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The Non-Abelian Tensor and Exterior Products of Crossed Modules of Lie Algebras

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Abstract. In this article, the notions of non-abelian tensor and exterior products of two ideal crossed submodules of a given crossed module of Lie algebras are introduced and some of their fundamental properties are established. Using the obtained results, we also give a new description of the second homology of Lie algebra crossed modules.
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1. Introduction

Throughout the article, all Lie algebras are considered over some fixed field Λ *and* $[\ ,\ ]$ *denotes the Lie bracket.*

In [7,8], Ellis introduces the notions of the non-abelian tensor and exterior products of Lie algebras and gives some of their basic properties. He investigates their relations to the low dimensional homology of Lie algebras; in particular, he proves that for any Lie algebra L, $H_2(L)$, the second homology of L with coefficients in Λ, is isomorphic to the kernel of the commutator map $L \wedge L \longrightarrow L$ and shows how the tensor product is related to the universal central extensions. There are a series of papers (for instance, see [7,10,11,12,14,15]) advertising the relevance of these notions to the development and exposition of the basic theory of the second homology of Lie algebras.

In [3] Casas, Inassaridze, and Ladra, using the general theory of cotriple homology of Barr and Beck, define the homology crossed modules $H_n(T, L, \partial)$ of a Lie algebra crossed module (T, L, ∂) as the simplicial derived functors of the abelianization functor from the category of Lie algebra crossed modules to the category of abelian crossed modules, which are generalizations of the Eilenberg-MacLane homologies of Lie algebras. Furthermore, they give a Hopf formula for the second integral homology of a crossed module. Edalatzadeh [6] presents the notion of the tensor product of abelian crossed modules of Lie algebras, which is analogous to the definition of the tensor product of abelian crossed modules of groups [13].

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Using this notion, he extends the five-term exact sequence on homology of Lie algebra crossed modules one term further and studies the connection between the capability of crossed modules with this sequence.

In this article, we introduce the notions of non-abelian tensor and exterior products of two ideal crossed submodules of some crossed module of Lie algebras, which are somehow a generalization of the definition given by Edalatzadeh [6]. We study some of their important properties and, as an application, describe the second homologies of crossed modules as central crossed submodules of their exterior products. In particular, we show that the second homology of the direct sum of two crossed modules is isomorphic to the direct sum of the second homologies of the factors and the tensor product of the two crossed modules abelianised.

2. Preliminaries on crossed modules of Lie algebras

In this section, we briefly review some standard facts about the category of crossed modules and the non-abelian tensor products of Lie algebras.

Let M and P be two Lie algebras. By an action of P on M we mean a Λ-bilinear map $P \times M \longrightarrow M$, $(p,m) \longmapsto {}^{p}m$, satisfying the conditions ${}^{[p,p']}m = {}^{p}({}^{p'}m) - {}^{p'}({}^{p}m)$ and ${}^{p}[m,m'] = [{}^{p}m, m'] + [m, {}^{p}m']$ for all $m, m' \in M$ and $p, p' \in P$. Obviously, if P is a subalgebra of some Lie algebra L and M is an ideal in L, then Lie multiplication in L induces an action of P on M given by ${}^{p}m = [p,m]$. A *crossed module* (M,P,∂) of Lie algebras is a Lie homomorphism $\partial : M \longrightarrow P$ with an action of P on M such that (i) $\partial({}^{p}m) = [p, \partial(m)]$, (ii) ${}^{\partial(m)}m' = [m,m']$ for all $m, m' \in M$, $p \in P$. If M is an ideal of P, then (M,P,i) is a crossed module in which i is the inclusion map. This way, every Lie algebra P can be regarded as crossed module in the two obvious ways $(0,P,i)$ or (P,P,id).

A *morphism of crossed modules* $(\theta_1, \theta_2) : (M,P,\partial) \longrightarrow (N,Q,\sigma)$ is a pair of Lie homomorphisms $\theta_1 : M \longrightarrow N$ and $\theta_2 : P \longrightarrow Q$ such that $\sigma\theta_1 = \theta_2\partial$ and, for all $p \in P$, $m \in M$, $\theta_1({}^{p}m) = {}^{\theta_2(p)}\theta_1(m)$. The corresponding category of crossed modules of Lie algebras is denoted by **XLie**. In this category we have the obvious notions of injection, surjection, (ideal) subobject, kernel, image, exact sequence, etc.; most of them can be found in detail in [2,4].

Let (M,P,∂) be a crossed module of Lie algebras with an ideal crossed submodule (U,Q,∂). The following is a list of notations which will be used:

- $Z(M,P,\partial) = (M^P, Z(P) \cap st_P(M), \partial)$ is the *center* of (M,P,∂), where $Z(P)$ is the center of P, $M^P = \{m \in M \mid {}^{p}m = 0 \text{ for all } p \in P\}$ and $st_P(M) = \{p \in P \mid {}^{p}m = 0 \text{ for all } m \in M\}$.
- $(M,P,\partial)' = ([P,M], P', \partial)$ is the *commutator crossed submodule* of (M,P,∂), where $P' = [P,P]$ and $[P,M] = \langle {}^{p}m \mid m \in M, p \in P \rangle$ is the displacement subalgebra of P.
- $[(U,Q,\partial),(M,P,\partial)]$ is the ideal crossed submodule $([P,U]+[Q,M],[Q,P],\partial)$ of (M,P,∂).
- $(M,P,\partial)_{ab} = (M/[P,M], P_{ab}, \overline{\partial})$ denotes the *abelianization* of (M,P,∂), where $P_{ab} = P/P'$ and $\overline{\partial}$ is induced by ∂.

Recall that a crossed module (M,P,∂) is *perfect* when it coincides with its commutator crossed submodule, and is *abelian* if it coincides with its center.

Let (M, P, ∂) and (N, P, σ) be two crossed modules. There are actions of M on N and of N on M given by ${}^{m}n = {}^{\partial(m)}n$ and ${}^{n}m = {}^{\sigma(n)}m$. We take M (and N) to act on itself by Lie multiplication. The non-abelian tensor product $M \otimes N$ is defined in [8] as the Lie algebra generated by symbols $m \otimes n$ for $m \in M$, $n \in N$, subject to the relations

(i) $\lambda(m \otimes n) = \lambda m \otimes n = m \otimes \lambda n$,
(ii) $(m + m') \otimes n = m \otimes n + m' \otimes n$,
$m \otimes (n + n') = m \otimes n + m \otimes n'$,
(iii) $[m, m'] \otimes n = m \otimes ({}^{m'}n) - m' \otimes ({}^{m}n)$,
$m \otimes [n, n'] = ({}^{n'}m) \otimes n - ({}^{n}m) \otimes n'$,
(iv) $[(m \otimes n), (m' \otimes n')] = -({}^{n}m) \otimes ({}^{m'}n')$,

for $\lambda \in \Lambda$, $m, m' \in M$, $n, n' \in N$.

The tensor product $M \otimes N$ actually satisfies the usual universal property as follows: Given a Lie algebra Q and a Λ-bilinear function $h : M \times N \longrightarrow Q$, we say that h is a Lie pairing if

$$\begin{aligned} h([m, m'], n) =& h(m, {}^{m'}n) - h(m', {}^{m}n), \\ h(m, [n, n']) =& h({}^{n'}m, n) - h({}^{n}m, n'), \\ h({}^{n}m, {}^{m'}n') =& -[h(m, n), h(m', n')], \end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$; the function $t : M \times N \longrightarrow M \otimes N$, $(m, n) \longmapsto m \otimes n$ is the universal Lie pairing from $M \times N$, that is, for any other Lie pairing $t' : M \times N \longrightarrow Q$ there is a unique Lie homomorphism $\theta : M \otimes N \longrightarrow Q$ such that $\theta t = t'$.

Let $M \square N$ be the submodule of $M \otimes N$ generated by the elements $m \otimes n$ with $\partial(m) = \sigma(n)$. It is straightforward to see that $M \square N$ lies in the centre of $M \otimes N$. The non-abelian exterior product $M \wedge N$ is defined to be the quotient $(M \otimes N)/(M \square N)$. We write $m \wedge n$ to denote the image in $M \wedge N$ of the generator $m \otimes n$.

The following proposition summarizes the rather elementary properties of the non-abelian tensor and exterior products, the proof of which is left to the reader (see also [8]).

Proposition 2.1. *Let (M, P, ∂) and (N, P, σ) be two crossed modules. Then*

(i) *There are Lie homomorphisms*

$$\begin{aligned} &\lambda_P : M \otimes N \longrightarrow P \quad , \quad m \otimes n \longmapsto [\partial(m), \sigma(n)] \\ &\lambda_M : M \otimes N \longrightarrow M \quad , \quad m \otimes n \longmapsto -{}^{n}m \\ &\lambda_N : M \otimes N \longrightarrow N \quad , \quad m \otimes n \longmapsto {}^{m}n \end{aligned}$$

Furthermore, these homomorphisms factor through $M \wedge N$.

(ii) *These homomorphisms are crossed modules in which the actions of P on $M \otimes N$ (resp. $M \wedge N$) given by ${}^{p}(m \otimes n) = {}^{p}m \otimes n + m \otimes {}^{p}n$ (resp. ${}^{p}(m \wedge n) = {}^{p}m \wedge n + m \wedge {}^{p}n$) for all $m \in M$, $n \in N$, $p \in P$, and M and N act on $M \otimes N$ (resp. $M \wedge N$) via ∂ and σ.*

(iii) *If M and N act trivially on each other then $M \otimes N \cong M_{\mathrm{ab}} \otimes N_{\mathrm{ab}}$ (resp. $M \wedge N \cong M_{\mathrm{ab}} \wedge N_{\mathrm{ab}}$), where $M_{\mathrm{ab}} = M/M'$ and $N_{\mathrm{ab}} = N/N'$.*

(iv) There is an isomorphism $M \otimes N \xrightarrow{\cong} N \otimes M$, $m \otimes n \longmapsto -n \otimes m$.
(v) There is an isomorphism

$$(M \oplus N) \wedge (M \oplus N) \xrightarrow{\cong} (M \wedge M) \oplus (N \wedge N) \oplus (M_{ab} \otimes N_{ab}).$$
$$(m_1, n_1) \wedge (m_2, n_2) \longmapsto (m_1 \wedge m_2, n_1 \wedge n_2, \bar{m}_1 \otimes \bar{n}_2 - \bar{m}_2 \otimes \bar{n}_1)$$

(vi) If (M, P, ∂) *is perfect, then* $P \wedge M \cong P \otimes M$ *and* $P \wedge P \cong P \otimes P$.
(vii) If Q *is an ideal of* P, *then there is an exact sequence of Lie algebras*

$$Q \wedge P \longrightarrow P \wedge P \longrightarrow\!\!\!\!\rightarrow \frac{P}{Q} \wedge \frac{P}{Q}.$$

(viii) There is a natural exact sequence of Lie algebras

$$\Gamma(M_{ab}) \xrightarrow{\ \psi\ } M \otimes M \xrightarrow{\ \pi\ } M \wedge M,$$

where $\psi(\gamma(\bar{m})) = m \otimes m$ *and* $\pi(m_1 \otimes m_2) = m_1 \wedge m_2$.

Here $\Gamma(M_{ab})$ is the generalized version of Whitehead's universal quadratic functor [17], which is defined by Simson and Tye in [16] (see also [8]) as follows: For any Λ-module A, $\Gamma(A)$ is the Λ-module generated by the symbols $\gamma(a)$ with $a \in A$, subject to the relations

$$\lambda^2\gamma(a) = \gamma(\lambda a),$$
$$\gamma(\lambda a + b) + \lambda\gamma(a) + \lambda\gamma(b) = \lambda\gamma(a + b) + \gamma(\lambda a) + \gamma(b),$$
$$\gamma(a + b + c) + \gamma(a) + \gamma(b) + \gamma(c) = \gamma(a + b) + \gamma(a + c) + \gamma(b + c),$$

for all $\lambda \in \Lambda$, $a, b, c \in A$. Note that the last two conditions imply that the map $\Delta\gamma : A \times A \longrightarrow \Gamma(A)$, taking $(a, b) \in A \times A$ to the element $\gamma(a + b) - \gamma(a) - \gamma(b) \in \Gamma(A)$, is a Λ-bilinear map, whence we get a Λ-module homomorphism $\Delta : A \otimes A \longrightarrow \Gamma(A)$.

3. The tensor and exterior products of crossed modules

In this section, we define and study the non-abelian tensor and exterior products of ideal crossed submodules.

Let (M, P, ∂) and (N, Q, ∂) be two ideal crossed submodules of a crossed module (T, L, ∂). We can form the non-abelian tensor products $M \otimes Q$, $P \otimes N$, and $P \otimes Q$. According to Proposition 2.1(i), there are homomorphisms $\lambda_P : P \otimes N \longrightarrow P$ and $\lambda'_P : P \otimes Q \longrightarrow P$ from which we find the actions of $P \otimes N$ on $M \otimes Q$, $P \otimes Q$, and of $P \otimes Q$ on $M \otimes Q$, $P \otimes N$. We can now construct the semidirect sum $(M \otimes Q) \rtimes (P \otimes N)$. An argument similar to that employed in proving [8; Proposition 9] may be used to show that the following maps are Lie homomorphisms.

$$\alpha : M \otimes N \longrightarrow (M \otimes Q) \rtimes (P \otimes N) \quad , \quad \beta : (M \otimes Q) \rtimes (P \otimes N) \longrightarrow P \otimes Q$$
$$m \otimes n \longmapsto (m \otimes \partial(n), -\partial(m) \otimes n) \qquad\qquad (v, w) \longmapsto \beta_1(v) + \beta_2(w)$$

in which $\beta_1 : M \otimes Q \longrightarrow P \otimes Q$ and $\beta_2 : P \otimes N \longrightarrow P \otimes Q$ are the functorial homomorphisms.

The following lemma plays a fundamental role in our investigation.

Lemma 3.1. *With the above assumptions and notations, we have*

(i) The image of α is an ideal of $(M\otimes Q)\rtimes(P\otimes N)$.

(ii) There is an action of $P\otimes Q$ on $(M\otimes Q)\rtimes(P\otimes N)$ defined by

$$^{x}(\sum c_i(m_i\otimes q_i),\sum d_j(p_j\otimes n_j))=(\sum c_i\ {}^{\lambda'_P(x)}(m_i\otimes q_i),\sum d_j\ {}^{\lambda'_P(x)}(p_j\otimes n_j)).$$

(iii) The homomorphism $\delta: Coker\alpha\longrightarrow P\otimes Q$ induced by β together with the action induced by the part (ii) is a crossed module.

Proof. (*i*) Let $m,m'\in M$, $n,n'\in N$, $p\in P$, $q\in Q$. Using the defining conditions of crossed module and the relation (*iii*) of the tensor product, we have

$$\begin{aligned}{}^{(-\partial(m)\otimes n)}(m'\otimes q)&={}^{({}^{n}\partial(m))}m'\otimes q+m'\otimes {}^{({}^{n}\partial(m))}q={}^{\partial({}^{n}m)}m'\otimes q+m'\otimes {}^{\partial({}^{n}m)}q\\&=[{}^{n}m,m']\otimes q+m'\otimes {}^{\partial({}^{n}m)}q={}^{n}m\otimes {}^{m'}q={}^{\partial(n)}m\otimes {}^{m'}q,\end{aligned}$$

and similarly, ${}^{(p\otimes n')}(m\otimes\partial(n))={}^{p}n'\otimes {}^{m}\partial(n)$. Therefore,

$$\begin{aligned}&[(m\otimes\partial(n),-\partial(m)\otimes n),(m'\otimes q,p\otimes n')]\\&=([m\otimes\partial(n),m'\otimes q]+{}^{(-\partial(m)\otimes n)}(m'\otimes q)-{}^{(p\otimes n')}(m\otimes\partial(n)),[-\partial(m)\otimes n,p\otimes n'])\\&=(-{}^{\partial(n)}m\otimes {}^{m'}q+{}^{\partial(n)}m\otimes {}^{m'}q-{}^{p}n'\otimes {}^{m}\partial(n)\ ,\ -{}^{n'}p\otimes {}^{\partial(m)}n)\\&=(-{}^{p}n'\otimes {}^{m}\partial(n)\ ,\ -{}^{n'}p\otimes {}^{m}n)\\&=(-{}^{p}n'\otimes\partial({}^{m}n)\ ,\ \partial({}^{p}n')\otimes {}^{m}n)\in \mathrm{Im}\alpha.\end{aligned}$$

(*ii*) It is readily seen that ${}^{[x,x']}y={}^{x}({}^{x'}y)-{}^{x'}({}^{x}y)$ for all $y\in(M\otimes Q)\rtimes(P\otimes N)$, $x,x'\in P\otimes Q$. We now show that ${}^{x}[y,y']=[{}^{x}y,y']+[y,{}^{x}y']$ for all $y,y'\in(M\otimes Q)\rtimes(P\otimes N)$, $x\in P\otimes Q$. Without loss of generality we may assume that $x=(p\otimes q)$, $y=(m_1\otimes q_1,p_1\otimes n_1)$, and $y'=(m_2\otimes q_2,p_2\otimes n_2)$. Then we have

$$\begin{aligned}{}^{(p\otimes q(p_1\otimes n_1))}(m_2\otimes q_2)+{}^{p_1\otimes n_1}({}^{p\otimes q}(m_2\otimes q_2))&={}^{({}^{p}q\otimes {}^{p_1}n_1)}(m_2\otimes q_2)+{}^{(p_1\otimes n_1)}({}^{p}q\otimes {}^{m_2}q_2)\\&={}^{({}^{p}q)}({}^{p_1}n_1)\otimes {}^{m_2}q_2+{}^{p_1}n_1\otimes {}^{({}^{p}q)}({}^{m_2}q_2)\\&={}^{(p\otimes q)}({}^{p_1}n_1\otimes {}^{m_2}q_2)={}^{p\otimes q}\ ({}^{p_1\otimes n_1}(m_2\otimes q_2)).\end{aligned}$$

In the same way, ${}^{(p\otimes q(p_2\otimes n_2))}(m_1\otimes q_1)+{}^{p_2\otimes n_2}({}^{p\otimes q}(m_1\otimes q_1))={}^{p\otimes q}({}^{p_2\otimes n_2}(m_1\otimes q_1))$. Therefore

$$\begin{aligned}[{}^{x}y,y']+[y,{}^{x}y']&=([{}^{p\otimes q}(m_1\otimes q_1),m_2\otimes q_2]+{}^{p\otimes q(p_1\otimes n_1)}(m_2\otimes q_2)\\&\quad-{}^{p_2\otimes n_2}({}^{p\otimes q}(m_1\otimes q_1))\ ,\ [{}^{p\otimes q}(p_1\otimes n_1),p_2\otimes n_2])\\&\quad+([m_1\otimes q_1,{}^{p\otimes q}(m_2\otimes q_2)]+{}^{p_1\otimes n_1}({}^{p\otimes q}(m_2\otimes q_2))\\&\quad-{}^{p\otimes q(p_2\otimes n_2)}(m_1\otimes q_1)\ ,\ [p_1\otimes n_1,{}^{p\otimes q}(p_2\otimes n_2)])\\&=({}^{p\otimes q}[m_1\otimes q_1,m_2\otimes q_2]+{}^{p\otimes q}({}^{p_1\otimes n_1}(m_2\otimes q_2))\\&\quad-{}^{p\otimes q}({}^{p_2\otimes n_2}(m_1\otimes q_1))\ ,\ {}^{p\otimes q}[p_1\otimes n_1,p_2\otimes n_2])\\&={}^{p\otimes q}[(m_1\otimes q_1,p_1\otimes n_1),(m_2\otimes q_2,p_2\otimes n_2)]={}^{x}[y,y'].\end{aligned}$$

(*iii*) First note that $\mathrm{Im}\,\alpha$ lies in the kernel of β and thus the homomorphism δ induced by β is well-defined. It immediately follows, by the direct calculations, that $\delta({}^{x}y) = [x, \delta(y)]$ for all $x \in P \otimes Q$, $y \in Coker\alpha$. We now indicate that ${}^{\delta(y)}y' = [y, y']$ for all $y, y' \in Coker\alpha$. Again, without loss of generality, $y = (m \otimes q, p \otimes n)+\mathrm{Im}\alpha$ and $y' = (m' \otimes q', p' \otimes n')+\mathrm{Im}\alpha$. Then

$$\begin{aligned}
{}^{\delta(y)}y' =&({}^{\partial(m)\otimes q}(m' \otimes q') + {}^{p\otimes\partial(n)}(m' \otimes q')\ ,\ {}^{\partial(m)\otimes q}(p' \otimes n') + {}^{p\otimes\partial(n)}(p' \otimes n')) + \mathrm{Im}\alpha \\
=&(-{}^{q}m \otimes {}^{m'}q' + {}^{p}n \otimes {}^{m'}q'\ ,\ -{}^{n'}p' \otimes {}^{q}m - {}^{n}p \otimes {}^{p'}n') + \mathrm{Im}\alpha \\
=&(-{}^{q}m \otimes {}^{m'}q' + {}^{p}n \otimes {}^{m'}q' - {}^{p'}n' \otimes {}^{m}q\ , -{}^{n}p \otimes {}^{p'}n') \\
&+({}^{p'}n' \otimes {}^{m}q\ , -{}^{n'}p' \otimes {}^{q}m) + \mathrm{Im}\alpha \\
=&[y, y'] + (-{}^{p'}n' \otimes \partial({}^{q}m)\ ,\ \partial({}^{p'}n') \otimes {}^{q}m) + \mathrm{Im}\alpha = [y, y'].
\end{aligned}$$

The proof is complete. ∎

Let I be the subalgebra of $Coker\alpha$ generated by the elements of the form $(x \otimes y, y \otimes x + \partial(z) \otimes z)+\mathrm{Im}\alpha$, in which $x, z \in M \cap N$, $y \in P \cap Q$. Then I is an ideal in $Coker\alpha$; since for the generators $(m \otimes q, p \otimes n)+\mathrm{Im}\alpha \in Coker\alpha$ and $(x \otimes y, y \otimes x + \partial(z) \otimes z)+\mathrm{Im}\alpha \in I$, we have

$$\begin{aligned}
&[(x \otimes y, y \otimes x + \partial(z) \otimes z)+\mathrm{Im}\alpha, (m \otimes q, p \otimes n) + \mathrm{Im}\alpha] \\
&\qquad =([x \otimes y, m \otimes q] + {}^{y\otimes x}(m \otimes q) + {}^{\partial(z)\otimes z}(m \otimes q) \\
&\qquad -{}^{p\otimes n}(x \otimes y)\ ,\ [y \otimes x + \partial(z) \otimes z, p \otimes n]) + \mathrm{Im}\alpha \\
&\qquad =({}^{\partial(z)}z \otimes {}^{m}q - {}^{p}n \otimes {}^{x}y\ , -{}^{x}y \otimes {}^{p}n - {}^{z}\partial(z) \otimes {}^{p}n) + \mathrm{Im}\alpha \\
&\qquad =(-{}^{p}n \otimes {}^{x}y\ , -{}^{x}y \otimes {}^{p}n) + \mathrm{Im}\alpha \in I.
\end{aligned}$$

It is apparent that $\delta(I) \subseteq P\Box Q$ and $(I, P\Box Q, \delta)$ is an ideal crossed submodule of $(Coker\alpha, P \otimes Q, \delta)$. Put $(M, P, \partial)\Box(N, Q, \partial) = (I, P\Box Q, \delta)$. We are now ready to give the following main definition.

Definition 3.2. The *non-abelian tensor and exterior products* of ideal crossed submodules (M, P, ∂) and (N, Q, ∂) are defined, respectively, as

$$\begin{aligned}
(M, P, \partial) \otimes (N, Q, \partial) =&\ (Coker\alpha, P \otimes Q, \delta), \\
(M, P, \partial) \wedge (N, Q, \partial) =& \frac{(Coker\alpha, P \otimes Q, \delta)}{(M, P, \partial)\Box(N, Q, \partial)} = (\frac{Coker\alpha}{I}, P \wedge Q, \bar{\delta}).
\end{aligned}$$

Note that the above generalizes somehow the definition given by Edalatzadeh [6] for the abelian cases.

In the following proposition, we investigate some special cases of the above definitions.

Proposition 3.3. *(a) For any two ideals P and Q of a Lie algebra L, we have*

(i) $(P, P, id) \otimes (Q, Q, id) \cong (P \otimes Q, P \otimes Q, id)$.

(ii) $(P, P, id) \wedge (Q, Q, id) \cong (P \wedge Q, P \wedge Q, id)$.

(iii) $(0, P, i) \otimes (0, Q, i) \cong (0, P \otimes Q, i)$.

(iv) $(0, P, i) \wedge (0, Q, i) \cong (0, P \wedge Q, i)$.

(b) For any crossed module (T, L, ∂) of Lie algebras,

$$(T, L, \partial) \wedge (T, L, \partial) \cong (L \wedge T, L \wedge L, id \wedge \partial).$$

Proof. (a) We only prove (i), and leave the analogous proofs of (ii)-(iv) to the reader.

Put $(P,P,id)\otimes(Q,Q,id)=(Coker\alpha, P\otimes Q,\delta)$, where the image of α is generated by the elements $(p\otimes q,-p\otimes q)$ with $p\in P, q\in Q$, and the homomorphism δ is induced by the epimorphism $\beta:(P\otimes Q)\rtimes(P\otimes Q)\longrightarrow P\otimes Q$, $\beta(v,w)=v+w$. Evidently, $\ker\beta=\mathrm{Im}\alpha$ and consequently δ is an isomorphism. Now, it is easy to check that $(\delta,id):(Coker\alpha,P\otimes Q,\delta)\longrightarrow(P\otimes Q,P\otimes Q,id)$ is a morphism of crossed modules.

(b) By Definition 3.2, $(T,L,\partial)\wedge(T,L,\partial)=(Coker\alpha/I,L\wedge L,\bar{\delta})$, in which $\bar{\delta}$ is induced by the homomorphism $\delta: Coker\alpha\longrightarrow L\otimes L$ defined by $\delta((t_1\otimes l_1,l_2\otimes t_2)+\mathrm{Im}\alpha)=\partial(t_1)\otimes l_1+l_2\otimes\partial(t_2)$. By Proposition 2.1$(iv)$, the map $\theta:T\otimes L\longrightarrow L\otimes T$, $t\otimes l\longmapsto -l\otimes t$, is an isomorphism. We now define an epimorphism $\varphi:(T\otimes L)\rtimes(L\otimes T)\longrightarrow L\otimes T$ by $\varphi(v,w)=\theta(v)+w$. Since

$$\varphi(t_1\otimes\partial(t_2),-\partial(t_1)\otimes t_2)=-\partial(t_2)\otimes t_1-\partial(t_1)\otimes t_2=-\partial(t_1+t_2)\otimes(t_1+t_2)\in L\Box T,$$

for any $t_1,t_2\in T$, φ gives rise to an epimorphism $\bar{\varphi}:Coker\alpha\longrightarrow L\wedge T$ which in turn induces an epimorphism $\tilde{\varphi}:Coker\alpha/I\longrightarrow L\wedge T$, because $\bar{\varphi}(I)\subseteq L\Box T$. We now prove that $\tilde{\varphi}$ is an isomorphism by showing that $\ker\bar{\varphi}=I$. Suppose $\bar{\varphi}((v,w)+\mathrm{Im}\alpha)=\theta(v)+w+L\Box T=0$. Then $w=-\theta(v)+x$ for some $x\in L\Box T$ and so, we have $(v,w)+\mathrm{Im}\alpha=(v,-\theta(v)+x)+\mathrm{Im}\alpha\in I$. Finally, one gets that $(\tilde{\varphi},id):(Coker\alpha/I,L\wedge L,\bar{\delta})\longrightarrow(L\wedge T,L\wedge L,id\wedge\partial)$ is a morphism of crossed modules, as required. ∎

Plainly, by combining the morphism $(\tilde{\varphi},id)$ constructed in Proposition 3.3(b) with the natural morphism $(\lambda_T,\lambda_L):(L\wedge T,L\wedge L,id\wedge\partial)\longrightarrow(T,L,\partial)$, one gets a morphism $(T,L,\partial)\wedge(T,L,\partial)\longrightarrow(T,L,\partial)$ whose image is equal to $(T,L,\partial)'$. In Theorem 4.2, we will show that the kernel of this morphism is isomorphic to the second homology of (T,L,∂).

In continuation, we give some basic properties of the tensor and exterior products of crossed modules.

Proposition 3.4. *Let (M,P,∂) and (N,Q,∂) be two ideal crossed submodules of a crossed module (T,L,∂). Then*

(i) *if $[(M,P,\partial),(N,Q,\partial)]=0$, then*

$$(M,P,\partial)\otimes(N,Q,\partial)\cong(M,P,\partial)_{ab}\otimes(N,Q,\partial)_{ab}.$$

(ii) *There is an exact sequence of crossed modules*

$$(M,P,\partial)\wedge(T,L,\partial)\xrightarrow{\ \eta\ }(T,L,\partial)\wedge(T,L,\partial)\xrightarrow{\ \pi\ }\frac{(T,L,\partial)}{(M,P,\partial)}\wedge\frac{(T,L,\partial)}{(M,P,\partial)}.$$

Proof. (i) Straightforward.

(ii) By the definition of the exterior product of crossed modules, we set

$$\begin{aligned}(M,P,\partial)\wedge(T,L,\partial)&=(Coker\alpha_1/I_1,P\wedge L,\delta_1),\\(T,L,\partial)\wedge(T,L,\partial)&=(Coker\alpha/I,L\wedge L,\delta),\\\frac{(T,L,\partial)}{(M,P,\partial)}\wedge\frac{(T,L,\partial)}{(M,P,\partial)}&=(Coker\bar{\alpha}/\bar{I},\frac{L}{P}\wedge\frac{L}{P},\bar{\delta}).\end{aligned}$$

Clearly, $\pi = (\pi_1, \pi_2)$ is a surjective morphism where $\pi_1 : Coker\alpha/I \longrightarrow Coker\bar{\alpha}/\bar{I}$ and $\pi_2 : L \wedge L \longrightarrow L/P \wedge L/P$ are the Lie homomorphisms induced by the natural epimorphisms $T \longrightarrow T/M$ and $L \longrightarrow L/P$. Letting $\beta_1 : M \otimes L \longrightarrow T \otimes L$ and $\beta_2 : P \otimes T \longrightarrow L \otimes T$ be the functional homomorphisms, the map $\varphi : M \otimes L \rtimes P \otimes T \longrightarrow T \otimes L \rtimes L \otimes T$ given by $\varphi(v, w) = (\beta_1(v), \beta_2(w))$ is a Lie homomorphism with $\varphi(\mathrm{Im}\alpha_1) \subseteq \mathrm{Im}\alpha$. Hence φ induces the Lie homomorphism $\widetilde{\varphi} : Coker\alpha_1 \longrightarrow Coker\alpha$ with $\widetilde{\varphi}(I_1) \subseteq I$, which in turn induces a Lie homomorphism $\eta_1 : Coker\alpha_1/I_1 \longrightarrow Coker\alpha/I$. Considering the functional homomorphism $\eta_2 : P \wedge L \longrightarrow L \wedge L$, one easily sees that $\eta = (\eta_1, \eta_2)$ is a morphism of crossed modules. It now remains to show that $\mathrm{Im}\eta = \ker\pi$. Invoking Proposition 2.1(vii), $\mathrm{Im}\eta_2 = \ker\pi_2$. Also, an easy verification indicates that $\mathrm{Im}\eta_1$ is an ideal in $Coker\alpha/I$ contained in $ker\pi_1$. To prove that this inclusion is an equality, we construct an isomorphism $\widetilde{\kappa} : Coker\bar{\alpha}/\bar{I} \longrightarrow Coker\eta_1$. Let us define the maps

$$e_1 : T/M \times L/P \longrightarrow Coker\eta_1 \qquad \text{and} \qquad e_2 : L/P \times T/M \longrightarrow Coker\eta_1$$

by $e_1(t + M, l + P) = \overline{(t \otimes l, 0)} + \mathrm{Im}\eta_1$ and $e_2(l + P, t + M) = \overline{(0, l \otimes t)} + \mathrm{Im}\eta_1$. For $t_1, t_2 \in T$, $l_1, l_2 \in L$, $m \in M$, $p \in P$, if $t_1 = t_2 + m$ and $l_1 = l_2 + p$, then we have

$$\begin{aligned} e_1(t_1 + M, l_1 + P) &= \overline{((t_2 + m) \otimes (l_2 + p), 0)} + \mathrm{Im}\eta_1 \\ &= \overline{(t_2 \otimes l_2, 0)} + \overline{(t_2 \otimes p, 0)} + \overline{(m \otimes l_2, 0)} + \overline{(m \otimes p, 0)} + \mathrm{Im}\eta_1 \\ &= \overline{(t_2 \otimes l_2, 0)} + \overline{(t_2 \otimes p, 0)} + \mathrm{Im}\eta_1, \end{aligned}$$

since $\overline{(m \otimes l_2, 0)}, \overline{(m \otimes p, 0)} \in \mathrm{Im}\eta_1$. But we know that $\overline{(t_2 \otimes p, p \otimes t_2)} = 0$ in $Coker\alpha/I$, implying that $\overline{(t_2 \otimes p, 0)} = -\overline{(0, p \otimes t_2)} \in \mathrm{Im}\eta_1$. Then

$$e_1(t_1 + M, l_1 + P) = \overline{(t_2 \otimes l_2, 0)} + \mathrm{Im}\eta_1 = e_1(t_2 + M, l_2 + P).$$

Therefore e_1 and, similarly, e_2 are correctly defined. It is readily checked that e_1 and e_2 are Lie pairings, and the universal property of the tensor product then yields the Lie homomorphisms $\bar{e}_1 : T/M \otimes L/P \longrightarrow Coker\eta_1$ and $\bar{e}_2 : L/P \otimes T/M \longrightarrow Coker\eta_1$. We can now obtain a Lie homomorphism

$$\kappa : (T/M \otimes L/P) \rtimes (L/P \otimes T/M) \longrightarrow Coker\eta_1,$$

defined by $\kappa(x, y) = \bar{e}_1(x) + \bar{e}_2(y)$. As $\mathrm{Im}\bar{\alpha}$ is annihilated by κ, this gives rise to a Lie homomorphism $\bar{\kappa} : Coker\bar{\alpha} \longrightarrow Coker\eta_1$ with $\bar{\kappa}(\bar{I}) = 0$. We thus get a Lie homomorphism $\widetilde{\kappa} : Coker\bar{\alpha}/\bar{I} \longrightarrow Coker\eta_1$. Evidently, $\widetilde{\kappa}$ is an isomorphism with inverse induced by π_2. The proof is complete. ■

Theorem 3.5. *Let (T_1, L_1, ∂_1) and (T_2, L_2, ∂_2) be arbitrary crossed modules. Then*

$$\begin{aligned} &((T_1, L_1, \partial_1) \oplus (T_2, L_2, \partial_2)) \wedge ((T_1, L_1, \partial_1) \oplus (T_2, L_2, \partial_2)) \cong \\ &((T_1, L_1, \partial_1) \wedge (T_1, L_1, \partial_1)) \oplus ((T_2, L_2, \partial_2) \wedge (T_2, L_2, \partial_2)) \oplus ((T_1, L_1, \partial_1)_{ab} \otimes (T_2, L_2, \partial_2)_{ab}). \end{aligned}$$

Proof. Set $\bar{T}_i = T_i/[L_i, T_i]$ and $\bar{L}_i = L_i/[L_i, L_i]$ for $i = 1, 2$. Using the definition of the tensor product of abelian crossed modules given in [6], we suppose that $(T_1, L_1, \partial_1)_{ab} \otimes (T_2, L_2, \partial_2)_{ab} = (Coker\alpha, \bar{L}_1 \otimes \bar{L}_2, \delta)$. Note that $\mathrm{Im}\alpha$ is an ideal of $(\bar{T}_1 \otimes \bar{L}_2) \oplus (\bar{L}_1 \otimes \bar{T}_2)$ generated by elements $(\bar{t}_1 \otimes \overline{\partial_2(t_2)}, -\overline{\partial_1(t_1)} \otimes \bar{t}_2)$ with $t_1 \in T_1$, $t_2 \in T_2$, and δ is induced on $Coker\alpha$ by the homomorphism $\beta : (\bar{T}_1 \otimes \bar{L}_2) \oplus (\bar{L}_1 \otimes \bar{T}_2) \longrightarrow \bar{L}_1 \otimes \bar{L}_2$, $(\bar{t}_1 \otimes \bar{l}_2, \bar{l}_1 \otimes \bar{t}_2) \longmapsto \overline{\partial_1(t_1)} \otimes \bar{l}_2 + \bar{l}_1 \otimes \overline{\partial_2(t_2)}$ (here the bar $\bar{.}$ denotes the equivalence class in each case). By virtue of Proposition 3.3(b), it is sufficient to define an isomorphism (γ_1, γ_2)

$$\begin{array}{ccc} (L_1 \oplus L_2) \wedge (T_1 \oplus T_2) & \xrightarrow{\gamma_1} & (L_1 \wedge T_1) \oplus (L_2 \wedge T_2) \oplus Coker\alpha \\ \Big\downarrow {\scriptstyle id \wedge (\partial_1 \oplus \partial_2)} & & \Big\downarrow {\scriptstyle (id \wedge \partial_1) \oplus (id \wedge \partial_2) \oplus \delta} \\ (L_1 \oplus L_2) \wedge (L_1 \oplus L_2) & \xrightarrow{\gamma_2} & (L_1 \wedge L_1) \oplus (L_2 \wedge L_2) \oplus (\bar{L}_1 \otimes \bar{L}_2). \end{array} \qquad (1)$$

Let us define γ_1 and γ_2 on generators as follows:

$$\gamma_1((l_1, l_2) \wedge (t_1, t_2)) = (l_1 \wedge t_1, l_2 \wedge t_2, (-\bar{t}_1 \otimes \bar{l}_2, \bar{l}_1 \otimes \bar{t}_2) + \mathrm{Im}\alpha),$$
$$\gamma_2((l_1, l_2) \wedge (l'_1, l'_2)) = (l_1 \wedge l'_1, l_2 \wedge l'_2, \bar{l}_1 \otimes \bar{l}'_2 - \bar{l}'_1 \otimes \bar{l}_2).$$

According to Proposition 2.1(v), γ_2 is an isomorphism. Also, it is easy to verify that γ_1 is correctly defined and preserves the defining relations of the exterior product. For instance, we indicate that

$$\gamma_1([(l_1, l_2), (l'_1, l'_2)] \wedge (t_1, t_2)) = \gamma_1((l_1, l_2) \wedge {}^{(l'_1, l'_2)}(t_1, t_2)) - \gamma_1((l'_1, l'_2) \wedge {}^{(l_1, l_2)}(t_1, t_2)). \quad (2)$$

We have

$$\begin{aligned} \gamma_1([(l_1, l_2), (l'_1, l'_2)] \wedge (t_1, t_2)) &= \gamma_1(([l_1, l'_1], [l_2, l'_2]) \wedge (t_1, t_2)) \\ &= ([l_1, l'_1] \wedge t_1, [l_2, l'_2] \wedge t_2, (-\bar{t}_1 \otimes \overline{[l_2, l'_2]}, \overline{[l_1, l'_1]} \otimes \bar{t}_2) + \mathrm{Im}\alpha) \\ &= ([l_1, l'_1] \wedge t_1, [l_2, l'_2] \wedge t_2, \mathrm{Im}\alpha). \end{aligned}$$

On the other hand,

$$\begin{aligned} \gamma_1((l_1, l_2) \wedge {}^{(l'_1, l'_2)}(t_1, t_2)) &- \gamma_1((l'_1, l'_2) \wedge {}^{(l_1, l_2)}(t_1, t_2)) \\ &= (l_1 \wedge {}^{l'_1}t_1 - l'_1 \wedge {}^{l_1}t_1, l_2 \wedge {}^{l'_2}t_2 - l'_2 \wedge {}^{l_2}t_2, (-\overline{{}^{l'_1}t_1} \otimes \bar{l}_2 \\ &+ \overline{{}^{l_1}t_1} \otimes \bar{l}'_2, \bar{l}_1 \otimes \overline{{}^{l'_2}t_2} - \bar{l}'_1 \otimes \overline{{}^{l_2}t_2}) + \mathrm{Im}\alpha) \\ &= ([l_1, l'_1] \wedge t_1, [l_2, l'_2] \wedge t_2, \mathrm{Im}\alpha), \end{aligned}$$

because ${}^{l_1}t_1, {}^{l'_1}t_1 \in [L_1, T_1]$ and ${}^{l_2}t_2, {}^{l'_2}t_2 \in [L_2, T_2]$. Then the equality (2) holds. We now prove that γ_1 is an isomorphism by giving an inverse for it. Consider the canonical homomorphisms

$$\begin{aligned} &\psi_1 : L_1 \wedge T_1 \longrightarrow (L_1 \oplus L_2) \wedge (T_1 \oplus T_2), && l_1 \wedge t_1 \longmapsto (l_1, 0) \wedge (t_1, 0) \\ &\psi_2 : L_2 \wedge T_2 \longrightarrow (L_1 \oplus L_2) \wedge (T_1 \oplus T_2), && l_2 \wedge t_2 \longmapsto (0, l_2) \wedge (0, t_2) \\ &\psi_3 : Coker\alpha \longrightarrow (L_1 \oplus L_2) \wedge (T_1 \oplus T_2), && (\bar{t}_1 \otimes \bar{l}_2, \bar{l}_1 \otimes \bar{t}_2) + \mathrm{Im}\alpha \longmapsto \\ & && \quad - (0, l_2) \wedge (t_1, 0) + (l_1, 0) \wedge (0, t_2). \end{aligned}$$

Evidently ψ_1 and ψ_2 are well-defined. Since, for all $t_1 \in T_1$, $t_2 \in T_2$, $l_1, l'_1 \in L_1$ and $l_2, l'_2 \in L_2$, we have

$$(0, l_2) \wedge ({}^{l_1}t_1, 0) = [(0, l_2), (l_1, 0)] \wedge (t_1, 0) + (l_1, 0) \wedge {}^{(0, l_2)}(t_1, 0) = 0$$

and similarly, $(l_1,0)\wedge(0,{}^{l_2}t_2)=0$, $(0,{}^{l_2}l'_2)\wedge(t_1,0)=0$, $({}^{l_1}l'_1,0)\wedge(0,t_2)=0$ and $(0,\partial_2(t_2))\wedge(t_1,0)+(\partial_1(t_1),0)\wedge(0,t_2)=0$, ψ_3 is also well-defined. Then the homomorphism $\psi=\langle\psi_1,\psi_2,\psi_3\rangle$ in the coproduct

$$\psi:(L_1\wedge T_1)\oplus(L_2\wedge T_2)\oplus Coker\alpha\longrightarrow(L_1\oplus L_2)\wedge(T_1\oplus T_2)$$

is evidently an inverse to γ_1. So, (γ_1,γ_2) is a pair of isomorphisms such that the diagram (1) is commutative. To complete the proof we need to indicate that $\gamma_1({}^{x}y)={}^{\gamma_2(x)}\gamma_1(y)$ for all $x\in(L_1\oplus L_2)\wedge(L_1\oplus L_2)$, $y\in(L_1\oplus L_2)\wedge(T_1\oplus T_2)$. Without loss of generality, we may assume that $x=(l_1,l_2)\wedge(l'_1,l'_2)$ and $y=(l''_1,l''_2)\wedge(t_1,t_2)$. Then we have

$$\begin{aligned}\gamma_1({}^{x}y)&=\gamma_1({}^{[(l_1,l_2),(l'_1,l'_2)]}(l''_1,l''_2)\wedge(t_1,t_2)+(l''_1,l''_2)\wedge{}^{[(l_1,l_2),(l'_1,l'_2)]}(t_1,t_2))\\&=\gamma_1([(l_1,l_2),(l'_1,l'_2)]\wedge{}^{(l''_1,l''_2)}(t_1,t_2))\\&=({}^{l_1}l'_1\wedge{}^{l''_1}t_1,{}^{l_2}l'_2\wedge{}^{l''_2}t_2,(-\overline{{}^{l''_1}t_1}\otimes\overline{{}^{l_2}l'_2},\overline{{}^{l_1}l'_1}\otimes\overline{{}^{l''_2}t_2})+\mathrm{Im}\alpha)\\&=({}^{l_1}l'_1\wedge{}^{l''_1}t_1,{}^{l_2}l'_2\wedge{}^{l''_2}t_2,\mathrm{Im}\alpha),\end{aligned}$$

since ${}^{l''_1}t_1\in[L_1,T_1]$, ${}^{l''_2}t_2\in[L_2,T_2]$. On the other hand,

$$\begin{aligned}{}^{\gamma_2(x)}\gamma_1(y)&={}^{(l_1\wedge l'_1,l_2\wedge l'_2,\bar{l}_1\otimes\bar{l}'_2-\bar{l}'_1\otimes\bar{l}_2)}(l''_1\wedge t_1,l''_2\wedge t_2,(-\bar{t}_1\otimes\bar{l}''_2,\bar{l}''_1\otimes\bar{t}_2)+\mathrm{Im}\alpha)\\&=({}^{l_1\wedge l'_1}(l''_1\wedge t_1),{}^{l_2\wedge l'_2}(l''_2\wedge t_2),{}^{\bar{l}_1\otimes\bar{l}'_2-\bar{l}'_1\otimes\bar{l}_2}(-\bar{t}_1\otimes\bar{l}''_2,\bar{l}''_1\otimes\bar{t}_2)+\mathrm{Im}\alpha)\\&=({}^{l_1}l'_1\wedge{}^{l''_1}t_1,{}^{l_2}l'_2\wedge{}^{l''_2}t_2,(-\overline{{}^{l''_2}t_1}\otimes\overline{{}^{l_1}l'_2}+\overline{{}^{l''_2}t_1}\otimes\overline{{}^{l'_1}l_2},\overline{{}^{l_1}l'_2}\otimes\overline{{}^{l''_1}t_2}-\overline{{}^{l'_1}l_2}\otimes\overline{{}^{l''_1}t_2})+\mathrm{Im}\alpha)\\&=({}^{l_1}l'_1\wedge{}^{l''_1}t_1,{}^{l_2}l'_2\wedge{}^{l''_2}t_2,\mathrm{Im}\alpha),\end{aligned}$$

since ${}^{l_1}l'_2,{}^{l'_1}l_2\in[L_1,L_2]=0$ (note that L_1 and L_2 act on each other trivially). Thus (γ_1,γ_2) is a morphism and the proof of the theorem is complete. ∎

We end this section by analyzing the kernel of the quotient morphism

$$\phi=(\phi_1,\phi_2):(T,L,\partial)\otimes(T,L,\partial)\longrightarrow(T,L,\partial)\wedge(T,L,\partial),$$

under some conditions. To do this, we need the following definition, which is analogous to the definition of the Whitehead's quadratic functor of abelian crossed modules of groups (see [13]).

Definition 3.6. Let (A,B,∂) be an abelian crossed module of Lie algebras, and $B\underline{\otimes}A$ be the quotient of $B\otimes A$ by the subalgebra generated by the elements $\partial(a_1)\otimes a_2-\partial(a_2)\otimes a_1$ with $a_1,a_2\in A$. Then we define $\Gamma(A,B,\partial)$ to be the abelian crossed module $(\bar{\Gamma}(A,B,\partial),\Gamma(B),\partial_\Gamma)$, in which $\bar{\Gamma}(A,B,\partial)$ is the cokernel of the Lie homomorphism $f:A\otimes A\longrightarrow(B\underline{\otimes}A)\oplus\Gamma(A)$ given by $f(a_1\otimes a_2)=(\overline{\partial(a_1)\otimes a_2},-\Delta(a_1\otimes a_2))$, and $\partial_\Gamma(\overline{\overline{b\otimes a},\gamma(a_1)})=\Delta(b\otimes\partial(a))+\gamma(\partial(a_1))$.

Theorem 3.7. *Let (T,L,∂) be a crossed module such that ∂ is onto or L acts trivially on T. Then there is a natural exact sequence*

$$\Gamma((T,L,\partial)_{ab})\xrightarrow{(\psi_1,\psi_2)}(T,L,\partial)\otimes(T,L,\partial)\xrightarrow{(\phi_1,\phi_2)}(T,L,\partial)\wedge(T,L,\partial).$$

To prove this, we use the following lemma.

Lemma 3.8. *With the assumptions of Theorem* 3.7*, we have*
(*i*) $\partial({}^{l}t_1)\otimes t_2 = -\partial(t_2)\otimes {}^{l}t_1$*, for all* $t_1,t_2\in T$ *and* $l\in L$*.*
(*ii*) $(T,L,\partial)\square(T,L,\partial)\subseteq Z((T,L,\partial)\otimes(T,L,\partial))$*.*

Proof. We prove the lemma for the case of ∂ is onto. The proof for the other case is analogous.

(*i*) Take $t_1,t_2\in T$ and $l\in L$. By hypothesis, there is an element $t\in T$ in the pre-image of l via ∂. We then have

$$\begin{aligned}\partial({}^{l}t_1)\otimes t_2 &= [\partial(t),\partial(t_1)]\otimes t_2 = \partial(t)\otimes {}^{t_1}t_2 - \partial(t_1)\otimes {}^{t}t_2\\ &= {}^{t_2}\partial(t)\otimes t_1 - {}^{t_1}\partial(t)\otimes t_2 + {}^{t}\partial(t_1)\otimes t_2 - {}^{t_2}\partial(t_1)\otimes t\\ &= -{}^{t}\partial(t_2)\otimes t_1 + {}^{t_1}\partial(t_2)\otimes t + 2\partial({}^{l}t_1)\otimes t_2\\ &= \partial(t_2)\otimes {}^{t}t_1 + 2\partial({}^{l}t_1)\otimes t_2 = \partial(t_2)\otimes {}^{l}t_1 + 2\partial({}^{l}t_1)\otimes t_2.\end{aligned}$$

So, $\partial({}^{l}t_1)\otimes t_2 = -\partial(t_2)\otimes {}^{l}t_1$ (notice that if the field Λ is of characteristic two, then $\partial(t_2)=-\partial(t_2)$ and the result again holds).

(*ii*) Taking into account that $L\square L\subseteq Z(L\otimes L)$ and that $L\square L$ acts trivially on $Coker\alpha$, it is referred that $L\square L\subseteq Z(L\otimes L)\cap st_{L\otimes L}(Coker\alpha)$. We now verify that ${}^{x}y=0$ for all $x\in L\otimes L$ and $y\in I$. Let $x=l_1\otimes l_2$ and $y=(t\otimes l, l\otimes t+\partial(t_0)\otimes t_0)+\mathrm{Im}\alpha$ for $l,l_1,l_2\in L$, $t_0,t\in T$. By the assumption, $l_1=\partial(t_1)$ for some $t_1\in T$ and then

$$\begin{aligned}{}^{l_1\otimes l_2}(t\otimes l, l\otimes t+\partial(t_0)\otimes t_0) &= ({}^{l}t\otimes[l_1,l_2],[l_1,l_2]\otimes {}^{l}t + [l_1,l_2]\otimes {}^{\partial(t_0)}t_0)\\ &= ({}^{l}t\otimes[l_1,l_2],[l_1,l_2]\otimes {}^{l}t) = ({}^{l}t\otimes[\partial(t_1),l_2],[\partial(t_1),l_2]\otimes {}^{l}t)\\ &= -({}^{l}t\otimes\partial({}^{l_2}t_1),\partial({}^{l_2}t_1)\otimes {}^{l}t)\\ &= -({}^{l}t\otimes\partial({}^{l_2}t_1),-\partial({}^{l}t)\otimes {}^{l_2}t_1)\in \mathrm{Im}\alpha.\end{aligned}$$

So, I is contained in $Coker\alpha^{L\otimes L}$, as desired. ■

Proof of Theorem 3.7. We only prove the result for the case of ∂ is onto. The proof for the other case is identical. As $\ker(\phi_1,\phi_2)=(T,L,\partial)\square(T,L,\partial)$, we try to define a natural surjective morphism (ψ_1,ψ_2)

$$\begin{array}{ccc}\overline{\Gamma}((T,L,\partial)_{ab}) & \xrightarrow{\psi_1} & I\\ \downarrow{\scriptstyle \partial_\Gamma} & & \downarrow{\scriptstyle \delta}\\ \Gamma(L_{ab}) & \xrightarrow{\psi_2} & L\square L.\end{array}$$

The second component ψ_2 is the epimorphism given in Proposition 2.1(*viii*). Assuming that $\bar{T}=T/[L,T]$ and $\bar{L}=L_{ab}$, we now construct ψ_1, which will be induced on $\overline{\Gamma}((T,L,\partial)_{ab})=Coker f$ by a homomorphism $\langle\widetilde{g_1},\widetilde{g_2}\rangle$

$$\begin{array}{ccccc}\bar{T}\otimes\bar{T} & \xrightarrow{f} & (\bar{L}\underline{\otimes}\bar{T})\oplus\Gamma(\bar{T}) & \twoheadrightarrow & \overline{\Gamma}((T,L,\partial)_{ab}).\\ & & \downarrow{\scriptstyle \langle\widetilde{g_1},\widetilde{g_2}\rangle} & \swarrow{\scriptstyle \psi_1} & \\ & & I & & \end{array}$$

Here f is the homomorphism introduced in Definition 3.6.

We first define the map $g_1 : \bar{L} \times \bar{T} \longrightarrow I$ by $g_1(\bar{l}, \bar{t}) = (t \otimes l, l \otimes t) + \mathrm{Im}\alpha$. Then g_1 is well-defined, because if $l_1 = l_2 + x$, $t_1 = t_2 + y$ for some $x \in [L, L]$, $y \in [L, T]$, then

$$\begin{aligned} g_1(\bar{l}_1, \bar{t}_1) &= ((t_2 + y) \otimes (l_2 + x), (l_2 + x) \otimes (t_2 + y)) + \mathrm{Im}\alpha \\ &= (t_2 \otimes l_2, l_2 \otimes t_2) + (y \otimes l_2, l_2 \otimes y) + (t_2 \otimes x, x \otimes t_2) + (y \otimes x, x \otimes y) + \mathrm{Im}\alpha. \end{aligned}$$

The surjectivity of ∂ implies that $l_2 = \partial(t_2')$ for some $t_2' \in T$ and consequently, invoking Lemma 3.8(i), it is easy to check that

$$(y \otimes l_2, l_2 \otimes y) = (y \otimes \partial(t_2'), \partial(t_2') \otimes y) = (y \otimes \partial(t_2'), -\partial(y) \otimes t_2') \in \mathrm{Im}\alpha.$$

Analogously, $(t_2 \otimes x, x \otimes t_2) \in \mathrm{Im}\alpha$. We claim that $(y \otimes x, x \otimes y) = 0$. Note that the element y may be expressed as a finite sum of elements of the form ${}^{l}t$ with $l \in L, t \in T$. The triviality of the action of $L \otimes L$ on I yields that $({}^{l}t \otimes x, x \otimes {}^{l}t) = {}^{x'}(t \otimes l, l \otimes t) = 0$, where x' is any element in the pre-image x via the commutator map $\lambda_L : L \otimes L \longrightarrow L$. So we must have $(y \otimes x, x \otimes y) = 0$, as claimed. Then $g_1(\bar{l}_1, \bar{t}_1) = (t_2 \otimes l_2, l_2 \otimes t_2) + \mathrm{Im}\alpha = g_1(\bar{l}_2, \bar{t}_2)$. According to Lemma 3.8$(ii)$, the ideal I is abelian, contained in $Coker\alpha^{L\otimes L}$. So g_1 is a Lie pairing and the universal property of $\bar{L} \otimes \bar{T}$ thus yields a Lie homomorphism $\bar{g}_1 : \bar{L} \otimes \bar{T} \longrightarrow I$. As $\bar{g}_1$ annihilates the ideal generated by the elements $\overline{\partial(t_1)} \otimes \bar{t}_2 - \overline{\partial(t_2)} \otimes \bar{t}_1$, $t_1, t_2 \in T$, we obtain a Lie homomorphism $\widetilde{g_1} : \bar{L} \underline{\otimes} \bar{T} \longrightarrow I$ induced by $\bar{g}_1$.

We now define the map $g_2 : \bar{T} \longrightarrow I$ by $g_2(\bar{t}) = (0, \partial(t) \otimes t) + \mathrm{Im}\alpha$. Then g_2 is well-defined, since if $t_1 = t_2 + y$ for some $y \in [L, T]$, then

$$\begin{aligned} g_2(\bar{t}_1) &= (0, \partial(t_2 + y) \otimes (t_2 + y)) + \mathrm{Im}\alpha \\ &= (0, \partial(t_2) \otimes t_2) + (0, \partial(t_2) \otimes y) + (0, \partial(y) \otimes t_2) + (0, \partial(y) \otimes y) + \mathrm{Im}\alpha. \end{aligned}$$

As above, using the surjectivity of ∂ and applying Lemma 3.8(i), one derives that $\partial(y) \otimes y = 0$ and $\partial(t_2) \otimes y = -\partial(y) \otimes t_2$. Then $g_2(\bar{t}_1) = g_2(\bar{t}_2)$ and so g_2 induces a well-defined map $\widetilde{g_2} : \Gamma(\bar{T}) \longrightarrow I$. Obviously, $\widetilde{g_2}$ preserves the defining relations of $\Gamma(-)$ and so is a Λ-module homomorphism. Again Lemma 3.8(ii) shows that $\widetilde{g_2}$ is a Lie homomorphism.

We therefore conclude the homomorphism $\widetilde{g} = \langle \widetilde{g_1}, \widetilde{g_2} \rangle$ in the coproduct

$$\widetilde{g} : (\bar{L} \otimes \bar{T}) \oplus \Gamma(\bar{T}) \longrightarrow I.$$

Since $\widetilde{g}(\mathrm{Im} f) = 0$, $\widetilde{g}$ induces the Lie homomorphism ψ_1. It is routine to see that ψ_1 is onto, and the pair (ψ_1, ψ_2) is a crossed module morphism. The proof of the theorem is complete. ■

The main achievement in [5] is the identification of the universal central extension of a perfect crossed module in terms of the tensor product of Lie algebras. This observation, together with the above theorem, provides the following interesting corollary.

Corollary 3.9. *With the assumptions of Theorem* 3.7*, if* (T, L, ∂) *is perfect, then*

$$\ker(\tau_1, \tau_2) \rightarrowtail (T, L, \partial) \otimes (T, L, \partial) \xrightarrow{(\tau_1, \tau_2)} (T, L, \partial)$$

is a universal central extension of (T, L, ∂)*, where the commutator morphism*

(τ_1, τ_2) is given by $\tau_1((t_1 \otimes l_1, l_2 \otimes t_2)+Im\alpha) = {}^{l_2}t_2 - {}^{l_1}t_1$ and $\tau_2(l_1 \otimes l_2) = [l_1, l_2]$ for all $t_1, t_2 \in T$, $l_1, l_2 \in L$.

Proof. In view of [5, Theorem 9], we need only to show that $(T, L, \partial) \otimes (T, L, \partial)$ is isomorphic to $(L \otimes T, L \otimes L, id \otimes \partial)$. But this isomorphism follows readily from Propositions 2.1(vi), 3.3(b) and Theorem 3.7, as required. ■

4. Applications to the second homology of crossed modules

This section is devoted to deal with the connection between the exterior products with the second homologies of crossed modules, and generalize some known results of the second homology of Lie algebras to the second homology of crossed modules.

Casas et al. [3] prove that the underlying set functor $\mathcal{U} : \mathbf{XLie} \longrightarrow \mathbf{Set}$, $\mathcal{U}(T, L, \partial) = T \times L$, which assigns to any crossed module (T, L, ∂) the cartesian product of the underlying sets of the Lie algebras T and L is tripleable. In addition, $\mathcal{U}$ has a left adjoint $\mathcal{F} : \mathbf{Set} \longrightarrow \mathbf{XLie}$ given by

$$\mathcal{F}(X) = (\overline{FX}, FX * FX, i),$$

where FX denotes the free Lie algebra over the set X, $FX * FX$ is the coproduct of FX on itself, with the natural inclusions $i_1, i_2 : FX \longrightarrow FX * FX$, and $\overline{FX}$ is the kernel of the retraction $p_2 : FX * FX \longrightarrow FX$ determined by the conditions $p_2 i_1 = 0$ and $p_2 i_2 = id_{FX}$. So, for any set X, the free crossed module $\mathcal{F}(X)$ on X is a projective object with respect to regular epimorphisms in the category of crossed modules of Lie algebras, and consequently every crossed module (T, L, ∂) admits a projective presentation by means of the counit of the adjunction $\mathcal{FU}(T, L, \partial) \longrightarrow (T, L, \partial)$. Also, a Lie algebra version of [1; Proposition 3] shows that every projective crossed module (Y, F, μ) is aspherical and the Lie algebras Y, F, $F/\mu(Y)$ are free. We can therefore consider the homomorphism μ of the projective crossed module (Y, F, μ) as an inclusion, and identify the elements of Y with their images in F.

Let $(V, R, \mu) \rightarrowtail (Y, F, \mu) \twoheadrightarrow (T, L, \partial)$ be a projective presentation of the crossed module (T, L, ∂). It is proved in [3] that the second homology crossed module of (T, L, ∂) is, up to isomorphism, the abelian crossed module

$$\left(\frac{V \cap [F, Y]}{[R, Y] + [F, V]}, \frac{R \cap F'}{[F, R]}, \bar{\mu}\right).$$

Regarding a Lie algebra L as a crossed module in the two usual ways, we have $H_2(L, L, id) = (H_2(L), H_2(L), id)$ or $H_2(0, L, i) = (0, H_2(L), i)$, which gives the classical Hopf's formula [9]. Moreover, if (Y, F, μ) is a projective crossed module, then $H_2(Y, F, \mu) = 0$.

The following proposition is fundamental in proving Theorem 4.2

Proposition 4.1. *If (Y, F, μ) is a projective crossed module, then*

$$(Y, F, \mu) \wedge (Y, F, \mu) \cong ([F, Y], F', \mu).$$

Proof. Since F/Y is a free Lie algebra, the kernel of the epimorphism $\lambda_Y : F \wedge Y \longrightarrow [F, Y]$ is trivial, thanks to [8; Theorems 34 and 35(iii)]. Also, invoking [7; Proposition 1.2], the homomorphism $\lambda_F : F \wedge F \longrightarrow F'$ is an isomorphism. Now, it is straightforward to see that the pair (λ_Y, λ_F) is an isomorphism from $(F \wedge Y, F \wedge F, id \wedge \mu)$ onto $([F, Y], F', \mu)$, as desired. ■

Given an inclusion crossed module (N, L, i), Ellis [8] proves that the second relative Chevalley-Eilenberg homology $H_2(L; N)$ is isomorphic to the kernel of the commutator map $L \wedge N \longrightarrow L$. In particular, $H_2(L) \cong \ker(L \wedge L \longrightarrow L)$. We therefore deduce from [3; Propositions 2.3.1 and 2.3.2] that

$$H_2(N, L, i) \cong (\ker(L \wedge N \longrightarrow L), \ker(L \wedge L \longrightarrow L), id \wedge i).$$

In the following result, we observe that the above isomorphism holds for all crossed modules.

Theorem 4.2. *Let $(V, R, \mu) \rightarrowtail (Y, F, \mu) \twoheadrightarrow (T, L, \partial)$ be a projective presentation of the crossed module (T, L, ∂). Then*

$$(T, L, \partial) \wedge (T, L, \partial) \cong \frac{(Y, F, \mu)'}{[(Y, F, \mu), (V, R, \mu)]}.$$

In particular, $H_2(T, L, \partial) \cong \ker((T, L, \partial) \wedge (T, L, \partial) \longrightarrow (T, L, \partial))$.

Proof. Assume that $(\widetilde{\varphi}, id)$, (η_1, η_2) and (λ_Y, λ_F) are the morphisms introduced in the proofs of Propositions 3.3(b), 3.4(ii) and 4.1, respectively. The composite morphism

$$\psi = (\psi_1, \psi_2) : (V, R, \mu) \wedge (Y, F, \mu) \xrightarrow{(\eta_1, \eta_2)} (Y, F, \mu) \wedge (Y, F, \mu) \xrightarrow{(\widetilde{\varphi}, id)} (F \wedge Y, F \wedge F, id \wedge \mu)$$

yields an exact sequence

$$(V, R, \mu) \wedge (Y, F, \mu) \xrightarrow{\psi} (F \wedge Y, F \wedge F, id \wedge \mu) \twoheadrightarrow (L \wedge T, L \wedge L, id \wedge \partial).$$

As $\mathrm{Im}\psi_1$ is an ideal of $F \wedge Y$ generated by all $f \wedge v$, $r \wedge y$ with $f \in F$, $v \in V$, $r \in R$, $y \in Y$, and $\mathrm{Im}\psi_2$ is an ideal of $F \wedge F$ generated by all $r \wedge f$ with $r \in R$, $f \in F$, the morphism (λ_Y, λ_F) maps the crossed submodule $(\mathrm{Im}\psi_1, \mathrm{Im}\psi_2, id \wedge \mu)$ isomorphically onto $([F, V] + [R, Y], [R, F], \mu)$. We therefore conclude from Proposition 4.1 that

$$(T, L, \partial) \wedge (T, L, \partial) \cong \frac{(F \wedge Y, F \wedge F, id \wedge \mu)}{\mathrm{Im}\psi} \cong \frac{(Y, F, \mu)'}{[(Y, F, \mu), (V, R, \mu)]},$$

and the proof of the theorem is complete. ■

As an immediate consequence of the above theorem, we deduce that if the crossed module (T, L, ∂) is abelian, then $H_2(T, L, \partial) \cong (T, L, \partial) \wedge (T, L, \partial)$.

The following important corollary is a Lie algebra crossed module analogue of a result of Pirashvili [13].

Corollary 4.3. *Let (T_1, L_1, ∂_1) and (T_2, L_2, ∂_2) be arbitrary crossed modules. Then $H_2((T_1, L_1, \partial_1) \oplus (T_2, L_2, \partial_2))$ is isomorphic to*

$$H_2(T_1, L_1, \partial_1) \oplus H_2(T_2, L_2, \partial_2) \oplus ((T_1, L_1, \partial_1)_{ab} \otimes (T_2, L_2, \partial_2)_{ab}).$$

Proof. This follows from Theorems 3.5 and 4.2. ■

From the above corollary and the proof of Theorem 3.5, one sees that for inclusion crossed modules (N_1, L_1, i_1) and (N_2, L_2, i_2), there are isomorphisms

$$\begin{aligned} H_2(L_1 \oplus L_2) &\cong H_2(L_1) \oplus H_2(L_2) \oplus (\overline{L}_1 \otimes \overline{L}_2), \\ H_2(L_1 \oplus L_2; N_1 \oplus N_2) &\cong H_2(L_1; N_1) \oplus H_2(L_2; N_2) \\ &\quad \oplus \Big(\frac{(\overline{N}_1 \otimes \overline{L}_2) \oplus (\overline{L}_1 \otimes \overline{N}_2)}{\langle(\overline{n_1} \otimes \overline{i_2(n_2)}, -\overline{i_1(n_1)} \otimes \overline{n_2}) | n_1 \in N_1, n_2 \in N_2\rangle}\Big). \end{aligned}$$

It is established by Casas and Ladra [4] that any crossed module (T, L, ∂) with ideal crossed submodule (M, P, ∂) gives rise to a natural five term exact sequence in homology of crossed modules

$$H_2(T, L, \partial) \longrightarrow H_2(\tfrac{T}{M}, \tfrac{L}{P}, \bar{\partial}) \longrightarrow \tfrac{(M,P,\partial)}{[(M,P,\partial),(T,L,\partial)]} \longrightarrow H_1(T, L, \partial) \twoheadrightarrow H_1(\tfrac{T}{M}, \tfrac{L}{P}, \bar{\partial}). \quad (3)$$

In [6], this sequence is extended one term to the left when (M, P, ∂) is central. We now generalize the result by proving the following

Theorem 4.4. *The exact sequence* (3) *is extended one term to the left by the natural exact sequence*

$$\ker((T, L, \partial) \wedge (M, P, \partial) \longrightarrow (T, L, \partial)) \longrightarrow H_2(T, L, \partial) \longrightarrow H_2(\frac{T}{M}, \frac{L}{P}, \bar{\partial}).$$

Proof. We have the following commutative diagram

$$\begin{array}{ccccc} \ker((T,L,\partial) \wedge (M,P,\partial) \longrightarrow (T,L,\partial)) & \rightarrowtail & (T,L,\partial) \wedge (M,P,\partial) & \twoheadrightarrow & (M,P,\partial) \cap (T,L,\partial)' \\ & & \downarrow & & \downarrow \\ \ker((T,L,\partial) \wedge (T,L,\partial) \longrightarrow (T,L,\partial)) & \rightarrowtail & (T,L,\partial) \wedge (T,L,\partial) & \twoheadrightarrow & (T,L,\partial)' \\ & & \downarrow & & \downarrow \\ \ker(\frac{(T,L,\partial)}{(M,P,\partial)} \wedge \frac{(T,L,\partial)}{(M,P,\partial)} \longrightarrow \frac{(T,L,\partial)}{(M,P,\partial)}) & \rightarrowtail & \frac{(T,L,\partial)}{(M,P,\partial)} \wedge \frac{(T,L,\partial)}{(M,P,\partial)} & \twoheadrightarrow & (\frac{(T,L,\partial)}{(M,P,\partial)})', \end{array}$$

in which, the rows and, thanks to Proposition 3.4(ii), the columns are exact. We thus get an exact sequence

$$\begin{aligned} \ker((T, L, \partial) \wedge (M, P, \partial) \longrightarrow (T, L, \partial)) &\longrightarrow \ker((T, L, \partial) \wedge (T, L, \partial) \longrightarrow (T, L, \partial)) \\ &\longrightarrow \ker(\frac{(T, L, \partial)}{(M, P, \partial)} \wedge \frac{(T, L, \partial)}{(M, P, \partial)} \longrightarrow \frac{(T, L, \partial)}{(M, P, \partial)}). \end{aligned}$$

The result now follows from Theorem 4.2. ■

Viewing a Lie algebra L as a crossed module in any of the two usual ways, we get the following exact sequence in integral homology of Lie algebras [8]

$$\ker(L \wedge P \longrightarrow L) \to H_2(L) \to H_2(L/P) \to P/[L, P] \longrightarrow H_1(L) \twoheadrightarrow H_1(L/P),$$

in which P is an ideal in L.

Corollary 4.5. *Let* (T, L, ∂) *be a perfect crossed module. If the Lie algebras* T *and* L *are finite dimensional, then* $H_2(L \otimes T, L \otimes L, id \otimes \partial) = 0$.

Proof. By [5; Lemma 8 and Theorem 9], the crossed module $(L{\otimes}T, L{\otimes}L, id{\otimes}\partial)$ is perfect, and

$$\ker(\lambda_T,\lambda_L) \rightarrowtail (L\otimes T, L\otimes L, id\otimes\partial) \xrightarrow{(\lambda_T,\lambda_L)} (T,L,\partial) \qquad (4)$$

is the universal central extension by (T,L,∂), where $\ker(\lambda_T,\lambda_L) \cong H_2(T,L,\partial)$ thanks to Proposition 2.1(vi) and Theorem 4.2. Also, using Proposition 3.4(i), $(L\otimes T, L\otimes L, id\otimes\partial)\wedge\ker(\lambda_T,\lambda_L) = 0$. Applying Theorem 4.4 to the sequence (4) and using the above results, it now follows that $H_2(L\otimes T, L\otimes L, id\otimes\partial) = 0$, as required. ■

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