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Cartan Decompositions and Semigroups of Simple Lie Groups

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Abstract. Let G be a split real connected simple Lie group and S a semigroup of G that contains a subgroup $G(\alpha)$ for an arbitrary root α, isomorphic to $\mathrm{SL}(2,\mathbb{R})$. We present a Cartan decomposition of the Lie algebra of G, *related to* α, invariant by the adjoint action of the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ that allows to characterize some properties of the Lie saturate of the semigroup S. We give necessary and sufficient conditions for S to be equal to the whole group G.
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1. Introduction

In this paper, we give necessary and sufficient conditions for a semigroup S of a real finite-dimensional connected simple Lie group G, to be equal to the whole group G, where S is generated by $\{\exp tX : X \in \Gamma, t \geq 0\}$ and Γ is a subset of the Lie algebra of G. The problem of semigroups that generate the whole group has several motivations, in particular in control theory, where the controllability problem for invariant systems is translated into the semigroup generation problem. Important work was performed to obtain controllability conditions on Γ (i.e., such that $S = G$) on various particular classes of Lie groups ([2], [10], [15] and [20]). It is known that the Lie algebra generated by Γ, $\mathrm{Lie}(\Gamma)$ is equal to the Lie algebra L of G is a necessary condition to have S generates G and if the group G is compact or Γ is symmetric ($\Gamma = -\Gamma$), then it is also sufficient. It is also known that S generates G if and only if the Lie saturate of Γ, $LS(\Gamma)$ is equal to the Lie algebra of G ([10], [16], [17] and [20]). The Lie saturate of Γ, is the largest subset of $\mathrm{Lie}(\Gamma)$ such that the closures of the semigroups generated by Γ and by $LS(\Gamma)$ are equal. Previous papers ([6], [7], and [8]) give sufficient conditions on a subset Γ of the form $\Gamma = \{A, B, -B\}$, where the pair (A, B) is generic in $L \times L$ and satisfies $\mathrm{Lie}(A, B) = L$, so that S generates G, where G is a semi-simple Lie group with finite-center. In those papers the conditions satisfied by the pair

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(A, B) imply the fact that there exists a root α (maximal root, which is long) such that the Lie saturate of Γ contains the plane $\mathbb{R}E_\alpha \oplus \mathbb{R}E_{-\alpha}$. In [18] it was proved for a complex simple Lie group using the geometry of flag manifolds that if S has nonempty interior and S contains a subgroup $G(\alpha)$, for an arbitrary root α then S is equal to G. In another direction, this problem of semigroups that generate semi-simple Lie groups was studied in [12] and [13] and recently in [1].

Simple Lie algebras are classified using Dynkin diagrams, which encode the geometric structure of root systems. Every Dynkin diagram with only simple edges, namely $\mathbf{A_n}$, $n \geq 1$; $\mathbf{D_n}$, $n \geq 4$; $\mathbf{E_6}$; $\mathbf{E_7}$ or $\mathbf{E_8}$, admits only long roots. For the remaining Dynkin diagrams with double (resp. triple) edges, namely $\mathbf{B_n}$, $n \geq 2$; $\mathbf{C_n}$, $n \geq 3$ and $\mathbf{F_4}$ (resp. $\mathbf{G_2}$), there are two orbits which are given by the long and short roots, respectively.

In this paper, we consider the subgroup of the Weyl group $\mathcal{W}_\alpha$, generated by the reflexions r_α and $r_{-\alpha}$, for a root α and we show that the action of $\mathcal{W}_\alpha$ on the root system $\mathcal{R}$ allows to decompose $\mathcal{R}$ in disjoint unions, $\mathcal{R} = \widehat{\mathcal{R}}'_1(\alpha) \cup \widetilde{\mathcal{R}}'_1(\alpha) \cup \mathcal{R}'_2(\alpha) \cup \mathcal{R}'_3(\alpha) \cup \mathcal{R}''(\alpha) \cup \{\alpha, -\alpha\}$ (see Definition 1), using the Cartan integers $\beta(H_\alpha)$, for a root β. This decomposition of $\mathcal{R}$ gives a decomposition of the Lie algebra $L = \mathfrak{p}(\alpha) \oplus \mathfrak{k}(\alpha)$, which is invariant by the adjoint action of the Lie algebra $\mathrm{sl}_\alpha(2, \mathbb{R})$ spanned by $\mathbb{R}E_\alpha$ and $\mathbb{R}E_{-\alpha}$, where $\mathbb{R}E_\alpha$ is the root subspace of L for α, and L is the split real form (noncompact) of a complex simple Lie algebra. We present a Cartan decomposition of L *related to* α, which allows to characterize some properties of the Lie saturate of semigroups S that contain the subgroup $G(\alpha)$, generated by the union of the one-parameter groups $\{\exp tE_\alpha : t \in \mathbb{R}\}$ and $\{\exp tE_{-\alpha} : t \in \mathbb{R}\}$, and to study S. We also present some conditions on a subset Γ of L which ensure the fact that $\mathrm{Lie}(\Gamma) = L$ and thus the semigroup S generated by Γ has nonempty interior.

It is proved that if S contains a subgroup $G(\alpha)$, then S generates the group G if and only if Γ generates its Lie algebra L, for any long root α when L is of any type, and for any short root α in the case of type $\mathbf{G_2}$. In the remaining cases, namely $\mathbf{B_n}$, $n \geq 2$, $\mathbf{C_n}$, $n \geq 3$ and $\mathbf{F_4}$, it is proved that if S contains a subgroup $G(\alpha)$ for a short root α and $\mathbb{R}(LS(\Gamma) \cap \mathfrak{p}_2^+(\alpha))$ is contained in $LS(\Gamma)$, where $\mathfrak{p}_2^+(\alpha)$ is the abelian subalgebra of L generated by the root spaces $\mathbb{R}E_\delta$, such that $\delta(H_\alpha) = -2$ (see Definition 2), then S generates G if and only if Γ generates L. In the case of type $\mathbf{C_n}$, $n \geq 2$ and G with finite-center, we show that if S contains a subgroup $G(\alpha)$, for a short root α and S contains a one-parameter group $\{\exp tE_\delta,\ t \in \mathbb{R}\}$, for a root $\delta \in \mathcal{R}_2(\alpha) \cup \widetilde{\mathcal{R}}_1(\alpha)$ (see Definition 1), then S generates G if and only if Γ generates L.
We illustrate all these results of Cartan decomposition related to a root α and semigroups, when G is the real symplectic Lie group of type $\mathbf{C_n}$, $n \geq 2$.

These results generalize the recent results in [5] concerning the case when $\Gamma = \{A + u_1E_\alpha + u_2E_{-\alpha},\ A \in L, u_1, u_2 \in \mathbb{R}\}$ and the previous results in [4] when $\Gamma = \{A, B, -B\}$ for a generic pair $(A, B) \in L \times L$.
Recently, in [19], using the fundamental groups of flag manifolds the same result is proved for a long root α when L is of type $\mathbf{A_n}$, $n \geq 2$ or L of type $\mathbf{C_n}$, $n \geq 3$, and for a short root α in the case of type $\mathbf{G_2}$. Also, in the case of type $\mathbf{C_n}$, the author give an example of semigroup S that has nonempty interior and S contains

a semigroup $G(\alpha)$ for a short root α, but S is not equal to G.

Here, we give in the cases $\mathbf{B_2}$ and $\mathbf{B_3}$ concrete examples of semigroups S generated by some sets Γ, S has nonempty interior and it contains a semigroup $G(\alpha)$ for a short root α but $S \neq G$. Also, we indicate how this example in $\mathbf{B_2}$ induces the existence of such semigroups S in the cases $\mathbf{B_n}$, $\mathbf{C_n}$, $n \geq 3$ and $\mathbf{F_4}$, using the fact that every Dynkin diagram of those cases contains the Dynkin diagram $\mathbf{B_2} = \mathbf{C_2}$.

The structure of this paper is as follows. In Section 2, we summarize some basic results of semi-simple Lie algebras and the associated root systems. In Section 3, we present some definitions and properties concerning the Cartan decomposition related to an arbitrary root, of a complex simple Lie algebra and of its split real form. We also specify the notations used in this paper. In Section 4, we give the statement and the proofs of the main results and present some examples.

2. Basic notions of semi-simple Lie algebras

Let L be a real semi-simple finite-dimensional Lie algebra and $L_{\mathbb{C}} = L \otimes_{\mathbb{R}} \mathbb{C}$, its complexification Lie algebra. Consider a Cartan subalgebra $\mathfrak{h}$ of $L_{\mathbb{C}}$ and $\mathcal{R}$ a root system associated to $(L_{\mathbb{C}}, \mathfrak{h})$. We refer to [3], [9] and [21] for more details.

The root space decomposition of the complex semi-simple Lie algebra $L_{\mathbb{C}}$ is given by $L_{\mathbb{C}} = \mathfrak{h} \oplus \sum_{\delta \in \mathcal{R}} L_\delta$, where $L_\delta = \{X \in L_{\mathbb{C}} : \forall H \in \mathfrak{h}, [H, X] = \delta(H)X, \delta(H) \in \mathbb{C}\}$ is the root subspace for the root δ. Clearly, $[\mathfrak{h}, L_\delta] \subset L_\delta$. It is known that $dim_{\mathbb{C}} L_\delta = 1$, $[L_\delta, L_{-\delta}] \subset \mathfrak{h}$ and for any $\delta, \gamma \in \mathcal{R}$, $[L_\delta, L_\gamma] = L_{\delta+\gamma}$ (resp. $\{0\}$) if $\delta + \gamma \in \mathcal{R}$ (resp. $\delta + \gamma \notin \mathcal{R}$ and $\delta + \gamma \neq 0$). For every $\delta \in \mathcal{R} \subset \mathfrak{h}^*$, the dual space of $\mathfrak{h}$, there exists a unique element $h_\delta \in \mathfrak{h}$ such that for any $H \in \mathfrak{h}$, $\delta(H) = \mathrm{Kill}(h_\delta; H)$ and $\delta(h_\delta) \neq 0$, where $\mathrm{Kill}(\ ;\ )$ is the Cartan-Killing symmetric and nondegenerate bilinear form on $L_{\mathbb{C}}$. Moreover, for every $\delta \in \mathcal{R}$, there exist unique elements $H_\delta \in \mathfrak{h}$ and $E_\delta \in L_\delta$ such that $\delta(H_\delta) = 2$, $\mathrm{Kill}(E_\delta; E_{-\delta}) = 1$ and $[E_\delta, E_{-\delta}] = H_\delta$. Also, for any root δ, γ if $\delta + \gamma \in \mathcal{R}$ then $[E_\delta, E_\gamma] = N_{\delta,\gamma} E_{\delta+\gamma}$, where $N_{\delta,\gamma}$ is a nonzero constant.

The reflection r_δ on $\mathfrak{h}^*$, defined by $r_\delta(\lambda) = \lambda - \lambda(H_\delta)\delta$, for every $\lambda \in \mathfrak{h}^*$, satisfies that for any root γ, $r_\delta(\gamma) = \gamma - \gamma(H_\delta)\delta$ is a root. The constants $\gamma(H_\delta)$ are integer, called the Cartan integers and satisfy $-3 \leq \gamma(H_\delta) \leq 3$. We know that $\gamma(H_\delta) = 2\frac{<\delta,\gamma>}{<\delta,\delta>}$, where $<,>$ denotes the nondegenerate symmetric bilinear form on $\mathfrak{h}^*$, defined by $<\delta, \gamma> = \mathrm{Kill}(h_\delta; h_\gamma)$. Moreover, for nonproportional roots δ and γ, there exist two unique nonnegative integers $p(\delta, \gamma), q(\delta, \gamma) \leq 3$ such that $\gamma(H_\delta) = p(\delta, \gamma) - q(\delta, \gamma)$ and for any integer m, $-p(\delta, \gamma) \leq m \leq q(\delta, \gamma)$, we have $m\delta + \gamma$ is a root.

Simple Lie algebras are classified using Dynkin diagrams, which encode the geometric structure of root systems. A Dynkin diagram for a simple Lie algebra is a connected graph, where the vertices are primitive roots (a primitive root is a positive root which can not be written as a sum of two positive roots) and the number of edges between two primitive roots δ and γ is equal to $\delta(H_\gamma)\gamma(H_\delta)$, which belongs to $\{0, 1, 2, 3\}$. For the Dynkin diagrams that have only simple edges, namely $\mathbf{A_n}$, $n \geq 1$; $\mathbf{D_n}$, $n \geq 4$; $\mathbf{E_6}$; $\mathbf{E_7}$ or $\mathbf{E_8}$, the action of the Weyl group $\mathcal{W}$, generated by the reflections r_δ, $\delta \in \mathcal{R}$, on the set of roots $\mathcal{R}$ is transitive.

Also, every root is long. In this case, for nonproportional roots δ and γ we have $-1 \leq \gamma(H_\delta) \leq 1$. For the remaining Dynkin diagrams with two (resp. three) edges $\mathbf{B_n}$, $n \geq 2$; $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$ (resp. $\mathbf{G_2}$) there are two orbits which are given by the short and long roots respectively, and for nonproportional roots δ and γ we have $-2 \leq \gamma(H_\delta) \leq 2$ (resp. $-3 \leq \gamma(H_\delta) \leq 3$); also, if δ is long and γ is short, then $<\delta,\delta>= 2<\gamma,\gamma>$ (resp. $<\delta,\delta>= 3<\gamma,\gamma>$).

There exists one split real form of $L_{\mathbb{C}}$ (which is not compact) isomorphic to the following real semi-simple Lie algebra:

$$\mathcal{L} = \mathfrak{h}_{\mathbb{R}} \oplus \sum_{\delta\in\mathcal{R}} \mathcal{L}_\delta, \ \textit{where} \ \ \mathcal{L}_\delta = \mathbb{R}E_\delta \ \ \textit{and} \ \ \mathfrak{h}_{\mathbb{R}} = \sum_{\delta\in\mathcal{R}} \mathbb{R}H_\delta.$$

$\mathcal{R}$ is a root system associated to $(\mathcal{L}, \mathfrak{h}_{\mathbb{R}})$.

3. Cartan decomposition related to an arbitrary root

For an arbitrary root α, we consider the subgroup of the Weyl group $\mathcal{W}_\alpha$, generated by the reflexions r_α and $r_{-\alpha}$. We will show how $\mathcal{W}_\alpha$ acts on the root system $\mathcal{R}$ using the Cartan integers $\beta(H_\alpha)$, for a root β, namely the fact that $r_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha$, with $\beta(H_\alpha) \in \{0, \pm1, \pm2, \pm3\}$ and $\beta(H_\alpha) = p(\alpha,\beta) - q(\alpha,\beta)$, with $p(\alpha,\beta), q(\alpha,\beta) \leq 3$. The action of $\mathcal{W}_\alpha$ on $\mathcal{R}$ allows to give a decomposition of the Lie algebra $L = \mathfrak{p}(\alpha) \oplus \mathfrak{k}(\alpha)$, which is invariant by the adjoint action of the Lie algebra $\mathrm{sl}_\alpha(2,\mathbb{R})$. We will precise when we obtain a Cartan decomposition of L. This Cartan decomposition related to a root α will be used to study semigroups S of G, when S contains the subgroup $G(\alpha)$ whose Lie algebra is $\mathrm{sl}_\alpha(2,\mathbb{R})$.

We consider the following sets of roots associated to a root α. Those sets have been considered in [6] and [7] to study semigroups and also in [14] to characterize the minimum orbit of the adjoint representation, when α is a maximal root, which is a long root. Recently in [4] and [5] some properties of those sets for an arbitrary long or short root are developed, mainly the existence of a Cartan decomposition of L related to an arbitrary root. Here, we give some other properties of this Cartan decomposition which are essential for our proofs.

3.1. Root properties related to an arbitrary root.

Definition 1. Let α be an arbitrary root. Consider the following:

$$\mathcal{R}'(\alpha) = \{\beta \in \mathcal{R} : p(\alpha,\beta) \geq 1 \ \textit{or} \ q(\alpha,\beta) \geq 1\}.$$

$$\mathcal{R}''(\alpha) = \{\beta \in \mathcal{R} : p(\alpha,\beta) = q(\alpha,\beta) = 0\}.$$

$$\mathcal{R}'_1(\alpha) = \{\beta \in \mathcal{R}'(\alpha) : p(\alpha,\beta) = 1 \ \textit{or} \ q(\alpha,\beta) = 1 \ \textit{and} \ p(\alpha,\beta)q(\alpha,\beta) \leq 1\}.$$

$$\widetilde{\mathcal{R}}'_1(\alpha) = \{\beta \in \mathcal{R}'_1(\alpha) : p(\alpha,\beta) = q(\alpha,\beta) = 1\},$$

$$\widehat{\mathcal{R}}'_1(\alpha) = \{\beta \in \mathcal{R}'_1(\alpha) : p(\alpha,\beta)q(\alpha,\beta) = 0\}.$$

$$\mathcal{R}'_2(\alpha) = \{\beta \in \mathcal{R}'(\alpha) : p(\alpha,\beta) = 2 \ \textit{or} \ q(\alpha,\beta) = 2 \ \textit{and} \ p(\alpha,\beta) \leq 2, \ q(\alpha,\beta) \leq 2\}.$$

$$\mathcal{R}'_3(\alpha) = \{\beta \in \mathcal{R}'(\alpha) : p(\alpha,\beta) = 3 \ \textit{or} \ q(\alpha,\beta) = 3\}.$$

Recall that the integers $p(\alpha,\beta)$ and $q(\alpha,\beta)$ are defined for two nonproportional roots α and β, (i.e., $\beta \neq \alpha$ and $\beta \neq -\alpha$). Therefore, a root $\beta \in \mathcal{R}'(\alpha)$ (resp. $\beta \in \mathcal{R}''(\alpha)$) signifies that β is not proportional to α and $\alpha+\beta$ or $-\alpha+\beta$ is a root (resp. $\alpha+\beta$ and $-\alpha+\beta$ are both not roots).

Also, a root $\beta \in \widetilde{\mathcal{R}}'_1(\alpha)$ (resp. $\beta \in \widehat{\mathcal{R}}'_1(\alpha)$) means that $\beta \in \mathcal{R}'_1(\alpha)$ and $\alpha+\beta$, $-\alpha+\beta$ are both roots (resp. either $\alpha+\beta \in \mathcal{R}$ or $-\alpha+\beta \in \mathcal{R}$).

We verify easily the following disjoint unions:

$$\mathcal{R} = \mathcal{R}'(\alpha) \,\dot\cup\, \mathcal{R}''(\alpha) \,\dot\cup\, \{\alpha, -\alpha\}\ ,$$

$$\mathcal{R}'(\alpha) = \mathcal{R}'_1(\alpha) \,\dot\cup\, \mathcal{R}'_2(\alpha) \,\dot\cup\, \mathcal{R}'_3(\alpha) \textit{ and}$$

$$\mathcal{R}'_1(\alpha) = \widetilde{\mathcal{R}}'_1(\alpha) \,\dot\cup\, \widehat{\mathcal{R}}'_1(\alpha).$$

Also, since $\mathcal{R} = -\mathcal{R}$ (i.e., $\delta \in \mathcal{R}$ if and only if $-\delta \in \mathcal{R}$), we have $\alpha+\beta \in \mathcal{R}$ is equivalent to $-\alpha-\beta \in \mathcal{R}$ and $-\alpha+\beta \in \mathcal{R}$ is equivalent to $\alpha-\beta \in \mathcal{R}$. Then by definition we get the following:

$$\mathcal{R}'(\alpha) = -\mathcal{R}'(\alpha) = \mathcal{R}'(-\alpha),\ \mathcal{R}''(\alpha) = -\mathcal{R}''(\alpha) = \mathcal{R}''(-\alpha),$$

$$\mathcal{R}'_i(\alpha) = -\mathcal{R}'_i(\alpha) = \mathcal{R}'_i(-\alpha),\ \textit{for every}\ \ 1 \leq i \leq 3,$$

$$\widetilde{\mathcal{R}}'_1(\alpha) = -\widetilde{\mathcal{R}}'_1(\alpha) = \widetilde{\mathcal{R}}'_1(-\alpha)\ \textit{ and }\ \widehat{\mathcal{R}}'_1(\alpha) = -\widehat{\mathcal{R}}'_1(\alpha) = \widehat{\mathcal{R}}'_1(-\alpha).$$

On the other hand, for any $w \in \mathcal{W}$, we know that β is a root if and only if $w\beta$ is a root, then we deduce that

$$\mathcal{R}'(w\alpha) = w\mathcal{R}'(\alpha),\ \mathcal{R}''(w\alpha) = w\mathcal{R}''(\alpha) \textit{ and } \mathcal{R}'_i(w\alpha) = w\mathcal{R}'_i(\alpha),\ \textit{for } 1 \leq i \leq 3.$$

Those sets $\mathcal{R}''(\alpha)$, $\mathcal{R}'(\alpha)$ and $\mathcal{R}'_i(\alpha)$, $1 \leq i \leq 3$, are characterized in the following lemmas for an arbitrary long or short root α. The proofs are given in [4] and [5].

Recall that for every Dynkin diagram with only simple edges namely, $\mathbf{A_n}$, $n \geq 1$; $\mathbf{D_n}$, $n \geq 4$; $\mathbf{E_6}$; $\mathbf{E_7}$ or $\mathbf{E_8}$, it is known that any root is long. However, for the remaining Dynkin diagrams with double (resp. triple) edges namely, $\mathbf{B_n}$, $n \geq 2$; $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$ (resp. $\mathbf{G_2}$), there are two orbits (by the action of the Weyl group on the set of roots) which are given by the long and short roots, respectively.

Lemma 3.1. *Let α be an arbitrary* **long** *root and $\beta \in \mathcal{R}$.*

1. *If $\alpha+\beta \in \mathcal{R}$ then $-\alpha+\beta \notin \mathcal{R}$, and $2\alpha+\beta \in \mathcal{R}$ if and only if $\beta = -\alpha$.*
2. *$\beta \in \mathcal{R}'(\alpha)$ if and only if $\beta(H_\alpha) = \pm 1$.*
3. *$\beta \in \mathcal{R}''(\alpha)$ if and only if $\beta(H_\alpha) = 0$ (i.e., $\mathcal{R}''(\alpha)$ is the orthogonal to α).*
4. *For any $\beta \in \mathcal{R}$, if $\alpha+\beta \in \mathcal{R}$ then $\alpha+\beta$ is long (resp. short) if β is long (resp. short).*

Notice that the assertion 1 gives that for any long root α, we have $\widetilde{\mathcal{R}}'_1(\alpha) = \mathcal{R}'_2(\alpha) = \mathcal{R}'_3(\alpha) = \emptyset$ and then $\mathcal{R}'(\alpha) = \mathcal{R}'_1(\alpha) = \widehat{\mathcal{R}}'_1(\alpha)$. Also, for any short root α, in the cases of types $\mathbf{B_n}$, $n \geq 2$, $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$ (resp. $\mathbf{G_2}$), it is known that $3\alpha + \beta$ is not a root for any root β and thus $\mathcal{R}'_3(\alpha) = \emptyset$ (resp. $3\alpha + \beta$ can be a root and then $\mathcal{R}'_3(\alpha) \neq \emptyset$).

Lemma 3.2. *Let α be a* **short** *root and $\beta \in \mathcal{R}$.*

1. *In the case of type $\mathbf{G_2}$, we have $\mathcal{R}'_1(\alpha) = \emptyset$ and*
   - *i. $\beta \in \mathcal{R}'_3(\alpha)$ if and only if $\beta(H_\alpha) = \pm 3$,*
   - *ii. $\beta \in \mathcal{R}'_2(\alpha)$ if and only if $\beta(H_\alpha) = \pm 1$,*
   - *iii. $\beta \in \mathcal{R}''(\alpha)$ if and only if $\beta(H_\alpha) = 0$,*
   - *iv. if $\beta \in \mathcal{R}'_2(\alpha)$ such that $2\alpha + \beta \in \mathcal{R}$, then $-\alpha + \beta \in \mathcal{R}'_3(\alpha)$.*
2. *In the case of type $\mathbf{B_n}$, $n \geq 2$, we have $\widehat{\mathcal{R}}'_1(\alpha) = \emptyset$, $\mathcal{R}'_1(\alpha) = \widetilde{\mathcal{R}}'_1(\alpha)$ and*
   - *i. $\beta \in \mathcal{R}'_2(\alpha)$ if and only if β is long and $\beta(H_\alpha) = \pm 2$,*
   - *ii. $\beta \in \widetilde{\mathcal{R}}'_1(\alpha)$ if and only if β is short and $\beta(H_\alpha) = 0$. Also, $\pm\alpha + \beta \in \mathcal{R}'_2(\alpha)$,*
   - *iii. $\beta \in \mathcal{R}''(\alpha)$ if and only if β is long and $\beta(H_\alpha) = 0$.*

Lemma 3.3. *In the case $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$ let α be a* **short** *root and $\beta \in \mathcal{R}$.*

1. *$\beta \in \mathcal{R}'_2(\alpha)$ if and only if β is long and $\beta(H_\alpha) = \pm 2$.*
2. *$\beta \in \widehat{\mathcal{R}}'_1(\alpha)$ if and only if $\beta(H_\alpha) = \pm 1$.*
3. *If $\beta \in \widetilde{\mathcal{R}}'_1(\alpha)$ then β is short and $\beta(H_\alpha) = 0$. Also, $\pm\alpha + \beta \in \mathcal{R}'_2(\alpha)$.*
4. *If $\beta \in \mathcal{R}''(\alpha)$ then $\beta(H_\alpha) = 0$ and β is long (resp. long or short) in the case $\mathbf{F_4}$ (resp. in the case $\mathbf{C_n}$, $n \geq 3$).*
5. *Set $\mathcal{R} = \widehat{\mathcal{R}}'_1(\alpha) \dot{\cup} \overline{\mathcal{R}}(\alpha)$, where $\overline{\mathcal{R}} = \widetilde{\mathcal{R}}'_1(\alpha) \dot{\cup} \mathcal{R}'_2(\alpha) \dot{\cup} \mathcal{R}''(\alpha) \dot{\cup} \{\pm\alpha\}$. Then $(\widehat{\mathcal{R}}'_1(\alpha) + \widehat{\mathcal{R}}'_1(\alpha)) \cap \mathcal{R} \subset \overline{\mathcal{R}}(\alpha)$, $(\widehat{\mathcal{R}}'_1(\alpha) + \overline{\mathcal{R}}(\alpha)) \cap \mathcal{R} \subset \widehat{\mathcal{R}}'_1(\alpha)$ and $(\overline{\mathcal{R}}(\alpha) + \overline{\mathcal{R}}(\alpha)) \cap \mathcal{R} \subset \overline{\mathcal{R}}(\alpha)$.*

The assertion 5 follows from the above assertions, because for any $\beta \in \widehat{\mathcal{R}}'_1(\alpha)$ and $\gamma \in \overline{\mathcal{R}}(\alpha)$, we have $\beta(H_\alpha) \in \{1, -1\}$ and $\gamma(H_\alpha) \in \{0, 2, -2\}$ and then we get $(\beta + \gamma)(H_\alpha)$.

Using the above lemmas and the fact that $r_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha$ with $\beta(H_\alpha) = 2\frac{<\alpha,\beta>}{<\alpha,\alpha>}$, one can verify the following remark.

Remark 3.4. Let α be an arbitrary root. The action of $\mathcal{W}_\alpha$ on $\mathcal{R}$ leaves fixed every root $\beta \in \mathcal{R}''(\alpha) \cup \widetilde{\mathcal{R}}'_1(\alpha)$ which is orthogonal to α (as $\beta(H_\alpha) = 0$). Then, $\mathcal{W}_\alpha \mathcal{R}''(\alpha) = \mathcal{R}''(\alpha)$ and $\mathcal{W}_\alpha \widetilde{\mathcal{R}}'_1(\alpha) = \widetilde{\mathcal{R}}'_1(\alpha)$. Also, we have the following:

$$\mathcal{W}_\alpha \mathcal{R}'_3(\alpha) = \mathcal{R}'_3(\alpha),\ \mathcal{W}_\alpha \mathcal{R}'_2(\alpha) = \mathcal{R}'_2(\alpha) \text{ and } \mathcal{W}_\alpha \widehat{\mathcal{R}}'_1(\alpha) = \widehat{\mathcal{R}}'_1(\alpha).$$

3.2. Root space decomposition related to an arbitrary root.

The root space decomposition of the complex simple Lie algebra $L_{\mathbb{C}}$ is:

$$L_{\mathbb{C}} = \mathfrak{h} \oplus \sum_{\delta \in \mathcal{R}} L_\delta, \ where\ L_\delta = \{X \in L_{\mathbb{C}} : \forall H \in \mathfrak{h}, [H, X] = \delta(H)X, \delta(H) \in \mathbb{C}\}.$$

If L is the split real form of a complex simple Lie algebra then

$$L = \mathfrak{h}_{\mathbb{R}} \oplus \sum_{\delta \in \mathcal{R}} L_\delta, \ \ where\ \ L_\delta = \mathbb{R}E_\delta \ \ and \ \ \mathfrak{h}_{\mathbb{R}} = \sum_{\delta \in \mathcal{R}} \mathbb{R}H_\delta.$$

$\mathcal{R}$ is a root system associated to $(L, \mathfrak{h}_{\mathbb{R}})$. To simplify we denote $\mathfrak{h}_{\mathbb{R}}$ by $\mathfrak{h}$. Since $\mathcal{R} = \mathcal{R}'(\alpha) \dot\cup \mathcal{R}''(\alpha) \dot\cup \{\pm\alpha\}$, the root space decomposition of $X \in L$ *related to* α is given by:

$$X = X_0 + X_\alpha + X_{-\alpha} + X'(\alpha) + X''(\alpha), \ \ where$$

$$X'(\alpha) = \sum_{\delta \in \mathcal{R}'(\alpha)} X_\delta, \ \ X''(\alpha) = \sum_{\delta \in \mathcal{R}''(\alpha)} X_\delta, \ \ X_\delta \in \mathbb{R}E_\delta \ \ and \ \ X_0 \in \mathfrak{h}.$$

Since $\mathcal{R}'(\alpha) = \mathcal{R}'_1(\alpha) \dot\cup \mathcal{R}'_2(\alpha) \dot\cup \mathcal{R}'_3(\alpha)$, we can write

$$X'(\alpha) = X'_1(\alpha) + X'_2(\alpha) + X'_3(\alpha), \ where\ \ X'_i(\alpha) = \sum_{\delta \in \mathcal{R}'_i(\alpha)} X_\delta, \ for\ i \in \{1, 2, 3\}.$$

Also, since $\mathcal{R}'_1(\alpha) = \widehat{\mathcal{R}}'_1(\alpha) \dot\cup \widetilde{\mathcal{R}}'_1(\alpha)$ we present

$$X'_1(\alpha) = \widehat{X}'_1(\alpha) + \widetilde{X}'_1(\alpha), \ where\ \ \widehat{X}'_1(\alpha) = \sum_{\delta \in \widehat{\mathcal{R}}'_1(\alpha)} X_\delta \ \ and \ \ \widetilde{X}'_1(\alpha) = \sum_{\delta \in \widetilde{\mathcal{R}}'_1(\alpha)} X_\delta.$$

Remark 3.5. Observe that if α is long, by Lemma 3.1.1, $\mathcal{R}'_2(\alpha) = \mathcal{R}'_3(\alpha) = \widetilde{\mathcal{R}}'_1(\alpha) = \emptyset$ and then $\mathcal{R}'(\alpha) = \mathcal{R}'_1(\alpha) = \widehat{\mathcal{R}}'_1(\alpha)$. Therefore, for any $X \in L$, in the root space decomposition related to α, we have $X'_2(\alpha) = X'_3(\alpha) = \widetilde{X}'_1(\alpha) = 0$ and thus $X'(\alpha) = X'_1(\alpha) = \widehat{X}'_1(\alpha)$. Moreover, if α is short in the case of type $\mathbf{B_n}$, $n \geq 2$, $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$, then we have $\mathcal{R}'_3(\alpha) = \emptyset$ and thus $X'_3(\alpha) = 0$. Also, in the case of type $\mathbf{B_n}$, $n \geq 2$, we have by Lemma 3.2 that $\widehat{\mathcal{R}}'_1(\alpha) = \emptyset$ and $\mathcal{R}'_1(\alpha) = \widetilde{\mathcal{R}}'_1(\alpha)$. Therefore, $\widehat{X}'_1(\alpha) = 0$ and $X'_1(\alpha) = \widetilde{X}'_1(\alpha)$. Finally, if α is short in the case $\mathbf{G_2}$, by Lemma 3.2, we get $\mathcal{R}'_1(\alpha) = \emptyset$ and then $X'_1(\alpha) = 0$.

Notation Write $\widehat{X}'_1(\alpha) = \widehat{X}'_1(\alpha)^+ + \widehat{X}'_1(\alpha)^-$, where $\widehat{X}'_1(\alpha)^+$ (resp. $\widehat{X}'_1(\alpha)^-$) is the sum of the (nonzero) components X_β, with $\beta \in \widehat{\mathcal{R}}'_1(\alpha)$ such that $\alpha + \beta \in \mathcal{R}$ and thus $\beta(H_\alpha) = -1$ (resp. $-\alpha + \beta \in \mathcal{R}$ and then $\beta(H_\alpha) = 1$).
Similarly, for $m \in \{2, 3\}$, write $X'_m(\alpha) = X'_m(\alpha)^+ + X'_m(\alpha)^-$, where $X'_m(\alpha)^+$ (resp. $X'_m(\alpha)^-$) is the sum of the (nonzero) components X_β, with $\beta \in \mathcal{R}'_m(\alpha)$ such that $m\alpha + \beta \in \mathcal{R}$ (resp. $-m\alpha + \beta \in \mathcal{R}$). The case $m = 3$ concerns only the case when α is short in the case $\mathbf{G_2}$, in the other cases we have $X'_3(\alpha) = 0$.

3.3. Cartan decomposition related to an arbitrary root.

Definition 2. Let α be an arbitrary root. Consider the following decomposition of L related to α, $L = \mathfrak{p}(\alpha) \oplus \mathfrak{k}(\alpha)$, where

$$\mathfrak{p}(\alpha) = \sum_{\beta \in \mathcal{R}'(\alpha)} \mathbb{R}E_\beta \quad \textit{and} \quad \mathfrak{k}(\alpha) = \mathfrak{h} \oplus \sum_{\beta \in \mathcal{R}''(\alpha) \cup \{\pm\alpha\}} \mathbb{R}E_\beta.$$

$$\textit{Set} \;\; \mathfrak{p}(\alpha) = \sum_{1 \le i \le 3} \mathfrak{p}_i(\alpha), \; \textit{where} \;\; \mathfrak{p}_i(\alpha) = \sum_{\beta \in \mathcal{R}'_i(\alpha)} \mathbb{R}E_\beta, \;\; 1 \le i \le 3 \; \textit{and}$$

$$\mathfrak{p}_1(\alpha) = \widehat{\mathfrak{p}}_1(\alpha) + \widetilde{\mathfrak{p}}_1(\alpha), \; \textit{where} \;\; \widehat{\mathfrak{p}}_1(\alpha) = \sum_{\beta \in \widehat{\mathcal{R}}'_1(\alpha)} \mathbb{R}E_\beta \;\; \textit{and} \;\; \widetilde{\mathfrak{p}}_1(\alpha) = \sum_{\beta \in \widetilde{\mathcal{R}}'_1(\alpha)} \mathbb{R}E_\beta$$

Set $\widehat{\mathfrak{p}}_1(\alpha) = \widehat{\mathfrak{p}}_1^+(\alpha) \oplus \widehat{\mathfrak{p}}_1^-(\alpha)$ and $\mathfrak{p}_m(\alpha) = \mathfrak{p}_m^+(\alpha) \oplus \mathfrak{p}_m^-(\alpha)$ for $m \in \{2,3\}$, where

$$\widehat{\mathfrak{p}}_1^+(\alpha) = \sum_{\beta \in \widehat{\mathcal{R}}'_1(\alpha)/\alpha+\beta \in \mathcal{R}} \mathbb{R}E_\beta \quad , \quad \widehat{\mathfrak{p}}_1^-(\alpha) = \sum_{\beta \in \widehat{\mathcal{R}}'_1(\alpha)/-\alpha+\beta \in \mathcal{R}} \mathbb{R}E_\beta \; ,$$

$$\mathfrak{p}_m^+(\alpha) = \sum_{\beta \in \mathcal{R}'_m(\alpha)/m\alpha+\beta \in \mathcal{R}} \mathbb{R}E_\beta \quad \textit{and} \quad \mathfrak{p}_m^-(\alpha) = \sum_{\beta \in \mathcal{R}'_m(\alpha)/-m\alpha+\beta \in \mathcal{R}} \mathbb{R}E_\beta.$$

Remark 3.6. Let us precise that for an arbitrary root α, $\mathfrak{p}(\alpha) = \{0\}$ if and only if $\dim L = 3$. Clearly, $\dim L = 3$ is equivalent to the fact that the Dynkin diagram of G is of type $\mathbf{A_1}$. In this case, we have $\mathcal{R} = \{\alpha, -\alpha\}$ and the sets $\mathcal{R}'(\alpha)$ and $\mathcal{R}''(\alpha)$ are empty. For any $X \in L$, we have $X = X_0 + X_\alpha + X_{-\alpha}$. L is isomorphic to $\mathrm{sl}_\alpha(2,\mathbb{R}) = \mathbb{R}H_\alpha \oplus \mathbb{R}E_\alpha \oplus \mathbb{R}E_{-\alpha}$, where $H_\alpha = [E_\alpha, E_{-\alpha}]$. In this case $\mathfrak{p}(\alpha) = \{0\}$ and $L = \mathfrak{k}(\alpha)$. But in this case if $LS(\Gamma)$ contains the subalgebra $\mathrm{sl}_\alpha(2,\mathbb{R}) = \mathrm{Lie}(E_\alpha, E_{-\alpha}) = L$, then clearly $LS(\Gamma) = L$ and hence $S = G$. We will be interested in the case where $\dim L \neq 3$.

If $\dim L \neq 3$, then $\mathcal{R}'(\alpha) \neq \emptyset$ and thus $\dim \mathfrak{p}(\alpha) \ge 2$. Indeed, if $\dim L \neq 3$, then the Dynkin diagram of L is not of type $\mathbf{A_1}$ and it contains at least two primitive roots. It satisfies that for every primitive root δ, there exists a primitive root β with at least one edge between δ and β. The number of edges between two primitive roots δ and β is equal to $\delta(H_\beta)\beta(H_\delta)$. This implies that $\delta(H_\beta) \neq 0$ and (since $\delta - \beta$ is not a root) then $\delta + \beta$ is a root. Hence, $\beta \in \mathcal{R}'(\delta)$. Moreover, we know that for a root α there exist a primitive root δ and $w \in \mathcal{W}$ (the Weyl group) such that $\alpha = w\delta$. Therefore, $\mathcal{R}'(\alpha) = \mathcal{R}'(w\delta) = w\mathcal{R}'(\delta)$. We deduce that $w\beta \in \mathcal{R}'(\alpha)$ and then $-w\beta \in \mathcal{R}'(\alpha)$. By Definition 2, $\dim \mathfrak{p}(\alpha) \ge 2$.

The following result was proved in [5] by using Lemma 3.1, 3.2 and 3.3. It shows that for any root α, there exists a Cartan decomposition of L, related to α. Here, we develop some properties of this Cartan decomposition of L that we will use in the proofs of our results.

Theorem 3.7. *[5] Let L be the split real form of a complex simple Lie algebra such that $\dim L \neq 3$. For any root α we have:*

1. $\dim \mathfrak{p}(\alpha) \neq 0$.

2. $L = \mathfrak{p}(\alpha) \oplus \mathfrak{k}(\alpha)$, $[\mathfrak{k}(\alpha), \mathfrak{k}(\alpha)] \subset \mathfrak{k}(\alpha)$ *and* $[\mathfrak{p}(\alpha), \mathfrak{k}(\alpha)] = \mathfrak{p}(\alpha)$.

3. $[\mathfrak{p}(\alpha), \mathfrak{p}(\alpha)] \subset \mathfrak{k}(\alpha)$ *and then* $L = \mathfrak{p}(\alpha) \oplus \mathfrak{k}(\alpha)$ *is a Cartan decomposition for:*

   i. *any long root* α.

   ii. *any short root* α *in the case of type* $\mathbf{G_2}$ *or* $\mathbf{B_n}$, $n \geq 2$.

4. $L = \widehat{\mathfrak{p}}_1(\alpha) \oplus \widehat{\mathfrak{k}}(\alpha)$ *is a Cartan decomposition of* L, *for any short root* α, *in the case of type* $\mathbf{C_n}$, $n \geq 3$ *or* $\mathbf{F_4}$, *where* $\widehat{\mathfrak{k}}(\alpha) = \widetilde{\mathfrak{p}}_1(\alpha) \oplus \mathfrak{p}_2(\alpha) \oplus \mathfrak{k}(\alpha)$ *is a Lie subalgebra of* L *and* $\dim \widehat{\mathfrak{p}}_1(\alpha) \neq 0$.

We briefly recall the idea of the proof of Theorem 3.7. The assertion 1 is given in Remark 3.6. The assertion 2 is just a consequence of the Jacobi identity and using Definition 1 and Definition 2 . Indeed, let $\beta \in \mathcal{R}'(\alpha)$ and $\gamma, \delta \in \mathcal{R}''(\alpha)$. If $\gamma + \delta$ is a root then it belongs to $\mathcal{R}''(\alpha)$, because $[E_{\pm\alpha}, [E_\gamma, E_\delta]] = 0$. If $\beta + \gamma$ is a root then it can not belong to $\mathcal{R}''(\alpha)$, because if not, we get $\beta = (\beta + \gamma) - \gamma \in \mathcal{R}''(\alpha)$, which contradicts the fact that $\mathcal{R}'(\alpha) \cap \mathcal{R}''(\alpha) = \emptyset$. Also, $\beta + \gamma \neq \pm\alpha$, otherwise we get $\gamma = \beta \pm \alpha$ which contradicts the fact that $\gamma \in \mathcal{R}''(\alpha)$. The assertion 4 arises immediately from the assertion 5 of Lemma 3.3. The assertion 3 can be deduced by the following proposition that shows when the decomposition of L related to α is a Cartan decomposition, using the above Lemmas.

Proposition 3.8. *1. For any long root* α, *when* L *is of any type and for any short root* α *in the case of type* $\mathbf{B_n}$, $n \geq 2$, $\mathbf{C_n}$, $n \geq 3$ *or* $\mathbf{F_4}$ *we have:*

   i. $[E_\alpha, \widehat{\mathfrak{p}}_1^+(\alpha)] = \widehat{\mathfrak{p}}_1^-(\alpha)$, $[E_{-\alpha}, \widehat{\mathfrak{p}}_1^-(\alpha)] = \widehat{\mathfrak{p}}_1^+(\alpha)$, $[E_{\pm\alpha}, \widetilde{\mathfrak{p}}_1(\alpha)] = \mathfrak{p}_2(\alpha)$ *and* $[E_{\pm\alpha}, \mathfrak{p}_2(\alpha)] = \widetilde{\mathfrak{p}}_1(\alpha)$.

   ii. $[\widehat{\mathfrak{p}}_1^+(\alpha), \widehat{\mathfrak{p}}_1^+(\alpha)] \subset \mathfrak{p}_2^+(\alpha) \oplus \mathbb{R}E_{-\alpha}$ *and* $[\widehat{\mathfrak{p}}_1^+(\alpha), \widetilde{\mathfrak{p}}_1(\alpha)] \subset \widehat{\mathfrak{p}}_1^+(\alpha)$.

   iii. $[\mathfrak{p}_2^+(\alpha), \mathfrak{p}_2^+(\alpha)] = \{0\}$, $[\mathfrak{p}_2^+(\alpha), \widetilde{\mathfrak{p}}_1(\alpha)] \subset \mathfrak{p}_2^+(\alpha) \oplus \mathbb{R}E_{-\alpha}$ *and* $[\mathfrak{p}_2^+(\alpha), \widehat{\mathfrak{p}}_1^-(\alpha)] \subset \widehat{\mathfrak{p}}_1^+(\alpha)$.

   iv. $\widehat{\mathfrak{p}}_1^+(\alpha) = \widehat{\mathfrak{p}}_1^-(-\alpha)$ *and* $\widehat{\mathfrak{p}}_1^-(\alpha) = \widehat{\mathfrak{p}}_1^+(-\alpha)$.

2. *For any long root* α, $\mathfrak{h}(\alpha) = \mathrm{Lie}(\widehat{\mathfrak{p}}_1^+(\alpha) \oplus \mathbb{R}E_{-\alpha})$ *is a Heisenberg algebra. Also,* $\mathfrak{h}(-\alpha) = \mathrm{Lie}(\widehat{\mathfrak{p}}_1^-(\alpha) \oplus \mathbb{R}E_\alpha)$.

3. *For any short root* α *in the case of type* $\mathbf{G_2}$, *we have*

   i. $[E_{\pm\alpha}, \mathfrak{p}_2(\alpha)] = \mathfrak{p}_3(\alpha)$ *and* $[E_{\pm\alpha}, \mathfrak{p}_3(\alpha)] = \mathfrak{p}_2(\alpha)$,

   ii. $[\mathfrak{p}_2^+(\alpha), \mathfrak{p}_2^+(\alpha)] = \mathbb{R}E_{-\alpha}$, $[\mathfrak{p}_3^+(\alpha), \mathfrak{p}_3^+(\alpha)] = \{0\}$ *and* $[\mathfrak{p}_3^+(\alpha), \mathfrak{p}_2^-(\alpha)] = \mathbb{R}E_{-\alpha}$,

   iii. $[\mathfrak{p}_m^+(\alpha), \mathfrak{p}_m^-(\alpha)] \subset \mathfrak{k}(\alpha)$, *for* $m \in \{2, 3\}$.

4. *For any root* α, $[\mathrm{sl}_\alpha(2, \mathbb{R}), \mathfrak{p}(\alpha)] = \mathfrak{p}(\alpha)$ *and* $[\mathrm{sl}_\alpha(2, \mathbb{R}), \mathfrak{k}(\alpha)] \subset \mathrm{sl}_\alpha(2, \mathbb{R})$.

Proof. Let $\beta \in \mathcal{R}'(\alpha)$, for a root α. Then by definition $\alpha + \beta$ or $-\alpha + \beta$ is a root. If $\alpha + \beta \in \mathcal{R}$, then since $-\alpha + (\alpha + \beta) = \beta$ we get $\alpha + \beta \in \mathcal{R}'(\alpha)$. Then $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \in \mathfrak{p}(\alpha)$. If α is long then clearly $\widetilde{\mathfrak{p}}_1(\alpha) = \mathfrak{p}_2(\alpha) = \{0\}$,

and for any $\beta \in \mathcal{R}'(\alpha) = \widehat{\mathcal{R}}'_1(\alpha)$, we have $\beta(H_\alpha) = \pm 1$ and then $(\alpha+\beta)(H_\alpha) = 2+\beta(H_\alpha) = 1$ or 3. Since $(\alpha+\beta)(H_\alpha) = 3$ is impossible, we get $\alpha+\beta \in \widehat{\mathcal{R}}'_1(\alpha)$. This implies that $E_\beta \in \widehat{\mathfrak{p}}_1^+(\alpha)$ and then $E_{\alpha+\beta} \in \widehat{\mathfrak{p}}_1^-(\alpha)$, (as $-\alpha+(\alpha+\beta) = \beta$). Similarly if $-\alpha+\beta$ is a root then $E_\beta \in \widehat{\mathfrak{p}}_1^-(\alpha)$ and then $E_{-\alpha+\beta} \in \widehat{\mathfrak{p}}_1^+(\alpha)$.

If α is short, $(\alpha+\beta)(H_\alpha) = 2+\beta(H_\alpha)$ and by Lemma 3.1, 3.2 and 3.3 we know $\beta(H_\alpha)$ in each case. Then we deduce the assertion 1.i in this case.

For 1.ii, let $\beta, \gamma \in \mathcal{R}'(\alpha)$ such that $\beta(H_\alpha) = \gamma(H_\alpha) = -1$. Then $(\beta+\gamma)(H_\alpha) = -2$ and thus if $\beta+\gamma$ is a root then it is equal to $-\alpha$ or it belongs to $\mathcal{R}'_2(\alpha)$. Hence, $[\widehat{\mathfrak{p}}_1^+(\alpha), \widehat{\mathfrak{p}}_1^+(\alpha)] \subset \mathfrak{p}_2^+(\alpha) \oplus \mathbb{R}E_{-\alpha}$. As above, for any root β, γ if $\beta+\gamma \in \mathcal{R}$, one can calculate $(\beta+\gamma)(H_\alpha) = \beta(H_\alpha)+\gamma(H_\alpha)$ and use the above Lemmas in each case to obtain the remaining result of 1.ii and also to obtain 1.iii.

To show 1.iv, just use Definition 2 and Definition 1 and the fact that $\beta(H_\alpha) = -1$ if and only if $\beta(H_{-\alpha}) = 1$.

For 2, if α is long then $\mathfrak{p}_2(\alpha) = \{0\}$, and then by 1.ii) we get $[\widehat{\mathfrak{p}}_1^+(\alpha), \widehat{\mathfrak{p}}_1^+(\alpha)] = \mathbb{R}E_{-\alpha}$, mainly, for any $E_\beta, E_\gamma \in \widehat{\mathfrak{p}}_1^+(\alpha)$, $[E_\beta, E_{-\alpha-\beta}] = aE_{-\alpha}$, with $a \neq 0$ and $[E_\beta, E_\gamma] = 0$ if $\gamma \neq -\alpha-\beta$. Clearly $[\mathbb{R}E_{-\alpha}, \widehat{\mathfrak{p}}_1^+(\alpha)] = \{0\}$. Therefore, $\mathfrak{h}(\alpha)$ is a Heisenberg subalgebra of L which is not equal to L. Obviously, $\mathfrak{h}(-\alpha) = \mathrm{Lie}(\widehat{\mathfrak{p}}_1^+(-\alpha) \oplus \mathbb{R}E_\alpha)$ and by 1.iv) we get $\mathfrak{h}(-\alpha) = \mathrm{Lie}(\widehat{\mathfrak{p}}_1^-(\alpha) \oplus \mathbb{R}E_\alpha)$.

To prove 3, one can use as above the definitions and Lemma 3.2. For 4, the fact $[\mathrm{sl}_\alpha(2,\mathbb{R}), \mathfrak{p}(\alpha)] = \mathfrak{p}(\alpha)$ becomes from the above assertions and since for any root δ, $[H_\alpha, E_\delta] = \delta(H_\alpha)E_\delta$. Also, since $[\mathrm{sl}_\alpha(2,\mathbb{R}), \mathfrak{h}] \subset \mathrm{sl}_\alpha(2,\mathbb{R})$, $\mathrm{sl}_\alpha(2,\mathbb{R})$ is a Lie algebra, $\mathfrak{h}$ is abelian and for any $X \in L$, $[E_{\pm\alpha}, X''(\alpha)] = 0$ and $[H_\alpha, X''(\alpha)] = 0$ we get $[\mathrm{sl}_\alpha(2,\mathbb{R}), \mathfrak{k}(\alpha)] \subset \mathrm{sl}_\alpha(2,\mathbb{R})$. ■

Notice that for a short root α, in the case of type $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$, the decomposition $L = \mathfrak{p}(\alpha) \oplus \mathfrak{k}(\alpha)$ is not a Cartan decomposition of L. Indeed, by Proposition 3.8.4 we have $[\mathrm{sl}_\alpha(2,\mathbb{R}), \mathfrak{p}(\alpha)] = \mathfrak{p}(\alpha)$. Also, we get that $[\mathfrak{p}(\alpha), \mathfrak{p}(\alpha)] \subset \mathfrak{k}(\alpha)$ if $\mathfrak{p}_2(\alpha) = \{0\}$ that means α is long and in this case $\widetilde{\mathfrak{p}}_1(\alpha) = \{0\}$, or α is short and $\widehat{\mathfrak{p}}_1(\alpha) = \{0\}$ this case corresponds to the case of type $\mathbf{B_n}$, $n \geq 2$ or $\mathbf{G_2}$. Moreover, in the case $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$ for a short root α, we have $\mathfrak{h}(\alpha)$ is not a Heisenberg subalgebra of L because $[\widehat{\mathfrak{p}}_1^+(\alpha), \widehat{\mathfrak{p}}_1^+(\alpha)] \cap \mathfrak{p}_2^+(\alpha) \neq \{0\}$, (see [4], [5] or the example in Subsection 4.3).

We deduce immediately the following corollaries that determine when $LS(\Gamma)$ is equal to L which implies that S has nonempty interior.

Corollary 3.9. *Let L be the split real form of a complex simple Lie algebra,* $\dim L \neq 3$ *and $\Gamma \subset L$. We have $LS(\Gamma) \neq L$ and hence $S \neq G$ in the following:*

1. *if $\Gamma \subset \mathfrak{k}(\alpha)$ or $\Gamma \subset \mathfrak{h}(\alpha)$, for a long root α.*
2. *if $\Gamma \subset \mathfrak{k}(\alpha)$ for a short root α, in the case of type $\mathbf{G_2}$ or $\mathbf{B_n}$, $n \geq 2$.*
3. *if $\Gamma \subset \widehat{\mathfrak{k}}(\alpha)$ for a short root α, in the case of type $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$.*

Indeed, by definition $LS(\Gamma) \subset \mathrm{Lie}(\Gamma)$ and if $\Gamma \subset \mathfrak{k}(\alpha)$ (resp. $\Gamma \subset \widehat{\mathfrak{k}}(\alpha)$ or $\Gamma \subset \mathfrak{h}(\alpha)$), then $\mathrm{Lie}(\Gamma) \subset \mathfrak{k}(\alpha)$ (resp. $\widehat{\mathfrak{k}}(\alpha)$ or $\mathfrak{h}(\alpha)$) and clearly every subalgebra $\mathfrak{k}(\alpha)$, $\widehat{\mathfrak{k}}(\alpha)$ or $\mathfrak{h}(\alpha)$ is not equal to L.

Corollary 3.10. *Let L be the split real form of a complex simple Lie algebra,* $\dim L \neq 3$ *and* $\Gamma \subset L$. *If* $\mathrm{Lie}(\Gamma) = L$ *then we have:*

1. $\Gamma \cap \widehat{\mathfrak{p}}_1(\alpha) \neq \{0\}$ *for any long root* α.
2. $\Gamma \cap \widehat{\mathfrak{p}}_1(\alpha) \neq \{0\}$ *for any short root* α *in the case of type* $\mathbf{C_n}$, $n \geq 3$ *or* $\mathbf{F_4}$.
3. $\Gamma \cap \mathfrak{p}(\alpha) \neq \{0\}$, *for any short root* α *in the case of type* $\mathbf{G_2}$ *or* $\mathbf{B_n}$, $n \geq 2$.

4. Lie saturate, Cartan decomposition and semigroups

Let $\Gamma \subset L$ and consider S the semigroup of G generated by $\{\exp tX : X \in \Gamma, t \geq 0\}$. Recall that the Lie saturate of Γ, $LS(\Gamma)$ is the set of all elements X of $\mathrm{Lie}(\Gamma)$ such that $\{\exp tX,\ t \geq 0\} \subset clS$, the closure of S. That is the largest subset of $\mathrm{Lie}(\Gamma)$ such that the closure of the semigroup generated by $LS(\Gamma)$ is equal to the closure of S. Some known properties of $LS(\Gamma)$ that we will often use in our proofs ([11] and [15]) are presented in the following lemma.

Lemma 4.1. 1. $LS(\Gamma)$ *is a closed convex cone in* L.

2. *If a subspace* $V \subset LS(\Gamma)$, *then* $\mathrm{Lie}(V) \subset LS(\Gamma)$.
3. *If* $\mathbb{R}X \subset LS(\Gamma)$ *and* $Y \in LS(\Gamma)$, *then for any* $v \in \mathbb{R}$, $e^{v\mathrm{ad}X}(Y) \in LS(\Gamma)$.
4. *If* $\mathbb{R}X \subset LS(\Gamma)$, $Y \in LS(\Gamma)$ *and* $\mathbb{R}\mathrm{ad}^2X(Y) \subset LS(\Gamma)$, *then* $\mathbb{R}[X,Y] \subset LS(\Gamma)$.

We briefly recall the proof of 4. $\forall v \in \mathbb{R}, e^{v\mathrm{ad}X}(Y) - \sum_{k\geq 2} \frac{v^k}{k!}\mathrm{ad}^k X(Y) = Y + v[X,Y] \in LS(\Gamma)$. Therefore $\lim_{v\to\pm\infty} \frac{1}{|v|}(X + v[X,Y]) = \pm[X,Y] \in LS(\Gamma)$.

Consider the decomposition of L related to a root α, $L = \mathfrak{p}(\alpha) \oplus \mathfrak{k}(\alpha)$. Recall that, if α is long then $\mathfrak{p}(\alpha) = \mathfrak{p}_1(\alpha) = \widehat{\mathfrak{p}}_1(\alpha)$ and $\mathfrak{p}_2(\alpha) = \{0\}$. If α is short in the case of type $\mathbf{B_n}$, $n \geq 2$, $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$, then $\mathfrak{p}(\alpha) = \mathfrak{p}_1(\alpha) \oplus \mathfrak{p}_2(\alpha)$. If α is short in the case of type $\mathbf{G_2}$, then $\mathfrak{p}(\alpha) = \mathfrak{p}_2(\alpha) \oplus \mathfrak{p}_3(\alpha)$ and $\mathfrak{p}_1(\alpha) = \{0\}$.

Notation Denote by $LS(\Gamma)_{\mathfrak{k}(\alpha)}$, $LS(\Gamma)_{\mathfrak{p}(\alpha)}$ and $LS(\Gamma)_{\mathfrak{p}_2^{\pm}(\alpha)}$ the projection of $LS(\Gamma)$ on $\mathfrak{k}(\alpha)$, $\mathfrak{p}(\alpha)$ and $\mathfrak{p}_2^{\pm}(\alpha)$, respectively.

The following proposition illustrate the proof of the following theorem.

Proposition 4.2. *Let L be the split real form of a complex simple Lie algebra and* $\Gamma \subset L$ *such that* $\mathbb{R}E_\alpha \oplus \mathbb{R}E_{-\alpha} \subset LS(\Gamma)$, *for an arbitrary root* α.

1. *If* α *is a long root when* L *is of any type or* α *is a short root in the case of type* $\mathbf{B_n}$, $n \geq 2$, $\mathbf{C_n}$, $n \geq 3$ *or* $\mathbf{F_4}$ *then for any* $X \in LS(\Gamma)$ *we have:*
   - *i.* $X_2'(\alpha)^+ \in LS(\Gamma)$ *and* $X_2'(\alpha)^- \in LS(\Gamma)$. *Then* $X_2'(\alpha) \in LS(\Gamma)$.
   - *ii. if* $-X_2'(\alpha)^+ \in LS(\Gamma)$ *(resp.* $-X_2'(\alpha)^- \in LS(\Gamma)$*), then* $\mathbb{R}\widetilde{X}_1'(\alpha)$ *and* $\mathbb{R}\widehat{X}_1'(\alpha)^+$ *(resp.* $\mathbb{R}\widehat{X}_1'(\alpha)^-$*) are contained in* $LS(\Gamma)$.

iii. $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2^+(\alpha)} \subset LS(\Gamma)$ *if and only if* $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2^-(\alpha)} \subset LS(\Gamma)$. *Also,* $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2^+(\alpha)} \subset LS(\Gamma)$ *if and only if* $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2(\alpha)} \subset LS(\Gamma)$.

2. *If* α *is long when* L *is of any type, or* α *is short in the case of type* $\mathbf{G_2}$ *then we have* $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2(\alpha)^\pm} \subset LS(\Gamma)$ *and* $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_3(\alpha)} \subset LS(\Gamma)$.

3. *For any root* α, *if* $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2^+(\alpha)} \subset LS(\Gamma)$ *then we have the following:*

   i. $\mathbb{R}LS(\Gamma)_{\mathfrak{p}(\alpha)} \subset LS(\Gamma)$ *and* $LS(\Gamma)_{\mathfrak{k}(\alpha)} \subset LS(\Gamma)$,

   ii. $[\mathrm{sl}_\alpha(2,\mathbb{R}), LS(\Gamma)] \subset LS(\Gamma)$,

   iii. $\mathbb{R}[LS(\Gamma)_{\mathfrak{p}(\alpha)}, LS(\Gamma)_{\mathfrak{k}(\alpha)}] \subset LS(\Gamma)$.

Theorem 4.3. *Let* L *be the split real form of a complex simple Lie algebra and* $\Gamma \subset L$ *such that* $\mathbb{R}E_\alpha \oplus \mathbb{R}E_{-\alpha} \subset LS(\Gamma)$, *for an arbitrary root* α. *If* $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2^+(\alpha)} \subset LS(\Gamma)$ *then* $LS(\Gamma) = L$ *if and only if* $\mathrm{Lie}(\Gamma) = L$.

Proof. To prove Theorem 4.3, we consider the Lie algebra $\mathcal{I}$ generated by the elements of $\mathrm{sl}_\alpha(2,\mathbb{R})$, $LS(\Gamma)_{\mathfrak{p}(\alpha)}$ and $[LS(\Gamma)_{\mathfrak{p}(\alpha)}, LS(\Gamma)_{\mathfrak{k}(\alpha)}]$. By Proposition 4.2.3, we get that $\mathcal{I} \subset LS(\Gamma)$. We show that $\mathcal{I}$ is an ideal (nontrivial) of $\mathrm{Lie}(\Gamma)$. To do this it is enough to show that $[\mathcal{I}, \Gamma] \subset \mathcal{I}$. Clearly, $\Gamma \subset LS(\Gamma)$ and $[\mathrm{sl}_\alpha(2,\mathbb{R}), \Gamma] \subset LS(\Gamma)$, by Proposition 4.2.3. Also, $[\mathrm{sl}_\alpha(2,\mathbb{R}), \Gamma] \subset \mathrm{sl}_\alpha(2,\mathbb{R}) \oplus \mathfrak{p}(\alpha)$ because $\Gamma \subset L = \mathfrak{p}(\alpha) \oplus \mathfrak{k}(\alpha)$ and Proposition 3.8.4 arises that $[\mathrm{sl}_\alpha(2,\mathbb{R}), \mathfrak{p}(\alpha)] \subset \mathfrak{p}(\alpha)$ and $[\mathrm{sl}_\alpha(2,\mathbb{R}), \mathfrak{k}(\alpha)] \subset \mathrm{sl}_\alpha(2,\mathbb{R})$. Therefore, $[\mathrm{sl}_\alpha(2,\mathbb{R}), \Gamma] \subset \mathrm{sl}_\alpha(2,\mathbb{R}) \oplus LS(\Gamma)_{\mathfrak{p}(\alpha)} \subset \mathcal{I}$. Let $X'(\alpha) \in LS(\Gamma)_{\mathfrak{p}(\alpha)}$ and $Y \in \Gamma$. We have $[X'(\alpha), Y] = [X'(\alpha), Y_\alpha + Y_{-\alpha} + Y'(\alpha)] + [X'(\alpha), Y_0 + Y''(\alpha)]$. Clearly, $[X'(\alpha), Y_\alpha + Y_{-\alpha} + Y'(\alpha)] \in \mathcal{I}$ and $Y_0 + Y''(\alpha) = Y - (Y_\alpha + Y_{-\alpha} + Y'(\alpha)) \in LS(\Gamma)$. Since $Y_0 + Y''(\alpha) \in \mathfrak{k}(\alpha)$, we get $Y_0 + Y''(\alpha) \in LS(\Gamma)_{\mathfrak{k}(\alpha)}$ and then we have $[X'(\alpha), Y_0 + Y''(\alpha)] \in \mathcal{I}$. Therefore, $[X'(\alpha), Y] \in \mathcal{I}$. Consider $W = [X'(\alpha), Y_0 + Y''(\alpha)] \in [LS(\Gamma)_{\mathfrak{p}(\alpha)}, LS(\Gamma)_{\mathfrak{k}(\alpha)}]$. By Theorem 3.7 $[X'(\alpha), Y_0 + Y''(\alpha)] \in \mathfrak{p}(\alpha)$. Then $W = W'(\alpha)$ and hence as above $[W'(\alpha), Z] = [[X'(\alpha), Y_0 + Y''(\alpha)], Z] \in \mathcal{I}$, for any $Z \in \Gamma$. Therefore, $\mathcal{I}$ is an ideal of $\mathrm{Lie}(\Gamma)$. We get that $\mathcal{I}$ is included in $LS(\Gamma)$ and it is an ideal (nontrivial) of $\mathrm{Lie}(\Gamma)$. We deduce that if $\mathrm{Lie}(\Gamma) = L$, which is simple then $\mathcal{I} = L$ and thus $LS(\Gamma) = L$. Clearly if $LS(\Gamma) = L$ then $\mathrm{Lie}(\Gamma) = L$. ∎

4.1. Proof of Proposition 4.2.

For any $X \in LS(\Gamma)$ since $\mathbb{R}E_{\pm\alpha} \subset LS(\Gamma)$ in the root decomposition of X related to a root α, we can assume that $X_\alpha = X_{-\alpha} = 0$ (as $X_{\pm\alpha} \in \mathbb{R}E_{\pm\alpha} \subset LS(\Gamma)$ and it suffices to take $X - (X_\alpha + X_{-\alpha}) \in LS(\Gamma)$ instead of X). Then, in our proofs we will take $X = X_0 + X'(\alpha) + X''(\alpha)$. Recall that

$X'(\alpha) = X_1'(\alpha) + X_2'(\alpha) + X_3'(\alpha)$, $X_1'(\alpha) = \widehat{X}_1'(\alpha) + \widetilde{X}_1'(\alpha)$, $\widehat{X}_1'(\alpha) = \widehat{X}_1'(\alpha)^+ + \widehat{X}_1'(\alpha)^-$ and for any $m \in \{2,3\}$, $X_m'(\alpha) = X_m'(\alpha)^+ + X_m'(\alpha)^-$.

We can present X in the general case as follows and use Remark 3.5.

$$X = X_0 + X''(\alpha) + \widetilde{X}_1'(\alpha) + \widehat{X}_1'(\alpha)^+ + \widehat{X}_1'(\alpha)^- + X_2'(\alpha)^+ + X_2'(\alpha)^- + X_3'(\alpha)^+ + X_3'(\alpha)^-.$$

To prove Proposition 4.2 we use the following lemma and remark.

Lemma 4.4. *Let $H_\alpha = [E_\alpha, E_{-\alpha}]$ and $v \in \mathbb{R}$.*

1. *If α is long then $e^{v\mathrm{ad}H_\alpha}(X) = X_0 + X''(\alpha) + e^{-v}\widehat{X}_1'(\alpha)^+ + e^{v}\widehat{X}_1'(\alpha)^-$.*
2. *If α is short in the case $\mathbf{G_2}$, then*
$$e^{v\mathrm{ad}H_\alpha}(X) = X_0 + X''(\alpha) + e^{-v}X_2'(\alpha)^+ + e^{v}X_2'(\alpha)^- + e^{-3v}X_3'(\alpha)^+ + e^{3v}X_3'(\alpha)^-.$$
3. *If α is short in the case $\mathbf{B_n}$, $n \geq 2$, then*
$$e^{v\mathrm{ad}H_\alpha}(X) = X_0 + X''(\alpha) + \widetilde{X}_1'(\alpha) + e^{-2v}X_2'(\alpha)^+ + e^{2v}X_2'(\alpha)^-.$$
4. *If α is short in the case $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$, then $e^{v\mathrm{ad}H_\alpha}(X) =$*
$$X_0 + X''(\alpha) + \widetilde{X}_1'(\alpha) + e^{-v}\widehat{X}_1'(\alpha)^+ + e^{v}\widehat{X}_1'(\alpha)^- + e^{-2v}X_2'(\alpha)^+ + e^{2v}X_2'(\alpha)^-.$$

Proof. Just use the fact that $[H_\alpha, X_0] = 0$ and for any root β, we have $[H_\alpha, E_\beta] = \beta(H_\alpha)E_\beta$ and then $e^{v\mathrm{ad}H_\alpha}(E_\beta) = e^{v\beta(H_\alpha)}E_\beta$. By Lemma 3.1, 3.2 and 3.3 we know the value of $\beta(H_\alpha)$ in each case and we use the definition of $\widehat{X}_1'(\alpha)^+$, $\widehat{X}_1'(\alpha)^-$, $X_m'(\alpha)^+$ and $X_m'(\alpha)^-$ for $m \in \{2, 3\}$. ■

In the following remark we present some technical properties that we will often use in our proofs.

Remark 4.5. Let α be a root and $X \in L$. We have:

1. $[E_\alpha, X] = [E_\alpha, X_0] + [E_\alpha, \widetilde{X}_1'(\alpha)] + [E_\alpha, \widehat{X}_1'(\alpha)^+] + [E_\alpha, X_2'(\alpha)^+] + [E_\alpha, X_3'(\alpha)^+]$ and $[E_\alpha, [E_\alpha, X]] = [E_\alpha, [E_\alpha, X_2'(\alpha)^+]] + [E_\alpha, [E_\alpha, X_3'(\alpha)^+]]$.
2. For any $\beta \in \mathcal{R}$ such that $\alpha + \beta \in \mathcal{R}$ and $-\alpha + \beta \notin \mathcal{R}$ we have $[E_{-\alpha}, [E_\alpha, E_\beta]] = -\beta(H_\alpha)E_\beta$. Therefore, $[E_{-\alpha}, [E_\alpha, \widehat{X}_1'(\alpha)^+]] = \widehat{X}_1'(\alpha)^+$, $[E_{-\alpha}, [E_\alpha, X_2'(\alpha)^+]] = 2X_2'(\alpha)^+$ and $[E_{-\alpha}, [E_\alpha, X_3'(\alpha)^+]] = 3X_3'(\alpha)^+$. Also, $\mathrm{ad}^2E_{-\alpha}(\mathrm{ad}^2E_\alpha(X_2'(\alpha)^+)) = aX_2'(\alpha)^+$, with $a \neq 0$.
3. For any $\beta \in \widetilde{\mathcal{R}}_1'(\alpha)$, we have $[E_{-\alpha}, [E_\alpha, E_\beta]] = [E_\alpha, [E_{-\alpha}, E_\beta]] = 2E_\beta$ and then $[E_{-\alpha}, [E_\alpha, \widetilde{X}_1'(\alpha)]] = [E_\alpha, [E_{-\alpha}, \widetilde{X}_1'(\alpha)]] = 2\widetilde{X}_1'(\alpha)$.
4. In the case $\mathbf{G_2}$, we have the following:
   i. for any $\beta \in \mathcal{R}_2'(\alpha)$ such that $2\alpha + \beta \in \mathcal{R}$, $[E_\alpha, [E_{-\alpha}, E_\beta]] = 3E_\beta$. Also, $[E_\alpha, [E_{-\alpha}, X_2'(\alpha)^+]] = 3X_2'(\alpha)^+$.
   ii. for any $\beta \in \mathcal{R}_3'(\alpha)$ such that $3\alpha + \beta \in \mathcal{R}$, $\mathrm{ad}^3E_{-\alpha}(\mathrm{ad}^3E_\alpha(E_\beta)) = aE_\beta$. Also, $\mathrm{ad}^3E_{-\alpha}(\mathrm{ad}^3E_\alpha(X_3'(\alpha)^+)) = aX_3'(\alpha)^+$, with $a \neq 0$.

Indeed, by definition we have $[E_\alpha, X''(\alpha)] = 0$, $[E_\alpha, \widehat{X}_1'(\alpha)^-] = 0$, $[E_\alpha, X_0] \in \mathbb{R}E_\alpha$ and $Y = [E_\alpha, \widehat{X}_1'(\alpha)^+] = \widehat{Y}_1'(\alpha)^-$. We also have $[E_\alpha, [E_\alpha, \widetilde{X}_1'(\alpha)]] = 0$ and $[E_\alpha, [E_\alpha, \widehat{X}_1'(\alpha)^+]] = 0$. Then we get the assertion 1.

For 2, if $\alpha + \beta \in \mathcal{R}$ and $-\alpha + \beta \notin \mathcal{R}$ then $[E_{-\alpha}, E_\beta] = 0$. We get $[E_{-\alpha}, [E_\alpha, E_\beta]] = [E_\alpha, [E_{-\alpha}, E_\beta]] + [E_\beta, [E_\alpha, E_{-\alpha}] = [E_\beta, H_\alpha] = -\beta(H_\alpha)E_\beta$. By

Lemma 3.1, 3.2 and 3.3 we know that $\beta(H_\alpha) \in \{-1,-2,-3\}$. By definition of $\widehat{X}_1'(\alpha)^+$, $X_2'(\alpha)^+$ and $X_3'(\alpha)^+$ we obtain the result. Using the Jacobi identity, one can show that for any $\beta \in \mathcal{R}_2'(\alpha)$, such that $\beta(H_\alpha) = -2$ we have $\mathrm{ad}^2 E_{-\alpha}(\mathrm{ad}^2 E_\alpha(E_\beta)) = aE_\beta$.

For 3, let $\beta \in \widetilde{\mathcal{R}}_1'(\alpha)$. We have $[E_{-\alpha},[E_\alpha,E_\beta]] = [E_\alpha,[E_{-\alpha},E_\beta]] + [E_\beta,H_\alpha]$ and $[E_\beta,H_\alpha] = -\beta(H_\alpha)E_\beta = 0$ (by Lemma 3.2 and 3.3 we know that $\beta(H_\alpha)=0$). Also, observe that $[E_{-\alpha},[E_\alpha,E_\beta]] = aE_\beta$ with $a \neq 0$. Then $[E_\alpha,[E_{-\alpha},[E_\alpha,E_\beta]]] = a[E_\alpha,E_\beta]$. By the Jacobi identity, $[E_\alpha,[E_{-\alpha},[E_\alpha,E_\beta]]] = [E_{-\alpha},[E_\alpha,[E_\alpha,E_\beta]]] + [H_\alpha,[E_\alpha,E_\beta]]$. But $[E_\alpha,[E_\alpha,E_\beta]] = 0$ since $2\alpha+\beta$ is not root, and $[H_\alpha,[E_\alpha,E_\beta]] = (\alpha+\beta)(H_\alpha)[E_\alpha,E_\beta] = \alpha(H_\alpha)[E_\alpha,E_\beta] = 2[E_\alpha,E_\beta]$. This implies that $a=2$ and thus $[E_{-\alpha},[E_\alpha,E_\beta]] = 2E_\beta$.

For 4.i, just use the fact that $\gamma = -\alpha+\beta \in \mathcal{R}_3'(\alpha)$ and γ satisfies $\alpha+\gamma \in \mathcal{R}$ and $-\alpha+\gamma \notin \mathcal{R}$ by Lemma 3.2. By using the assertion 1 for γ, and since $\gamma(H_\alpha) = -3$, we get the result. 4.ii follows from 4.i (as $-\alpha+3\alpha+\beta = 2\alpha+\beta \in \mathcal{R}_2'(\alpha)$) and 2.

Notice that one can prove some assertions of Remark 4.5 by using the following property of roots (see [3], [9] and [21]):
For any root α,β we have $[E_{-\alpha},[E_\alpha,E_\beta]] = q(\alpha,\beta)(p(\alpha,\beta)+1)E_\beta$.

Now, we give the proof of Proposition 4.2, in the case when α is a long root of any type, or α is a short root in the case of type $\mathbf{B_n}$, $n \geq 2$, $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$. The case when α is a short root in the case $\mathbf{G_2}$ will be given separately after.

• Let α be a long root when L is of any type, or a short root in the case of type $\mathbf{B_n}$, $n \geq 2$, $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$. Clearly if α is long then $\pm X_2'(\alpha) = 0 \in LS(\Gamma)$. If α is short we know that $Y = e^{v\mathrm{ad}H_\alpha}(X) \in LS(\Gamma)$, for any $v \in \mathbb{R}$ and then by using Lemma 4.4 we get

$$\lim_{v\to+\infty} e^{-2v}Y = +X_2'(\alpha)^- \in LS(\Gamma) \textit{ and } \lim_{v\to-\infty} e^{2v}Y = +X_2'(\alpha)^+ \in LS(\Gamma).$$

Notice that if $\pm X \in LS(\Gamma)$ then $\pm X_2'(\alpha)^+ \in LS(\Gamma)$ and $\pm X_2'(\alpha)^- \in LS(\Gamma)$.

For 1.ii, let $X \in LS(\Gamma)$. If $-X_2'(\alpha)^+ \in LS(\Gamma)$ then $\mathbb{R}X_2'(\alpha)^+ \subset LS(\Gamma)$. Since $\mathrm{ad}^2E_\alpha(X) = \mathrm{ad}^2E_\alpha(X_2'(\alpha)^+)$ (by Remark 4.5.1), we get that $\mathbb{R}\mathrm{ad}^2E_\alpha(X) \subset LS(\Gamma)$ and Lemma 4.1.4 gives $\mathbb{R}[E_\alpha,X] \subset LS(\Gamma)$. Using Remark 4.5.1 we present $Z = [E_\alpha,X] = [E_\alpha,X_0] + [E_\alpha,\widetilde{X}_1'(\alpha)] + [E_\alpha,\widehat{X}_1'(\alpha)^+] + [E_\alpha,X_2'(\alpha)^+]$. Clearly, $\pm[E_\alpha,X_0] \in \mathbb{R}E_\alpha \subset LS(\Gamma)$ and $\pm[E_\alpha,X_2'(\alpha)^+] \in LS(\Gamma)$. By Proposition 3.8.1, $[E_\alpha,\widetilde{X}_1'(\alpha)] = Z_2'(\alpha)^-$ and $[E_\alpha,\widehat{X}_1'(\alpha)^+] = \widehat{Z}_1'(\alpha)^-$. Using 1.i, we get $\pm Z_2'(\alpha)^- = \pm[E_\alpha,\widetilde{X}_1'(\alpha)] \in LS(\Gamma)$ and thus $\pm[E_\alpha,\widehat{X}_1'(\alpha)^+] \in LS(\Gamma)$. But by Remark 4.5, $[E_{-\alpha},[E_\alpha,\widetilde{X}_1'(\alpha)]] = 2\widetilde{X}_1'(\alpha)$ and $[E_{-\alpha},[E_\alpha,\widehat{X}_1'(\alpha)^+]] = 2\widehat{X}_1'(\alpha)^+$. Therefore, $\mathbb{R}\widetilde{X}_1'(\alpha) \subset LS(\Gamma)$ and $\mathbb{R}\widehat{X}_1'(\alpha)^+ \subset LS(\Gamma)$.

Similarly, if $-X_2'(\alpha)^- \in LS(\Gamma)$ just use $E_{-\alpha}$ instead of E_α and as above we obtain that $\mathbb{R}\widetilde{X}_1'(\alpha) \subset LS(\Gamma)$ and $\mathbb{R}\widehat{X}_1'(\alpha)^- \subset LS(\Gamma)$.

For 1.iii, let $X \in LS(\Gamma)$. If α is long then, $X_2'(\alpha) = 0$ and the equivalences are obvious. If α is short, we consider $Y = [E_\alpha,[E_\alpha,X_2'(\alpha)^+]]$, (resp. $Y = [E_{-\alpha},[E_{-\alpha},X_2'(\alpha)^-]]$) and we have $Y = Y_2'(\alpha)^-$ (resp. $Y = Y_2'(\alpha)^+$). Clearly, $X_2'(\alpha)^+ \neq 0$ if and only if $Y_2'(\alpha)^- \neq 0$ and $\pm X_2'(\alpha)^+ \in LS(\Gamma)$ if and only if

$\pm Y_2'(\alpha)^- \in LS(\Gamma)$ (because by Remark 4.5.2, $[E_{-\alpha},[E_{-\alpha},Y_2'(\alpha)^-]] = aX_2'(\alpha)^+$, $a \neq 0$). Then $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2^+(\alpha)} \subset LS(\Gamma)$ if and only if $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2^-(\alpha)} \subset LS(\Gamma)$. Also, since $LS(\Gamma)_{\mathfrak{p}_2(\alpha)} = LS(\Gamma)_{\mathfrak{p}_2^+(\alpha)} \oplus LS(\Gamma)_{\mathfrak{p}_2^-(\alpha)}$, we deduce that $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2(\alpha)} \subset LS(\Gamma)$ if and only if $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2^+(\alpha)} \subset LS(\Gamma)$.

For 2, if α is long we know that for any $X \in L$, $X_2'(\alpha) = X_3'(\alpha) = 0$ and then the result is evident.

For 3, assume that $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2(\alpha)^+} \subset LS(\Gamma)$. Then we have for any $X \in LS(\Gamma)$, $\pm X_2'(\alpha) \in LS(\Gamma)$. By the assertion 1, we get $\mathbb{R}X'(\alpha) \subset LS(\Gamma)$ and thus $X - X'(\alpha) \in LS(\Gamma)$, we know that $X - X'(\alpha) \in \mathfrak{k}(\alpha)$. Then $\mathbb{R}LS(\Gamma)_{\mathfrak{p}(\alpha)} \subset LS(\Gamma)$ and $LS(\Gamma)_{\mathfrak{k}(\alpha)} \subset LS(\Gamma)$. Also, since $[E_{\pm\alpha}, X] = [E_{\pm\alpha}, X_0] + [E_{\pm\alpha}, X'(\alpha)]$, $[E_{\pm\alpha}, X_0] \in \mathbb{R}E_{\pm\alpha}$ and $\mathbb{R}X'(\alpha) \subset LS(\Gamma)$, we have that $\mathbb{R}[E_{\pm\alpha}, X] \subset LS(\Gamma)$. As $H_\alpha = [E_\alpha, E_{-\alpha}]$ we get $\mathbb{R}[H_\alpha, X] \subset LS(\Gamma)$. Then, $[\mathrm{sl}_\alpha(2,\mathbb{R}), LS(\Gamma)] \subset LS(\Gamma)$.

To prove that $\mathbb{R}[LS(\Gamma)_{\mathfrak{p}(\alpha)}, LS(\Gamma)_{\mathfrak{k}(\alpha)}] \subset LS(\Gamma)$, we consider $X, Y \in LS(\Gamma)$ and we show that $\mathbb{R}[X'(\alpha), Y_0 + Y''(\alpha)] \subset LS(\Gamma)$. To do this, we show that $\mathbb{R}[X_1'(\alpha), Y_0 + Y''(\alpha)] \subset LS(\Gamma)$ and $\mathbb{R}[X_2'(\alpha), Y_0 + Y''(\alpha)] \subset LS(\Gamma)$ as follows: Clearly, $Y - Y'(\alpha) = Y_0 + Y''(\alpha) \in LS(\Gamma)$. We know that $\pm(\widehat{X}_1'(\alpha)^+ + X_2'(\alpha)^+) \in LS(\Gamma)$. By Definition 2, $Y_0 + Y''(\alpha) \in \mathfrak{k}(\alpha)$, $\widehat{X}_1'(\alpha)^+ \in \widehat{\mathfrak{p}}_1^+(\alpha)$ and $X_2'(\alpha)^+ \in \mathfrak{p}_2^+(\alpha)$. Using Proposition 3.8, we have $\mathrm{ad}^2(\widehat{X}_1'(\alpha)^+ + X_2'(\alpha)^+)(Y_0 + Y''(\alpha)) \in \mathbb{R}E_{-\alpha} \oplus \mathfrak{p}_2^+(\alpha)$ (because $[\widehat{X}_1'(\alpha)^+ + X_2'(\alpha)^+, Y_0 + Y''(\alpha)] \in \widehat{\mathfrak{p}}_1^+(\alpha) \oplus \mathfrak{p}_2^+(\alpha)$), and then $\mathrm{ad}^3(\widehat{X}_1'(\alpha)^+ + X_2'(\alpha)^+)(Y_0 + Y''(\alpha)) = 0$. We deduce that $\mathbb{R}\mathrm{ad}^2(\widehat{X}_1'(\alpha)^+ + X_2'(\alpha)^+)(Y_0 + Y''(\alpha)) \subset LS(\Gamma)$. Lemma 4.1.4 gives that $\mathbb{R}[\widehat{X}_1'(\alpha)^+ + X_2'(\alpha)^+, Y_0 + Y''(\alpha)] \subset LS(\Gamma)$. Similarly, we show that $\mathbb{R}[\widehat{X}_1'(\alpha)^- + X_2'(\alpha)^-, Y_0 + Y''(\alpha)] \subset LS(\Gamma)$. We deduce that $\mathbb{R}[\widehat{X}_1'(\alpha) + X_2'(\alpha), Y_0 + Y''(\alpha)] \subset LS(\Gamma)$.

Now we prove that $\mathbb{R}[\widetilde{X}_1'(\alpha), Y_0 + Y''(\alpha)] \subset LS(\Gamma)$. Take $W = [E_{-\alpha}, \widetilde{X}_1'(\alpha)]$. We have $\pm W \in LS(\Gamma)$ and by Proposition 3.8.1, $W = W_2'(\alpha)^+$. We obtain $\mathbb{R}[W_2'(\alpha)^+, Y_0 + Y''(\alpha)] \subset LS(\Gamma)$. The Jacobi identity gives that $[[E_{-\alpha}, \widetilde{X}_1'(\alpha)], Y_0 + Y''(\alpha)] = [[E_{-\alpha}, Y_0 + Y''(\alpha)], \widetilde{X}_1'(\alpha)] + [E_{-\alpha}, [\widetilde{X}_1'(\alpha), Y_0 + Y''(\alpha)]]$. But $\mathbb{R}[[E_{-\alpha}, Y_0 + Y''(\alpha)], \widetilde{X}_1'(\alpha)] = \mathbb{R}[[E_{-\alpha}, Y_0], \widetilde{X}_1'(\alpha)] \subset LS(\Gamma)$. Therefore, $\mathbb{R}[E_{-\alpha}, [\widetilde{X}_1'(\alpha), Y_0 + Y''(\alpha)]] \subset LS(\Gamma)$. We deduce that $\mathbb{R}[E_\alpha, [E_{-\alpha}, [\widetilde{X}_1'(\alpha), Y_0 + Y''(\alpha)]]] \subset LS(\Gamma)$. Just observe that $[\widetilde{X}_1'(\alpha), Y_0 + Y''(\alpha)] \in \widetilde{\mathfrak{p}}_1(\alpha)$ and thus by Remark 4.5.3, $[E_\alpha, [E_{-\alpha}, [\widetilde{X}_1'(\alpha), Y_0 + Y''(\alpha)]]] = 2[\widetilde{X}_1'(\alpha), Y_0 + Y''(\alpha)]$. Then $\mathbb{R}[\widetilde{X}_1'(\alpha), Y_0 + Y''(\alpha)] \subset LS(\Gamma)$. We obtain $\mathbb{R}[X'(\alpha), Y_0 + Y''(\alpha)] \subset LS(\Gamma)$.

• Now, we prove Proposition 4.2 when α is short in the case $\mathbf{G_2}$. Let $X \in LS(\Gamma)$. For any $v \in \mathbb{R}$, $Y = e^{v\mathrm{ad}H_\alpha}(X) \in LS(\Gamma)$. Lemma 4.4 gives $Y = X_0 + X''(\alpha) + e^{-v}X_2'(\alpha)^+ + e^v X_2'(\alpha)^- + e^{-3v}X_3'(\alpha)^+ + e^{3v}X_3'(\alpha)^-$. Then,

$$\lim_{v\to+\infty} e^{-3v}Y = X_3'(\alpha)^- \in LS(\Gamma) \textit{ and } \lim_{v\to-\infty} e^{3v}Y = X_3'(\alpha)^+ \in LS(\Gamma).$$

Since $4\alpha + \beta$ is not a root for any root β we have $\mathrm{ad}^4 E_\alpha = 0$. Then for any $v \in \mathbb{R}$,

$$e^{v\mathrm{ad}E_\alpha}(X_3'(\alpha)^+) = X_3'(\alpha)^+ + v[E_\alpha, X_3'(\alpha)^+] + \frac{v^2}{2}\mathrm{ad}^2 E_\alpha(X_3'(\alpha)^+) + \frac{v^3}{6}\mathrm{ad}^3 E_\alpha(X_3'(\alpha)^+).$$

Therefore, we get

$$\lim_{v\to\pm\infty} 6\frac{1}{|v|^3}e^{v\mathrm{ad}E_\alpha}(X_3'(\alpha)^+) = \pm\mathrm{ad}^3 E_\alpha(X_3'(\alpha)^+) \in LS(\Gamma).$$

By Remark 4.5.4, we have $\mathrm{ad}^3 E_{-\alpha}(\mathrm{ad}^3 E_\alpha(X_3'(\alpha)^+)) = aX_3'(\alpha)^+$, $a \neq 0$. Then $\mathbb{R}X_3'(\alpha)^+ \subset LS(\Gamma)$. Similarly, by taking $e^{v\mathrm{ad}E_{-\alpha}}(X_3'(\alpha)^-))$ we get $\mathbb{R}X_3'(\alpha)^- \subset LS(\Gamma)$. Therefore, $\mathbb{R}X_3'(\alpha) \subset LS(\Gamma)$ and then $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_3(\alpha)} \subset LS(\Gamma)$. Consider $Z = X - X_3'(\alpha) = X_0 + X_2'(\alpha) + X''(\alpha) \in LS(\Gamma)$. Lemma 4.4 gives that

$$\lim_{v\to+\infty} e^v e^{v\mathrm{ad}H_\alpha}(Z) = X_2'(\alpha)^+ \in LS(\Gamma) \textit{ and } \lim_{v\to+\infty} e^{-v} e^{v\mathrm{ad}H_\alpha}(Z) = X_2'(\alpha)^- \in LS(\Gamma).$$

Also, we have $\mathrm{ad}^2 E_{-\alpha}(X_2'(\alpha)^+) = 0$ because $-2\alpha+\beta$ is not a root for any $\beta \in \mathcal{R}_2(\alpha)$ such that $2\alpha+\beta \in \mathcal{R}$ (otherwise $4\alpha + (-2\alpha+\beta) = 2\alpha+\beta$, which is a root, this contradicts the fact that $4\alpha+\gamma$ is not a root for any root γ). By Lemma 4.1.4, we get $\mathbb{R}[E_{-\alpha}, X_2'(\alpha)^+] \subset LS(\Gamma)$ and by Remark 4.5.4, we have $[E_\alpha, [E_{-\alpha}, X_2'(\alpha)^+]] = 3X_2'(\alpha)^+$. Therefore, $\mathbb{R}X_2'(\alpha)^+ \subset LS(\Gamma)$. Similarly, we show that $\mathbb{R}X_2'(\alpha)^- \subset LS(\Gamma)$. We obtain that $\mathbb{R}X_2'(\alpha)^\pm \subset LS(\Gamma)$ and then $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2(\alpha)^\pm} \subset LS(\Gamma)$. This proves the assertion 2 in the case $\mathbf{G_2}$.

For 3, since $\mathfrak{p}(\alpha) = \mathfrak{p}_2(\alpha) \oplus \mathfrak{p}_3(\alpha)$, we deduce that $\mathbb{R}LS(\Gamma)_{\mathfrak{p}(\alpha)} \subset LS(\Gamma)$. Clearly, $[\mathrm{sl}_\alpha(2,\mathbb{R}), LS(\Gamma)] \subset LS(\Gamma)$ (as $[E_{\pm\alpha}, X] = [E_{\pm\alpha}, X_0] + [E_{\pm\alpha}, X'(\alpha)]$). To prove that $\pm[X'(\alpha), Y] \in LS(\Gamma)$ for any $X, Y \in LS(\Gamma)$, it suffices to show that $\mathbb{R}[X_2'(\alpha), Y_0 + Y''(\alpha)] \subset LS(\Gamma)$ and $\mathbb{R}[X_3'(\alpha), Y_0 + Y''(\alpha)] \subset LS(\Gamma)$, because $Y = Y_0 + Y''(\alpha) + Y'(\alpha)$ and $\pm Y'(\alpha) \in LS(\Gamma)$. We know that $\pm X_m'(\alpha)^+ \in LS(\Gamma)$ for $m \in \{2,3\}$. Since $Y_0 + Y''(\alpha) \in \mathfrak{k}(\alpha)$, we have using Proposition 3.8 that $[X_3'(\alpha)^+, Y_0 + Y''(\alpha)] \in [\mathfrak{p}_3^+(\alpha), \mathfrak{k}(\alpha)] \subset \mathfrak{p}_3^+(\alpha)$ and then, $\mathrm{ad}^2 X_3'(\alpha)^+(Y_0 + Y''(\alpha)) = 0$. Also, we get $\mathrm{ad}^2 X_2'(\alpha)^+(Y_0 + Y''(\alpha)) \subset \mathbb{R}E_{-\alpha}$. Lemma 4.1.4 gives that $\mathbb{R}[X_m'(\alpha)^+, Y_0 + Y''(\alpha)] \subset LS(\Gamma)$. Similarly we show that $\mathbb{R}[X_m'(\alpha)^-, Y_0 + Y''(\alpha)] \subset LS(\Gamma)$. We deduce that $\mathbb{R}[X'(\alpha), Y_0 + Y''(\alpha)] \subset LS(\Gamma)$. This ends the proof of Proposition 4.2.

To simplify the notation, observe that $\mathbb{R}LS(\Gamma)_{\mathfrak{p}_2^+(\alpha)} = \mathbb{R}(LS(\Gamma) \cap \mathfrak{p}_2^+(\alpha))$. Because by Proposition 4.2, for any $X \in LS(\Gamma)$ we have $X_2'(\alpha)^+ \in LS(\Gamma)$.

4.2. Generating semigroups of simple Lie groups.

Let G be real Lie group, whose Lie algebra L is the split real form of a complex simple Lie algebra and S a semigroup infinitesimally generated by a set $\Gamma \subset L$. The following proposition is an immediate consequence of Theorem 4.3.

Proposition 4.6. *If S contains a subgroup $G(\alpha)$ for an arbitrary root α and $\mathbb{R}(LS(\Gamma) \cap \mathfrak{p}_2^+(\alpha)) \subset LS(\Gamma)$, then S generates G if and only if Γ generates L.*

Theorem 4.7. *If S contains a subgroup $G(\alpha)$ for a root α, then S generates G if and only if Γ generates L as a Lie algebra for*

1. *any long root α when L is of any type,*
2. *any short root α in the case of type $\mathbf{G_2}$.*

Proof. Proposition 4.2.2 gives that $\mathbb{R}(LS(\Gamma) \cap \mathfrak{p}_2^+(\alpha)) \subset LS(\Gamma)$ in those cases and by Proposition 4.6 we get the result. ∎

Recently, in [19], this result is proved for a long root α when L is of type $\mathbf{A_n}$, $n \geq 2$ or $\mathbf{C_n}$, $n \geq 3$ and for a short root α in the case of type $\mathbf{G_2}$. Also, in the

case of type $\mathbf{C_n}$, the authors give an example for a semigroup S with nonempty interior and S contains a subgroup $G(\alpha)$ for a short root α, but $S \neq G$.

In this paper, we study the case of type $\mathbf{C_n}$, $n \geq 2$, where $\mathfrak{p}_2(\alpha)^+$, has the smallest dimension equal to 2 (see Lemma 4.8), and we precise when the condition of Proposition 4.6 saying that $\mathbb{R}(LS(\Gamma) \cap \mathfrak{p}_2^+(\alpha)) \subset LS(\Gamma)$ is satisfied.

According to the fact that each Dynkin diagram of type $\mathbf{B_n}$, $n \geq 2$, $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$ contains the Dynkin diagram $\mathbf{B_2} = \mathbf{C_2}$, then the conditions obtained in the case $\mathbf{C_n}$, in particular in $\mathbf{C_2}$ will certainly appear in those cases. These cases where $\dim\mathfrak{p}_2^+(\alpha) \geq 4$ (see lemma 4.8), will be studied in a future work.

Also, to compare with [19] we will present in the cases $\mathbf{B_2}$ and $\mathbf{B_3}$ a semigroup S, generated by a set Γ, S has nonempty interior and contains a subgroup $G(\alpha)$ for a short root α, but $S \neq G$. We also indicate how this example in $\mathbf{B_2}$ induces the existence of such semigroups S in the cases $\mathbf{B_n}$, $\mathbf{C_n}$, $n \geq 3$ and $\mathbf{F_4}$.

Recall that in the case of type $\mathbf{C_n}$, $n \geq 3$ (resp. $\mathbf{F_4}$) we have just one (resp. two) long primitive root. In the case $\mathbf{B_n}$, $n \geq 2$ we have $(n-1)$ primitive long roots. Also, we have $\mathbf{C_2} = \mathbf{B_2}$. In [4], the sets $\mathcal{R}_2'(\alpha)$ and $\widetilde{\mathcal{R}}_1'(\alpha)$ are given explicitly for a primitive short root α. Since the Weyl group $\mathcal{W}$ acts transitively on the set of short roots and (as in Subsection 3.1) for any $w \in \mathcal{W}$, $\mathcal{R}_2'(w\alpha) = w\mathcal{R}_2'(\alpha)$ and $\widetilde{\mathcal{R}}_1'(w\alpha) = w\widetilde{\mathcal{R}}_1'(\alpha)$, we deduce the following lemma, for more details see [4].

Lemma 4.8. *Let α be a short root.*

1. *In the case of type $\mathbf{C_n}$, $n \geq 3$, there exist a unique long root β such that $\mathcal{R}_2'(\alpha) = \{\pm\beta,\ \pm(2\alpha+\beta)\}$ and $\widetilde{\mathcal{R}}_1'(\alpha) = \{\pm(\alpha+\beta)\}$. Also, $\dim\mathfrak{p}_2^+(\alpha) = 2$.*

2. *In the case of type $\mathbf{F_4}$, there exist three long roots β_i, $1 \leq i \leq 3$ such that $\mathcal{R}_2'(\alpha) = \{\pm\beta_i,\ \pm(2\alpha+\beta_i), 1 \leq i \leq 3\}$ and $\widetilde{\mathcal{R}}_1'(\alpha) = \{\pm(\alpha+\beta_i), 1 \leq i \leq 3\}$. Also, $\dim\mathfrak{p}_2^+(\alpha) = 6$.*

3. *In the case $\mathbf{B_n}$, $n \geq 2$, there exist $(n-1)$ long roots β_i, $1 \leq i \leq n-1$ such that $\mathcal{R}_2'(\alpha) = \{\pm\beta_i,\ \pm(2\alpha+\beta_i), 1 \leq i \leq n-1\}$ and $\widetilde{\mathcal{R}}_1'(\alpha) = \{\pm(\alpha+\beta_i), 1 \leq i \leq n-1\}$. Also, $\dim\mathfrak{p}_2^+(\alpha) = 2(n-1)$.*

Lemma 4.9. *Let α be a short root in the case $\mathbf{C_n}$, $n \geq 2$ and G has finite-center. If one of the following equivalent conditions holds*

1. *there exists $X \in LS(\Gamma)$ such that $X_2'(\alpha)^+ \neq 0$ and $-X_2'(\alpha)^+ \in LS(\Gamma)$,*

2. *there exists $X \in LS(\Gamma)$ such that $\mathrm{Kill}(X_2'(\alpha); X_2'(\alpha)) < 0$,*

then $\mathbb{R}(LS(\Gamma) \cap \mathfrak{p}_2^+(\alpha)) \subset LS(\Gamma)$.

Proof. In our proof we use the following known property: if $X \in LS(\Gamma)$ and $\mathrm{Kill}(X;X) = \mathrm{Trace}\ \mathrm{ad}X \circ \mathrm{ad}X < 0$, then X is compact as soon as the group G has a finite-center and thus $\mathbb{R}X \subset LS(\Gamma)$ (see [7], [15]).

The assertion 1 implies 2. Indeed, if there exists $X \in LS(\Gamma)$ such that $X_2'(\alpha)^+ \neq 0$ and $-X_2'(\alpha)^+ \in LS(\Gamma)$ then just take $Y_2'(\alpha)^- = [E_\alpha, [E_\alpha, X_2'(\alpha)^+]]$

and we get

$$\mathrm{Kill}(aX_2'(\alpha)^+ + Y_2'(\alpha)^-; aX_2'(\alpha)^+ + Y_2'(\alpha)^-) = 2\mathrm{Kill}(aX_2'(\alpha)^+; Y_2'(\alpha)^-) < 0,$$

for $a = 1$ *or* -1. The assertion 2 implies 1, because if there exists $X \in LS(\Gamma)$ such that $\mathrm{Kill}(X_2'(\alpha); X_2'(\alpha)) < 0$ then $X_2'(\alpha) \neq 0$ and $\mathbb{R}X_2'(\alpha) \subset LS(\Gamma)$. We get by Proposition 4.2.1 that $\pm X_2'(\alpha)^+$ and $\pm X_2'(\alpha)^-$ belong to $LS(\Gamma)$. If $X_2'(\alpha)^+ \neq 0$ we get the result, if not just take $Y_2'(\alpha)^+ = [E_{-\alpha}, [E_{-\alpha}, X_2'(\alpha)^-]]$, $Y_2'(\alpha)^+ \neq 0$ and $-Y_2'(\alpha)^+ \in LS(\Gamma)$.

Assume the assertion 1 and let $Z \in LS(\Gamma)$. If $Z_2'(\alpha)^+$ is proportional to $X_2'(\alpha)^+$, then $\pm Z_2'(\alpha)^+ \in LS(\Gamma)$. If $Z_2'(\alpha)^+$ is not proportional to $X_2'(\alpha)^+$, by Lemma 4.8 we can write $X_2'(\alpha)^+ = x_\beta E_\beta + x_{-2\alpha-\beta}E_{-2\alpha-\beta}$ and then we get that $a = x_\beta z_{-2\alpha-\beta} - x_{-2\alpha-\beta}z_\beta \neq 0$, where $Z_2'(\alpha)^+ = z_\beta E_\beta + z_{-2\alpha-\beta}E_{-2\alpha-\beta}$. We can assume that $x_\beta \neq 0$ (if not we take $x_{-2\alpha-\beta} \neq 0$). Also, we can assume that $x_\beta > 0$ (otherwise we take $-X_2'(\alpha)^+$ instead of $X_2'(\alpha)^+$). Therefore, as $Z_2'(\alpha)^+ \in LS(\Gamma)$, we deduce that $x_\beta Z_2'(\alpha)^+ - z_\beta X_2'(\alpha)^+ = aE_{-2\alpha-\beta} \in LS(\Gamma)$ and then $[E_\alpha, [E_\alpha, aE_{-2\alpha-\beta}]] = bE_{-\beta} \in LS(\Gamma)$, with $b \neq 0$ ($b = aN_{\alpha,-\alpha-\beta}N_{\alpha,-2\alpha-\beta}$).

If $b > 0$, then $\mathrm{Kill}(-X_2'(\alpha)^+ + bE_{-\beta}; -X_2'(\alpha)^+ + bE_{-\beta}) = -2bx_\beta < 0$. Also, we get that $\pm(-X_2'(\alpha)^+ + bE_{-\beta}) \in LS(\Gamma)$. If $b < 0$ then $\mathrm{Kill}(X_2'(\alpha)^+ + bE_{-\beta}; X_2'(\alpha)^+ + bE_{-\beta}) = 2bx_\beta < 0$. We obtain $\pm(cX_2'(\alpha)^+ + bE_{-\beta}) \in LS(\Gamma)$, for $c = 1$ or -1. We deduce that $\pm bE_{-\beta} \in LS(\Gamma)$ and then $\mathbb{R}[E_{-\alpha}, [E_{-\alpha}, E_{-\beta}]] = \mathbb{R}E_{-2\alpha-\beta} \subset LS(\Gamma)$. Then $\pm x_\beta E_\beta \in LS(\Gamma)$ and thus $\pm Z_2'(\alpha)^+ \in LS(\Gamma)$. We obtain that for any $Z \in LS(\Gamma)$, $\pm Z_2'(\alpha)^+ \in LS(\Gamma)$ and thus $\mathbb{R}(LS(\Gamma) \cap \mathfrak{p}_2^+(\alpha)) \subset LS(\Gamma)$. ■

Notice that if $\forall X \in LS(\Gamma)$, $\mathrm{Kill}(X_2'(\alpha); X_2'(\alpha)) \geq 0$ then $LS(\Gamma) \neq L$.

Proposition 4.10. *Let α be a short root in the case $\mathbf{C_n}$, $n \geq 2$ and G has finite-center, then $\mathbb{R}(LS(\Gamma) \cap \mathfrak{p}_2^+(\alpha)) \subset LS(\Gamma)$ if we have one of the following conditions:*

1. *there exists $X \in LS(\Gamma)$ such that $X_2'(\alpha)^+ = 0$ and*
   i. *$\widetilde{X}_1'(\alpha) \neq 0$, or*
   ii. *$\widehat{X}_1'(\alpha)^+ \neq 0$ and $[\widehat{X}_1'(\alpha)^+, [E_\alpha, \widehat{X}_1'(\alpha)^+]] \notin \mathfrak{h}$.*
2. *there exist $X, Y \in LS(\Gamma)$ such that $\mathrm{Kill}(X_2'(\alpha)^+; Y_2'(\alpha)^-) < 0$.*
   In particular, for $Y = Y_2'(\alpha)^- = [E_\alpha, [E_\alpha, X_2'(\alpha)^+]]$.

Proof. If $X \in LS(\Gamma)$ and $X_2'(\alpha)^+ = 0$ then $-X_2'(\alpha)^+ \in LS(\Gamma)$ and Proposition 4.2.1 gives that $\mathbb{R}\widetilde{X}_1'(\alpha) \subset LS(\Gamma)$ and $\mathbb{R}\widehat{X}_1'(\alpha)^\pm \subset LS(\Gamma)$. Therefore, if $\widetilde{X}_1'(\alpha) \neq 0$ we take $Y = [E_{-\alpha}, \widetilde{X}_1'(\alpha)]$ we clearly have $Y \neq 0$ and $Y = Y_2'(\alpha)^+$. Obviously, $\pm Y_2'(\alpha)^+ \in LS(\Gamma)$. By Lemma 4.9 we get $\mathbb{R}(LS(\Gamma) \cap \mathfrak{p}_2^+(\alpha)) \subset LS(\Gamma)$.

If ($\widetilde{X}_1'(\alpha) = 0$) and $\widehat{X}_1'(\alpha)^+ \neq 0$, we consider $Z = [\widehat{X}_1'(\alpha)^+, [E_\alpha, \widehat{X}_1'(\alpha)^+]]$, we have $\pm Z \in LS(\Gamma)$. On can verify using Proposition 3.8 that $Z = Z_0 + \widetilde{Z}_1'(\alpha) + Z''(\alpha)$. Therefore, if $\widetilde{Z}_1'(\alpha) \neq 0$ as above we get $\mathbb{R}(LS(\Gamma) \cap \mathfrak{p}_2^+(\alpha)) \subset LS(\Gamma)$. If $\widetilde{Z}_1'(\alpha) = 0$, then since $Z \notin \mathfrak{h}$, we have $Z''(\alpha) \neq 0$. We take

$W = [\widehat{X'}_1^+(\alpha), [\widehat{X'}_1^+(\alpha), Z_0 + Z''(\alpha)]]$. We clearly obtain $\pm W \in LS(\Gamma)$ with $W = W_2'(\alpha)^+ \neq 0$. Then $\pm W_2'(\alpha)^+ \in LS(\Gamma)$ and Lemma 4.9 gives the result.

To show the assertion 2, just observe that

$$\mathrm{Kill}(X_2'(\alpha)^+ + Y_2'(\alpha)^-; X_2'(\alpha)^+ + Y_2'(\alpha)^-) = 2\mathrm{Kill}(X_2'(\alpha)^+; Y_2'(\alpha)^-) < 0$$

and if $X, Y \in LS(\Gamma)$ then $Z_2'(\alpha) = X_2'(\alpha)^+ + Y_2'(\alpha)^- \in LS(\Gamma)$. We get $\pm Z_2'(\alpha) \in LS(\Gamma)$ and then Lemma 4.9 gives the result. ■

Theorem 4.11. *Let L be of type $\mathbf{C_n}$, $n \geq 2$ and G with finite-center. If S contains a subgroup $G(\alpha)$, for a short root α then S generates G if and only if Γ generates L and S contains a one-parameter group $\{\exp tE_\delta,\ t \in \mathbb{R}\}$, for a root $\delta \in \mathcal{R}_2(\alpha) \cup \widetilde{\mathcal{R}}_1(\alpha)$.*

Proof. If there exists a root $\delta \in \mathcal{R}_2(\alpha)$ so that $\mathbb{R}E_\delta \subset LS(\Gamma)$, then by Lemma 4.9 we get $\mathbb{R}(LS(\Gamma) \cap \mathfrak{p}_2^+(\alpha)) \subset LS(\Gamma)$ and Proposition 4.6 gives the result. If $\delta \in \widetilde{\mathcal{R}}_1(\alpha)$ just take $\alpha + \delta \in \mathcal{R}_2(\alpha)$ and $\mathbb{R}[E_\alpha, E_\delta] = \mathbb{R}E_{\alpha+\delta} \subset LS(\Gamma)$ to obtain the result. ■

Notice that if $\forall \delta \in \mathcal{R}_2(\alpha) \cup \widetilde{\mathcal{R}}_1(\alpha)$, $\mathbb{R}E_\delta \not\subset LS(\Gamma)$, then $LS(\Gamma) \neq L$.

4.3. Example: Symplectic Lie algebra $\mathrm{sp}(2n, \mathbb{R})$ of type $\mathbf{C_n}$.

Consider the real symplectic Lie group $G = \mathrm{SP}(2n, \mathbb{R})$ and its Lie algebra $L = \mathrm{sp}(2n, \mathbb{R})$. It is known that L is the set of matrix $X = \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}$, where B, C are (n, n) symmetric matrix (${}^tB = B$ and ${}^tC = C$) and X satisfies ${}^tXJ + JX = 0$, where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

The symplectic Lie group G is the set of matrix Z such that ${}^tZ.J.Z = J$. We start by giving the root space decomposition of L and for a short root α we present the root space decomposition of L related to α.

Denote by $E_{r,s}$ the $(2n, 2n)$ matrix $(c_{i,j})$ such that $c_{r,s} = 1$ and all the other coefficients are zero.

Take the n diagonal matrix which form a basis of a Cartan subalgebra $\mathfrak{h}$

$$E_{r,r} - E_{n+r,n+r},\ for\ \ 1 \leq r \leq n$$

and the $2n^2$ matrix

$$E_{n+r,s} + E_{n+s,r},\ \ E_{r,n+s} + E_{s,n+r}\ \ 1 \leq r \leq s \leq n$$

$$E_{r,s} - E_{n+s,n+r},\ \ 1 \leq r,\ \ s \leq n,\ \ r \neq s$$

Denote by ε_r the linear form on $\mathfrak{h}$ given by

$$\varepsilon_r(E_{s,s} - E_{n+s,n+s}) = \delta_{rs}\ \ for\ \ 1 \leq r,\ \ s \leq n.$$

One can verify that for any $H \in \mathfrak{h}$, we get:

$$[H, E_{n+r,s} + E_{n+s,r}] = -(\varepsilon_r(H) + \varepsilon_s(H))(E_{n+r,s} + E_{n+s,r}),\ \ 1 \leq r \leq s \leq n$$

$$[H, E_{r,n+s} + E_{s,n+r}] = (\varepsilon_r(H) + \varepsilon_s(H)(E_{r,n+s} + E_{s,n+r}), \quad 1 \leq r \leq s \leq n$$
$$[H, E_{r,s} - E_{n+s,n+r}] = (\varepsilon_r(H) - \varepsilon_s(H))(E_{r,s} - E_{n+s,n+r}), \quad 1 \leq r, \quad s \leq n, \quad r \neq s.$$

This implies that the $2n^2$ roots are given by

$$\pm 2\varepsilon_r \text{ for } 1 \leq r \leq n \quad \text{and} \quad \pm\varepsilon_r \pm \varepsilon_s \text{ for } 1 \leq r < s \leq n.$$

Consider $\alpha_r = \varepsilon_r - \varepsilon_{r+1}$ for $1 \leq r \leq n-1$ and $\alpha_n = 2\varepsilon_n$. We verify that

$$\varepsilon_r - \varepsilon_s = \alpha_r + \alpha_{r+1} + ... + \alpha_{s-1}, \; for \; r < s,$$

$$\varepsilon_r + \varepsilon_s = \alpha_r + \alpha_{r+1} + ... + \alpha_{s-1} + 2\alpha_s + ... + 2\alpha_{n-1} + \alpha_n \; for \; r < s \leq n-1,$$
$$\varepsilon_r + \varepsilon_n = \alpha_r + \alpha_{r+1} + ... + \alpha_{n-1} + \alpha_n \; for \; r \leq n-1,$$
$$2\varepsilon_r = 2\alpha_r + 2\alpha_{r+1} + ... + 2\alpha_{n-1} + \alpha_n \; for \; r \leq n-1.$$

Therefore, α_r, for $1 \leq r \leq n-1$ and α_n are primitive roots. The positive roots are $2\varepsilon_r$, for $1 \leq r \leq n$ and $\varepsilon_r \pm \varepsilon_s$ for $1 \leq r < s \leq n$.

• In the case $\mathbf{C_3}$, let α_1, α_2 (resp. α_3) be the primitive short (resp. long) roots. Denote by $\mathcal{R}^+$ the set of positive roots. We take $\alpha = \alpha_1$. We obtain that

$$\mathcal{R}''(\alpha) \cap \mathcal{R}^+ = \{\alpha_3\}, \; \widehat{\mathcal{R}}'_1(\alpha) \cap \mathcal{R}^+ = \{\alpha_2, \; \alpha_2 + \alpha_3, \; \alpha_1 + \alpha_2, \; \alpha_1 + \alpha_2 + \alpha_3\},$$

$$\widetilde{\mathcal{R}}'_1(\alpha) \cap \mathcal{R}^+ = \{\alpha_1 + 2\alpha_2 + \alpha_3\} \; and \; \mathcal{R}'_2(\alpha) \cap \mathcal{R}^+ = \{2\alpha_2 + \alpha_3, \; 2\alpha_1 + 2\alpha_2 + \alpha_3\}.$$

Using the above properties one can verify that:

$$E_{\alpha_1} = E_{1,2} - E_{5,4}\ , \quad E_{\alpha_2} = E_{2,3} - E_{6,5}\ , \quad E_{\alpha_1+\alpha_2} = E_{1,3} - E_{6,4}\ , \quad E_{\alpha_3} = 2E_{3,6}\ ,$$

$$E_{\alpha_2+\alpha_3} = E_{2,6} + E_{3,5}\ , \quad E_{2\alpha_2+\alpha_3} = 2E_{2,5}\ , \quad E_{\alpha_1+2\alpha_2+\alpha_3} = E_{1,5} + E_{2,4}\ ,$$
$$E_{2\alpha_1+2\alpha_2+\alpha_3} = 2E_{1,4}\ , \quad E_{\alpha_1+\alpha_2+\alpha_3} = E_{1,6} + E_{3,4}\ .$$

Clearly, for a root δ if $E_\delta = E_{i,j} + aE_{k,l}$, with $a = \pm 1$ then $E_{-\delta} = E_{j,i} + aE_{l,k}$ and $H_{\alpha_i} = [E_{\alpha_i}, E_{-\alpha_i}]$, for $1 \leq i \leq 3$.

The root decomposition of $X \in L$ is given by: $X = X_0 + \sum_{\delta \in \mathcal{R}} x_\delta E_\delta$, and $X_0 = \sum_{1 \leq i \leq 3} x_i H_{\alpha_i} = \sum_{1 \leq i \leq 3} h_i(E_{i,i} - E_{3+i,3+i})$.

$$X = X_0 + X''(\alpha) + \widetilde{X}'_1(\alpha) + \widehat{X}'_1(\alpha)^+ + \widehat{X}'_1(\alpha)^- + X'_2(\alpha)^+ + X'_2(\alpha)^-.$$

Write $X = \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}$, where

$$A = \begin{pmatrix} h_1 & x_{\alpha_1} & x_{\alpha_1+\alpha_2} \\ x_{-\alpha_1} & h_2 & x_{\alpha_2} \\ x_{-\alpha_1-\alpha_2} & x_{-\alpha_2} & h_3 \end{pmatrix}, \; B = \begin{pmatrix} 2x_{2\alpha_1+2\alpha_2+\alpha_3} & x_{\alpha_1+2\alpha_2+\alpha_3} & x_{\alpha_1+\alpha_2+\alpha_3} \\ x_{\alpha_1+2\alpha_2+\alpha_3} & 2x_{2\alpha_2+\alpha_3} & x_{\alpha_2+\alpha_3} \\ x_{\alpha_1+\alpha_2+\alpha_3} & x_{\alpha_2+\alpha_3} & 2x_{\alpha_3} \end{pmatrix}$$

and $C = \begin{pmatrix} 2x_{-2\alpha_1-2\alpha_2-\alpha_3} & x_{-\alpha_1-2\alpha_2-\alpha_3} & x_{-\alpha_1-\alpha_2-\alpha_3} \\ x_{-\alpha_1-2\alpha_2-\alpha_3} & 2x_{-2\alpha_2-\alpha_3} & x_{-\alpha_2-\alpha_3} \\ x_{-\alpha_1-\alpha_2-\alpha_3} & x_{-\alpha_2-\alpha_3} & 2x_{-\alpha_3} \end{pmatrix}$.

Observe that the matrix $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ contains the following components of X: $X_2'(\alpha)^+$, $X_2'(\alpha)^-$, $\widetilde{X}_1(\alpha)$, $X''(\alpha)$ and the remaining components which are part of $\widehat{X}_1'(\alpha)$. The matrix $\begin{pmatrix} A & 0 \\ O & -{}^tA \end{pmatrix}$ contains the components $X_{\pm\alpha_1}$, and the other part of $\widehat{X}_1'(\alpha)$ which is $X_{\pm\alpha_2}$, $X_{\pm(\alpha_1+\alpha_2)}$, (with $\alpha = \alpha_1$, α_2 and $\alpha_1+\alpha_2$ are short). Notice that $\begin{pmatrix} A & 0 \\ O & -{}^tA \end{pmatrix}$ belongs to the subalgebra of L generated by $\mathfrak{h} \oplus \mathbb{R}E_{\pm\alpha_1} \oplus \mathbb{R}E_{\pm\alpha_2}$ which is not equal to L.

Therefore, if for any $X \in \Gamma$ we have $B = C = 0$, then $\mathrm{Lie}(\Gamma) \neq L$. We deduce that, if $\mathrm{Lie}(\Gamma) = L$ then there exists an element $Z \in \Gamma$ such that the associated matrix $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ is nonzero and then if Z satisfies the conditions of Proposition 4.10 we get $S = G$. The following example illustrates this.

Consider $\Gamma = \{Z + u_1E_{\alpha_1} + u_2E_{-\alpha_1},\ Z \in L,\ u_1, u_2 \in \mathbb{R}\}$, where $Z = E_{\alpha_2} + E_{-\alpha_2} + E_{\alpha_1+2\alpha_2+\alpha_3} + E_{-2\alpha_2-\alpha_3}$. We have $\mathbb{R}E_{\alpha_1} \oplus \mathbb{R}E_{-\alpha_1} \subset LS(\Gamma)$, because for $u_2 = 0$ we have $\lim\limits_{v\to\pm\infty} \frac{1}{|u_1|}(Z + u_1E_{\alpha_1}) = \pm E_{\alpha_1} \in LS(\Gamma)$. Similarly, $\pm E_{-\alpha_1} \in LS(\Gamma)$. By Lemma 4.4 and since for any $X \in \mathrm{Lie}(\Gamma)$ we have $\pm X \in \mathrm{Lie}(\Gamma)$, we get that $Z_2'(\alpha)^-$, $\widetilde{Z}_1'(\alpha)$ and $\widehat{Z}_1'(\alpha)^\pm$ belong to $\mathrm{Lie}(\Gamma)$. Then, $E_{-2\alpha_2-\alpha_3}$, $E_{\alpha_1+2\alpha_2+\alpha_3}$ and $E_{\pm(\alpha_1+\alpha_2+\alpha_3)}$ belong to $\mathrm{Lie}(\Gamma)$. Therefore, $\mathrm{Lie}(\Gamma) = L$. Since $Z \in LS(\Gamma)$, $Z_2'(\alpha_1)^+ = 0$ and $\widetilde{Z}_1'(\alpha_1) \neq 0$, by Proposition 4.10 and Proposition 4.6, $S = G$.

4.4. Examples: Short roots in $\mathbf{B_n}$, $n \geq 2$, $\mathbf{C_n}$, $n \geq 3$ or $\mathbf{F_4}$.

We know (see [19] for more details) that for the real symplectic Lie group, of type $\mathbf{C_n}$, $n \geq 2$, there exists a semigroup S which is a compression semigroup, with nonempty interior and S contains a subgroup $G(\alpha)$ for a short root α, but $S \neq G$. Precisely, S contains $\{\exp tQ : t \geq 0\}$, where Q is a quadratic form given in the standard basis of $\mathbb{R}^{2n}$ by the matrix, $Q = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$, where $B = C$ is the (n,n) identity matrix and S contains $\{\exp tX : t \geq 0\}$, for any matrix X of the form $X = \begin{pmatrix} A & 0 \\ O & -{}^tA \end{pmatrix}$. Here we use this result in the cases $n = 3$ and $n = 2$.

• In the case $\mathbf{C_3}$, consider $\Gamma = \{Z + u_1E_{\alpha_1} + u_2E_{-\alpha_1},\ u_1, u_2 \in \mathbb{R}\}$, where $Z = E_{\alpha_2} + E_{-\alpha_2} + 2Q$ and $2Q = E_{2\alpha_2+\alpha_3} + E_{-2\alpha_2-\alpha_3} + E_{\alpha_3} + E_{-\alpha_3} + E_{2\alpha_1+2\alpha_2+\alpha_3} + E_{-2\alpha_1-2\alpha_2-\alpha_3} = Z_2'(\alpha_1) + Z''(\alpha_1)$ and $\widehat{Z}_1'(\alpha_1) = E_{\alpha_2} + E_{-\alpha_2}$. The semigroup S_Γ generated by Γ contains $G(\alpha_1)$ and $S_\Gamma \subset S \neq G$. We show that $\mathrm{Lie}(\Gamma) = L$ and then S_Γ has nonempty interior.

Indeed, $E_{\pm\alpha_1} \in \mathrm{Lie}(\Gamma)$ and as above by Lemma 4.4 we obtain that $\mathrm{Lie}(\Gamma)$ contains E_{α_2}, $E_{-\alpha_2}$, $E_{\alpha_3} + E_{-\alpha_3}$. Since $[E_{-\alpha_2}, [E_{\alpha_2}, E_{\alpha_3} + E_{-\alpha_3}]] = aE_{\alpha_3}$ with $a \neq 0$ we get that $E_{\alpha_3} \in \mathrm{Lie}(\Gamma)$. Similarly, $E_{-\alpha_3} \in \mathrm{Lie}(\Gamma)$.

• In the case $\mathbf{B_2} = \mathbf{C_2}$, let α_1 and α_2 be the primitive roots where α_1 is short and α_2 is long. The positive short roots are α_1 and $\alpha_1+\alpha_2$ and the positive long roots are α_2 and $2\alpha_1+\alpha_2$. Consider the Cartan decomposition of L related

to a short root α, $L = \mathfrak{k}(\alpha) \oplus \mathfrak{p}(\alpha)$. Using Lemma 4.8, there exists a unique root β such that $\mathcal{R}'_2(\alpha) = \{\pm\beta,\ \pm(2\alpha+\beta)\}$ and $\widetilde{\mathcal{R}}'_1(\alpha) = \{\pm(\alpha+\beta)\}$. Also, $\mathcal{R}''(\alpha) = \widehat{\mathcal{R}}'_1(\alpha) = \emptyset$. We have $\mathfrak{p}(\alpha) = \mathfrak{p}_2(\alpha) \oplus \widetilde{\mathfrak{p}}_1(\alpha)$ and $\mathfrak{k}(\alpha) = \mathrm{sl}_\alpha(2,\mathbb{R}) \oplus \mathbb{R}H_\beta$.

Clearly, if S contains $G(\alpha_1)$ and $G(\alpha_1+\alpha_2)$ then $S = G$. Therefore, if $S \neq G$ then it contains exactly one subgroup $G(\alpha)$ with $\alpha \in \{\alpha_1, \alpha_1+\alpha_2\}$.

Notice that if $\alpha = \alpha_1$ then $\beta = \alpha_2$ and if $\alpha = \alpha_1+\alpha_2$, then $\beta = -\alpha_2$. In both cases $\alpha = \alpha_1$ or $\alpha = \alpha_1+\alpha_2$, we have $Q = \frac{1}{2}(E_\beta + E_{-2\alpha-\beta} + E_{-\beta} + E_{2\alpha+\beta})$.

Consider $\Gamma = \{H_\beta + E_\beta + E_{-2\alpha-\beta} + u_1E_\alpha + u_2E_{-\alpha},\ u_1, u_2 \in \mathbb{R}\}$ and denote by S_Γ the semigroup generated by Γ. Clearly, $S_\Gamma \subset S \neq G$. Also, we have $\mathrm{Lie}(\Gamma) = L$. Because by Lemma 4.4 we obtain $H_\beta,\ E_\beta + E_{-2\alpha-\beta} \in \mathrm{Lie}(\Gamma)$ and then $E_{2\alpha+\beta} + E_{-\beta} \in \mathrm{Lie}(\Gamma)$. We have $[H_\beta, E_\beta + E_{-2\alpha-\beta}] = 2E_\beta$ and $[H_\beta, E_{-\beta} + E_{2\alpha+\beta}] = -2E_{-\beta}$ (because $(2\alpha+\beta)(H_\beta) = 0$ as $\alpha(H_\beta) = -1$).

Claim For any $X \in LS(\Gamma)$, we have $\mathrm{Kill}(X'_2(\alpha)^+; X'_2(\alpha)^-) \geq 0$.

Indeed, if there exists $X \in LS(\Gamma)$ such that $\mathrm{Kill}(X'_2(\alpha)^+; X'_2(\alpha)^-) < 0$, then by Lemma 4.9 and Proposition 4.6 we get $S_\Gamma = G$ and thus $S = G$, which is absurd.

Using this claim and the fact that the Dynkin diagram $\mathbf{B_3}$ contains $\mathbf{B_2}$ we deduce that in $\mathbf{B_3}$ there exists a semigroup S_Σ generated by a set Σ, with nonempty interior and $S_\Sigma \neq G$.

• In the case $\mathbf{B_3}$, let α_3, α_2 be the primitive long roots and α_1 the primitive short root. Take $\alpha = \alpha_1$. We have

$$\widehat{\mathcal{R}}'_1(\alpha) = \emptyset,\ \ \mathcal{R}''(\alpha) \cap \mathcal{R}^+ = \{\alpha_3,\ 2\alpha_1 + 2\alpha_2 + \alpha_3\},$$

$$\widetilde{\mathcal{R}}'_1(\alpha) \cap \mathcal{R}^+ = \{\alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3\}\ \ \textit{and}$$

$$\mathcal{R}'_2(\alpha) \cap \mathcal{R}^+ = \{\alpha_2,\ \alpha_2+\alpha_3,\ 2\alpha_1+\alpha_2,\ 2\alpha_1+\alpha_2+\alpha_3\}.$$

Let $\Sigma = \{Z + u_1E_{\alpha_1} + u_2E_{-\alpha_1}, u_1, u_2 \in \mathbb{R}\}$, where $Z = E_{\alpha_2} + E_{-2\alpha_1-\alpha_2} + E_{2\alpha_1+\alpha_2} + E_{\alpha_3} + E_{-\alpha_3}$. The semigroup S_Σ generated by Σ contains $G(\alpha_1)$, has nonempty interior and $S_\Sigma \neq G$. Indeed, we have $\mathrm{Lie}(\Gamma) = L$ because by using Lemma 4.4 we obtain that $\mathrm{Lie}(\Gamma)$ contains $Z'_2(\alpha_1)^+ = E_{\alpha_2} + E_{-2\alpha_1-\alpha_2}$, $Z'_2(\alpha_1)^- = E_{2\alpha_1+\alpha_2}$, $Z''(\alpha_1) = E_{\alpha_3} + E_{-\alpha_3}$ and $[E_{\alpha_1}[E_{\alpha_1}, Z'_2(\alpha_1)^+]] = 2(E_{2\alpha_1+\alpha_2} + E_{-\alpha_2})$. Then $E_{-\alpha_2} \in \mathrm{Lie}(\Gamma)$. We deduce that E_{α_2} and then $E_{\pm\alpha_3}$ belong to $\mathrm{Lie}(\Gamma)$.

We show that $\forall X \in LS(\Sigma)$, $\mathrm{Kill}(X'_2(\alpha); X'_2(\alpha)) \geq 0$ and then $S_\Sigma \neq G$. To do this, assume that there exists $X \in LS(\Sigma)$ so that $\mathrm{Kill}(X'_2(\alpha); X'_2(\alpha)) < 0$. By Lemma 4.8, let $\beta_1 = \alpha_2$ and $\beta_2 = \alpha_2 + \alpha_3$ and write $X'_2(\alpha)^+ = X_{\beta_1} + X_{-2\alpha_1-\beta_1} + X_{\beta_2} + X_{-2\alpha_1-\beta_2}$ and $X'_2(\alpha)^- = X_{-\beta_1} + X_{2\alpha_1+\beta_1} + X_{-\beta_2} + X_{2\alpha_1+\beta_2}$. We have $\mathrm{Kill}(X'_2(\alpha); X'_2(\alpha)) = 2\mathrm{Kill}(X'_2(\alpha_1)^+; X'_2(\alpha_1)^-) = 2\mathrm{Kill}(X_{\beta_1} + X_{-2\alpha_1-\beta_1}; X_{-\beta_1} + X_{2\alpha_1+\beta_1}) + 2\mathrm{Kill}(X_{\beta_2} + X_{-2\alpha_1-\beta_2}; X_{-\beta_2} + X_{2\alpha_1+\beta_2}) < 0$. But by using the claim we have that $\mathrm{Kill}(X_{\beta_1} + X_{-2\alpha_1-\beta_1}; X_{-\beta_1} + X_{2\alpha_1+\beta_1}) \geq 0$. Then

$$\mathrm{Kill}(X_{\beta_2} + X_{-2\alpha_1-\beta_2}; X_{-\beta_2} + X_{2\alpha_1+\beta_2}) < 0.$$

Therefore, if G has finite-center we get that $\pm(X_{\beta_2} + X_{-2\alpha_1-\beta_2} + X_{-\beta_2} + X_{+2\alpha_1+\beta_2}) \in LS(\Sigma)$. By Proposition 4.2 we get $\pm(X_{\beta_2} + X_{-2\alpha_1-\beta_2})$ and $\pm(X_{-\beta_2} + X_{2\alpha_1+\beta_2})$ belong to $LS(\Sigma)$. As in the proof of Lemma 4.9 we obtain that $\pm E_{\beta_2}, \pm E_{-\beta_2} \in LS(\Sigma)$. Then S_Σ contains $G(\beta_2)$, and β_2 is long. By Theorem 4.7, we get $S_\Sigma = G$

and thus $S = G$ which is absurd.

We can generalize this example to $\mathbf{B_n}$, $n \geq 4$, by taking $\alpha = \alpha_1$ the primitive short root and S_Σ with $Z = E_{\alpha_2} + E_{-2\alpha_1-\alpha_2} + E_{2\alpha_1+\alpha_2} + Z''(\alpha_1) \in \Sigma$, where $Z''(\alpha_1) = \sum_{3\leq i\leq n}(E_{\alpha_i} + E_{-\alpha_i})$.

Observe that if we take of the long (resp. short) primitive roots α_i, for $i = n, ..., 3$ in $\mathbf{B_n}$ (resp. $\mathbf{C_n}$), $n \geq 3$ we obtain $\mathbf{B_2}$. Also, if we take of the short (resp. long) primitive root at the end of the Dynkin diagram $\mathbf{F_4}$ we obtain $\mathbf{B_3}$ (resp. $\mathbf{C_3}$). Then (as in $\mathbf{B_2} = \mathbf{C_2}$), there exists in those cases semigroups S with nonempty interior that contain $G(\alpha)$, for a short root α, but $S \neq G$. Those cases will be studied in a future work.

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