}) = \bigoplus_{\overline{p} \in \Lambda} L(\widetilde{V})(\overline{p}).$$

Then by Proposition 8.4, we see that there exists $S \subset \Lambda$ such that

$$L(W) = \bigoplus_{\overline{p} \in S} L(\widetilde{V})(\overline{p}).$$

This means $L(\widetilde{V})(\overline{p}) \subset L(W)$ for some $\overline{p}$. This means $\overline{v}_k \otimes t^{k+p} \in L(W)$, which means $\overline{v}_k \in W$. By 8.5, the module generated by $\overline{v}_k$ is $\widetilde{V}$. Hence $W = \widetilde{V}$. This proves the other part of the theorem. ■

(8.6) The aim is to find suitable irreducible $\widetilde{L}$ submodule of $\widetilde{V}$. Fix a $0 \neq \overline{p} \in \Lambda$ from $p \in \mathbb{Z}^n$. Suppose θ is a Λ-graded $\widetilde{L}$-module automorphism of degree $\overline{p}$. Let N_p be the order of θ.

(8.7) Let $\mathbb{Z}_p$ be the cyclic group generated by $\overline{p}$ inside Λ. Let $\Lambda_p = \Lambda/\mathbb{Z}_p$. Consider

$$\mathbb{Z}^n \to \mathbb{Z}^n/\Gamma = \Lambda \to \Lambda_p.$$

Let Γ_p be the kernal of the above map. For $0 \leq i < N_p$ define

$$\overline{v}_k(i) = \overline{v}_k + \zeta^i \theta(\overline{v}_k) + \cdots \zeta^{i(N_p-1)} \theta^{N_p-1} v_k$$

Then clearly $\theta(\overline{v}_k(i)) = \zeta^{-i} \overline{v}_k(i)$
Let $\widetilde{M_i} = \{v \in \widetilde{V} \ | \theta(v) = \zeta^{-i} v\}$ which can seen as a proper submodule of $\widetilde{V}$. Further $\widetilde{V} = \bigoplus_{i=0}^{N_p-1} \widetilde{M}_i$. By earlier Proposition 8.12, all $\widetilde{M}_i$ are isomorphic as $\widetilde{L}$-modules and Λ_p-graded. Let $m_p = |\Lambda_p|$. As earlier let $0 = q_1, q_2, \cdots q_{m_p}$ be a set of coset representatives of Λ_p. Now, for $1 \leq l \leq m_p$, we have

$$L(\widetilde{V})(q_l) \cong L(\widetilde{V})(q_l + p) \cdots \cong L(\widetilde{V})(q_l + (N_p - 1)p).$$

Let $W(q_l) = \bigoplus_{j=0}^{(N_p-1)} L(\widetilde{V})(q_l + jp)$

Let $M_i^l = \{\overline{v}_k(i) \otimes t^{k+r+q_l}, \overline{v}_k \in \widetilde{V}_{\overline{k}}, k \in \mathbb{Z}^n, r \in \Gamma\}$. It is easy to see that M_i^l is a L-module and contained in $W(q_l)$.

Claim $\overline{v}_k(i) \otimes t^{k+r+q_l-jp} \in M_i^l$. To see this consider for any $j, \overline{v}_k(i) = \zeta^i \theta(\overline{v}_k(i)) = \zeta^{ij} \theta^j (\overline{v}_k(i)) = \overline{w}_{k-jp}(i)$ for some $\overline{w}_{k-jp} \in \widetilde{V}_{k-jp}$. The claim now follows from definition. Thus in the definition of M_l^i, Γ can be replaced by Γ_p. It is easy to see that $L(\widetilde{M}_i) = \bigoplus_{l=1}^{m_p} M_i^l$. $(*)$

By Proposition 8.12(2) we know that each $\widetilde{M}_i$ is Λ_p-irreducible. Now by arguments similar to Lemma 8.5 and Proposition 8.6 it follows that each M_i^l is irreducible L-module. We have already noted that $M_i^l \subseteq W(q_l)$ which is sum of isomorphic L-modules. It now follows that $M_i^l \cong L(\widetilde{V})(q_l)$. Now it also follows the $\bigoplus_{i=0}^{N_p-1} M_i^l = W(q_l)$ as the number of components are same each side and they are all isomorphic.

The situation $*$ is similar to Theorem 8.13. We can now prove the following, whose proof is similar to Theorem 8.13.

Theorem 8.14. *We fix i. $\widetilde{M}_i$ is irreducible as $\widetilde{L}$ -module if and only if $M_i^l, 0 \leq l < m_p$ are mutually non-isomorphic as L-modules.*

Proof of the Theorem 8.3. Recall that $\widetilde{V} = \bigoplus_{i=0}^{N_p-1} \widetilde{M}_i$ and all components are isomorphic as $\widetilde{L}$-modules, which follows from Proposition 8.12(2). Further dim $\widetilde{M}_i <$ dim $\widetilde{V}$. Suppose $\widetilde{M}_i$ is reducible as $\widetilde{L}$-module, then we can repeat the above process. This process has to stop for dimensions reasons. Hence the theorem follows. ∎

(8.23) To avoid more complex notation we assume that each $\widetilde{M}_i$ is irreducible $\widetilde{L}$-module.

9. Final Theorem

In this section we will describe the $\widetilde{L}$-module structure of $\widetilde{M}_i$. We are assuming that each $\widetilde{M}_i$ is irreducible $\widetilde{L}$-module. We will actually prove that $\widetilde{M}_i$ is an irreducible module for the direct sum $gl_n \oplus \overset{0}{\mathfrak{g}}$. Recall that gl_n is a quotient of I from Lemma 7.2(c) and $\overset{0}{\mathfrak{g}}$ is a quotient of $L(\overset{0}{\mathfrak{g}}, \sigma)$ by the map $X(k) \to X \in \overset{0}{\mathfrak{g}}_{\overline{k}}$ (See Lemma 7.1(c)). Here $X(\overline{k})$ is identified by X. We will start with a simple Lemma.

Suppose S is a vector space such that $S = \bigoplus_{\overline{k} \in \Lambda} S_{\overline{k}}$.

An operator $T : S \to S$ is called degree $\overline{k}$ operator if $T(S_{\overline{l}}) \subseteq S_{\overline{l}+\overline{k}}$. The following lemma is trivial to see.

Lemma 9.1. *Suppose T is a degree $\overline{k}$ operator such that $\overline{k} \neq 0$. Suppose T acts as a scalar λ. Then $\lambda = 0$.*

The following is well known. See Proposition 19.1(b) of [17].

Lemma 9.2. *Let $\mathfrak{g}'$ be a Lie algebra which need not be finite dimensional. Let (V_1, ρ) be an irreducible finite dimensional module for $\mathfrak{g}'$.*

We have a map $\rho : \mathfrak{g}' \to End\ V_1$. Then $\rho(\mathfrak{g}')$ is a reductive Lie-algebra with at most one dimensional center.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be infinite dimensional Lie algebra such that $\mathfrak{g}_1$ acts on $\mathfrak{g}_2$. Let $\mathfrak{g}' = \mathfrak{g}_1 \ltimes \mathfrak{g}_2$ be the natural semi-direct Lie algebra. Let J be an abelian ideal of $\mathfrak{g}_2$ which will not be assumed to be a ideal of $\mathfrak{g}'$.

Proposition 9.3. *Suppose (V', ρ) is an irreducible finite dimensional module for $\mathfrak{g}'$. We have $\rho : \mathfrak{g}' \to End(V')$. Then $\rho(J)$ is central ideal in $\rho(\mathfrak{g}')$.*

Proof. From above Lemma 9.2, it follows that $\rho(\mathfrak{g}')$ is a reductive Lie algebra. Since $\mathfrak{g}_2$ is an ideal in $\mathfrak{g}'$ we have $\rho(\mathfrak{g}_2)$ is an ideal in $\rho(\mathfrak{g}')$. Thus $\rho(\mathfrak{g}_2)$ is reductive Lie algebra. Now we know that J is abelian ideal in $\mathfrak{g}_2$ and hence $\rho(J)$ is contained in the center of $\rho(\mathfrak{g}_2)$. This proves $\rho(J)$ is actually contained in the center of $\rho(\mathfrak{g}')$. In particular $\rho(J)$ is an ideal in $\rho(\mathfrak{g}')$. ■

Theorem 9.4. *$\widetilde{M}_i$ is actually an irreducible finite dimensional module for the direct sum $gl_n \oplus \overset{0}{\mathfrak{g}}$.*

We need the following Lemma.

Lemma 9.5. *Let $k \in \mathbb{Z}^n, r_1, r_2, \cdots, r_d \in \Gamma$ and $d \geq 1$. Suppose $X(k, r_1, \cdots, r_d)$ acts as a scalar λ on $\widetilde{V}$. Then the scalar λ is zero.*

Proof. First we note that if $X \in \overset{0}{\mathfrak{g}} \cap \mathfrak{g}(\overline{0}, \overline{0})$ then $X \in \mathfrak{h}(0)$. This follows from the theory of finite dimensional simple Lie algebras. Suppose $\overline{k} = 0$, then $X \in \mathfrak{h}(0)$. From Lemma 6.2(a), it follows that $X(k, r) = X(k) - X(k + r)$ is zero on $\widetilde{V}$. Since $X(k, r_1, r_2, \cdots, r_d)$ is spanned by $X(k, r), r \in \Gamma$, it follows that λ is zero. Now suppose $\overline{k} \neq 0$, then from Lemma 9.1, it follows that λ is zero. This completes the proof of the Lemma. ∎

Proof of the Theorem 9.4. Let ρ denote the $\widetilde{L}$-module action on $\widetilde{M}_i$ and note that ρ is independent of i as all $\widetilde{M}_i$ are isomorphic. Consider $(\ker \rho) \cap I$ which is a co-finite ideal of I. Thus from Lemma 7.2(b), it follows that $(\ker \rho) \cap I$ contains I_d for large d. Consider

$$[I(u, k, r_1, r_2, \cdots, r_d), X(l)] = X(k + l, r_1, r_2, \cdots, r_d) \in \ker \rho. \quad (**)$$

This proves F_d acts trivially on $\widetilde{M}_i$. But we know that $[F_{d-1}, F_{d-1}] \subseteq F_d$ by Lemma 7.1(b). Thus $\rho(F_{d-1})$ is an abelian ideal in $\rho(L(\overset{0}{\mathfrak{g}}, \sigma))$. By Proposition 9.3 it follows that $\rho(F_{d-1})$ is a central ideal in $\rho(\widetilde{L})$. It is well known that center acts as scalars on a finite dimensional irreducible module. Thus $X(k, r_1, r_2, \cdots, r_d)$ acts as scalar on each $\widetilde{M}_i$ and the scalar is independent of i. Thus $X(k, r_1, r_2, \cdots, r_d)$ acts as a single scalar on $\widetilde{V}$. Now by Lemma 9.5, scalar is zero. Thus F_{d-1} acts trivially on each $\widetilde{M}_i$. It now follows that $\rho(I_{d-1})$ is an ideal in $\rho(\widetilde{L})$ by $(**)$. By Lemma 4.2 of [11], I_{d-1} acts trivially. By repeating this argument finitely many times we see that I_2 acts trivially and F_2 acts trivially. Now from above argument we see that F_1 acts trivially. Thus $I_2 \oplus F_1$ acts trivially on $\widetilde{M}_i$. As noted in the beginning of the section we see that each $\widetilde{M}_i$ is a module for $gl_n \oplus \overset{0}{\mathfrak{g}}$ and hence the theorem is proved. ∎

(9.1) By Lemma 2.7 of [18], there exists an irreducible module $\widetilde{V}_1$ of gl_n and an irreducible module $\widetilde{V}_2$ for $\overset{0}{\mathfrak{g}}$ such that $\widetilde{V}_1 \otimes \widetilde{V}_2 \cong \widetilde{V}$ as $gl_n \oplus \overset{0}{\mathfrak{g}}$-module. Regarding gradation, note that all vectors of I are grade zero and hence we can assume $\widetilde{V}_1$ is zero graded and $\widetilde{V}_2$ is Λ_p-graded.

(9.2) We will now describe a class of modules for τ. For that we will define modules for τ^0. Recall that

$$\tau^0 = L^0(\mathfrak{g}, \sigma) \oplus Z^0 \oplus D^0(m_0, m)$$

where

$$Z^0 = \bigoplus_{\substack{0 \leq i < n \\ r \in \Gamma}} \mathbb{C} t^r d_i$$

and

$$D^0(m_0, m) = \mathrm{Der}A(m) \oplus \sum_{r \in \Gamma} \mathbb{C} t^r d_0$$

Let W_1, W_2 be irreducible finite dimensional modules for gl_n and $\overset{0}{\mathfrak{g}}$ respectively. We further assume that W_2 is Λ-graded which is compatible with Λ-gradation on $\overset{0}{\mathfrak{g}}$. Let $\{E_{ij} | 1 \leq i, j \leq n\}$ be standard basis for gl_n. Now define τ^0 module structure on

$$W_1 \otimes W_2 \otimes A_n$$

Let $r = \sum m_i r_i e_i \in \Gamma, \lambda, u, \alpha \in \mathbb{C}^n, u_1 \in W_1$ and $u_2 \in W_2, l, k \in \mathbb{Z}^n, X \in \overset{0}{\mathfrak{g}}$.
Define

$$\begin{aligned}
D(u,r) \cdot v_1 \otimes v_2 \otimes t^k) &= (u, k+\alpha) v_1 \otimes v_2 \otimes t^{k+r} \\
&+ \sum (u_i r_j m_j E_{ji} v_1) \otimes v_2 \otimes t^{k+r} \\
X \otimes t^l \cdot (v_1 \otimes v_2 \otimes t^k) &= v_1 \otimes X v_2 \otimes t^{k+l} \\
t^r d_0 \cdot v_1 \otimes v_2 \otimes t^k &= v_1 \otimes v_2 \otimes t^{k+r} \\
\frac{1}{C_0} t^r K_0 \cdot v_1 \otimes v_2 \otimes t^k &= v_1 \otimes v_2 \otimes t^{k+r} \\
t^r K_p v_1 \otimes v_2 \otimes t^k &= 0, 1 \leq p \leq n
\end{aligned}$$

It is easily checked that $W_1 \otimes W_2 \otimes A$ is a module for $\overset{0}{\mathfrak{g}} \otimes A_n \oplus D^0(m_0, m) \oplus Z^0$. It is also a module for τ^0 as $L(\overset{0}{\mathfrak{g}}, \sigma) \subseteq \overset{0}{\mathfrak{g}} \otimes A$.
Let $W_2 = \bigoplus_{\overline{k} \in \Lambda} W_{2,\overline{k}}$ be the Λ-gradation.
Now consider the submodule $T' = \bigoplus_{k \in \Lambda} W_1 \otimes W_{2,\overline{k}} \otimes t^k \subseteq W_1 \otimes W_2 \otimes A_n$ for τ^0.
It is easy checking that the above module is irreducible for τ^0. Now consider the induced modules for τ^0.

$$M = \mathrm{Ind}_{\tau^0 \oplus \tau^+} T'$$

where τ^+ acts trivially or T'. Now by standard arguments there is unique maximal module M^{rad} for M. Consider the irreducible module M/M^{rad} for τ.

Theorem 9.6. *Let V be irreducible integrable module for τ with finite dimensional weight spaces. Assume that K_0 acts as $C_0 > 0$ and k_i acts trivially for $i \neq 0$. Then $V \cong M/M^{rad}$ where M is defined as above.*

Remark 9.7. The assumption on the central elements are not serious. It is sufficient to assume that some central element acts non trivially. In that case there is exists another twisted full toroidal Lie algebra which is isomorphic to given algebra where the assumption on the central elements hold. This is explained in (3.2).

Proof of the Theorem. We already have $\widetilde{V_1}$ and $\widetilde{V_2}$ modules for gl_n and $\overset{0}{\mathfrak{g}}$. We take $W_1 = \widetilde{V_1}$ and $W_2 = \widetilde{V_2}$. It will also follows that $L(M_i) \cong \widetilde{V_1} \otimes \widetilde{V_2} \otimes A_n$. Then the submodule $\bigoplus_{k \in \mathbb{Z}^n} \widetilde{V}_1 \otimes \widetilde{V}_{2,\overline{k}} \otimes t^k$ is isomorphic to M_i^1 for all i. We have already seen that $T \cong L(\widetilde{V})(0)(\cong M_i^1)$ as L-modules for all i. Note that it is difficult to give a direct map between M_i^1 and T as there is a twisting taking place. Taking $T' = T$ the theorem follows. ■

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Punita Batra
Harish-Chandra Research Institute, HBNI
Allahabad 211019, India
batra@hri.res.in

S. Eswara Rao
School of Mathematics
Tata Institute
of Fundamental Research
Mumbai, India
senapati@math.tifr.res.in

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